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Sets of simple observables in the operational approach to quantum theory

by

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ABSTRACT. — In the operational approach to the theory of statistical physical systems, the set $\mathscr{Q}$ of simple observables is represented by the order unit interval in the dual of a complete base norm space. Three subsets of $\mathscr{Q}$ are introduced in the abstract framework. It is shown that, in the C*-algebra model, these lead to operators in the C*-algebra itself and in its envelopping Σ^* and Baire* algebras respectively.

RÉSUMÉ. — *Ensembles d'observables simples dans l'approche operationnelle à la theorie quantique.* — Dans l'approche opérationnelle à la théorie des systémes physiques statistiques, l'ensemble $\mathscr{Q}$ des observables simples est représenté par l'intervalle d'unité d'ordre dans le dual d'un espace complet par rapport à une norme de base. On introduit, dans le cadre abstrait, trois sous-ensembles de $\mathscr{Q}$. On démontre que dans le modèle C*-algèbre ces sous-ensembles deviennent des opérateurs dans la C*-algèbre même et respectivement dans la Σ^*-algèbre et la Baire* algèbre qui l'enveloppent.

§ 1. INTRODUCTION

In previous papers [*5*] [*6*] [*7*] the operational approach to the theory of statistical physical systems, originally suggested by Haag and Kastler [*12*] and recently formulated by Davies and Lewis [*2*] in terms of partially ordered vector spaces, was discussed in some detail and applied to the

Von Neumann algebra and C*-algebra models for classical and quantum probability theories. Throughout this paper the notation of [5] [6] will be preserved. In particular, Postulates 1-6 of § 3 of [5] will be supposed to hold for a given physical system.

It will be recalled that the set of states of the system is represented by a norm closed generating cone K in a complete base norm space (V, B), where B is a base for K, and that the set of simple observables of the system is represented by the set $\mathcal{Q} = [0, e]$ in the dual space (V^*, e) of (V, B). (V^*, e) is a complete order unit space Archimedean ordered by the cone K^* dual to K and $\mathcal{Q}$ is the intersection of the unit ball $[-e, e]$ in (V^*, e) with K^*.

In the conventional approach to quantum theory, K is chosen to be the set of positive normal linear functionals on the Von Neumann algebra $\mathfrak{L}(Y)$ of bounded linear operators on the separable Hilbert space Y. Then, $\mathcal{Q}$ is the set of positive operators A on Y such that $A \leqq I$, the identity operator on Y. In the conventional approach to classical theory, K is chosen to be the set of positive regular Borel measures of finite total variation on the separable locally compact Hausdorff space Ω. In this case $\mathcal{Q}$ no longer coincides with the conventional set of simple random variables, the set of positive bounded Borel functions A on Ω such that $A(\omega) \leqq 1$, $\forall \omega \in \Omega$, and is in fact a far larger set.

In this paper various subsets of $\mathcal{Q}$ are defined and identified in the C*-algebra model in an attempt to explain the anomaly described above. The basis of their definitions is the notion of physical equivalence suggested by Haag and Kastler [*12*].

§ 2. THE PHYSICAL TOPOLOGY

In [*12*] it was pointed out that, in making measurements on a statistical physical system, the physical limitations on the system often made measurements on different states indistinguishable. In effect, this produces a topology τ on B, K or V, which for convenience is referred to as the physical topology. By assuming that all physically relevant information concerning a separating subset $\mathcal{Q}'$ of $\mathcal{Q}$ could be obtained in a finite number of experiments, Gunsen [*11*] showed that τ may be supposed to be locally convex Hausdorff and that B may be chosen τ compact. Results of Klee [*13*] [*14*] show that this is equivalent to supposing that K is τ locally compact. A result of D. A. Edwards [*4*] shows that this implies that (V, B) is the dual of a complete order unit space (U, *e*) Archimedean ordered by a norm closed cone C, C and K are compatible cones and the topologies τ

and $\sigma(V, U)$ coincide on norm bounded subsets of V and on K. In fact, U may be identified with the space of all τ continuous affine functionals on B. Notice that V* may also be identified with the space of all bounded affine functionals on B.

The set $\mathcal{Q}^B = [0, e] \subset U$ is said to be the set of *basic simple observables* of the system. Notice that the relationship between $\mathcal{Q}'$ and $\mathcal{Q}^B$ is as follows. Clearly, $\mathcal{Q}' \subset \mathcal{Q}^B$ and for $f \in B$, an ε-neighbourhood of f for the topology τ is defined by

$$\{ g : g \in B, \mid A_i(f) - A_i(g) \mid < \varepsilon, A_i \in \mathcal{Q}', i = 1, 2, \ldots, r \}.$$

Hence, it follows that the weak topologies of B and hence of K, defined by $\mathcal{Q}'$ and $\mathcal{Q}^B$ respectively are both identical to τ. Therefore, there is no loss in generality in identifying $\mathcal{Q}'$ with $\mathcal{Q}^B$.

