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L. K. PATEL

P. C. VAIDYA

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A generalized plane wave metric

by

L. K. PATEL and P. C. VAIDYA

Department of Mathematics,
Gujarat University, Ahmedabad-9 (India).

1. INTRODUCTION

The theory of plane gravitational waves in general relativity has been discussed by many investigators. H. Takeno [1] has discussed the mathematical theory of plane gravitational waves in detail. A Peres [2] has studied the plane wave like line-element

$$(1.1) \quad ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x, y, u)(dt - dz)^2$$

where $u = t - z$ and f is a function of x, y and u . The line-element (1.1) can also be expressed as

$$(1.2) \quad ds^2 = -dx^2 - dy^2 + 2dudz + (1 - 2f)du^2$$

Vaidya and Pandya [3] have studied the metric (1.2) in connection with gravitational and electromagnetic radiation. In fact the solution of Peres is a particular case of a more general solution obtained by Pandya and Vaidya [4].

In Peres' solution, all components of the metric tensor are not functions of u . The object of the present investigation is to generalize Peres' metric in such a way that all the components of the metric tensor g_{ik} become functions of u . Of course, some of these components do depend upon x and y also.

2. GRAVITATIONAL WAVES

In Minkowskian space, consider an arbitrary smooth world line L that is every where time-like. Let u be the parameter along the world line. Let λ^i be the unit tangent vector at any point of L . Let A^i, B^i, C^i

be the three mutually orthogonal space-like unit vectors lying in the 3-space orthogonal to λ^i at the point under consideration. Thus we have the following relations:

$$(2.1) \quad \lambda^i \lambda_i = 1, \quad A^i A_i = B^i B_i = C^i C_i = -1$$

and

$$(2.2) \quad \lambda^i A_i = \lambda^i B_i = \lambda^i C_i = 0,$$

Here it should be noted that the raising and lowering of vector indices of λ^i , A^i , B^i and C^i is carried out with respect to the Minkowskian metric

$$n_{ik} = \text{diag} (-1, -1, -1, 1).$$

Let us define the new co-ordinates x , y , z and t in terms of the co-ordinates x^i by the following relations.

$$(2.3) \quad x = x^i A_i, \quad y = x^i B_i, \quad z = x^i C_i, \quad t = x^i \lambda_i$$

Then clearly

$$(2.4) \quad x_{,k} = A_k, \quad y_{,k} = B_k, \quad z_{,k} = C_k, \quad t_{,k} = \lambda_k$$

Here and in what follows a comma followed by a lower index will imply partial differentiation with respect to that index. Let

$$(2.5) \quad Z_i = \lambda_i - C_i \quad \text{and} \quad p_i = A_i - B_i$$

It follows from (2.5) that

$$(2.6) \quad Z_i Z^i = 0, \quad p_i p^i = -2.$$

Thus Z^i is a null vector with respect to the Minkowskian metric and p^i is a space-like vector. In this paper we shall confine ourselves to the case in which λ^i , A^i , B^i and C^i are all constant vectors.

Consider a Riemannian 4-space whose metric is given by

$$(2.7) \quad ds^2 = g_{ik} dx^i dx^k$$

where the metric tensor g_{ik} is expressed by the following equation.

$$(2.8) \quad g_{ik} = \eta_{ik} + H p_i p_k + S Z_i Z_k$$

Here H is a function of $u = t - z$ and S is a function of x , y and u . The determinant g of the metric tensor g_{ik} can be easily computed. It is given by

$$(2.9) \quad g = |g_{ik}| = -(1 - 2H).$$

As g is negative $1 - 2H$ should be positive. The vectors p_i and Z_i are orthogonal to each other.

$$(2.10) \quad Z_i p^i = 0$$

The contravariant components of the metric tensor g_{ik} are given by

$$(2.11) \quad g^{ik} = \eta^{ik} - \frac{H}{1 - 2H} p^i p^k - S Z^i Z^k$$

It follows from (2.9), (2.10) and (2.11) that

$$(2.12) \quad g^{ik} Z_i = \eta^{ik} Z_i$$

and

$$(2.13) \quad g^{ik} Z_i Z_k = \eta^{ik} Z_i Z_k = Z^i Z_i = 0$$

Thus raising and lowering of the vector indices of Z_i can be carried out with the Riemannian or Minkowskian metric. Also the null character of the congruence Z_i with respect to the Minkowskian metric implies its null character with respect to the Riemannian metric.

