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Section A :
Physique théorique.

Differential geometry of canonical quantization

by

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ABSTRACT. — The differential geometric objects from classical mechanics in the theory of canonical quantization, in particular those used in Souriau's solution of his « problème de la quantification d'un système dynamique », are identified.

RÉSUMÉ. — Dans la théorie de la quantification canonique, les objets de la géométrie différentielle de la mécanique classique, en particulier ceux que M. Souriau emploie dans la solution de son « problème de la quantification d'un système dynamique », sont identifiés.

INTRODUCTION

In the last two decades there has been considerable interest in the differential geometry of canonical quantization (e. g. [*46*] [*35*] [*36*] [*38*] [*41*] [*13*] [*19*] [*37*] [*23*]-[*24*] [*8*] [*15*]-[*16*], etc.). We recall here that Souriau's program considers a symplectic manifold (B, Ω) and the R-module of smooth functions $U^0(B)$ on B, which is a Lie algebra with respect to the Lie-Poisson bracket, $[f, g]_P$. So $U^0(B)$ is called the Lie algebra of dynamical variables. Souriau's problem of Dirac is the construction of a

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Hilbert space E and a linear representation ρ of $U^0(B)$ on the algebra of hermitian operators $\mathbf{H}(E)$ on E; so $\rho: f \to \rho(f): U^0(B) \to \mathbf{H}(E)$ satisfies $\rho(f)\rho(g) - \rho(g)\rho(f) = [\rho(f), \rho(g)] = -\sqrt{-1}\rho([f, g]_P)$ and $\rho(1) = Id_E$. If B is quantifiable, then there is an espace fibré quantifiant $T \to M \xrightarrow{p} B$ specified by (M, ω, Z); in this case Souriau gives a solution of the Dirac problem by $\rho(f) = -\sqrt{-1}\delta(f)$ for a certain class of functions $f \in S$; here $\delta(f) \in V(M)$ is the vector field defined by $\omega(\delta(f)) = f$ and locally $S \mid U = \{ f \in U^0(M) \mid i(p\delta(f))\Omega_B = -d(pf) \}$ where $p^*\Omega_B = d\omega$, for open U of M. Then $\delta(f)$ has the properties $\delta(1) = Z$ and $[\delta(f), \delta(g)] = \delta([f, g]_P)$.

In [*15*], we have shown that there is a well-known analogue of the espace fibré quantifiant (M, ω, Z) in classical mechanics, namely the regular contact manifolds. The aim of this paper is to show that $\delta(f)$ has an analogue in the differential geometry of classical mechanics, which clarifies several points in Souriau's works.

The program of Kostant considers a symplectic manifold (B, Ω) homogeneous with respect to group G with Lie algebra $\mathfrak{g}$ and the set L_Ω of equivalence classes of line bundles over B with connection ω such that the curvature class of ω is Ω. Associated to each element of L_Ω is a map $\varphi_C : \Lambda(B) \to C^*$ (where $\Lambda(B)$ is the loop space of B). Let H_Ω be the subset of L_Ω for which $\varphi_C(\Lambda(B)) \subseteq T$ $(= S^1)$. Then for each element of H_Ω construct the Hilbert space E of measurable sections of this line bundle. Noting that there is a natural representation of $U^0(B)$ on E, we have the desired representation of $\mathfrak{g}$, so G, if they lift: i. e. if we can complete the diagram

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] & U^0(B) \arrow[r] & H(B) \arrow[r] & 0 \\ & & & \mathfrak{g} \arrow[u] \arrow[ul, dotted] & \end{tikzcd}$$

(where $H(B)$ is the set of all Hamiltonian transformations, v. i.). As a second goal we show that the language of quantizable dynamical systems is also natural for Kostant's program.

Finally, the program of van Hove considers the « group » of transformations preserving contact structure ω, v. i., defines Hilbert spaces E, E′ and representations of the « infinitesimal transformations » of this « group »: namely

$$H(f)\Phi = \sqrt{-1}\left(f - \Sigma p_i \frac{\partial f}{\partial p_i}\right)\frac{\partial \Phi}{\partial s} + \sqrt{-1}[f, \Phi]_P$$

for $\Phi \in E$ and

$$H^a(f)\varphi = a\left(f - \Sigma p_i \frac{\partial f}{\partial p_i}\right)\varphi + \sqrt{-1}[f, \varphi]_P$$

for $\varphi \in \mathrm{E}'$; van Hove shows that $\rho(f) = a^{-1}\mathrm{H}^{a}(f)$ with $a = 2\pi$ is a solution to the Dirac problem. This is clearly related to Souriau's program and again the formulation below is shown to be natural for van Hove's program, clarifying several points of his work.

We denote by Z, the ring of integers, R, C the real and complex fields, $\mathrm{U}^{p}(\mathrm{M})$, the R-module of p-forms on manifold M, $\mathrm{C}^{p}(\mathrm{M})$, the submodule of closed forms, V(M), the R-module of smooth vector fields on M, $i(\ )$, the inner product, $\mathscr{L}(\ )$, the Lie derivative; finally, given a group or R-module F, we denote by **F**, the sheaf of germs of F.