In the following discussion U will be identified with its canonical embedding in its second dual V* and both U and V* will be regarded merely as spaces of affine functionals on B. Clearly $\mathcal{Q}^B \subset \mathcal{Q}$. Physically $\mathcal{Q}$ is the set of « ideal » simple observables of the system which would be measurable were it not for the limitations of the experiments. However, recall that an observable $\mathscr{A}$ is a $\mathcal{Q}$-valued measure on some Borel space $(\mathscr{S}, \mathscr{B})$ such that $\mathscr{A}(\mathscr{S}) = e$. It follows that, for each sequence $\{ A_r \} \subset \mathcal{Q}$ such that

$$\sum_{r=1}^{n} A_r \in \mathcal{Q}, \qquad \forall n,$$

then

$$A = \sum_{r=1}^{\infty} A_r$$

is well-defined and in $\mathcal{Q}$. The topology for which this convergence is defined is that of pointwise convergence on B, the weak* topology of V*. However, $\mathcal{Q}$ being the intersection of the weak* compact unit ball in V* and the weak* closed cone K* in V* is itself weak* compact. Therefore the conditions above are always satisfied for $\mathcal{Q}$. However, it need not follow that if $\{ A_r \} \subset \mathcal{Q}^B$ that $A \in \mathcal{Q}^B$. Hence, in order that a satisfactory notion of observable may be obtained, the set $\mathcal{Q}^B$ must be widened in some way. The remarks above show that one of the conditions which must be satisfied by the enlarged subset $\mathcal{Q}''$ of $\mathcal{Q}$ is that the pointwise limit of any monotone increasing sequence $\{ A_n \} \subset \mathcal{Q}''$ must itself lie in $\mathcal{Q}''$.

Led by these considerations certain properties of functions on B are examined. Let $\mathscr{F}(B)$ be the space of all real-valued bounded functions on B equipped with the norm $\|\,.\,\|$ defined for $T \in \mathscr{F}(B)$ by

$$\| T \| = \sup_{f \in B} | T(f) |$$

and let $\mathscr{F}^+(B)$ be the subset of $\mathscr{F}(B)$ consisting of non-negative functions. Clearly $\mathscr{F}(B)$ is a Banach space and $\mathscr{F}^+(B)$ is a cone in $\mathscr{F}(B)$. The spaces U and V* are norm closed subspaces of $\mathscr{F}(B)$.

For $\{T_n\} \subset \mathscr{F}(B)$ a sequence (resp. a monotone sequence) and $T \in \mathscr{F}(B)$, write $T_n \overset{w}{\to} T$ (resp. $T_n \overset{m}{\to} T$) if and only if $\| T_n \| \leqq k < \infty$, $\forall n$ and $T_n(f) \to T(f)$, $\forall f \in B$. A subset $\mathscr{V}$ of $\mathscr{F}(B)$ is said to be *w*-(resp. *m*-)closed if for each sequence (resp. monotone sequence) $\{T_n\} \subset \mathscr{V}$ such that that $T_n \overset{w}{\to} T$ (resp. $T_n \overset{m}{\to} T$) then $T \in \mathscr{V}$. If $\mathscr{V} \subset \mathscr{F}(B)$ is any subset a smallest *w*-(resp. m-)closed subset $\mathscr{V}^w$ (resp. $\mathscr{V}^m$) containing $\mathscr{V}$ exists and is called the *w*-(resp. *m*-)closure of $\mathscr{V}$. The main properties of *w*- and *m*-convergence are given below. The proofs which owe much to [*1*] are given in § 3.

PROPOSITION 2.1. — (i) *If $\mathscr{V} \subset \mathscr{F}(B)$ is w-closed then $\mathscr{V}$ is m-closed and norm closed.*

(ii) *If $\mathscr{V} \subset \mathscr{F}(B)$ is convex then $\mathscr{V}^w$, $\mathscr{V}^m$ are convex.*
(iii) *If $\mathscr{V} \subset \mathscr{F}^+(B)$ is a cone then $\mathscr{V}^w$, $\mathscr{V}^m$ are cones in $\mathscr{F}^+(B)$.*
(iv) *If $\mathscr{V} \subset \mathscr{F}(B)$ is a subspace then $\mathscr{V}^w$, $\mathscr{V}^m$ are subspaces.*

PROPOSITION 2.2. — V* *is a w- and m-closed subspace of* $\mathscr{F}(B)$ *and any* $\sigma(V^*, V)$ *closed subset of* V* *is w- and m-closed.*

Let U^w (resp. U^m) be the *w*-(resp. *m*-)closure of U. Then, Propn. 2.1 and Propn. 2.2 show that U^w and U^m are subspaces of V* and that U^w is norm closed. Let $\mathscr{C}(B)$ be the space of real-valued τ continuous functions on B. Then it is well known that $\mathscr{C}(B)^w = \mathscr{C}(B)^m = \mathscr{B}(B)$, the space of bounded Baire functions on B. Since $U \subset \mathscr{C}(B)$, it follows that elements of U^w and U^m are affine Baire functionals on B.

PROPOSITION 2.3. — (i) (U^w, e), (U^m, e) *are complete order unit spaces Archimedean ordered by the cones* $U^w \cap K^*$, $U^m \cap K^*$ *respectively and with their order unit norms identical to their norms as subspaces of* (V^*, e).