From (2.8), (2.10) and (2.11) we also have

$$(2.14) \quad g^{ik} p_i = \eta^{ik} p_i + \frac{2H p^k}{1 - 2H} = \frac{p^k}{1 - 2H}$$

We shall continue to use the Minkowskian metric η_{ik} for raising and lowering of indices and any dependence on g_{ik} will be explicitly written out as in (2.14). The result (2.6) will be frequently used without mention.

The 3-index symbols for the metric (2.8) are given by

$$(2.15) \quad \Gamma_{i k}^n = \frac{1}{2} \left[\frac{2p^n H_{, (i} p_{k)} }{1 - 2H} + 2Z^n S_{, (i} Z_{k)} - n^l H_{, l} p_i p_k - \eta^{nl} S_{, l} Z_i Z_k + \frac{H(S_y - S_x)}{1 - 2H} p^n Z_i Z_k \right]$$

Throughout this paper the following conventions are used:

Indices range and sum over 1, 2, 3, 4; a semicolon indicates covariant differentiation; round index brackets indicate symmetrization over the indices enclosed; square brackets indicate antisymmetrization; and the lower suffixes attached to functional symbols denote the derivatives of the function with respect to the corresponding variable, e. g.

$$S_y = \frac{\partial S}{\partial y}, \quad S_{xy} = \frac{\partial^2 S}{\partial y \partial x}, \quad H_{uu} = \frac{\partial^2 H}{\partial u^2}, \text{ etc.}$$

It is clear from (2.15) that

$$(2.16) \quad \Gamma_{ik}^n Z_n = 0$$

The result (2.16) imply that the null congruence Z_i is geodetic.

In our case the expression for the Ricci tensor reduces to

$$(2.17) \quad R_{ik} = \frac{1}{1-2H} \left[-H_{uu} - \frac{H_u^2}{1-2H} - \frac{1-H}{2} (S_{xx} + S_{yy}) + HS_{xy} \right] Z_i Z_k$$

The Riemann curvature tensor R_{hijk} for the metric (2.8) is given by

$$(2.18) \quad \begin{aligned} R_{hijk} = 2 & \left[H_{uu} + \frac{H_u^2}{1-2H} \right] p_{[i} Z_{j]} p_{[h} Z_{k]} \\ & + 2S_{xx} A_{[j} Z_{i]} A_{[k} Z_{h]} + 2S_{yy} B_{[j} Z_{i]} B_{[k} Z_{h]} \\ & + 2S_{xy} \{ A_{[i} Z_{j]} B_{[k} Z_{h]} + B_{[i} Z_{j]} A_{[k} Z_{h]} \} \end{aligned}$$

For gravitational waves we have

$$(2.19) \quad R_{ik} = 0.$$

The results (2.17) and (2.19) imply that

$$(2.20) \quad S_{xx} + S_{yy} - \frac{2H}{1-H} S_{xy} = -\frac{2}{1-H} \left(H_{uu} + \frac{H_u^2}{1-2H} \right)$$

For gravitational waves, S and H have to satisfy the equation (2.20). The choice of any one of S and H is at our disposal.

If $\dot{S} = 0$, then from (2.18), (2.19) and (2.20) we obtain $R_{hijk} = 0$ and the space-time becomes flat.

If $H = 0$, then $R_{ik} = 0$ implies $S_{xx} + S_{yy} = 0$ and we get the space-time of Peres.

Thus it is clear that if we choose H in such a way that $H \neq 0$, $1-2H > 0$ and $H_{uu} + (H_u^2/1-2H) \neq 0$, then we get the gravitational field which is different from that discussed by Peres.

