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Group representations by automorphisms of a proposition system (*)

by

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SUMMARY. — A representation of a group G is defined to be a homomorphism of G into the group of automorphisms of the proposition system of the « quantum logic » approach to axiomatic quantum mechanics. After a systematic formulation of the concepts of homomorphism between proposition systems and of direct union thereof, the decomposition theory of group representations is dealt with: results analogous to Schur's lemma and theorem are derived and the uniqueness of a decomposition into irreducibles is shown. A physical interpretation of the results is discussed briefly.

RÉSUMÉ. — On appelle représentation d'un groupe G un homomorphisme de G dans le groupe des automorphismes du système des propositions relatif à l'approche « logique » à l'axiomatique de la mécanique quantique. Après avoir donné une formulation systématique des concepts de homomorphisme entre systèmes de propositions et de leur union directe, on affronte la théorie de la décomposition des représentations des groupes : on déduit des résultats analogues aux lemme et théorème de Schur et on montre l'unicité d'une décomposition en représentations irréductibles. On discute brièvement une interprétation physique des résultats.

INTRODUCTION

From the very beginning [*1*] of the philosophy that the « logical » structure of the system of propositions of a quantum system determines completely

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the character of quantum laws as well as the mathematical formalism employed in quantum theory, one of the most important problems in the axiomatic approach to quantum mechanics has been probably to find reasonable axioms for the structure of the system of propositions so that it may be represented by a lattice of closed subspaces (or projections, which is the same) of some Hilbert space [*2*]. This problem seems to be solved today as a result of the work of Piron [*3*]; in fact, assuming the proposition system to be a weakly modular, orthocomplemented and complete lattice (what is also called croc) equipped with atomicity and covering law, Piron is able to show that propositions may be represented by linear subspaces of a vector space over a field, if the proposition system is irreducible. If we want the field to contain the real numbers as a subfield, we are left to choose between the reals [*4*], complexes and quaternions [*5*]. If then we take into account a result of Amemiya and Araki [*6*], we get that a proposition system is isomorphic to the lattice of all projections from a family of Hilbert spaces $\{ \mathscr{H}_\zeta \}$ ($\zeta \in \mathrm{Z}$) [*7*]; two different Hilbert spaces of the family have not to be over the same field and nothing can be said about the index set Z.

Dropping the request of atomicity and of covering law, the natural result to expect is that the proposition system is then isomorphic to the lattice of all projections from a family $\{ \mathrm{A}(\zeta) \}$ ($\zeta \in \mathrm{Z}$) of von Neumann algebras in a family $\{ \mathscr{H}_\zeta \}$ of Hilbert spaces. We shall in fact define a proposition system to be a croc not necessarily atomic, to gain a little in generality.

If the system of propositions has to embody all the properties of a physical system, it seems very natural to call symmetry of a physical system an automorphism of its system of propositions [*8*], and to call representation of a symmetry group G a group-homomorphism of G into the group of all automorphisms of the system of propositions. This to be the right definition of symmetry is argued by a theorem of Emch and Piron (see ref. [*8*]) and by Theorems 7.27 and 7.29 of Varadarajan's book quoted in ref. [*2*] (these two theorems show in fact that, if a proposition system $\mathscr{L}$ is the lattice of all projections from a Hilbert space $\mathscr{H}$, then automorphisms of $\mathscr{L}$ and automorphisms of the ray Hilbert space underlying $\mathscr{H}$ coincide; Theorem 7.29 is in fact Wigner's theorem [*9*]).

A motivation to study representations of groups yet in the proposition system, without overpassing to Hilbert space, is furnished by two works of Mielnik [*10*], who is able to show that, if some « geometric properties » of a proposition system are taken into account, the physical reality could be too complex in order to fit in any Hilbert space. Another motivation to study group-homomorphisms of a group into the group of automorphisms of a proposition system is that this kind of representation is a part of a definition of symmetry group more involved than that given above [*11*]. The aim of the present paper is in fact to study representations of groups by automorphisms of proposition systems; to be definite, we shall study decomposition theory.

To perform this program, a theory of homomorphisms between crocs has to be settled (Sections 1 and 2); results like Schur's lemma and theorem for linear representations are then derived (Section 3) and, in the completely decomposable case, the uniqueness of a decomposition into irreducibles is shown (Section 4). Differences between Hilbert unitary representations and croc-representations of groups are easily understood if one takes into account that Hilbert space is in a sense « void », whilst a system of propositions represents a physical system (for instance superselection rules are embodied in it). These differences will be briefly discussed (Section 5).

An appendix is added, in which few well known facts and results are collected in order to introduce notations and basic definitions.

1. HOMOMORPHISMS AND SUBCROCS

Here and in the sequel, $\mathscr{L}$ (with or without any sign attached to it) is a croc (that is a proposition system). When we speak about lattices and sublattices, we mean that they are complete.

DEF. 1.1. — A homomorphism of $\mathscr{L}$ into $\mathscr{L}'$ is a mapping $\varphi : \mathscr{L} \to \mathscr{L}'$ such that for any family of elements of $\mathscr{L}$, $\{ x_i \}$ $(i \in \mathscr{I})$ with index set $\mathscr{I}$, and for each element x of $\mathscr{L}$:

a) $$\varphi(\bigcup_i x_i) = \bigcup_i{}' \varphi(x_i),$$

b) $$\varphi(\bigcap_i x_i) = \bigcap_i{}' \varphi(x_i),$$

c) $$\varphi(cx) = c'\varphi(x) \cap' \varphi(\mathrm{I}).$$

Being φ a homomorphism, it is easy to show that $\varphi(\Phi) = \Phi'$ and that

$$x \perp y \Rightarrow \varphi(x) \perp' \varphi(y),$$
$$x < y \Rightarrow \varphi(x) <' \varphi(y),$$

if φ is injective then

$$\varphi(x) <' \varphi(y) \Rightarrow x < y,$$

if φ is bijective then

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

If a homomorphism φ is bijective, it is called an isomorphism; the set of homomorphisms (isomorphisms) of $\mathscr{L}$ into (onto) $\mathscr{L}'$ is denoted by Hom $(\mathscr{L}, \mathscr{L}')$ (Is $(\mathscr{L}, \mathscr{L}')$).

When $\mathscr{L}' = \mathscr{L}$ it is usual to call endomorphism a homomorphism and automorphism an isomorphism. The set Aut $(\mathscr{L})$ of all the automorphisms of $\mathscr{L}$ is a group if the product $\alpha \circ \beta : (\alpha \circ \beta)(x) = \alpha(\beta(x))$, $\forall x \in \mathscr{L}$, for $\alpha, \beta \in \mathrm{Aut}\,(\mathscr{L})$, is assumed to be the composition law.

When dealing with a homomorphism φ, of paramount importance are the two subsets $\mathrm{Ker}\,\varphi \equiv \{ x \in \mathscr{L} ; \varphi(x) = \Phi' \} \subset \mathscr{L}$ and $\mathrm{Im}\,\varphi \equiv \varphi(\mathscr{L}) \subset \mathscr{L}'$.

A very important endomorphism of a croc $\mathscr{L}$ must be noticed: given $z \in \mathscr{C}(\mathscr{L})$, define $\mathrm{P}_z : \mathscr{L} \to \mathscr{L}$, $\mathrm{P}_z(x) = z \cap x$. Taking into account (A.6) and (A.7) it is easy to show that P_z is an idempotent endomorphism; moreover $\mathrm{Im}\,\mathrm{P}_z = [\Phi, z]$ and from (A.5) it follows that $\mathrm{Ker}\,\mathrm{P}_z = [\Phi, cz]$. The endomorphism P_z is called projection related to z, and we define $\mathscr{P}(\mathscr{L}) \equiv \{ \mathrm{P}_z ; z \in \mathscr{C}(\mathscr{L}) \}$. The null-endomorphism and the identity endomorphism are the projections related to Φ and I respectively. They will be called trivial projections.

DEF. 1.2. — We say that the croc $\mathscr{L}'$ is a subcroc of $\mathscr{L}$ and then we write $\mathscr{L}' \lhd \mathscr{L}$, when as a set $\mathscr{L}' \subset \mathscr{L}$ and the canonical injection

$$i : \mathscr{L}' \to \mathscr{L}, \; i(x) = x$$

is a homomorphism from $\mathscr{L}'$ into $\mathscr{L}$.