§ 1. DIFFERENTIAL SYSTEMS [3] [22] [18] [1] [14]

Let M be a smooth manifold of $2n + 1$ dimensions with a system of local coordinates $m^1, \ldots, m^{2n+1}$ on open U of M. Let $\mathrm{Z} \in \mathrm{V(M)}$ be a non-nul vector field on M with components Z^i, $i = 1, \ldots, 2n + 1$. Then the *differential system* S(Z) of trajectories of the local group defined on U by Z is $\dfrac{dm^1}{\mathrm{Z}^1} = \ldots = \dfrac{dm^{2n+1}}{\mathrm{Z}^{2n+1}}$. Given $\mathrm{Z} \in \mathrm{V(M)}$ as above and p-form $\eta \in \mathrm{U}^p(\mathrm{M})$, then Z is an *associated field* of η if $i(\mathrm{Z})\eta = 0$ (i. e. $\mathrm{Z} \in \ker \eta$); and η is said to be *semibasic* or *generates an integral relation of invariance* for S(Z) (i. e. $\displaystyle\int_\tau \eta = 0$ where τ is the p-chain tube of trajectories from $(p-1)$-chain or cycle W to $\mathrm{W}' = \exp(t\mathrm{Z})\mathrm{W}$). The p-form η is an *invariant form* of Z or for S(Z) if $\mathscr{L}(\mathrm{Z})\eta = 0$. Vector field Z is an *extremal field* of η if $\mathscr{L}(\mathrm{Z})\displaystyle\int_{\mathrm{C}} \eta = \int_{\mathrm{C}} i(\mathrm{Z})d\eta = 0$, where C is a smooth p-cycle on M; i. e. $i(\mathrm{Z})d\eta = 0$ or Z is an associated field of $d\eta$. In this case η is said to *define a relative integral invariant* for S(Z) (i. e. $\displaystyle\int_{\mathrm{C}} \eta = \int_{\mathrm{C}'} \eta$ for p-cycles C and $\mathrm{C}' = \exp(t\mathrm{Z})\mathrm{C}$ on M). Vector field Z is a *characteristic field* of η if $i(\mathrm{Z})\eta = 0$ and $\mathscr{L}(\mathrm{Z})\eta = i(\mathrm{Z})d\eta = 0$; and η is said to be *basic* or *defines an absolute integral invariant* for S(Z). Clearly if η is closed, then Z is a characteristic field of η iff Z is an associated field of η.

Semibasic closed 1-forms $\eta \in \mathrm{C}^1(\mathrm{M})$, or locally basic 0-forms $f \in \mathrm{U}^0(\mathrm{M})$, are the *first integrals* of Z or S(Z). That is, locally, $\eta = df$ and $\mathscr{L}(\mathrm{Z})f = i(\mathrm{Z})df = 0$. Then $f(m) =$ constant is locally a maximal integral submanifold for S(Z).

§ 2. DYNAMICAL SYSTEMS AND ALMOST CONTACT MANIFOLDS [3] [30] [5] [9] [32] [43]

A *dynamical system* (D. S.) is a pair (M, Ω), a smooth $2n+1$-dimensional manifold and a 2-form $\Omega \in U^2(M)$ of rank $2n$, i. e. $(\Omega)^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n} \neq 0$.

If $d\Omega = 0$, then (M, Ω) is a D. S. *with integral invariant* (D. S. I.). Triple (M, ω, Ω) with $\omega \in U^1(M)$ and $\Omega \in U^2(M)$ is an *almost contact manifold* if $\omega \wedge (\Omega)^n \neq 0$.

Lemma 2.1. — Ω has rank $2n$; and dim (ker Ω) = dim M − rank $\Omega = 1$.
Thus if (M, ω, Ω) is an almost contact manifold, then the associated pair (M, Ω) is a D. S. Furthermore, there is a single associated field Z of Ω which is defined up to a numerical function factor; so one further condition is needed to make it unique, which we take to be $i(Z)\omega = 1$.

Lemma (Cartan; Reeb; Takizawa [43]) *2.2.* — There is one and only one vector field Z in V(M) characterized uniquely by the conditions 1) $i(Z)\omega = 1$ and 2) $i(Z)\Omega = 0$, namely Z defined by $\eta(Z)\omega \wedge (\Omega)^n = \eta \wedge (\Omega)^n$ for $\eta \in U^1(M)$.

Sasaki [32] noted that M is an almost contact manifold (i. e. manifold with $U(n) \times 1$-structure, v. [5]) iff M carries a structure (Φ, Z, ω) where Φ is a tensor field of type (1, 1), $Z \in V(M)$ and $\omega \in U^1(M)$ under the axioms: 1) $\omega(Z) = 1$ and 2) $\Phi \circ \Phi = -\mathbf{1} + \omega \otimes Z$. Then

Lemma (Sasaki) *2.3.* — $\Phi(Z) = 0$, $\omega \otimes \Phi = 0$, $\Phi^3 + \Phi = 0$, and rank $\Phi = 2n$.

Proposition (Sasaki) *2.4.* — Every almost contact manifold (M, Φ, Z, ω) admits a positive definite Riemannian metric g such that $g(X, Z) = \omega(X)$, $g(\Phi X, \Phi Y) = g(X, Y) - \omega(X)\omega(Y)$, and $\Omega(X, Y) = g(X, \Phi Y)$ is the 2-form of rank $2n$. Thus (M, Ω) is a D. S.

And conversely we have

Proposition (Sasaki [32], Hatakeyama [10]) *2.5.* — If (M, Ω) is a D. S., then M admits an almost contact (metric) structure (Φ, Z, ω, g) such that $\Omega(X, Y) = g(X, \Phi Y)$.

An almost contact structure (Φ, Z, ω) on M is *normal* if the Nijenhuis' tensor field of type (1, 2)

$$N(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] + d\omega(X, Y) \otimes Z$$

vanishes for X, Y $\in$ V(M). An almost contact (metric) structure (Φ, Z, ω, g) such that Z is a Killing vector with respect to g, i. e. $\mathscr{L}(Z)g = 0$, is called a K-*almost contact structure.*

Proposition (Sasaki-Hatakeyama [*34*]) *2.6.* — Structure (Φ, Z, ω, g) is a K-almost contact structure iff $\mathscr{L}(Z)\Phi = 0$.