(ii) *Let* W (*resp.* M) *be the set of bounded linear functionals* $f^\wedge$ *on* (U^w, e) (*resp.* (U^m, e)) *such that if* $\{T_n\}$ *is a sequence* (*resp. monotone sequence*) *in* U^w (*resp.* U^m) *and* $T_n \overset{w}{\to} T$ (*resp.* $T_n \overset{m}{\to} T$), $f^\wedge(T_n) \to f^\wedge(T)$. *Let* K_w (*resp.* K_m) *be the subset of* W (*resp.* M) *consisting of elements* $f^\wedge$ *such that* $f^\wedge(U^w \cap K^*) \geqq 0$ (*resp.* $f^\wedge(U^m \cap K^*) \geqq 0$) *and let* B_w (*resp.* B_m)

be the subset of K_w *(resp.* K_m*) consisting of elements* $f^\wedge$ *such that* $f^\wedge(e) = 1$. *Then* (W, B_w) *(resp.* (M, B_m)*) is a complete base norm space with norm closed cone* K_w *(resp.* K_m*) order isomorphic to* (V, B) *and with base norm identical to its norm as a subspace of* (U^{w*}, A_w) *(resp.* (U^{m*}, A_m)*).*

(iii) *If* $\{T_n\}$ *is a sequence (resp. monotone sequence) in* U^w *(resp.* U^m*) then* $T_n \overset{w}{\to} T$ *(resp.* $T_n \overset{m}{\to} T$*) if and only if* $T_n(f) \to T(f)$, $\forall f \in B$.

Notice that in Propn. 2.3 (ii) a mapping ϕ between base norm spaces (V_1, B_1), (V_2, B_2) with cones K_1, K_2 respectively is an order isomorphism if and only if ϕ is an isomorphism between V_1 and V_2 mapping B_1 one-one onto B_2. Such a mapping is automatically isometric.

The order unit spaces (U^w, e), (U^m, e) are said to be the weak and monotone σ-envelopes of (U, e) respectively. If the order intervals $[0, e]$ in U^w and $[0, e]$ in U^m are denoted by $\mathscr{Q}^W$, $\mathscr{Q}^M$ respectively then clearly $\mathscr{Q}^B \subset \mathscr{Q}^W \subset \mathscr{Q}$, $\mathscr{Q}^B \subset \mathscr{Q}^M \subset \mathscr{Q}$ and both $\mathscr{Q}^W$ and $\mathscr{Q}^M$ satisfy the condition discussed earlier which allows a satisfactory definition of observable as a $\mathscr{Q}^W$ or $\mathscr{Q}^M$-valued measure to be given. $\mathscr{Q}^W$, $\mathscr{Q}^M$ are said to be the sets of *weak* and *monotone simple observables* respectively. The importance of Propn. 2.3 is in showing that, when $\mathscr{Q}^W$ or $\mathscr{Q}^M$ is regarded as the set of physically relevant simple observables, the set B of normalised states is precisely the set of possible normalised states satisfying the expected convergence properties.

§ 3. PROOFS

Proof of Propn. 2.1. — (*i*) Let $\mathscr{V} \subset \mathscr{F}(B)$ be w-closed and let

$$\{T_n\} \subset \mathscr{V},\ T \in \mathscr{F}(B),\ \|T_n - T\| \to 0.$$

Then, clearly $\{T_n\}$ is a bounded sequence and $T_n(f) \to T(f)$, $\forall f \in B$. Hence $T_n \overset{w}{\to} T$ and $T \in \mathscr{V}$. Therefore $\mathscr{V}$ is norm closed. If $\{T_n\}$ is monotone and $T_n \overset{m}{\to} T$, then $T_n \overset{w}{\to} T$ and, since $\mathscr{V}$ is w-closed, $T \in \mathscr{V}$. Therefore $\mathscr{V}$ is m-closed.

(*ii*) Let $\mathscr{V} \subset \mathscr{F}(B)$ be convex and let $T \in \mathscr{V}$, $t \in (0, 1)$. Let

$$Z = \{S : S \in \mathscr{F}(B),\ tT + (1-t)S \in \mathscr{V}^w\}.$$

If $\{S_n\} \subset Z$, $S_n \overset{w}{\to} S$ then $\|S_n\| \leqq k < \infty$. Further

$$\{tT + (1-t)S_n\} \subset \mathscr{V}^w,$$

$(tT + (1-t)S_n)(f) \to (tT + (1-t)S)(f)$, $\forall f \in B$ and

$$\|tT + (1-t)S_n\| \leqq t\|T\| + (1-t)k < \infty.$$

Therefore $t\mathrm{T} + (1-t)\mathrm{S}_n \xrightarrow{w} t\mathrm{T} + (1-t)\mathrm{S}$ and since $\mathscr{V}^w$ is w-closed $\mathrm{S} \in \mathrm{Z}$. Therefore Z is w-closed and since $\mathscr{V} \subset \mathrm{Z}$, $\mathscr{V}^w \subset \mathrm{Z}$. It follows that $t\mathrm{T} + (1-t)\mathrm{S} \in \mathscr{V}^w$, $\forall \mathrm{T} \in \mathscr{V}$, $\forall \mathrm{S} \in \mathscr{V}^w$, $\forall t \in (0, 1)$. Now let $\mathrm{S} \in \mathscr{V}^w$, $t \in (0, 1)$ and $\mathrm{Z}' = \{\mathrm{T} : \mathrm{T} \in \mathscr{F}(\mathrm{B}), t\mathrm{T} + (1-t)\mathrm{S} \in \mathscr{V}^w\}$. Then, Z′ is w-closed and from above $\mathscr{V} \subset \mathrm{Z}'$. Hence, $\mathscr{V}^w \subset \mathrm{Z}'$ and therefore $\mathscr{V}^w$ is convex. A similar proof applies to $\mathscr{V}^m$.