We notice that

$$\varphi \in \mathrm{Hom}\,(\mathscr{L}_1, \mathscr{L}_2), \qquad \psi \in \mathrm{Hom}\,(\mathscr{L}_2, \mathscr{L}_3) \Rightarrow \psi \circ \varphi \in \mathrm{Hom}\,(\mathscr{L}_1, \mathscr{L}_3),$$

where

$$\psi \circ \varphi : (\psi \circ \varphi)(x) = \psi(\varphi(x)), \qquad \forall x \in \mathscr{L}_1 ;$$

then it holds that

$$\varphi \in \mathrm{Hom}\,(\mathscr{L}, \hat{\mathscr{L}}), \qquad \mathscr{L}' \lhd \mathscr{L} \Rightarrow \varphi \restriction \mathscr{L}' \equiv \varphi \circ i \in \mathrm{Hom}\,(\mathscr{L}', \hat{\mathscr{L}}).$$

The notion of subcroc being very important in what follows, we characterize it by means of the following theorem:

PROP. 1.1. — If $\mathscr{L}' \lhd \mathscr{L}$, then $\mathscr{L}'$ is a sublattice of $\mathscr{L}$ and $cx \cap \mathrm{I}' \in \mathscr{L}'$, $\forall x \in \mathscr{L}'$.

Proof. — For any family $\{ x_k \}$ $(k \in \mathrm{K})$ of elements of $\mathscr{L}'$,

$$\bigcup_k{}' x_k = i(\bigcup_k{}' x_k) = \bigcup_k i(x_k) = \bigcup_k x_k$$

holds and, in the same way,

$$\bigcap_k{}' x_k = \bigcap_k x_k,$$

whence $\mathscr{L}'$ is a sublattice of $\mathscr{L}$; moreover,

$$\forall x \in \mathscr{L}' : \qquad cx \cap \mathrm{I}' = ci(x) \cap i(\mathrm{I}') = i(c'x) = c'x \in \mathscr{L}'.$$

A subcroc $\mathscr{L}' \lhd \mathscr{L}$ is said to be trivial if it is a trivial sublattice of $\mathscr{L}$. We can also show a sort of a converse statement of Prop. 1.1:

PROP. 1.2. — If the lattice $\mathscr{L}'$ is a sublattice of $\mathscr{L}$ and if $cx \cap \mathrm{I}' \in \mathscr{L}'$, $\forall x \in \mathscr{L}'$, then it is possible to define an orthocomplementation on $\mathscr{L}'$ such that with it $\mathscr{L}'$ is a croc and $\mathscr{L}' \lhd \mathscr{L}$.

Proof. — Since $\mathscr{L}'$ is a sublattice, it is a lattice; now, if we define $c' : \mathscr{L}' \to \mathscr{L}'$, $c'x = cx \cap \mathrm{I}'$, we can show that c' is an orthocomplementation on $\mathscr{L}'$; in fact, as a result of weak modularity (A.3) in $\mathscr{L}$, of property (A.6) and of the very definition of sublattice, we get for $x, y \in \mathscr{L}'$:

i) $c'(c'x) = c(cx \cap \mathrm{I}') \cap \mathrm{I}' = (x \cup c\mathrm{I}') \cap \mathrm{I}' = x$;

ii) $x <' y \Rightarrow x \cap' y = x \cap y = x \Rightarrow (cx \cup cy) \cap \mathrm{I}' = cx \cap \mathrm{I}'$
$\Rightarrow c'x \cup' c'y = c'x \Rightarrow c'y <' c'x$;

iii) $x \cap' c'x = (x \cap cx) \cap \mathrm{I}' = \Phi$, whence $\Phi \in \mathscr{L}'$; then $\Phi' = \Phi$ and $x \cap' c'x = \Phi'$; in the same way we get $x \cup' c'x = \mathrm{I}'$.

Moreover, taking into account weak modularity (A.4) in $\mathscr{L}$, we can show that it holds also in the orthocomplemented lattice $\mathscr{L}'$, which is therefore a croc: x and y being elements of $\mathscr{L}'$,

$$x <' y \Rightarrow x < y \Rightarrow x \cup (cx \cap y) = y$$
$$\Rightarrow x \cup ((cx \cap \mathrm{I}') \cap y) = y \Rightarrow x \cup' (c'x \cap' y) = y.$$

As a result of this theorem, the segment $[\Phi, a]$ (with the orthocomplementation $c' : [\Phi, a] \to [\Phi, a]$, $c'x = cx \cap a$) for any $a \in \mathscr{L}$, and the center $\mathscr{C}(\mathscr{L})$ (with the orthocomplementation $c' : \mathscr{C}(\mathscr{L}) \to \mathscr{C}(\mathscr{L})$, $c'x = cx$) are subcrocs of $\mathscr{L}$. As a corollary of Prop. 1.2, we have

PROP. 1.3. — $\varphi \in \operatorname{Hom}(\mathscr{L}, \mathscr{L}') \Rightarrow \operatorname{Im} \varphi \lhd \mathscr{L}'$.

Proof. — It is an immediate consequence of Prop. 1.2 if we notice that, because of *a*) and *b*) of Def. 1.1, $\hat{\mathscr{L}} \equiv \operatorname{Im} \varphi$ is a sublattice of $\mathscr{L}'$ with

$\hat{\Phi} = \varphi(\Phi) = \Phi'$ and $\hat{\mathrm{I}} = \varphi(\mathrm{I})$ and if we use *c*) of Def. 1.1. The orthocomplementation in Im φ results to be $\hat{c}x = c'x \cap' \varphi(\mathrm{I})$.

Then we can state

PROP. 1.4. — Given $\varphi \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{L}')$, if $x, y \in \mathscr{L}$ then

$$x \leftrightarrow y \Rightarrow \varphi(x) \leftrightarrow \varphi(y)$$

both in Im φ and in $\mathscr{L}'$.

Proof. — Compatibility in $\hat{\mathscr{L}} \equiv \mathrm{Im}\,\varphi$ follows straightforwardly from the very definition of homomorphism and from $\hat{\mathrm{I}} = \varphi(\mathrm{I})$, if one uses for instance property (A.8). It is then trivial to show that compatibility of two elements of a subcroc with respect to operations defined in it results into compatibility with respect to the total croc.

As an immediate consequence we have $\varphi(\mathscr{C}(\mathscr{L})) \subset \mathscr{C}\,(\mathrm{Im}\,\varphi)$. Finally we state a theorem which characterizes injective homomorphisms:

PROP. 1.5. — Given $\varphi \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{L}')$, then $\mathrm{Ker}\,\varphi = \{\Phi\}$ iff φ is injective.

Proof. — The « if » part of this theorem is trivial, and what we have in fact to show is the « only if » part. Let it be $\varphi(x) = \varphi(y)$: we want to show that $x = y$ then follows; set $z \equiv x \cap y$ and $\hat{\varphi} \equiv \varphi \upharpoonright [\Phi, x]$ (notice that $[\Phi, x] \equiv \hat{\mathscr{L}} \lhd \mathscr{L}$); taking into account *c*) of Def. 1.1 and

$$\varphi(x) = \varphi(y) = \varphi(z)$$

as well, we get

$$\varphi(\hat{c}z) = \hat{\varphi}(\hat{c}z) = c'\varphi(z) \cap' \varphi(x) = c'\varphi(z) \cap' \varphi(z) = \Phi',$$

whence $cz \cap x = \hat{c}z = \Phi$; on the other hand it is also true that $cz \leftrightarrow x$, because $z < x$ and (A.3) holds; then, because of (A.5), we have $x < c(cz) = z$ which, along with $z < x$, implies $x = z$. In the same way we can show $y = z$, from which $x = y$ follows.

2. DIRECT UNION

In this section we shall define direct union of crocs, both in an « external » and in an « internal » way; we shall prove the equivalence of the two definitions and we shall be able to define decomposition of a croc into components; we shall also define direct union of homomorphisms.

PROP. 2.1. — Let $\mathscr{I}$ be an index set and $\{ \mathscr{L}_i \}$ $(i \in \mathscr{I})$ a family of crocs. On the product set

$$\mathscr{L} = \prod_{i \in \mathscr{I}} \mathscr{L}^{(i)}$$

($\mathscr{L}^{(i)}$ here denotes the set underlying the croc $\mathscr{L}^{(i)}$) define the relation

$$x < y \qquad \text{iff} \qquad x(i) <^{(i)} y(i), \qquad \forall i \in \mathscr{I} \qquad (x, y \in \mathscr{L}),$$

and the mapping

$$c : \mathscr{L} \to \mathscr{L}, \qquad (cx)(i) = c^{(i)}x(i).$$

Then the relation $<$ is a partial ordering on $\mathscr{L}$, the poset $(\mathscr{L}, <)$ is a lattice (we shall indicate it shortly by $\mathscr{L}$) and the mapping c is an orthocomplementation on it. The lattice $\mathscr{L}$ with this orthocomplementation is a croc.