Proposition (Sasaki-Hatakeyama [*34*]) *2.7.* — If (Φ, Z, ω) is a normal almost contact structure, then $\mathscr{L}(Z)\Phi = 0$ and $\mathscr{L}(Z)\omega = i(Z)d\omega = 0$; thus a normal almost contact structure is a K-almost contact structure.

§ 3. ORBIT SPACES, CONNECTIONS AND FIBER SPACES [*4*] [*7*] [*29*] [*42*]

Let M be a smooth manifold of $2n + 1$ dimensions and let Z be a non-nul vector field on M. So Z defines a 1-dimensional differential system (distribution) S(Z), which is involutive (completely integrable); i. e. S(Z) is a foliation. Then by Frobenius' theorem on each chart (U, m^i), $i = 1, \ldots, 2n + 1$, centered on m in M, $Z^{2n+1} = \partial/\partial m^{2n+1}$ is a base for S(Z), i. e. every point m in M admits a neighborhood U such that S(Z) is defined on U by $dm^1 = \ldots = dm^{2n} = 0$. All submanifolds defined by $m^i =$ constant, $i = 1, \ldots, 2n$, are then maximal integral submanifolds; that is, these m^i are first integrals of S(Z) in the sense of § 1. The orbit space (or space of leaves) of S(Z) is defined as follows: two points m, m' of M are called equivalent, $m \sim m'$, iff they belong to the same trajectory or orbit of S(Z); this is an equivalence relation and the quotient space $M/\sim\ = B$ is the orbit space of $2n$ dimensions. The natural quotient map $p : M \to B : m \to$ orbit containing m is onto, continuous and open in the quotient topology. Clearly, B can be identified with the space of first integrals of S(Z), since any $2n$ independent first integrals $m^1, \ldots, m^{2n}$ represent a system of coordinates for point $pm = b$ in B. In fact, more generally, letting $U_0^p(M) = \{ \eta \in U^p(M) \mid i(Z)\eta = 0 \text{ and } \mathscr{L}(Z)\eta = i(Z)d\eta = 0 \}$ be the R-module of basic forms, so $U_0^0(M)$ is the R-module of first integrals of S(Z), we have

Lemma (Reeb [*30*]; Cartan [*4*]) *3.1.* — $U_0^p(M) \simeq p^*U^p(B)$. In particular, $U_0^0(M) \simeq p^*U^0(B)$.

However, even if M is Hausdorff, B need not be Hausdorff, so the fibers $p^{-1}(b)$, b in B, are not closed. Thus we assume the foliation S(Z) is transverse or *regular* in the sense of Palais [*29*]. That is, each point m in M has a cubical (flat) neighborhood $(U, m^1, \ldots, m^{2n+1})$ and each integral submanifold of S(Z) intersects U in at most one 1-dimensional slice or segment given by $m^i = c^i$, $i = 1, \ldots, 2n$, for sufficiently small constants c^i (cf. Souriau [*38*], § 0.15, [*39*], § 1.73).

If $p\colon M^{2n+1} \to B^{2n}$ is a smooth map of rank $2n = \dim B$, then the space of vertical vectors $V_m = \{ X \in T_m(M) \mid pX = 0 \}$ is one dimensional. Its complement, the space of horizontal vectors Q_m, i. e. $T(M) = V \oplus Q$, is a $2n$-dimensional submanifold of T(M), so a $2n$-differential system. Q is called the *connection*. Given a path $C \in P(B)$, path space of B, if there exists a path C^* with tangent vector in Q from $m \in p^{-1}(C(0))$ such that $C^*(0) = m$ and $pC^* = C$, then C is said to have a horizontal lift and there is a map $\varphi_C\colon p^{-1}(C(0)) \to p^{-1}(C(1))\colon \Lambda(B) \to$ holonomy group of the connection.

In the case above assume that S(Z) generates a global one parameter group G, i. e. $\exp(tZ)$ $(-\infty < t < \infty)$, of transformations of M. In this case S(Z) is said to be *proper*. Then, requiring the fibers to be G-spaces, a connection is given by 1-form $\omega \in U^1(M, g)$ (g, the Lie algebra of G) such that 1) $\omega \circ R(g) = \mathrm{ad}\,(g^{-1})\omega$ for g in G and 2) $\omega(mX) = X$ for m in M and X in g. Then $Q_m = \{ X \in T_m(M) \mid i(X)\omega = 0 \}$. The *curvature form* associated to connection ω is the 2-form $\Omega \in U^2(M, g)$ given by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$.

The following theorem is due to Ehresmann [*7*], Reeb [*31*], Palais [*29*], Hermann [*12*], Wolf [*47*], Earle and Eells [*6*], and Hurt [*17*] (cf. Souriau [*38*], § 0.17):

Theorem 3.2. — Given a smooth map $p\colon M \to B$ of rank = the dimension of B, with M and B connected, paracompact spaces, then the following two statements are equivalent:

1) $p\colon M \to B$ is a locally trivial smooth fiber bundle;

2) there exists a connection ω on M with respect to which every (sectionally) smooth path in M has horizontal lifts.

Returning to the case above, as shown by Ehresmann [*7*] and Palais [*29*], we have:

Proposition 3.3. — If S(Z) is a regular differential system, then p is of rank $2n$. And if M is a connected Hausdorff space and S(Z) is proper —or the leaves of S(Z) are closed (resp. compact, e. g. if M is compact)—, then B is a (resp. compact) Hausdorff manifold and $p\colon M \to B$ is a smooth fiber bundle with leaves = fibers being C^∞-isomorphic.