(*iii*) Let $\mathscr{V} \subset \mathscr{F}^+(\mathrm{B})$ be a cone and for $\mathrm{T} \in \mathscr{V}$, let

$$\mathrm{Z} = \{\mathrm{S} : \mathrm{S} \in \mathscr{F}(\mathrm{B}), \mathrm{T} + \mathrm{S} \in \mathscr{V}^w\}.$$

Then, as above, Z is w-closed and $\mathscr{V} \subset \mathrm{Z}$. Hence $\mathscr{V}^w \subset \mathrm{Z}$. Now suppose that $\mathrm{S} \in \mathscr{V}^w$ and let $\mathrm{Z}' = \{\mathrm{T} : \mathrm{T} \in \mathscr{F}(\mathrm{B}), \mathrm{T} + \mathrm{S} \in \mathscr{V}^w\}$. Then Z′ is w-closed and from above $\mathscr{V} \subset \mathrm{Z}'$. Therefore $\mathscr{V}^w \subset \mathrm{Z}'$ and so

$$\mathscr{V}^w + \mathscr{V}^w \subset \mathscr{V}^w.$$

For $\alpha \geqq 0$, let $\mathrm{Z}'' = \{\mathrm{S} : \mathrm{S} \in \mathscr{F}(\mathrm{B}), \alpha\mathrm{S} \in \mathscr{V}^w\}$. Then, Z″ is w-closed, $\mathscr{V} \subset \mathrm{Z}''$ and hence $\mathscr{V}^w \subset \mathrm{Z}''$. Therefore, $\alpha\mathscr{V}^w \subset \mathscr{V}^w$, $\forall \alpha \geqq 0$. Let $\mathrm{T} \in \mathscr{V}^w \cap -\mathscr{V}^w$. Clearly $\mathscr{F}^+(\mathrm{B})$ is w-closed and hence

$$\mathrm{T} \in \mathscr{F}^+(\mathrm{B}) \cap -\mathscr{F}^+(\mathrm{B}) = \{0\}.$$

Therefore $\mathscr{V}^w \cap -\mathscr{V}^w = \{0\}$ and $\mathscr{V}^w$ is a cone. Similarly, $\mathscr{V}^m$ is a cone.

(*iv*) Let $\mathscr{V}$ be a subspace of $\mathscr{F}(\mathrm{B})$. Then, a proof similar to that of (*ii*) shows that, $\forall \alpha, \beta \in \mathrm{R}$, the real line, $\forall \mathrm{T}, \mathrm{S} \in \mathscr{V}^w$, $\alpha\mathrm{T} + \beta\mathrm{S} \in \mathscr{V}^w$ and $\mathscr{V}^w$ is a subspace. A similar proof applies to $\mathscr{V}^m$.

Proof of Propn. 2.2. — Let $\{\mathrm{T}_n\} \subset \mathrm{V}^*$ and suppose $\|\mathrm{T}_n\| \leqq k < \infty$, $\mathrm{T}_n \xrightarrow{w} \mathrm{T} \in \mathscr{F}(\mathrm{B})$. Let $f, g \in \mathrm{B}$, $t \in (0, 1)$. Then, for $\varepsilon > 0$, $\exists n_0$ such that for $n \geqq n_0$,

$$|\mathrm{T}_n(f) - \mathrm{T}(f)| < \frac{1}{2}\varepsilon, \qquad |\mathrm{T}_n(g) - \mathrm{T}(g)| < \frac{1}{2}\varepsilon,$$

$$|\mathrm{T}_n(tf + (1-t)g) - \mathrm{T}(tf + (1-t)g)| < \frac{1}{2}\varepsilon.$$

It follows that

$$|\mathrm{T}(tf + (1-t)g) - t\mathrm{T}(f) - (1-t)\mathrm{T}(g)| < \varepsilon$$

and therefore T is an affine functional. Clearly T is bounded on B and hence $\mathrm{T} \in \mathrm{V}^*$. It follows that V^* is w-closed and Propn. 2.1 (*i*) shows that V^* is m-closed.

Let $\mathscr{V} \subset V^*$ be $\sigma(V^*, V)$ closed and let $\{T_n\} \subset \mathscr{V}, T_n \xrightarrow{w} T$. For $f \in V$, $\exists \alpha, \beta \geqq 0, g, h \in B$ such that $f = \alpha g - \beta h$. For $\varepsilon > 0$, $\exists n_0$ such that for $n \geqq n_0$,

$$| T_n(g) - T(g) | < \varepsilon/2\alpha, \qquad | T_n(h) - T(h) | < \varepsilon/2\beta.$$

Therefore, $| T_n(f) - T(f) | < \varepsilon$ and since $\mathscr{V}$ is $\sigma(V^*, V)$ closed and

$$\mathscr{V}^w \subset V^{*w} = V^*, \ T \in \mathscr{V}.$$

It follows that $\mathscr{V}$ is w-closed and Propn. 2.1 (*i*) shows that $\mathscr{V}$ is m-closed.