Proof. — It is clear that the relation $<$ turns out to be anti-symmetric, reflexive and transitive, namely a partial ordering. Moreover, if A is an index set and $\{ x_\alpha \}$ $(\alpha \in \mathrm{A})$ a family of elements of $\mathscr{L}$, then for all $x \in \mathscr{L}$

$$x < x_\alpha, \qquad \forall \alpha \in \mathrm{A} \qquad \text{iff} \qquad x(i) <^{(i)} x_\alpha(i), \qquad \forall \alpha \in \mathrm{A}, \qquad \forall i \in \mathscr{I}$$

iff

$$x(i) <^{(i)} \bigcap_{\alpha \in \mathrm{A}}{}^{(i)} x_\alpha(i), \qquad \forall i \in \mathscr{I}$$

iff $x < y$, where

$$y(i) \equiv \bigcap_{\alpha \in \mathrm{A}}{}^{(i)} x_\alpha(i);$$

then for $\{ x_\alpha \}$ the g. l. b. exists and equals y. In the same way we can show the existence of the l. u. b., with

$$(\bigcup_{\alpha \in \mathrm{A}} x_\alpha)(i) = \bigcup_{\alpha \in \mathrm{A}}{}^{(i)} x_\alpha(i).$$

Then $\mathscr{L}$ is a lattice with $\Phi(i) = \Phi^{(i)}$ and $\mathrm{I}(i) = \mathrm{I}^{(i)}$. It is now very easy to show that c satisfies properties (A.2) and that for $\mathscr{L}$ and c property (A.4) holds.

DEF. 2.1. — The croc $\mathscr{L}$ of Prop. 2.1 will be denoted by

$$\bigvee_{i \in \mathscr{I}}^{\oplus} \mathscr{L}^{(i)}$$

and it is referred to as the direct union of the family of crocs $\{ \mathscr{L}^{(i)} \}$ $(i \in \mathscr{I})$.

For the sake of clarity, the preceding definition of direct union will be called « external ». Parallel to this definition, we shall introduce another definition of direct union, that will be called « internal ». Before stating it, we need to define two classes of homomorphisms related to « external » direct union and which will prove useful in finding the equivalence of the two definitions of direct union.

DEF. 2.2. — Given $\mathscr{L} = \bigvee_i^{\oplus} \mathscr{L}^{(i)}$, we define *e*-projection onto $\mathscr{L}^{(i)}$ to be $\pi_i : \mathscr{L} \to \mathscr{L}^{(i)}$, $\pi_i(x) = x(i)$ and *e*-injection of $\mathscr{L}^{(i)}$ to be $j_i : \mathscr{L}^{(i)} \to \mathscr{L}$ such that $\pi_k \circ j_i$ is the identity mapping onto $\mathscr{L}^{(i)}$ when $k = i$ and the null-homomorphism from $\mathscr{L}^{(i)}$ into $\mathscr{L}^{(k)}$ otherwise.

It is easy to see that j_i is injective, $\pi_i \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{L}^{(i)})$, $j_i \in \mathrm{Hom}\,(\mathscr{L}^{(i)}, \mathscr{L})$.

We shall now characterize a class of families of elements of $\mathscr{L}$ which will play an important role in the definition of « internal » direct union.

PROP. 2.2. — $\mathscr{L}$ being a croc, let $\mathscr{I}$ be an index set and $\{ z_i \}$ $(i \in \mathscr{I})$ a family of elements of $\mathscr{C}(\mathscr{L})$; the following properties are equivalent:

a) $\qquad z_i \cap z_j = \Phi \quad \text{if} \quad i \neq j; \quad \bigcup_{i \in \mathscr{I}} z_i = \mathrm{I}.$

b) $\qquad \bigcup_{i \neq k} z_i = c z_k, \quad \forall k \in \mathscr{I}.$

c) If P_i is the projection related to z_i, then $\bigcup_i \mathrm{P}_i(x) = x$, $\forall x \in \mathscr{L}$, and $\mathrm{P}_i \circ \mathrm{P}_j$ is the null-endomorphism of $\mathscr{L}$ if $i \neq j$.

Proof.

a) $\Rightarrow$ *b)* Fix $k \in \mathscr{I}$; then $(\bigcup_{i \neq k} z_i) \cap z_k = \Phi$ as a consequence of (A.6) and $(\bigcup_{i \neq k} z_i) \cup z_k = \mathrm{I}$, whence $\bigcup_{i \neq k} z_i$ is a compatible complement of z_k; since in a croc the compatible complement is unique [*12*], we get $\bigcup_{i \neq k} z_i = c z_k$.

b) $\Rightarrow$ *a)* It follows simply from (A.5).

a) $\Rightarrow$ *c)* It follows simply from (A.6).

c) $\Rightarrow$ *a)* Write down *c)* with $x = \mathrm{I}$ and transform I by $\mathrm{P}_i \circ \mathrm{P}_j$.

DEF. 2.3. — A family of elements of $\mathscr{C}(\mathscr{L})$ for which properties of Prop. 2.2 hold is called a *d*-family; the family of projections related to the elements of a *d*-family is called a *d*-family of projections.

As an useful example we notice that in $\bigvee_i^{\oplus} \mathscr{L}^{(i)}$ the family $\{ j_i(\mathrm{I}^{(i)}) \}$ is a *d*-family. We notice also that an isomorphism maps a *d*-family into a *d*-family. We can now define the « internal » direct union.

DEF. 2.4. — $\mathscr{I}$ being an index set, if $\{ z_i \}$ $(i \in \mathscr{I})$ is a d-family of elements of a croc $\mathscr{L}$, then $\mathscr{L}$ is said to be the direct union of its subcrocs $\{ [\Phi, z_i] \}$ $(i \in \mathscr{I})$ and it is denoted by $\bigcup\limits_{i \in \mathscr{I}}^{\oplus} [\Phi, z_i]$.

We shall now show what is the link between the « internal » definition of direct union and the « external » one. As a first result, the remark after Def. 2.3 shows that every « external » direct union is in fact also an « internal » one. The next theorem states that the converse statement holds up to an isomorphism.

PROP. 2.3. — $\mathscr{I}$ being an index set, the following ones are equivalent properties for a croc $\mathscr{L}$:

a) $\{ z_i \}$ $(i \in \mathscr{I})$ is a d-family in $\mathscr{L}$.

b) The mapping

$$\varphi : \mathscr{L}' = \bigvee_i^{\oplus} [\Phi, z_i] \to \mathscr{L}, \qquad \varphi(x) = \bigcup_{i \in \mathscr{I}} \pi_i(x)$$

is such that

$$\varphi \in \mathrm{Is}\,(\mathscr{L}', \mathscr{L}).$$

Proof.

a) $\Rightarrow$ *b*) Let A be an index set and $\{ x_\alpha \}$ $(\alpha \in \mathrm{A})$ a family of elements of $\mathscr{L}'$; taking into account the homomorphic character of e-projections and of projections, we can prove:

$$\begin{aligned} \mathrm{P}_e(\varphi(\bigcap_\alpha{}' x_\alpha)) &= \mathrm{P}_e(\bigcup_i \pi_i(\bigcap_\alpha{}' x_\alpha)) = \pi_e(\bigcap_\alpha{}' x_\alpha) \\ &= \bigcap_\alpha \pi_e(x_\alpha) = \bigcap_\alpha \mathrm{P}_e(\varphi(x_\alpha)) = \mathrm{P}_e(\bigcap_\alpha \varphi(x_\alpha)), \qquad \forall e \in \mathscr{I}, \end{aligned}$$

whence $\varphi(\bigcap_\alpha{}' x_\alpha) = \bigcap_\alpha \varphi(x_\alpha)$; the similar result holds for the l. u. b.; in the same way we can find $\varphi(c'x) = c\varphi(x)$ for all $x \in \mathscr{L}'$; because $\varphi(\mathrm{I}') = \mathrm{I}$ holds, $\varphi \in \mathrm{Hom}\,(\mathscr{L}', \mathscr{L})$ then follows. Moreover φ is surjective: given $y \in \mathscr{L}$, take $w \equiv \bigcup_i{}' j_i(\mathrm{P}_i(y))$; it is easy to show that $\varphi(w) = y$. It is in fact injective as well: using Prop. 1.5 this is easily shown.

b) $\Rightarrow$ *a*) The family $\{ z_i \}$ $(i \in \mathscr{I})$ is the isomorphic image according to φ of the d-family $\{ j_i(z_i) \}$ $(i \in \mathscr{I})$ of $\mathscr{L}'$.

This theorem shows that any « internal » direct union is isomorphic to an « external » one, and accomplishes the proof of the equivalence of the two definitions. From now on, we shall simply speak of direct union and we shall always use the internal definition: the direct union of the family of crocs $\{ \mathscr{L}^{(i)} \}$ $(i \in \mathscr{I})$ will be an « internal » direct union of subcrocs of $\bigvee_i^{\oplus} \mathscr{L}^{(i)}$, namely $\bigcup_i^{\oplus} [\Phi, j_i(\mathrm{I}^{(i)})] = \bigcup_i^{\oplus} j_i(\mathscr{L}^{(i)})$, and it will be denoted shortly by $\bigcup_i^{\oplus} \mathscr{L}^{(i)}$ (notice that to replace $\bigcup_i^{\oplus} j_i(\mathscr{L}^{(i)})$ by $\bigcup_i^{\oplus} \mathscr{L}^{(i)}$ amounts to replace e-projections by projections and e-injections by injections); each $\mathscr{L}^{(i)}$ will be called a component of $\bigcup_i^{\oplus} \mathscr{L}^{(i)}$ and, given a subcroc $\hat{\mathscr{L}}$ of $\mathscr{L}$, we shall write $\hat{\mathscr{L}} < \mathscr{L}$ if there is a decomposition of $\mathscr{L}$ into a direct union of which $\hat{\mathscr{L}}$ is a component; $\hat{\mathscr{L}}$ will be said shortly to be a component of $\mathscr{L}$. A component is said to be trivial if it is a trivial subcroc.