As a corollary, if M is compact, then B is compact and all leaves are compact so homeomorphic to $T = S^1$; and $T \to M \to B$ is a principal toral bundle.

Lemma 3.4. — If S(Z), i. e. G, acts transitively on M, then S(Z) is regular.

If B is connected, then φ_C is a representation (continuous homomorphism) of $\Lambda(B)$ in group G, i. e. $\varphi_{C*C'} = \varphi_C \varphi_{C'}$, where * is composition in $P(B)$ (v. [*20*]). We state the following result of Ehresmann [*7*], Lashof [*20*], Hermann [*12*], and Hurt [*17*]:

Theorem 3.5. — Equivalence classes of principal bundles with group G and base B, arcwise and locally arcwise connected, with connection ω, are in one-one correspondence with conjugate classes of continuous homomorphisms of $\Lambda(B)$ into G: i. e. Hom $(\Lambda(B), G)/G \rightleftarrows H^1(B, G)$.

In particular every principal G-bundle $G \to P \to M$ can be constructed as follows: $P(\varphi) = P(B) \times G/\sim$, where $(C, g) \sim (C', g')$ iff the endpoint of C = the endpoint of C′ and $g' = \varphi_{C'*C} g$. If G is totally disconnected and M is semi-locally 1-connected, we have

Corollary 3.5. — Hom $(\pi_1(B), G)/G \rightleftarrows H^1(B, G)$, where $\pi_1(B)$ is the fundamental group of B.

§ 4. ALMOST SYMPLECTIC MANIFOLDS

A smooth manifold B of $2n$ dimensions is an *almost symplectic manifold* if there is a 2-form $\Omega \in U^2(M)$ of rank $2n$. If $d\Omega = 0$, then (B, Ω) is a *symplectic manifold.* An almost symplectic manifold with almost complex structure J is an *almost Hermitian manifold*; and a symplectic manifold with almost complex structure J is an *almost Kählerian manifold.*

§ 5. PROPER, REGULAR DYNAMICAL SYSTEMS

If (M, Ω) is a D. S. or equivalently M has almost contact structure (Φ, Z, ω), then the condition $\omega(Z) = 1$ shows that Z is non-nul everywhere on M; thus Z defines an involutive differential system S(Z) on M. If S(Z) is regular, resp. proper, then (M, Ω) is a *regular*, resp. *proper*, D. S.; or (Φ, Z, ω) is a regular, resp. proper, almost contact structure.

If (Φ, Z, ω) is a normal almost contact structure, then by proposition 2.7 $\mathscr{L}(Z)\Phi = 0$ and $\mathscr{L}(Z)\omega = i(Z)d\omega = 0$. This means Φ and ω

are invariant under the action of the group G generated by Z. Also it implies Z is the associated = characteristic field for $d\omega$, or extremal field for ω.

Define the period function of Z by

$$\lambda_Z(m) = \inf \{ t \mid (\exp (tZ))(m) = m, m \text{ in } M \}.$$

Then

Lemma (Boothby-Wang [*2*]) *5.1.* — $\lambda_Z(m)$ is a differentiable function on M.

Lemma (Tanno [*44*]) *5.2.* — If S(Z) is proper and regular, then the following are equivalent:

1) λ_Z is a constant (finite or infinite);

2) there exists a $\omega \in U^1(M)$ such that $\omega(Z) = 1$ and $\mathscr{L}(Z)\omega = 0$.

Clearly if λ_Z is infinite, G acts without fixed points on M and G = R; if λ_Z is finite, then G = T. Thus from proposition 3.3 and theorem 3.2 we have

Proposition (Morimoto [*26*]; Tanno [*44*]; Reeb [*30*]) *5.3.* — If (M, Ω, Z) is a proper, regular dynamical system or equivalently if (Φ, Z, ω) is a proper, regular almost contact structure, then

1) if λ_Z is infinite, G $\to$ M $\to$ B is a principal R-bundle with connection ω;

2) if λ_Z is finite, G $\to$ M $\to$ B is a principal toral bundle with connection ω.

The D. S. in the second case of this proposition has been called a fibered D. S. of type " by Reeb [*30*].

If M has a proper, regular almost contact structure (Φ, Z, ω), then we define a (1, 1)-tensor field J on B by $J_b(U) = dp(\Phi_m)(U^h_m)$, for m in M, b in B with $p(m) = b$, where U^h_m is the lift of vector field $U \in V(B)$ at m with respect to the connection ω (i. e. the unique vector field $U^h \in V(M)$ such that $\omega(U^h) = 0$ (i. e. U^h is horizontal) and $p(U^h_m) = U_{pm}$). Then $J^2(U) = -U$, so

Proposition (Ogiue [*28*], Hatakeyama [*10*]) *5.4.* — J is an almost complex structure on B.

Proposition (Morimoto [*22*], Hatakeyama [*11*]) *5.5.* — A regular, proper almost contact structure (Φ, Z, ω) on M is normal iff 1) J is integrable (i. e. a complex structure) and 2) $\Omega(U, V) = \Omega(JU, JV)$, U, V in V(B) where $p^*\Omega = d\omega$.

If M has a proper, regular K-almost contact metric structure (Φ, Z, ω, g), then we define the metric tensor field of type (0, 2) on B by $g_B(U, V) = g(U^h, V^h)$ for all U, V in V(B). Then

Proposition (Ogiue [*28*]; Hatakeyama [*10*]) *5.6.* — (J, g_B) is an almost Hermitian structure on B. If (Φ, Z, ω, g) is normal, then (J, g_B) is an Hermitian structure on B.