Proof of Propn. 2.3. — From Propn. 2.1, U^w is a norm closed subspace of V^* and $e \in U^w$. $U^w \cap K^*$ is clearly a cone in U^w and, for $T \in U^w$,

$$\frac{1}{2}(\| T \| e - T), \qquad \frac{1}{2}(\| T \| e + T) \in U^w \cap K^*.$$

It follows that $U^w \cap K^*$ generates U^w. In addition, for the ordering of U^w defined by $U^w \cap K^*$,

$$- \| T \| e \leqq T \leqq \| T \| e \tag{3.1}$$

and e is therefore an order unit. Let $T \in U^w$ and suppose $T \leqq \lambda e$, $\forall \lambda > 0$. Then, $T(f) \leqq \lambda$, $\forall f \in B$, $\forall \lambda > 0$ and hence $T \in - K^*$. It follows that the ordering is Archimedean. If, for $T \in U^*$,

$$\| T \|_w = \inf \{ \lambda \geqq 0 : - \lambda e \leqq T \leqq \lambda e \} \tag{3.2}$$

it follows that $\| \, . \, \|_w$ is a norm on U^w and that (U^w, e) is an order unit space Archimedean ordered by $U^w \cap K^*$. It follows from (3.1) and (3.2) that $\| T \|_w \leqq \| T \|$, $\forall T \in U^w$. If $\lambda > 0$ satisfies $\lambda e + T, \lambda e - T \in U^w \cap K^*$ where $T \in U^w$, then $- \lambda \leqq T(f) \leqq \lambda$, $\forall f \in B$, $| T(f) | \leqq \lambda$, $\forall f \in B$ and hence $\| T \| \leqq \lambda$. It follows that the reverse inequality $\| T \| \leqq \| T \|_w$ also holds and hence $\| T \|_w = \| T \|$. A similar proof shows that (U^m, e) is an order unit space Archimedean ordered by $U^m \cap K^*$ and with order unit norm identical to its norm as a subspace of (V^*, e).

Since U^w is norm closed in V^* it follows that (U^w, e) is complete in its order unit norm.

Before showing that (U^m, e) is also complete in its order unit norm (*ii*) is proved. Let (U^{w*}, A_w) be the complete base norm space dual to (U^w, e) and let C_w be the cone dual to $U^w \cap K^*$. Then, $W \subset U^{w*}$, $K_w \subset C_w$ and $B_w \subset A_w$. First, suppose $h \in W$ satisfies $h(T) = 0$, $\forall T \in U$ and let $Z_h = \{ T : T \in U^w, h(T) = 0 \}$. Let $\{T_n\} \subset Z_h, T_n \xrightarrow{w} T$. Since $h \in W$,

$h(\mathrm{T}) = \lim h(\mathrm{T}_n) = 0$ and therefore $\mathrm{T} \in \mathrm{Z}_h$ which implies that Z_h is w-closed. However, $\mathrm{U} \subset \mathrm{Z}_h$ and therefore $\mathrm{U}^w \subset (\mathrm{Z}_h)^w = \mathrm{Z}_h \subset \mathrm{U}^w$ which implies that $\mathrm{Z}_h = \mathrm{U}^w$ and $h(\mathrm{T}) = 0$, $\forall \mathrm{T} \in \mathrm{U}^w$.

For $f^\wedge \in \mathrm{W}$, let $\phi(f^\wedge)$ denote the restriction of $f^\wedge$ to U. Then ϕ is clearly a linear mapping from W into V, sending K_w into K and B_w into B. From the result above, it follows that ϕ is one-one. Let $f \in \mathrm{B}$ and define $f^\wedge$ on U^w by $f^\wedge(\mathrm{T}) = \mathrm{T}(f)$. Then $f^\wedge \in \mathrm{B}_w$ and $\phi(f^\wedge) = f$. It follows that ϕ maps B_w onto B and hence K_w onto K and W onto V. Therefore ϕ is an order isomorphism between complete base norm spaces $(\mathrm{W}, \mathrm{B}_w)$ and (V, B). To complete the proof of (*ii*) it remains to show that the base norm is identical to the norm as a subspace of $(\mathrm{U}^{w*}, \mathrm{A}_w)$. Let $f^\wedge \in \mathrm{W}$ and let $\| f^\wedge \|$ denote the norm of $f^\wedge$ regarded as an element of U^{w*}. Then, $\| \phi(f^\wedge) \| \leqq \| f^\wedge \|$. Since U is a closed subspace of $\mathscr{C}(\mathrm{B})$, $\phi(f^\wedge)$ has a Hahn-Banach extension f_1 a bounded linear functional on $\mathscr{C}(\mathrm{B})$ such that $\| \phi(f^\wedge) \| = \| f_1 \|$. Hence, there exists a regular Borel measure μ of finite total variation $\| \mu \|$ on B such that

$$f_1(\mathrm{T}) = \int_{\mathrm{B}} \mathrm{T}(g) d\mu(g), \qquad \forall \mathrm{T} \in \mathscr{C}(\mathrm{B})$$

and $\| f_1 \| = \| \mu \|$. Elements of U^w are bounded Baire functions on B and therefore μ-integrable. Let

$$h(\mathrm{T}) = f^\wedge(\mathrm{T}) - \int_{\mathrm{B}} \mathrm{T}(g) d\mu(g), \qquad \forall \mathrm{T} \in \mathrm{U}^w.$$

Then, using the properties of integrals, $h \in \mathrm{W}$ and $h(\mathrm{T}) = 0$, $\forall \mathrm{T} \in \mathrm{U}$. From the earlier result it follows that $h(\mathrm{T}) = 0$, $\forall \mathrm{T} \in \mathrm{U}^w$ and therefore,

$$f^\wedge(\mathrm{T}) = \int_{\mathrm{B}} \mathrm{T}(g) d\mu(g), \qquad \forall \mathrm{T} \in \mathrm{U}^w.$$

Hence, $\| f^\wedge \| \leqq \| \mu \| = \| f_1 \| = \| \phi(f^\wedge) \| \leqq \| f^\wedge \|$, which implies that $\| f^\wedge \| = \| \phi(f^\wedge) \|$. Therefore the base norm in $(\mathrm{W}, \mathrm{B}_w)$ coincides with its norm as a closed subspace of $(\mathrm{U}^{w*}, \mathrm{A}_w)$. A similar proof applies to $(\mathrm{M}, \mathrm{B}_m)$.