DEF. 2.5. — A croc $\mathscr{L}$ is called reducible if it is a direct union of non-trivial subcrocs; otherwise it is called irreducible.

From the very definitions it follows that a subcroc $\mathscr{L}' \lhd \mathscr{L}$ is a component iff $\mathscr{L}' = [\Phi, z]$ with $z \in \mathscr{C}(\mathscr{L})$, that a croc is irreducible iff its center is trivial, that $\mathscr{L} = \operatorname{Ker} \mathrm{P} \bigcup^{\oplus} \operatorname{Im} \mathrm{P}$, $\forall \mathrm{P} \in \mathscr{P}(\mathscr{L})$, and that reducibility or irreducibility is preserved through an isomorphism. In fact isomorphisms preserve much more, as it is shown by

PROP. 2.4. — $\mathscr{I}$ being an index set and $\{ \mathscr{L}^{(i)} \}$ $(i \in \mathscr{I})$ a family of crocs,

$$\mathscr{L} = \bigcup_i^{\oplus} \mathscr{L}^{(i)}, \qquad \varphi \in \operatorname{Is}(\mathscr{L}, \mathscr{L}') \Rightarrow \mathscr{L}' = \bigcup_i^{\oplus} [\Phi', \varphi(\mathrm{I}^{(i)})]$$

and the components are pairwise isomorphic.

Proof. — The family $\{ \varphi(\mathrm{I}^{(i)}) \}$ is an isomorphic image of the d-family $\{ \mathrm{I}^{(i)} \}$ of $\mathscr{L}$ and $\varphi \restriction \mathscr{L}^{(i)} \in \operatorname{Is}(\mathscr{L}^{(i)}, [\Phi', \varphi(\mathrm{I}^{(i)})])$.

Now a theorem follows in which it is shown how a croc decomposes if a homomorphism is defined on it and how, relating to this decomposition, the homomorphism results in fact to be the product of a projection and of an isomorphism.

PROP. 2.5. — Given $\varphi \in \operatorname{Hom}(\mathscr{L}, \mathscr{L}')$, $\operatorname{Ker} \varphi$ is a component of $\mathscr{L}$ and, if P is the projection such that $\operatorname{Ker} \mathrm{P} = \operatorname{Ker} \varphi$, then $\varphi = \tilde{\varphi} \circ \mathrm{P}$, where $\tilde{\varphi}$ is an isomorphism of Im P onto Im φ.

Proof. — If a is the l. u. b. of $\operatorname{Ker} \varphi$, $a \in \operatorname{Ker} \varphi$ is easily seen and then $\operatorname{Ker} \varphi = [\Phi, a]$ holds; we want now to show that $a \in \mathscr{C}(\mathscr{L})$. In fact, for any $x \in \mathscr{L}$, we have:

$$\begin{aligned} \varphi((cx \cup a) \cap x) &= ((c'\varphi(x) \cap' \varphi(\mathrm{I})) \cup' \varphi(a)) \cap' \varphi(x) \\ &= c'\varphi(x) \cap' \varphi(x) = \Phi' \Rightarrow (cx \cup a) \cap x < a; \end{aligned}$$

but $(cx \cup a) \cap x < x$ holds, whence $(cx \cup a) \cap x < a \cap x$. On the other hand $cx \cup a > a \Rightarrow (cx \cup a) \cap x > a \cap x$, and then $(cx \cup a) \cap x = a \cap x$, $\forall x \in \mathscr{L}$; hence $a \in \mathscr{C}(\mathscr{L})$ follows as a consequence of (A.9) and, along with it, Ker $\varphi < \mathscr{L}$ also follows. Let P be the projection related to ca (that is the projection for which Ker P $= [\Phi, a] =$ Ker φ) and

$$\tilde{\varphi} \equiv \varphi \restriction [\Phi, ca] = \varphi \restriction \mathrm{Im}\ \mathrm{P};$$

it is easy to show that $\tilde{\varphi} \in \mathrm{Is}\,(\mathrm{Im}\ \mathrm{P}, \mathrm{Im}\ \varphi)$; moreover, since $\{ a, ca \}$ is a d-family, if Q is the projection related to a then we have

$$\varphi(x) = \varphi(\mathrm{P}(x) \cup \mathrm{Q}(x)) = \varphi(\mathrm{P}(x)) = \tilde{\varphi}(\mathrm{P}(x)), \qquad \forall x \in \mathscr{L}.$$

We notice that Prop. 2.5 could be stated in this way: given $\varphi \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{L}')$, if φ is not the null-homomorphism, there is a projection P such that Im P is a non-null component of $\mathscr{L}$ and it is isomorphic to Im φ, which is a subcroc of $\mathscr{L}'$.

We turn now to the definition of direct union of homomorphisms. We need the following theorem:

PROP. 2.6. — Let $\{ \mathscr{L}^{(i)} \}$ $(i \in \mathscr{I})$ and $\{ \tilde{\mathscr{L}}^{(i)} \}$ $(i \in \mathscr{I})$ be two families of crocs with the same index set $\mathscr{I}$, $\{ \varphi_i \}$ $(i \in \mathscr{I})$ a family of mappings such that $\varphi_i \in \mathrm{Hom}\,(\mathscr{L}^{(i)}, \tilde{\mathscr{L}}^{(i)})$, $\forall i \in \mathscr{I}$, and define

$$\psi : \mathscr{L} \equiv \bigcup_i{}^{\oplus} \mathscr{L}^{(i)} \to \tilde{\mathscr{L}} \equiv \bigcup_i{}^{\oplus} \tilde{\mathscr{L}}^{(i)}, \qquad \psi(x) = \tilde{\bigcup_i} \varphi_i(\mathrm{P}_i(x)),$$

where P_i is the projection related to $\mathrm{I}^{(i)}$; then $\psi \in \mathrm{Hom}\,(\bigcup_i{}^{\oplus} \mathscr{L}^{(i)}, \bigcup_i{}^{\oplus} \tilde{\mathscr{L}}^{(i)})$.

Proof. — We shall prove this theorem by means of a technique similar to that used for Prop. 2.3; if $\tilde{\mathrm{P}}_e$ is the projection related to $\tilde{\mathrm{I}}^{(e)}$, for any family $\{ x_\alpha \}$ $(\alpha \in \mathrm{A})$ of elements of $\mathscr{L}$ with index set A we can prove

$$\tilde{\mathrm{P}}_e(\tilde{\bigcup_i} \varphi_i(\mathrm{P}_i(\bigcap_\alpha x_\alpha))) = \varphi_e(\mathrm{P}_e(\bigcap_\alpha x_\alpha)) = \tilde{\mathrm{P}}_e(\tilde{\bigcap_\alpha}(\tilde{\bigcup_i} \varphi_i(\mathrm{P}_i(x_\alpha)))), \qquad \forall e \in \mathscr{I},$$

from which $\psi(\bigcap_\alpha x_\alpha) = \tilde{\bigcap_\alpha} \psi(x_\alpha)$ follows; the similar result holds for the l. u. b. and moreover we have

$$\begin{aligned}\tilde{\mathrm{P}}_e\psi(cx) = \varphi_e(\mathrm{P}_e(cx)) &= \varphi_e(c_e(\mathrm{P}_e(x))) \\ &= \tilde{c}_e(\varphi_e(\mathrm{P}_e(x))) \tilde{\cap} \varphi_e(\mathrm{I}^{(e)}) = \tilde{\mathrm{P}}_e(\tilde{c}\psi(x) \tilde{\cap} \psi(\mathrm{I})),\end{aligned}$$

which proves the theorem.

DEF. 2.6. — The homomorphism of Prop. 2.6 will be denoted by $\bigcup_i{}^{\oplus} \varphi_i$ and it is referred to as the direct union of the family of homomorphisms $\{ \varphi_i \}$ $(i \in \mathscr{I})$.

It is easy to prove that $\bigcup_i^{\oplus} \varphi_i$ is an isomorphism (automorphism) iff it is a direct union of isomorphisms (automorphisms).

3. SCHUR'S LEMMA AND THEOREM

Now the theory previously developed will be used to study group representations; to define them, set:

DEF. 3.1. — Let G be a group and $\mathscr{L}$ a croc. A croc-representation (we could also call it a proposition system representation) $\alpha(\mathscr{L})$, or simply α, of G on $\mathscr{L}$ is a group-homomorphism of G into the group Aut $(\mathscr{L})$. The automorphism corresponding to g is written α_g.