§ 6. CONTACT MANIFOLDS

If (M, ω, Θ) is an almost contact manifold or M has an almost contact metric structure (Φ, Z, ω, g) with $\Theta(X, Y) = g(X, \Phi Y)$, then M is a *contact manifold* or manifold with *almost Sasakian structure* if $\Theta = d\omega$; if the structure is normal, then M is said to have a *Sasakian structure.* Clearly, if (M, ω) is a contact manifold, then (M, $\Theta = d\omega$) is a D. S. I. Vector field Z is uniquely determined by $i(Z)\omega = 1$ and $i(Z)d\omega = 0$; thus Z is a characteristic = associated field of $d\omega$, an extremal field of ω, etc. If (M, Θ) is a D. S. I. and B is the orbit space of Z, then

Lemma (Reeb [*30*]) *6.1.* — There exists a 2-form $\Omega \in U^2(B)$ such that $p^*\Omega = \Theta$.

Assume (Φ, Z, ω, g) is a regular, proper K-almost contact structure and let (J, g_B) be the associated almost Hermitian structure on B. Let $\Omega(U, V) = g_B(U, JV)$ for U, V in V(B). Then

Lemma (Ogiue [*28*]; Hatakeyama [*10*]; Reeb [*30*]) *6.2.* — $p^*\Omega = \Theta$; and if M is a contact manifold, then Θ is the curvature for of connection ω.

Proposition (Ogiue [*28*]; Tanno [*44*]) *6.3.* — If M is a regular, proper contact manifold, then (B, Ω) is a symplectic manifold and (J, g_B) is an almost Kählerian structure on B. M has a normal, regular, proper contact structure iff (J, g_B) is a Kählerian structure.

Proposition (Tanno [*44*]; Hatakeyama [*11*]) *6.4.* — If M is a proper, regular contact manifold and (B, g_B) is a Riemannian manifold, then (M, g) with $g = p^*g_B + \omega \otimes \omega$ is a Riemannian manifold; furthermore, $g(Z, Z) = 1$ and $\mathscr{L}(Z)g = 0$; so each trajectory of Z is a geodesic with arc length t and (M, ω, Z) is a K-contact manifold.

If M is a compact contact manifold, then Z is proper. Thus we have

Proposition (Boothby-Wang [*2*]; Takizawa [*43*]; Tanno [*44*]) *6.5.* — If (M, ω) is a compact regular contact manifold, then $T \to M \to B$ is a principal toral bundle over symplectic manifold (B, Ω) with connection ω such that $p^*\Omega = d\omega$ is the curvature form of ω; furthermore, Ω determines an integral cocycle on B.

The converse also holds; thus we state

Theorem (Boothby-Wang [*2*]; Takizawa [*43*]) *6.6.* — Let (B, Ω) be a symplectic manifold; then there exists a smooth (C^∞ resp. C^ω) principal toral bundle over B with (C^∞ resp. C^ω) connection form ω which determines a regular contact structure such that $p^*\Omega = d\omega$ iff closed 2-form Ω $\left(\text{resp. } \dfrac{1}{2\pi\sqrt{-1}}\Omega\right)$ represents an integral cohomology class on M.

Cf. Souriau [*38*], § 3.36-3.38, 4.3-4.5.

Proof. — In brief, from the fundamental diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Z} & \to & \mathrm{R} & \to & \mathrm{T} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Z} & \to & \mathbf{R} & \to & \mathbf{T} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathbf{Z} & \to & \mathbf{C} & \to & \mathbf{C}^* & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathrm{Z} & \to & \mathrm{C} & \xrightarrow{e} & \mathrm{C}^* & \to & 0
\end{array}$$

(where $e = \exp 2\pi\sqrt{-1}$), we have the cohomological diagram

$$\begin{array}{cccccccc}
 & & & \chi(\xi) & \to & \Omega & & \\
 & \mathrm{H}^1(\mathrm{B}, \mathrm{T}) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{Z}) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{R}) & \xrightarrow{\sim} & \Gamma(\mathrm{C}^2)/d\Gamma(\mathrm{U}^1) \\
 & \xi \to \chi(\xi) & & \wr\uparrow & & & & \\
 & \uparrow & & & & & & \\
\to & \mathrm{H}^1(\mathrm{B}, \mathbf{T}) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{Z}) & & & & \\
 & \uparrow & & & & & & \\
\xrightarrow{e} & \mathrm{H}^1(\mathrm{B}, \mathbf{C}^*) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{Z}) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{C}) & \xrightarrow{\sim} & \Gamma(\mathrm{B}^{0,2})/d''\Gamma(\mathrm{U}^{0,1}) \\
 & \downarrow & & \wr\downarrow & & & & \\
 & \xi \to c(\xi) & & & & & & \\
 & & & \mathrm{H}^2(\mathrm{B}, \mathrm{Z}) & \to & \mathrm{H}^2(\mathrm{B}, \mathrm{C}) & \xrightarrow{\sim} & \Gamma(\mathrm{C}^2)/d\Gamma(\mathrm{U}^1) \\
 & & & c(\xi) & \to & \dfrac{1}{2\pi\sqrt{-1}}\Omega & &
\end{array}$$

where $\mathrm{U}^{p,q}$ denotes the C^∞ differential (p, q)-forms, $\mathrm{B}^{p,q}$ denotes the d''-closed forms, $\chi(\xi)$ is the Euler-Poincaré class, and $c(\xi)$ is the first Chern class. The theorem follows by exactness.

Proposition (Hatakeyama [*11*]; Morimoto [*25*]) *6.7.* — A compact regular contact manifold admits an associated normal contact structure iff B is a Hodge manifold (i. e. Kähler manifold with integral fundamental 2-form Ω). So, e. g., a compact simply connected homogeneous contact manifold admits a normal almost contact structure.