It is now possible to complete the proof of (*i*) by showing that (U^m, e) is complete and therefore closed in (V^*, e). Let $\{ \mathrm{T}_r \} \subset \mathrm{U}^m$ be a sequence such that

$$\lim_{n \to \infty} \sum_{r=1}^{n} \| \mathrm{T}_r \| = k < \infty.$$

Since V^* is complete, there exists $T \in V^*$ such that

$$\lim_{n\to\infty} \left\| T - \sum_{r=1}^{n} T_r \right\| = 0$$

and to prove that (U^m, e) is complete it is sufficient to show that $T \in U^m$. Define

$$S_{1r} = \frac{1}{2}(\| T_r \| e + T_r), \qquad S_{2r} = \frac{1}{2}(\| T_r \| e - T_r).$$

Then, S_{1r}, $S_{2r} \in U^m \cap K^*$ and $T_r = S_{1r} - S_{2r}$. Let

$$R_{1n} = \sum_{r=1}^{n} S_{1r}, \qquad R_{2n} = \sum_{r=1}^{n} S_{2r}.$$

Then, $\{ R_{1n} \}$, $\{ R_{2n} \} \subset U^m \cap K^*$ are monotone sequences satisfying

$$\| R_{in} \| \leqq \sum_{r=1}^{n} \| S_{ir} \| \leqq \sum_{r=1}^{n} \| T_r \| \leqq k, \qquad i = 1, 2.$$

For $f \in B$,

$$\begin{aligned} | R_{in}(f) - R_{in'}(f) | &= \left| \sum_{r=n'+1}^{n} S_{ir}(f) \right| \\ &= \left| \sum_{r=n'+1}^{n} \frac{1}{2}(\| T_r \| + T_r(f)) \right| \\ &\leqq \sum_{r=n'+1}^{n} \| T_r \| \to 0 \quad \text{as} \quad n, n' \to \infty, \quad i = 1, 2. \end{aligned}$$

Therefore, $\{ R_{in}(f) \}$ converges for $f \in B$, $i = 1$, 2 and hence converges for $f \in V$. Therefore, $\exists R_i \in V^*$, $i = 1, 2$, such that $R_{in}(f) \to R_i(f)$, $\forall f \in B$, $i = 1, 2$. It follows that $R_{in} \overset{m}{\to} R_i$, $i = 1$, 2 and since $\{ R_{in} \}$ is monotone increasing that $R_i \in K^*$. Since U^m is m-closed, $R_i \in U^m$, $i = 1, 2$. Therefore, for $f \in B$,

$$\begin{aligned} | T(f) - (R_1 - R_2)(f) | \leqq \left| T(f) - \sum_{r=1}^{n} T_r(f) \right| + | R_{1n}(f) - R_1(f) | \\ + | R_{2n}(f) - R_2(f) | \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

It follows that $T(f) = (R_1 - R_2)(f)$, $\forall f \in B$ which implies that

$$T = R_1 - R_2 \in U^m.$$

This completes the proof of (*i*).

To prove (*iii*), notice that $T_n \xrightarrow{w} T$ if and only if $\| T_n \| \leqq k < \infty$ and $T_n(f) \to T(f)$, $\forall f \in B$. Conversely, let $\{ T_n \} \subset U^w$ and let $T_n(f) \to T(f)$, $\forall f \in B$. Then, $T_n(f) \to T(f)$, $\forall f \in V$ and the principle of uniform boundedness shows that $\| T_n \| \leqq k < \infty$ and hence $T_n \xrightarrow{\omega} T$. Similarly for $\{ T_n \} \subset U^m$ monotone, $T_n \xrightarrow{m} T$ if and only if $T_n(f) \to T(f)$, $\forall f \in B$.

§ 4. THE C*-ALGEBRA MODEL

For the general theory of C*-algebras and Von Neumann algebras the reader is referred to [*3*] [*4*] and for the order theory to [*9*] [*16*].