The croc $\mathscr{L}$ will be called (as suggested by Weyl's terminology for linear representations of groups) the substratum of α and, when necessary to avoid confusion, it will be written $\mathscr{L}_\alpha$. Croc-representations will be called simply representations. Most of the results that we shall get do not depend on the assumption for G to be a group; also in conventional representation theory many results may in fact be obtained with much more general conditions on G [*13*].

DEF. 3.2. — A representation $\alpha(\mathscr{L})$ of a group G is said reducible if there is a non-trivial component $\mathscr{L}' < \mathscr{L}$ such that $\alpha_g(\mathscr{L}') \subset \mathscr{L}'$, $\forall g \in G$; otherwise it is called irreducible.

Since this notion is a basic one in the theory of group representations, we characterize it by the following theorem:

PROP. 3.1. — $\alpha(\mathscr{L})$ being a representation of the group G, the following are equivalent properties:

a) $\alpha(\mathscr{L})$ is reducible.

b) There is a non-trivial component $\mathscr{L}' < \mathscr{L}$ such that $\alpha_g(\mathscr{L}') = \mathscr{L}'$, $\forall g \in G$.

c) There is a non-trivial projection P in $\mathscr{P}(\mathscr{L})$ such that $\alpha_g \circ P = P \circ \alpha_g$, $\forall g \in G$.

Proof.

a) $\Rightarrow$ *b*) Let $\mathscr{L}'$ be the component involved in the definition of reducibility; then there is $z \in \mathscr{C}(\mathscr{L})$ such that $\mathscr{L}' = [\Phi, z]$ and $x < z \Rightarrow \alpha_g(x) < z$, $\forall g \in G$; hence for each $g \in G$ we have $\alpha_g(z) < z$ along with $\alpha_{g^{-1}}(z) < z$, which in turn implies $z < \alpha_g(z)$; it follows that $\alpha_g(z) = z$, $\forall g \in G$; therefore *b*) holds because reducibility of α entails non-triviality of $\mathscr{L}'$.

b) $\Rightarrow$ *c*) z being the greatest element of $\mathscr{L}'$ and P the projection related to z, non-triviality of $\mathscr{L}'$ implies non-triviality of P; besides, for any $g \in G$ and $x \in \mathscr{L}$, we have $(\alpha_g \circ P)(x) = \alpha_g(z \cap x) = z \cap \alpha_g(x) = (P \circ \alpha_g)(x)$.

c) $\Rightarrow$ *a*) If z is the element of $\mathscr{C}(\mathscr{L})$ to which P is related, then $\{ z, cz \}$ is a d-family with respect to which $\mathscr{L}$ is reducible; if we denote P by P_1 and the projection related to cz by P_2, we get $P_i(\mathscr{L}) < \mathscr{L}$ and,

$$\forall g \in G, \qquad \alpha_g(P_i(\mathscr{L})) = P_i(\mathscr{L}) \qquad (i = 1, 2).$$

This result is nothing but a particular case of a more general situation; to tackle it, we need a couple of definitions:

DEF. 3.3. — Given a representation $\alpha(\mathscr{L})$ of a group G, if $\mathscr{L}' \lhd \mathscr{L}$ and $\alpha_g(\mathscr{L}') = \mathscr{L}'$, $\forall g \in G$, then $\alpha_g \restriction \mathscr{L}' \in \mathrm{Aut}(\mathscr{L}')$ is easily seen; then the mapping $\alpha' : G \to \mathrm{Aut}(\mathscr{L}')$, $\alpha'_g = \alpha_g \restriction \mathscr{L}'$, is a new representation $\alpha'(\mathscr{L}')$ of G; we say that α' is a subrepresentation of α and we write $\alpha' \lhd \alpha$ or $\alpha' = \alpha \restriction \mathscr{L}'$ to express this fact.

DEF. 3.4. — $\mathscr{I}$ being an index set, if $\{ \alpha^{(i)}(\mathscr{L}^{(i)}) \}$ $(i \in \mathscr{I})$ is a family of representations of a group G, construct the mapping

$$\sigma : G \to \mathrm{Aut}\,(\bigcup_i^{\oplus} \mathscr{L}^{(i)}), \qquad \sigma_g = \bigcup_i^{\oplus} \alpha_g^{(i)};$$

this to be a representation of G on $\bigcup_i^{\oplus} \mathscr{L}^{(i)}$ is easily proved; σ will be denoted by $\bigcup_i^{\oplus} \alpha^{(i)}$ and it will be called direct union of the family of representations $\{ \alpha^{(i)} \}$ $(i \in \mathscr{I})$. Each $\alpha^{(i)}$ $(i \in \mathscr{I})$ will be called a component of $\bigcup_i^{\oplus} \alpha^{(i)}$ and we shall write $\alpha' < \alpha$ to mean α' to be a component of a representation $\alpha(\mathscr{L})$. A component is said to be trivial if its substratum is a trivial subcroc of $\mathscr{L}$.

PROP. 3.2. — Given a representation $\alpha(\mathscr{L})$ of a group G, let $\mathscr{L}$ be a direct union $\mathscr{L} = \bigcup_i^{\oplus} \mathscr{L}^{(i)}$ with index set $\mathscr{I}$; then the following are equivalent properties:

a) $\alpha_g(\mathscr{L}^{(i)}) = \mathscr{L}^{(i)}$, $\forall g \in G$, $\forall i \in \mathscr{I}$.

b) P_i being the element of $\mathscr{P}(\mathscr{L})$ related to $I^{(i)}$,

$$\alpha_g \circ P_i = P_i \circ \alpha_g, \qquad \forall g \in G, \qquad \forall i \in \mathscr{I}.$$

If these properties hold, then $\alpha = \bigcup_i^{\oplus} \alpha^{(i)}$, where $\alpha^{(i)} = \alpha \restriction \mathscr{L}^{(i)}$.

Proof.

a) $\Rightarrow$ *b*) Since $\mathscr{L}^{(i)} = [\Phi, \mathrm{I}^{(i)}]$, then $[\Phi, \alpha_g(\mathrm{I}^{(i)})] = [\Phi, \mathrm{I}^{(i)}]$, $\forall i \in \mathscr{I}$, holds, whence $\alpha_g(\mathrm{I}^{(i)}) = \mathrm{I}^{(i)}$, $\forall i \in \mathscr{I}$ follows; now the proof runs exactly as the second part of the proof of Prop. 3.1.

b) $\Rightarrow$ *a*) Since $\mathscr{L}^{(i)} = \mathrm{P}_i(\mathscr{L})$, then $\alpha_g(\mathscr{L}^{(i)}) = \mathrm{P}_i(\alpha_g(\mathscr{L})) = \mathrm{P}_i(\mathscr{L}) = \mathscr{L}^{(i)}$, $\forall g \in \mathrm{G}$.

Finally, from *b*) we get easily for any $x \in \mathscr{L}$ and $g \in \mathrm{G}$:

$$\alpha_g(x) = \alpha_g(\bigcup_i \mathrm{P}_i(x)) = \bigcup_i \alpha_g(\mathrm{P}_i(x)) = \bigcup_i \alpha_g^{(i)}(\mathrm{P}_i(x)) = (\bigcup_i{}^{\oplus} \alpha_g^{(i)})(x).$$

The family $\{ \mathrm{P}_i \}$ $(i \in \mathscr{I})$ of projections in this theorem is said to reduce the representation α. The theorem now stated is a sort of a generalization of Prop. 3.1; the family $\{ \mathrm{P}_i \}$ $(i \in \mathscr{I})$ has in fact the same role as the projections P_i $(i = 1, 2)$ of *c*) $\Rightarrow$ *a*) part of Prop. 3.1, which can now be reexpressed by saying that a representation is reducible iff it is the direct union of two non-trivial components.

DEF. 3.5. — Given two representations $\alpha(\mathscr{L})$ and $\alpha'(\mathscr{L}')$ of the same group G, we define an intertwining homomorphism for α and α' to be $\varphi \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{L}')$ such that $\varphi \circ \alpha_g = \alpha'_g \circ \varphi$, $\forall g \in \mathrm{G}$. Being $\mathrm{R}(\alpha, \alpha')$ the set of all intertwining homomorphisms for α and α', we say that the two representations are equivalent if there is in $\mathrm{R}(\alpha, \alpha')$ an isomorphism from $\mathscr{L}$ onto $\mathscr{L}'$. In this case we write $\alpha \simeq \alpha'$.

Notice that this definition is reasonable, since the relation $\simeq$ now introduced is indeed an equivalence relation.

The next theorem shows how an intertwining homomorphism links representations together.