From (2.18) we have:

A necessary and sufficient condition that a space-time given by (2.8) be Minkowskian is

$$(2.21) \quad S_{ab} = 0 \quad \text{and} \quad H_{uu} + \frac{H_u^2}{1-2H} = 0, \quad a, b = x, y$$

Thus, when S is a linear function of x and y whose coefficients are functions of u and H satisfies $H_{uu} + \frac{H_u^2}{1-2H} = 0$, then the space-time discussed here becomes flat.

3. CO-EXISTANCE OF ELECTROMAGNETIC WAVES

In this section we shall show that the solution obtained in the previous section can be generalized to the case in which the electromagnetic waves co-exist with the gravitational waves. The field equations of electromagnetic field in general relativity are

$$(3.1) \quad R_{ik} = -8\pi E_{ik}$$

and the maxwell equations are

$$(3.2) \quad \begin{aligned} F_{ik,n} + F_{kn,i} + F_{ni,k} &= 0 \\ F^{ik}_{;k} &= 0 \end{aligned}$$

Here F_{ik} is the antisymmetric tensor describing electromagnetic field and E_{ik} is the electromagnetic energy tensor defined by

$$(3.3) \quad E_{ik} = \frac{1}{4} g_{ik} F_{lm} F_{ab} g^{la} g^{mb} - F_{il} F_{km} g^{lm}$$

If ϕ_i is the 4-potential of the electromagnetic field then

$$(3.4) \quad F_{ik} = \phi_{i,k} - \phi_{k,i}$$

Let us choose the 4-potential ϕ_i of the electromagnetic field as

$$(3.5) \quad \phi_i = D(x, y, u) Z_i$$

Looking to the nature of our problem this choice of ϕ_i seems appropriate. Now,

$$(3.6) \quad F_{ik} = D_{,k} Z_i - D_{,i} Z_k$$

Clearly

$$(3.7) \quad g^{im} h^{kn} F_{mn} F_{ik} = 0$$

Thus the electromagnetic field is null with respect to Riemannian metric.

(2.8). The electromagnetic energy tensor E_{ik} is given by

$$(3.8) \quad E_{ik} = \left[D_x^2 + D_y^2 + \frac{H(D_y - D_x)^2}{1 - 2H} \right] Z_i Z_k$$

The results (2.17), (3.1) and (3.8) imply that

$$(3.9) \quad 8\pi \left[D_x^2 + D_y^2 + \frac{H(D_y - D_x)^2}{1 - 2H} \right] \\ = \frac{1}{1 - 2H} \left[H_{uu} + \frac{H_u^2}{1 - 2H} + \frac{1 - H}{2} (S_{xx} + S_{yy}) - HS_{xy} \right]$$

The Maxwell equations (3.2) are equivalent to

$$(3.10) \quad D_{xx} + D_{yy} - \frac{2H}{1 - H} D_{xy} = 0$$

Hence, for electromagnetic waves D and S have to satisfy equations (3.9) and (3.10) and H remains arbitrary.

However if we consider S as a function of D , equations (3.9) and (3.10) imply

$$(3.11) \quad \frac{16\pi - \frac{d^2S}{dD^2}}{2} [(1 - H)(D_x^2 + D_y^2) - 2HD_xD_y] = H_{uu} + \frac{H_u^2}{1 - 2H}.$$

Let us consider a particular case in which

$$(3.12) \quad \frac{d^2S}{dD^2} = 16\pi \quad \text{i.e.} \quad S = 8\pi D^2 + \alpha D + \beta$$

where α and β are constants.

Equation (3.11) reduces to

$$(3.13) \quad H_{uu} + \frac{H_u^2}{1 - 2H} = 0 \quad \text{i.e.} \quad H = \frac{1 - (au + b)^2}{2}$$

where a and b are constants.

For electromagnetic waves in this case, H and D have to satisfy (3.10) and (3.13). When $H = 0$ (i.e. $a = 0$, $b = 1$), the electromagnetic field discussed above reduces to the electromagnetic field studied by Takeno [5]. It may however be noted that Takeno [5] has not taken S as a function of D .

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