Let $U(\mathfrak{A})$ denote the space of self-adjoint elements of a C*-algebra $\mathfrak{A}$ with identity e and let $C(\mathfrak{A})$ denote the subset of positive elements of $\mathfrak{A}$. Then, $(U(\mathfrak{A}), e)$ is a complete order unit space Archimedean ordered by the norm closed cone $C(\mathfrak{A})$ and with order unit norm identical to the C*-algebra norm. The dual space $(U^*(\mathfrak{A}), S(\mathfrak{A}))$ of $(U(\mathfrak{A}), e)$ with dual cone $C^*(\mathfrak{A})$ is a complete base norm space with closed unit ball $\operatorname{conv}(S(\mathfrak{A}) \cup (-S(\mathfrak{A})))$. $U^*(\mathfrak{A})$ is the space of bounded hermitean linear functionals on $\mathfrak{A}$ and $S(\mathfrak{A})$ is the set of states of $\mathfrak{A}$. If $\mathfrak{A}^*$ denotes the Banach dual space of $\mathfrak{A}$, $\mathfrak{A}^* = U^*(\mathfrak{A}) + iU^*(\mathfrak{A})$ and $\mathfrak{A}^*$ may be identified with the pre-dual of the Von Neumann envelope $\mathfrak{B} = \mathfrak{A}^{**}$ of $\mathfrak{A}$. In this identification, $U^*(\mathfrak{A})$ is the space $V(\mathfrak{B})$ of ultraweakly continuous hermitean linear functionals on $\mathfrak{B}$, $C^*(\mathfrak{A})$ is the cone $K(\mathfrak{B})$ of positive normal linear functionals on $\mathfrak{B}$ and $S(\mathfrak{A})$ is the set $B(\mathfrak{B})$ of normal states of $\mathfrak{B}$. The dual space $(V^*(\mathfrak{B}), e)$ of the complete base norm space $(V(\mathfrak{B}), B(\mathfrak{B}))$ is a complete order unitspace Archimedean ordered by the cone $K^*(\mathfrak{B})$ dual to $K(\mathfrak{B})$. $V^*(\mathfrak{B})$ is the space of self-adjoint elements of $\mathfrak{B}$, $K^*(\mathfrak{B})$ is the cone of positive elements of $\mathfrak{B}$ and e is the identity in $\mathfrak{B}$. Notice that if $\mathfrak{A}$ does not possess an identity the theory described above holds with slight changes in definitions for the C*-algebra obtained by adjoining an identity to $\mathfrak{A}$.

The model for a statistical physical theory in which the set of states of the system is represented by the cone $K(\mathfrak{B})$ is said to be the *C*-algebra model* corresponding to $\mathfrak{A}$. In this case the set $\mathcal{Q}$ of all simple observables may be identified with the set of elements A of $\mathfrak{B}$ such that $0 \leqq A \leqq e$. It is clear from the results of § 2 that the set $\mathcal{Q}^B$ of basic simple observables may be identified with the set of elements A of $\mathfrak{A}$ such that $0 \leqq A \leqq e$.

Recall that, according to Davies [1], a C*-algebra $\mathfrak{A}$ is said to be a Σ^*-algebra when there exists a set $\mathcal{G}$ of ordered pairs $\{ A_n, A \}$ consisting of a sequence $\{ A_n \} \subset \mathfrak{A}$ and an element $A \in \mathfrak{A}$ called the σ-convergent sequences such that if $S^\sigma(\mathfrak{A}) = \{ f : f \in S(\mathfrak{A}), \{ A_n, A \} \in \mathcal{G} \Rightarrow f(A_n) \to f(A) \}$ then,

(i) $\{ A_n, A \} \in \mathcal{G} \Rightarrow \| A_n \| \leqq k < \infty, \forall n$,

(ii) $\{ A_n, A \} \in \mathcal{G}, A' \in \mathfrak{A} \Rightarrow \{ A_n A', AA' \} \in \mathcal{G}$,

(iii) if $\{ A_n \} \subset \mathfrak{A}$ is such that $\{ f(A_n) \}$ converges, $\forall f \in S^\sigma(A)$, $\exists A \in \mathfrak{A}$ such that $\{ A_n, A \} \in \mathcal{G}$,

(iv) if $A \in \mathscr{A}$, $A \neq 0$, $\exists f \in S^\sigma(\mathfrak{A})$ such that $f(A) \neq 0$. $S^\sigma(\mathfrak{A})$ is said to be the set of σ-states of $\mathfrak{A}$.

A subset $\mathfrak{V}$ of the algebra $\mathfrak{L}(X)$ of bounded linear operators on the Hilbert space X is said to be σ-closed if every norm bounded weakly convergent sequence in V has its limit in V. The smallest σ-closed subset $\mathfrak{V}^\sigma$ containing an arbitrary subset $\mathfrak{V}$ of $\mathfrak{L}(X)$ is said to be the σ-closure of $\mathfrak{V}$. A σ-closed concrete C*-algebra $\mathfrak{A}$ is a Σ^*-algebra and the σ-closure $\mathfrak{A}^\sigma$ of an arbitrary concrete C*-algebra is a Σ^*-algebra.

For an arbitrary C*-algebra $\mathfrak{A}$ taken in its universal representation, the Σ^*-algebra $\mathfrak{A}^\sigma$ is said to be the σ-envelope of $\mathfrak{A}$. In this case $S^\sigma(\mathfrak{A}^\sigma) = S(\mathfrak{A})$ and $\mathfrak{A}^\sigma$ may be identified with the smallest family of bounded affine functionals on $S(\mathfrak{A})$ containing $\mathfrak{A}$ and such that every bounded sequence $\{ A_n \}$ in $\mathfrak{A}^\sigma$ converging pointwise on $S(\mathfrak{A})$ has its limit in $\mathfrak{A}^\sigma$.

It follows that, in the C*-algebra model, the set $\mathcal{Q}^W$ of weak simple observables can be identified with the set of elements A of $\mathfrak{A}^\sigma$ such that $0 \leqq A \leqq e$.

Recall that according to Pederson [15] and Kehlet [10], a C*-algebra $\mathfrak{A}$ is said to be a Baire*-algebra when

(i) For each monotone increasing sequence $\{ A_n \} \subset \mathfrak{A}$ such that $\| A_n \| \leqq k < \infty$, $\forall n$, $\exists A \in \mathfrak{A}$ such that $A_n \leqq A$, $\forall n$ and $A' \in \mathfrak{A}$, $A_n \leqq A'$, $\forall n$ implies $A \leqq A'$. In this case write $A_n \nearrow A$.