PROP. 3.3 (Schur's lemma). — $\alpha(\mathscr{L}_\alpha)$ and $\beta(\mathscr{L}_\beta)$ being two representations of the same group G, the following are equivalent assertions:

a) There are a component $\alpha' < \alpha$, the substratum of which is not the trivial croc, and a subrepresentation $\beta' \lhd \beta$ such that $\alpha' \simeq \beta'$.

b) There is a non-null homomorphism $\varphi \in \mathrm{R}(\alpha, \beta)$.

Proof.

a) $\Rightarrow$ *b*) $\alpha' \simeq \beta'$ means that there exists an isomorphism $\tilde{\varphi}$ of $\mathscr{L}'_\alpha$, the substratum of α', onto $\mathscr{L}'_\beta$, the substratum of β', such that $\tilde{\varphi} \circ \alpha'_g = \beta'_g \circ \tilde{\varphi}$, $\forall g \in \mathrm{G}$; if P is the projection related to α' by Prop. 3.1, then its range is $\mathscr{L}'_\alpha$ and $\alpha_g \circ \mathrm{P} = \mathrm{P} \circ \alpha_g$, $\forall g \in \mathrm{G}$, holds. P is not the null-homomorphism because $\mathscr{L}'_\alpha$ is not the trivial croc. If we define $\varphi : \mathscr{L}_\alpha \to \mathscr{L}_\beta$, $\varphi(x) = \tilde{\varphi}(P(x))$, then clearly $\varphi \in \operatorname{Hom}(\mathscr{L}_\alpha, \mathscr{L}_\beta)$, φ is non-null and for each $x \in \mathscr{L}_\alpha$ and $g \in G$ we get:

$$(\varphi \circ \alpha_g)(x) = \tilde{\varphi}(\alpha'_g(P(x))) = \beta'_g(\tilde{\varphi}(P(x)) = (\beta_g \circ \varphi)(x).$$

b) $\Rightarrow$ *a)* If a is the l. u. b. of Ker φ, then we have for each $g \in G$:

$$\varphi(\alpha_g(a)) = \beta_g(\varphi(a)) = \beta_g(\Phi_\beta) = \Phi_\beta,$$

whence $\alpha_g(a) < a$; as a particular case you get $\alpha_{g-1}(a) < a$, from which $a < \alpha_g(a)$ follows; in this way we get $\alpha_g(a) = a$, $\forall g \in G$. If P is then the projection related to ca as in Prop. 2.5, we get $\alpha_g \circ P = P \circ \alpha_g$, $\forall g \in G$: hence $\alpha' \equiv \alpha \restriction \operatorname{Im} P$ may be defined according to Prop. 3.1 and it is a component of α with non-trivial substratum because Ker φ does not equal $\mathscr{L}_\alpha$. Moreover, $\beta_g(\varphi(\mathscr{L}_\alpha)) = \varphi(\alpha_g(\mathscr{L}_\alpha)) = \varphi(\mathscr{L}_\alpha)$, $\forall g \in G$, so that $\beta' \equiv \beta \restriction \operatorname{Im} \varphi$ may be defined and it is a subrepresentation of β. If we set $\tilde{\varphi} \equiv \varphi \restriction \operatorname{Im} P$, then by Prop. 2.5 we get $\tilde{\varphi} \in \operatorname{Is}(\operatorname{Im} P, \operatorname{Im} \varphi)$; by direct computation we have for each $g \in G$

$$\tilde{\varphi}(\alpha'_g(x)) = \varphi(\alpha_g(x)) = \beta_g(\varphi(x)) = \beta'_g(\tilde{\varphi}(x)), \qquad \forall x \in \operatorname{Im} P,$$

whence $\alpha' \simeq \beta'$ follows.

Using this theorem, we can further characterize irreducible representations:

PROP. 3.4 (Schur's theorem). — $\alpha(\mathscr{L})$ being a representation of a group G, the following are equivalent properties:

a) α is irreducible.

b) If a projection belongs to $R(\alpha, \alpha)$, then it is trivial.

c) Each endomorphism in $R(\alpha, \alpha)$ is either the null-endomorphism or injective.

Proof. — The equivalence of *a)* and *b)* is shown by Prop. 3.1.

a) $\Rightarrow$ *c)* Let φ be an endomorphism in $R(\alpha, \alpha)$; if P is the projection such that Ker P = Ker φ, then from Prop. 3.3 it follows that $\alpha \restriction \operatorname{Im} P$ is a component of α and Im P has to be a trivial subcroc, because of irreducibility of α and Prop. 3.1; hence either Im P $= \{ \Phi \}$, from which Ker $\varphi = \mathscr{L}$ follows, or Im P $= \mathscr{L}$, from which Ker $\varphi = \{ \Phi \}$ follows; the result is then proved by Prop. 1.5.

c) $\Rightarrow$ *b)* Let P belong to $R(\alpha, \alpha)$; then either it is the null-endomorphism or Ker P $= \{ \Phi \}$; it is now easy to show that in the latter case P is the identity endomorphism: if z is the element of $\mathscr{C}(\mathscr{L})$ to which P is related then $\mathrm{Ker}\, P = \{\Phi, cz\}$.

We have called Prop. 3.3 and Prop. 3.4 Schur's lemma and theorem because they are the equivalents, in the theory of croc-representations, of well known propositions which were proved for finite dimensional linear representations by Schur [*14*] and for unitary representations by Mackey [*15*]. The analogue of the usual formulation of Schur's lemma (see for instance Kahan or Pontryagin [*16*]) may be obtained as an easy corollary from Prop. 3.3 and Prop. 3.4:

PROP. 3.5. — A representation $\alpha(\mathscr{L})$ of a group G is irreducible iff each homomorphism in $R(\alpha, \beta)$, β being any representation of the same group, is either null or injective.

4. THE UNIQUENESS OF THE DIRECT UNION DECOMPOSITION

Let us say that a representation is completely decomposable if it is a direct union of irreducible components; in the same way as in the theory of unitary representations of groups, for a reducible representation of a group G there is no need to be completely reducible, as it is shown by a simple example. Let $\mathscr{L}$ be a Boolean croc such that for each element x of $\mathscr{L}$ an element x' exists for which $x' < x$, $x' \neq x$, $x' \neq \Phi$ hold; otherwise stated, let $\mathscr{L}$ be a Boolean croc in which no atom exists (see ref. [*1*] and [*3*]). If $\alpha(\mathscr{L})$ is the trivial representation of G which maps every element of the group into the identity automorphism of $\mathscr{L}$, then it is highly reducible: any projection reduces this representation, and projections are in fact as many as the elements of $\mathscr{L}$; it is then easy to see that the lack of atoms entails that no irreducible subrepresentation can occur.

We shall show that a decomposition into irreducible components is quite unique. Before doing this, we will prove a theorem which shows how a reduction of a representation is transported through equivalence.

PROP. 4.1. — Let $\alpha = \bigcup_i^{\oplus} \alpha^{(i)}$ be a decomposition with index set $\mathscr{I}$ of a representation α of a group G and β another representation of the same group such that $\alpha \simeq \beta$. Then β admits of the decomposition $\beta = \bigcup_i^{\oplus} \beta^{(i)}$ with index set $\mathscr{I}$ and the components of α and β are pairwise equivalent, that is $\alpha^{(i)} \simeq \beta^{(i)}$, $\forall i \in \mathscr{I}$.

Proof. — Let φ be the isomorphism from $\mathscr{L}_\alpha$ onto $\mathscr{L}_\beta$ which sets up the equivalence; then by Prop. 2.4 we get $\mathscr{L}_\beta = \bigcup_i^\oplus \mathscr{L}_\beta^{(i)}$, with

$$\mathscr{L}_\beta^{(i)} = [\Phi_\beta, \varphi(\mathrm{I}_\alpha^{(i)})];$$

if $\mathrm{P}_\alpha^{(i)}$ is the projection related to $\mathrm{I}_\alpha^{(i)}$, the projection related to $\varphi(\mathrm{I}_\alpha^{(i)})$ is $\mathrm{P}_\beta^{(i)} = \varphi \circ \mathrm{P}_\alpha^{(i)} \circ \varphi^{-1}$; then, taking into account that $\varphi \in \mathrm{R}(\alpha, \beta) \Rightarrow \varphi^{-1} \in \mathrm{R}(\beta, \alpha)$ and that if α is reduced by a family $\{ \mathrm{P}_i \}$ $(i \in \mathscr{I})$ of projections then $\mathrm{P}_i \in \mathrm{R}(\alpha, \alpha)$, $\forall i \in \mathscr{I}$ (see Prop. 3.2), we get: $\beta \circ \mathrm{P}_\beta^{(i)} = \mathrm{P}_\beta^{(i)} \circ \beta$, $\forall i \in \mathscr{I}$. By Prop. 3.2 now we get $\beta = \bigcup_i^\oplus \beta^{(i)}$ if $\beta^{(i)} \equiv \beta \restriction \mathscr{L}_\beta^{(i)}$; moreover

$$\varphi^{(i)} \equiv \varphi \restriction \mathscr{L}_\alpha^{(i)} \in \mathrm{Is}\,(\mathscr{L}_\alpha^{(i)}, \mathscr{L}_\beta^{(i)})$$

and

$$\varphi^{(i)}(\alpha^{(i)}(x)) = \varphi(\alpha(x)) = \beta(\varphi(x)) = \beta^{(i)}(\varphi^{(i)}(x)), \qquad \forall x \in \mathscr{L}_\alpha^{(i)},$$

whence $\alpha^{(i)} \simeq \beta^{(i)}$, $\forall i \in \mathscr{I}$, follows.