Corollary 6.8. — If B is a compact Hodge manifold, then it has over it a canonically associated principal toral bundle which admits a normal (regular) contact metric structure.

§ 7. QUANTIZABLE DYNAMICAL SYSTEMS

If (M, ω) is a proper, regular contact manifold with a finite period, then $(M, \Theta = d\omega)$ is called a *quantizable* D. S. (Q. D. S.) (v. [*15*] [*16*]). Pair (M, ω), $\omega \in U^1(M)$, is an *espace fibré quantifiant* if $\dim(\ker(d\omega)) = 1$ where $\omega \neq 0$ on $\ker(d\omega)$ and the maximal characteristic curves of $Z \in \ker(d\omega)$ are compact sets (so homeomorphic to T).

Proposition (Hurt [*15*]) *7.1.* — $(M, \Theta = d\omega)$ is a Q. D. S. iff (M, ω) is an E. F. Q.

As a corollary, theorem 6.6 gives necessary and sufficient conditions for Souriau's « quantification d'une variété symplectique ».

We note that we can apply theorem 3.5 to classify Q. D. S. Cf. Souriau [*38*], § 4.15, [*40*], § 1.27-28; cf. Kostant [*19*]. In particular explicit construction of principal toral bundle associated to Q. D. S. (M, Θ) is given as above (v. [*17*] for details). Namely, if locally we denote the connection form by ω_i, then $\varphi_C = \exp\left(-2\pi\sqrt{-1}\int_C \omega_i\right) : \Lambda(B) \to T$; and $M(\varphi)$ is defined as in § 3. $M(\varphi)/T = B$ is just the orbit space. By Stokes theorem $\varphi_C = \exp\left(-2\pi\sqrt{-1}\int_K \Omega\right)$ where $C = \partial K$ and the characteristic class of the bundle is given by $\int_B \Omega \in H^2(B, Z)$. This we state as

Proposition (Hurt [*17*]) *7.2.* — (M, Θ) is a Q. D. S. iff the wave function is single-valued; and Q. D. S. *s* are classified by the loop space of the base space: $\mathrm{Hom}(\Lambda(B), T(C^*)) \rightleftarrows H^1(B, T(C^*))$.

Kostant has noted the following (cf. cor. 3.5):

Proposition (Kostant [*19*]) *7.3.* — If (M, Θ) is a Q. D. S., then $H^1(B, T) = \pi_1(B)^\wedge$ operates simply transitively on H_Ω; i. e. given $\xi \in H_\Omega$, then $H_\Omega = \pi_1(B)^\wedge \cdot \xi$.

§ 8. EXAMPLES

Clearly R^{2n+1} is a normal contact manifold for

$$\omega = dx^{2n+1} - \sum_{i=1}^{n} x^{n+i} dx^i.$$

Let S^{2n+1} be the hypersphere in R^{2n+2} defined by

$$\sum_{a=1}^{2n+2} (x^a)^2 = 4.$$

Then R^{2n+2} is an almost Hermitian space for

$$J = (J^a_b) = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} I_{n+1} & 0 \\ 0 & I_{n+1} \end{pmatrix}$$

(where I_{n+1} is the unit matrix of dimension $n+1$). Defining $Z^a = \frac{1}{2} J^a_b x^b$, $a, b = 1, \ldots, 2n+2$, and map π: $T_s(R^{2n+2}) \to T_s(S^{2n+1})$ for s in S^{2n+1}, then

$$\omega = \frac{1}{2} \sum_{i=1}^{n+1} (x^{n+1+i} dx^i - x^i dx^{n+1+i})$$

is a contact form for S^{2n+1}; and (Φ, Z, ω, g) is a normal contact metric structure where $\Phi = -\pi J$ (cf. [*9*] [*34*] [*41*], § 1.36).

If M^n with coordinates $m = (m^1, \ldots, m^n)$ is an n dimensional Finsler manifold defined by function F on the tangent space T(M) with coordinates $y = (y^1, \ldots, y^n)$, where $F(m, y)$ is positively homogeneous of degree one in y with the rank of matrix $(\partial^2 F/\partial y^i \partial y^j)$ being $n-1$ $(i, j = 1, \ldots, n)$. Define $p_i = \partial F/\partial y^i$ and map $f: T(M) \to T^*(M)$: $(m, y) \to (m, p)$. Then $\omega = \Sigma p_i dm^i$ defines a contact structure on manifold $N = f(T(M))$ (except for points where $F = 0$) (v. [*3*] [*30*] [*18*], etc.). The extremal curves, i. e. C such that $\mathscr{L}(Z) \int_C \omega = 0$, iff $i(Z) d\omega = 0$ iff $i(\partial/\partial p_i) d\omega = 0$ and $i(\partial/\partial m^i) d\omega = 0$, project on extremals of $\int F(m^i, y^i = dm^i/du) du$; and conversely all extremals of M can be obtained in this manner.

Assume contact manifold M is homogeneous with respect to connected Lie group G, then M is regular contact manifold and if M is compact simply connected, then M is toral bundle over homogeneous Hodge manifold [*2*].