(ii) If $S^\beta(\mathfrak{A}) = \{ f : f \in S(\mathfrak{A}), A_n \nearrow A \Rightarrow f(A_n) \to f(A) \}$ and $A \in \mathfrak{A}, A \neq 0$, $\exists f \in S^\beta(\mathfrak{A}), f(A) \neq 0$.

A subset $\mathfrak{V}$ of the space of bounded self-adjoint operators on the Hilbert space X is said to be β-closed if every norm bounded monotone increasing sequence $\{ A_n \} \subset \mathfrak{V}$ has its least upper bound in $\mathfrak{V}$. The smallest β-closed subset $\mathfrak{V}^\beta$ of the space of bounded self-adjoint operators on X containing an arbitrary subset $\mathfrak{V}$ is said to be the β-closure of $\mathfrak{V}$. A concrete C*-algebra $\mathfrak{A}$ such that $U(\mathfrak{A})$ is β-closed is a Baire*-algebra and for an arbitrary concrete C*-algebra $\mathfrak{A}$, $\mathfrak{A}^\beta = U(\mathfrak{A})^\beta + iU(\mathfrak{A})^\beta$ is a Baire*-algebra Every Σ*-algebra is a Baire*-algebra.

For an arbitrary C*-algebra $\mathfrak{A}$ taken in its universal representation, the Baire*-algebra $\mathfrak{A}^\beta$ is said to be the Baire envelope of $\mathfrak{A}$. In this case $S^\beta(\mathfrak{A}^\beta) = S(\mathfrak{A})$ and $U(\mathfrak{A})^\beta = U(\mathfrak{A}^\beta)$ can be identified with the smallest family of bounded affine functionals on $S(\mathfrak{A})$ containing $U(\mathfrak{A})$ and such that every bounded monotone increasing sequence $\{A_n\}$ in $U(\mathfrak{A})^\beta$ has its least upper bound in $U(\mathfrak{A})^\beta$.

It follows that, in the C*-algebra model, the set $\mathscr{Q}^M$ of monotone simple observables can be identified with the set of elements A of $\mathfrak{A}^\beta$ such that $0 \leqq A \leqq e$.

§ 5. EXAMPLES AND CONCLUDING REMARKS

(*i*) Let $\mathfrak{A} = \mathfrak{LC}(Y)$, the C*-algebra of compact operators on the Hilbert space Y. Then, as was described in [5], $\mathfrak{B} = \mathfrak{L}(Y)$ and $K(\mathfrak{B})$ can be identified with the set of positive trace class operators on Y. In this case $\mathfrak{A}^\sigma = \mathfrak{A}^\beta = \mathfrak{L}(Y)$. Hence in this example, the conventional model for quantum probability, the sets $\mathscr{Q}$, $\mathscr{Q}^W$ and $\mathscr{Q}^M$ coincide and are equal to the set of operators A on Y such that $0 \leqq A \leqq I$, the identity operator on Y. The set $\mathscr{Q}^B$ of basic simple observables is the subset of $\mathscr{Q}$ consisting of operators of the form $A + \lambda I$ where A is compact.

(*ii*) Let $\mathfrak{A} = \mathscr{C}_0(\Omega)$ the commutative C*-algebra of continuous functions on the separable locally compact Hausdorff space Ω which take arbitrarily small values outside compact subsets of Ω. Then, as was described in [5], $K(\mathfrak{B})$ can be identified with the set of positive regular Borel measures of finite total variation on Ω. In this case $\mathfrak{A}^\sigma = \mathfrak{A}^\beta = \mathscr{B}(\Omega)$, the space of bounded Borel functions on Ω. Hence, in this example, the conventional model for classical probability, the sets $\mathscr{Q}^W$ and $\mathscr{Q}^M$ coincide and are equal to the set of Borel functions A on Ω such that $0 \leqq A(\omega) \leqq 1$, $\forall \omega \in \Omega$. The set $\mathscr{Q}^B$ of basic simple observables is the subset of $\mathscr{Q}^W = \mathscr{Q}^M$ consisting of continuous functions.

The two examples above shed some light on which sets of simple observables should be regarded as more natural than others in the abstract theory. In both examples the set $\mathscr{Q}^B$ is smaller than that usually chosen for the set of simple observables. Whilst in general $\mathscr{Q}^M \subset \mathscr{Q}^W$, in the examples, $\mathscr{Q}^M = \mathscr{Q}^W$ and this will always be the case for a Type I C*-algebra [*15*]. However, these two coincident sets are precisely those in which the conventional simple observables, the projections in $\mathfrak{L}(Y)$ in (*i*) and the characteristic functions of Borel subsets of Ω in (*ii*), lie. To be more precise, the conventional simple observables form the set of extreme points of the set $\mathcal{Q}^W = \mathcal{Q}^M$ in both examples. Some discussion of the importance of extreme points of sets of simple observables has been given elsewhere [*5*] [*6*] and therefore will not be pursued here.

The inferences which can be gained from examples (*i*) and (*ii*) are therefore that the most natural class of simple observables must be chosen from $\mathcal{Q}^W$ and $\mathcal{Q}^M$. The motivation for studying these classes at all seems to point to $\mathcal{Q}^M$ as the most likely. However, when applied to the C*-algebra model, for a wide class of C*-algebras, $\mathcal{Q}^W = \mathcal{Q}^M$. Pederson [*15*] points out that it is still an open question whether the notions of Σ^*-algebra and Baire*-algebra coincide in general.

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