As a consequence of this theorem, if $\alpha \simeq \beta$ then α is irreducible iff β is.

PROP. 4.2. — Let $\{ \alpha^{(i)} \}$ $(i \in \mathscr{I})$ be a family of representations of a group G and $\tilde{\alpha}$ another representation (with non-trivial substratum) such that $\tilde{\alpha} < \alpha = \bigcup_i^\oplus \alpha^{(i)}$; then there is at least one value of the index i such that $\tilde{\alpha}$ and $\alpha^{(i)}$ have a common component with non-trivial substratum.

Proof. — Let $\{ \mathrm{P}_i \}$ $(i \in \mathscr{I})$ be the family of projections related by Prop. 3.2 to the reduction $\bigcup_i^\oplus \alpha^{(i)}$ of $\alpha(\mathscr{L})$, and let Q be the projection related by Prop. 3.1 to the component $\tilde{\alpha}$; as $\bigcup_i \mathrm{Q}(\mathrm{P}_i(x)) = \mathrm{Q}(x)$, $\forall x \in \mathscr{L}$, there is at least one index i such that $\mathrm{Q} \circ \mathrm{P}_i$ is a projection different from the null one. Moreover $\mathrm{Q} \circ \mathrm{P}_i$ is a non-null projection also with respect to $\mathscr{L}^{(i)} = \mathrm{P}_i(\mathscr{L})$ and $\tilde{\mathscr{L}} = \mathrm{Q}(\mathscr{L})$ and it is easy to get:

$$\tilde{\alpha}_g \circ (\mathrm{Q} \circ \mathrm{P}_i) = (\mathrm{Q} \circ \mathrm{P}_i) \circ \tilde{\alpha}_g \quad \text{and} \quad \alpha_g^{(i)}(\mathrm{Q} \circ \mathrm{P}_i) = (\mathrm{Q} \circ \mathrm{P}_i) \circ \alpha_g^{(i)}, \ \forall g \in \mathrm{G},$$

whence, by Prop. 3.2, $\tilde{\alpha} \restriction (\mathrm{Q} \circ \mathrm{P}_i)(\mathscr{L})$ is a component of $\tilde{\alpha}$ and $\alpha^{(i)} \restriction (\mathrm{Q} \circ \mathrm{P}_i)(\mathscr{L})$ is a component of $\alpha^{(i)}$; finally we notice that

$$\alpha^{(i)} \restriction (\mathrm{Q} \circ \mathrm{P}_i)(\mathscr{L}) = \alpha \restriction (\mathrm{Q} \circ \mathrm{P}_i)(\mathscr{L}) = \tilde{\alpha} \restriction (\mathrm{Q} \circ \mathrm{P}_i)(\mathscr{L}).$$

The above theorem enables us to state that if a representation is completely decomposable then its decomposition into irreducible components is quite unique; in fact the proofs of the next two theorems rely essentially on Prop. 4.2.

PROP. 4.3. — If for a completely decomposable representation α the two decompositions into irreducible components $\bigcup_i^{\oplus} \alpha^{(i)}$ and $\bigcup_k^{\oplus} \tilde{\alpha}^{(k)}$ hold, $\mathscr{I}$ and K being the respective index sets, then to each $i \in \mathscr{I}$ a $k \in \mathrm{K}$ corresponds such that $\alpha^{(i)} = \tilde{\alpha}^{(k)}$ and the same holds with K and $\mathscr{I}$ interchanged.

Proof. — It is an immediate consequence of Prop. 4.2, when one remembers the irreducibility of the components involved.

The uniqueness of a decomposition into irreducible components is retained also through equivalence, as it is shown by:

PROP. 4.4. — If $\alpha = \bigcup_i^{\oplus} \alpha^{(i)}$ and $\beta = \bigcup_k^{\oplus} \beta^{(k)}$ are two decompositions into irreducibles, with index set $\mathscr{I}$ and K respectively, of two representations α and β of a group G and if $\alpha \simeq \beta$ holds, then to each $i \in \mathscr{I}$ a $k \in \mathrm{K}$ corresponds such that $\alpha^{(i)} \simeq \beta^{(k)}$ and the same is true with $\mathscr{I}$ and K interchanged.

Proof. — By Prop. 4.1, β admits of the decomposition $\beta = \bigcup_i^{\oplus} \tilde{\beta}^{(i)}$ with index set $\mathscr{I}$, and $\alpha^{(i)} \simeq \tilde{\beta}^{(i)}$, $\forall i \in \mathscr{I}$; to get the result it is now sufficient to notice that $\bigcup_i^{\oplus} \tilde{\beta}^{(i)}$ and $\bigcup_k^{\oplus} \beta^{(k)}$ fulfil the requests of Prop. 4.3.

The meaning of the above theorem is that if two equivalent representations are decomposed into irreducibles, then the decomposition is essentially unique: there is in fact an up to equivalence uniqueness.

5. CONCLUDING REMARKS

In this last section we will draw a brief sketch of a possible interpretation of some of the results that we have obtained; without any claim of definiteness, we want to outline some physical differences between Hilbert unitary and croc-representations of groups. The main difference relies on the different role played by superselection rules [*17*], which have to be superimposed on the Hilbert space description of quantum mechanics, while they are completely embodied in the proposition system approach; a croc to be decomposable means in fact that superselection rules act in it [*18*].

We divide our discussion into two parts: in the first one the decomposition theory will be examined while in the second one a physical interpretation of Schur's lemma will be inquired.

To perform the first half of this program, we assume $\mathscr{L}$ to be the croc $\mathscr{L}(\mathscr{A})$ related to a von Neumann algebra $\mathscr{A}$ in some Hilbert space $\mathscr{H}$ [*19*], we take into account some results by von Neumann discussed by Jauch and Misra [20] and we assume superselection rules to be purely discrete (this assumption is useful in order to avoid some trouble, but it could be further justified: see for instance Antoine [21]). If we now examine the decomposition of the proposition system $\mathscr{L}$, that of a unitary representation of a group on $\mathscr{H}$ and that of a croc-representation of a group on $\mathscr{L}$, then the third decomposition is easily seen to embody in a sense the other two ones. In fact, the proposition system $\mathscr{L}$ is decomposed by superselection rules, that is by any projection from $\mathscr{P}(\mathscr{L})$ (see Def. 2.4; this is seen to be nothing else that Jauch and Misra's decomposition of $\mathscr{A}$ with respect to its center $\mathscr{C}(\mathscr{A})$, if one reminds that $\mathscr{C}(\mathscr{L}) = \mathscr{L}(\mathscr{C}(\mathscr{A}))$). Besides, a unitary representation $\mathrm{U}(\mathscr{H})$ in Hilbert space $\mathscr{H}$ is decomposed by any projection from R(U), which is the commutant of $\mathrm{U}(\mathscr{H})$ (see for instance ref. [15]). Finally, if the proposition system $\mathscr{L}$ bears a symmetry group G, namely if $\mathscr{L}$ is the substratum of some croc-representation of G, then in its decomposition both superselection rules and symmetry have to be taken into account: a theorem by Guenin [22] shows that the croc representation of G is implemented by a unitary representation $\mathrm{U}(\mathscr{H})$ of the same group (at least when a ray representation is reducible to a unitary vector representation) and the proposition system with symmetry is decomposed in fact by projections from $\mathscr{P}(\mathscr{L}) \cap \mathrm{R(U)}$. When $\mathscr{L} = \mathscr{L}(\mathscr{A})$ this is indeed the significance of Prop. 3.2 and we could say that to decompose a croc-representation amounts to restrict the decomposition of a unitary representation to the « superselected sectors » singled out by superselection rules. It is also possible to see that this is the reason for the complete uniqueness of a decomposition into irreducibles of a croc-representation (see Prop. 4.3): it should be noticed that in the unitary Hilbert case only an up to equivalence uniqueness holds (see Prop. 1.11 of Pozzi's paper quoted in ref. [15]).

Let us now consider Schur's lemma (Prop. 3.3); comparing it with Prop. 2.5 we become aware that these two propositions interchange if the set of projections and the narrower one of intertwining projections interchange: if compared with Prop. 2.5, Prop. 3.3 amounts then to taking into account, along with superselection rules, symmetries as well; moreover, inasmuch Prop. 2.5 characterize the imbedding of a physical system into another, the same is done by Prop. 3.3 when symmetries partecipate in the description of a physical system. On Prop. 3.3 relies then the construction of simplified physical systems, in the way to be now illustrated by an example.