§ 9. HAMILTONIAN TRANSFORMATIONS [*21*] [*43*] [*1*] [*42*]

Let (B, Ω) be an almost symplectic manifold where B and Ω are C^∞ resp. C^ω. Let k denote R or C, the field of constants. The *Poisson brackets* is a map P: $U^0(B) \to V(B)$ given by $\eta(P(f))(\Omega)^n = ndf \wedge \eta \wedge (\Omega)^{n-1}$ for η in $U^1(B)$ and f in $U^0(B)$; and P is characterized uniquely by $i(P(f))\Omega = -df$ (v. [*43*]). If (B, Ω) is a symplectic manifold, then $X \in V(B)$ is a *locally Hamiltonian vector field* if $\mathscr{L}(X)\Omega = 0$ or equivalently $i(X)\Omega = \omega_0$, for some closed 1-form $\omega_0 \in C^1(B)$; i. e. X is a Γ_H-vector field, v. App. $X \in V(B)$ is a *Hamiltonian transformation* if $i(X)\Omega = -df$ for $f \in U^0(B)$. Let $H_0(B)$, resp. $H(B)$, denote the k-module of all locally Hamiltonian vector fields, resp. Hamiltonian transformations. Let p: $U^1(B) \to V(B)$ denote the bijection given by $\eta(p(\theta))(\Omega)^n = n\theta \wedge \eta \wedge (\Omega)^{n-1}$ for η, $\theta \in U^1(B)$ and characterized uniquely by $i(p(\theta)) = -\theta$.

Lemma (Libermann [*21*]; Takizawa [*43*]) *9.1.* — $H_0(B)$ is a Lie subalgebra of V(B) and $H(B)$ is a Lie subalgebra of $H_0(B)$. The map α: $X \to i(X)\Omega$ gives the Lie algebra isomorphism between $H_0(B)$ and $C^1(B)$ and takes $H(B)$ onto $dU^0(B)$.

The Lie algebra structure on $C^1(B)$ is given by $[\eta, \eta'] = i[p(\eta), p(\eta')]\Omega$.

Corollary 9.2. — $0 \to H(B) \to H_0(B) \to H^1(B, k) \to 0$ is an exact sequence; thus $H(B) \simeq H_0(B)$ when $H^1(B, k) = 0$, e. g. if B is simply connected.

Proposition 9.3. — If (B, Ω) is a symplectic manifold, then $P(f) \in H_0(B)$ (in fact $H(B)$); and P gives a Lie algebra isomorphism of $U^0(B)$ and $H_0(B)$. This isomorphism gives $U^0(B)$ a Lie algebra structure for Poisson brackets $[f, g]_P = -i(P(f))i(P(g))\Omega$.

Corollary 9.4. — $H_0(B)$ is a Lie algebra of infinite dimensions (so the pseudogroup of Hamiltonian transformations is infinite).

Proposition 9.5. — $0 \to k \to U^0(B) \simeq H_0(B) \xrightarrow{P(f)} H(B) \to 0$ is an exact sequence of Lie algebras (central extension).

§ 10. CONTACT TRANSFORMATIONS

Let (M, ω, Ω, Z) be an almost contact manifold. Then the *Lagrange brackets* is a map L: $U^0(M) \to V(M)$ given by

$$\eta(L(f))\omega \wedge (\Omega)^n = ndf \wedge \eta \wedge \omega \wedge (\Omega)^{n-1}$$

for $\eta \in U^1(M)$, $f \in U^0(M)$ and characterized uniquely by 1) $i(L(f))\omega = 0$ and 2) $i(L(f))\Omega = Zf\omega - df$ (v. [*43*]). Define the *canonical vector field* of Cartan-Reeb $K(f) \in V(M)$ by $K(f) := fZ + L(f)$.

Proposition (Takizawa [*43*]) *10.1.* — $K(f)$ is characterized uniquely by 1) $i(K(f))\omega = f$ and 2) $i(K(f))\Omega = (Zf)\omega - df$.

If (M, ω) is a contact manifold, then $X \in V(M)$ is an *infinitesimal contact transformation* if $\mathscr{L}(X)\omega = f\omega$ for some $f \in U^0(M)$; i. e. X is a Γ_C-vector field. X is an *infinitesmal automorphism of contact structure* if $\mathscr{L}(X)\omega = 0$. Let $C(M)$, resp. $C_0(M)$, denote the R-module of all infinitesmal contact transformations, resp. automorphisms of contact structure. Clearly $C(M)$ is a Lie subalgebra of $V(M)$ with respect to the usual bracket structure; and $C_0(M)$ is a Lie subalgebra of $C(M)$. If M has almost contact structure (Φ, Z, ω), then $X \in V(M)$ is an *infinitesmal* (Φ, Z, ω)*-transformation* if $\mathscr{L}(X)\Phi = 0$ and $\mathscr{L}(X)\omega = 0$. Denote the R-module of such by A; then A is a Lie algebra for the usual bracket.

Proposition (Takizawa [*43*]) *10.2.* — If (M, ω, Z) is a contact manifold, then K is characterized uniquely by

$$1)\ i(K(f)) = f \quad \text{and} \quad 2)\ i(K(f))d\omega = (Zf)\omega - df;$$

the inverse of K is ω: $\omega(K(f)) = f$; and $K(1) = Z$.

Cf. Souriau [*38*], § 4.25.

Proposition (Gray [*9*]; Libermann [*21*]; Sasaki-Hatakeyama [*34*]; Takizawa [*43*]) *10.3.* — If (M, ω, Z) is a contact manifold, then $K(f) \in C(M)$; and K gives an R-module bijection, in fact Lie algebra isomorphism, between $U^0(M)$ and $C(M)$ with inverse ω; so K induces the isomorphism of $U^0_0(M) = \{ f \in U^0(M) \mid Z(f) = i(Z)df = 0 \}$, the R-submodule of $U^0(M)$ of first integrals of Z (cf. § 3), onto $C_0(M)$. Thus

$$\begin{tikzcd} 0 \arrow[r] & U^0_0(M) \arrow[r] \arrow[d, shift left, "K"] & U^0(M) \arrow[r] \arrow[d, shift left, "K"] & U^0(M) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C_0(M) \arrow[r] \arrow[u, shift left, "\omega"] & C(M) \arrow[r] \arrow[u, shift left, "\omega"] & U^0(M) \arrow[r] & 0 \end{tikzcd}$$

is an exact sequence of sheaves.