Let $\mathscr{L}$ be a croc and let it be the direct union of two elementary spin systems of spin s and s' respectively (this to be definite; what is important is in fact $\mathscr{L}$ to contain the elementary spin s system as a component) and $\mathscr{L}'$ the croc of a Galilean quantal particle with spin s; $\mathscr{L}$ and $\mathscr{L}'$ are the substrata of two croc-representations of SO(3), α and α' respectively, to be defined in a very natural way; a homomorphism from $\mathscr{L}$ into $\mathscr{L}'$ may now be constructed such that Prop. 3.3 holds; this is a very easy task because the superselection rule which is embodied in the construction of $\mathscr{L}$ is compatible with the representation of SO(3) defined on it. We notice that the subrepresentation of α', which is asserted by Prop. 3.3 to be the isomorphic image of the elementary system of spin s with SO(3) as symmetry group, is not a component; its substratum is not in fact a component of $\mathscr{L}'$, since it is the subcroc related to the von Neumann algebra of the spectral projections of spin observables and then it contains Φ' and I' but it does not equal $\mathscr{L}'$. This is a suggestion to use Prop. 3.3 to define simplified systems, as for instance a spin system if compared with a particle with the same spin. We notice that, as a consequence of Prop. 3.3, a subrepresentation of a croc-representation of a group may be imbedded into another one of the same group only if it is a component: this amounts to say that a subsystem of a physical system with symmetries may be considered to be a simplified physical system in the way previously explained only if it is a « superselected sector » of the complete system.

The procedure now discussed to introduce simplified physical systems has not an Hilbert analogue, as it is shown straightforwardly by this remark: the spectral projections of spin observables do not span an Hilbert subspace of the Hilbert space of states of a Galilean quantal particle, whence no Hilbert subrepresentation may be attached to them.

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APPENDIX

Here we give some accounts on the theory of proposition systems. Many of the results of this section may be found in ref. [3]. Many of the definitions we shall give (e. g. that of orthocomplementation) are not the most general and standard ones: in order to shorten at the highest degree this section, they are restricted according to the use we want to make of them.

A) A poset is a pair (X, $<$) where X is a set and $<$ is a partial ordering on it. A poset (X, $<$) is said to be a complete lattice if, for any family $\{ x_i \}$ $(i \in \mathscr{I})$ of elements of X with index set $\mathscr{I}$, both the least upper bound (l. u. b.) and the greatest lower bound (g. l. b.) exist; these two bounds will be denoted by $\bigcup_{i \in \mathscr{I}} x_i$ and $\bigcap_{i \in \mathscr{I}} x_i$ respectively or, when confusion can not occur, simply by $\bigcup_i x_i$ and $\bigcap_i x_i$; if the family consists of two elements, x and y, we shall write $x \cup y$ and $x \cap y$ respectively. The least and the greatest elements of X will be denoted by Φ and I respectively and a complete lattice by $\mathscr{L}$. By elements of $\mathscr{L}$ we shall mean elements of the underlying X. A complete lattice is called trivial if it contains only one element. A complete lattice is said to be distributive if for any three elements x, y, z of $\mathscr{L}$, the identities

$$\begin{aligned} x \cap (y \cup z) &= (x \cap y) \cup (x \cap z) \\ x \cup (y \cap z) &= (x \cup y) \cap (x \cup z) \end{aligned}, \qquad (A.1)$$

are satisfied.

Given the complete lattice $\mathscr{L} = (X, <)$, a complete sublattice of $\mathscr{L}$ is a complete lattice $\mathscr{L}' = (X', <')$ such that

i) $$X' \subset X,$$

ii) $$\bigcup_i{}' x_i = \bigcup_i x_i, \qquad \bigcap_i{}' x_i = \bigcap_i x_i$$

for any family $\{ x_i \}$ $(i \in \mathscr{I})$ of elements of X'. It is easy to show that, if $\mathscr{L}'$ is a sublattice of $\mathscr{L}$, then: X' is a subset of X closed with respect to $\cup$ and $\cap$ and $<'$ is the restriction of $<$ to X'. If conversely, given the complete lattice $\mathscr{L}$, X' is a subset of X closed with respect to $\cup$ and $\cap$, the prescription $x <' y$ iff $x \cap y = x$ defines a partial ordering relation on X' and (X', $<'$) turns out to be a complete sublattice of $\mathscr{L}$. In this way it is easy to make any segment $[a, b] \equiv \{ x \in \mathscr{L} ; a < x < b \}$ into a complete sublattice of $\mathscr{L}$, with least and greatest element a and b respectively. It is noteworthy that a complete lattice can be regarded to as an algebraic structure with $\cup$ and $\cap$ as binary operations [23]: then a complete sublattice turns out to be a subalgebra.

From now on we shall call shortly lattice (sublattice) a complete lattice (sublattice). A sublattice $\mathscr{L}'$ of $\mathscr{L}$ is called trivial if either $\mathscr{L}'$ equals $\mathscr{L}$ or $\mathscr{L}'$ is the trivial lattice.

B) An orthocomplementation on a lattice $\mathscr{L}$ is a mapping $c : \mathscr{L} \to \mathscr{L}$ such that, for $x, y \in \mathscr{L}$:

i) $$c(cx) = x,$$

ii) $$x < y \Rightarrow cy < cx, \qquad (A.2)$$

iii) $$x \cap cx = \Phi, \qquad x \cup cx = I.$$

Let $\mathscr{I}$ be an index set and $\{ x_i \}$ $(i \in \mathscr{I})$ a family of elements of $\mathscr{L}$; then it is $c(\bigcup_i x_i) = \bigcap_i cx_i$ and $c(\bigcap_i x_i) = \bigcup_i cx_i$.

An orthocomplemented lattice (that is a lattice on which an orthocomplementation is defined) is called Boolean algebra if it is distributive. Two elements x and y of $\mathscr{L}$ are said compatible if the sublattice generated by $S \equiv \{ x, cx, y, cy \}$ (that is the smallest sublattice which contains S) is a Boolean algebra, namely if it is a Boolean sublattice; in this case we write $x \leftrightarrow y$. It is trivial that $x \leftrightarrow y$ iff $x \leftrightarrow cy$ and that $x \leftrightarrow cx$, $\forall x \in \mathscr{L}$. The set $\mathscr{C}(\mathscr{L})$ of elements of $\mathscr{L}$ which are compatible with all the elements of $\mathscr{L}$ is called the center of $\mathscr{L}$. It is a Boolean sublattice of $\mathscr{L}$. The center is called trivial if $\mathscr{C}(\mathscr{L}) = \{ \Phi, I \}$.

Two elements x and y of $\mathscr{L}$ are said orthogonal if $x < cy$; in this case we write $x \perp y$.

C) A proposition system is an orthocomplemented lattice such that

$$x < y \Rightarrow x \leftrightarrow y. \tag{A.3}$$

In an orthocomplemented lattice, let x and y be such that $x < y$; then $x \leftrightarrow y$ iff $x \cup (cx \cap y) = y$ and $y \cap (cy \cup x) = x$. As a consequence, an orthocomplemented lattice is a proposition system iff

$$x < y \Rightarrow x \cup (cx \cap y) = y. \tag{A.4}$$

A proposition system will be called shortly croc (for the motivation of this name see Theorem VI of ref. [3]) and its elements are sometimes called propositions. Property (A.3), or equivalently (A.4), is called weak modularity.

In a croc these statements are true:

$$x \perp y \quad \text{iff} \quad x \leftrightarrow y \quad \text{and} \quad x \cap y = \Phi; \tag{A.5}$$

if $\{ x_i \}$ $(i \in \mathscr{I})$ is a family of propositions, then

$$x_i \leftrightarrow y, \quad \forall i \in \mathscr{I} \Rightarrow y \cap (\bigcup_i x_i) = \bigcup_i (y \cap x_i) \text{ and } y \cup (\bigcap_i x_i) = \bigcap_i (y \cup x_i); \tag{A.6}$$

$$x \leftrightarrow y \quad \text{iff} \quad cx \cap y = c(x \cap y) \cap y; \tag{A.7}$$

$$x \leftrightarrow y \quad \text{iff} \quad x \cup (cx \cap y) = y \cup (cy \cap x); \tag{A.8}$$

$$x \leftrightarrow y \quad \text{iff} \quad (cx \cup y) \cap x = y \cap x. \tag{A.9}$$

A croc is said to be trivial if it is the trivial lattice. In this paper, $\mathscr{L}$ is always meant for a croc. If a sign is needed to specify a croc (for instance $\hat{\mathscr{L}}$), the same sign is attached to anything relating to it (for instance $\hat{\cap}$, $\hat{\cup}$, $\hat{c}$ and also $\hat{<}$, $\hat{\Phi}$ and $\hat{I}$).

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