Cf. Souriau [*38*], § 0.24, 4.25; [*41*], § 1.19; van Hove [*46*], § 5.6.

Corollary (Gray [*9*]) *10.4.* —

$$0 \to H^0(M, C_0(M)) \to H^0(M, C(M)) \to H^0(M, U^0(M)) \to H^1(M, C_0(M)) \to 0$$

is an exact sequence; and $H^q(M, C_0(M)) = 0$ for $q \geqslant 2$.

Corollary (Gray [*9*]; Libermann [*21*]; Takizawa [*43*]) *10.5.* — If $X = K(f)$

and $Y = K(g)$, then bijection ω gives R-module $U^0(M)$ a Lie algebra structure for bracket

$$[f, g] = i([X, Y])\omega = \omega([X, Y]) = -i(X)i(Y)d\omega + fi(Z)dg - gi(Z)df = L(f)g + fZg - gZf;$$

that is $K([f, g]) = [K(f), K(g)]$.

Cf. Souriau [*38*], § 4.26; [*41*], § 1.22.

If $\omega = dt - \sum_{i=1}^{n} p_i dq^i$, then $i(Z)dh = Z(h) = \partial h/\partial t$ for h in $U^0(M)$ and we have

Corollary (Libermann [*21*]) *10.6.* —

$$[f, g] = [f, g]_P - Zf(g + \Sigma p_i \partial g/\partial p_i) + Zg(f + \Sigma p_i \partial f/\partial p_i)$$

where $[f, g]_P$ is the Poisson bracket for $-d\omega$.

Cf. van Hove [*46*], § 5.

Corollary 10.7. — If f and g are basic functions for Z, i. e. $i(Z)df = 0$ and $i(Z)dg = 0$, then $[f, g] = [f, g]_P$.

Corollary (Cartan) *10.8.* — Bijection K implies the Lie algebra $C(M)$ is infinite dimensional (so the pseudogroup Γ_C of contact transformations is infinite).

Proposition (Ogawa [*27*]; Morimoto [*26*]) *10.9.* — If M is a compact manifold with normal almost contact structure then A is finite dimensional and the associated group of diffeomorphisms preserving this structure is a Lie transformation group.

§ 11. CONCLUSIONS

As we noted in § 3, we can identify $U_0^0(M)$ with $U^0(B)$, i. e. $p^*U^0(B) \simeq U_0^0(M)$. Recall from the introduction, Souriau's data $(\delta(f), S)$; we see then from § 9-10 that

Proposition 11.1. — $K(f) = \delta(f)$ and $S = p^*U^0(B) = p^*H_0(B) = U_0^0(M)$.

Cf. Souriau [*38*], § 4.20, [*40*], § 4.9.

As a corollary we see the equivalence to the program of van Hove. Finally, the details of contact manifolds above lead to simplifications in Kostant's work. In particular the Borel-Weil-Tits theory has a natural formulation in the language of Q. D. S.

APPENDIX

Let Diff (M) be the set of local diffeomorphisms of manifold M. A pseudogroup Γ on M is a collection of elements of Diff (M) such that 1) if g is in Γ, then g^{-1} is in Γ; 2) if g and h are in Γ, and $g \circ h$ is defined, then $g \circ h$ is in Γ; 3) if g is in Γ, then $g \mid U$ is in Γ for any open U of M; 4) the identity diffeomorphism is in Γ; 5) (completeness) if g is in Diff (M) and $\{ U_i \}$ is a covering of M such that $g \mid U_i$ is in Γ, then g is in Γ. Γ is transitive if for every m, m' in M there is a g in Γ with $g(m) = m'$. If $\eta \in U^p(M)$, then define

1) $\Gamma(\eta) = \{ g$ in Diff (M) $\mid \eta(g(m)) = \eta(m)$;
2) $\Gamma_r(\eta) = \{ g$ in Diff (M) $\mid \eta(g(m)) = f_g(m)\eta(m)$ for f_g in $U^0(M) \}$ if $d\eta \neq 0$ and

$$\Gamma_r(\eta) = \{ g \text{ in Diff } (M) \mid \eta(g(m)) = c_g \eta(m) \}$$

if $d\eta = 0$. These are pseudogroups. Cartan showed that there are six classes of primitive transitive infinite continuous pseudogroups which include $\Gamma(\Omega)$, $\Omega \in C^2(M^{2n})$, the Hamiltonian pseudogroup Γ_H and $\Gamma_r(\omega)$, $\omega \in U^1(M^{2n+1})$, the contact pseudogroup Γ_C. Vector field $X \in V(M)$ is a Γ-vector field if 1) $\mathscr{L}(X)\eta = 0$ for $\Gamma = \Gamma(\eta)$, 2) $\mathscr{L}(X)\eta = f_g(m)\eta$ for $\Gamma = \Gamma_r(\eta)$ with $d\eta \neq 0$ and 3) $\mathscr{L}(X)\eta = c_g\eta$ for $\Gamma = \Gamma_r(\eta)$ with $d\eta = 0$. Or equivalently if the one parameter group, $G_t = \exp(tX)$, generated by X belongs to Γ; then if G_t is in Γ, then $X = dG_t/dt\,|_{t=0}$ is a Γ-vector field. Clearly Γ-vector fields are closed under addition and the usual bracket, so form a Lie algebra.

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