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An exponentiation theorem for unbounded derivations

by

J. F. GILLE (*)

Abstract — We give a sufficient (and necessary) condition to define the exponential of unbounded derivations in C*-algebras.

1. Definitions

Let $\mathcal{A}$ be a Banach algebra, a derivation is a linear function $D$ from a dense sub-algebra $\mathcal{A}^{(1)}$ of $\mathcal{A}$, into $\mathcal{A}$, such that

$$\forall x \in \mathcal{A}^{(1)} \forall y \in \mathcal{A}^{(1)} \quad D(xy) = D(x)y + xD(y)$$

For a *-Banach algebra $\mathcal{A}$, the derivation $D$ is said to be hermitian if:

$$\forall x \in \mathcal{A}^{(1)} \quad x^* \in \mathcal{A}^{(1)} \quad \text{and} \quad D(x^*) = (D(x))^*.$$ 

The set of the elements $x$ in $\mathcal{A}$ such that the function

$$\zeta \to \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} D^n(x)$$

exists and is analytic in some neighbourhood of 0, is called « the set of the analytic elements » with respect to this derivation and is written $\mathcal{A}^{(a)}$.

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2. THEOREM

Let $\mathcal{A}$ be a C*-algebra, $D$ an hermitian closed derivation of $\mathcal{A}$, such as $\mathcal{A}^{(a)}$ is dense in $\mathcal{A}$, then $D$ induces a strongly continuous group $\{\alpha_t | t \in \mathbb{R}\}$ of automorphisms of $\mathcal{A}$.

Proof. — If $x \in \mathcal{A}^{(a)}$, $\exists t_x > 0$ such that $t \in \mathbb{R}$, $|t| \leq t_x$ we can define:

$$\alpha_t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n(x)$$

which is absolutely convergent in $\mathcal{A}$.

$$\alpha_t(x) \in \mathcal{A}^{(a)},$$

since for $|t' - t| < t_x - |t|$ we shall show that:

$$\alpha_{t'}(\alpha_t(x)) = \alpha_{t+t'}(x).$$

We write

$$y = \alpha_t(x); \quad y_j = \sum_{n=0}^{j} \frac{D^n(x)}{n!} t^n; \quad y_j \in \mathcal{A}^{(1)}.$$

$$D(y_j - y_k) = \sum_{n=k+1}^{\infty} \frac{D^{n+1}(x)}{n!} t^n$$

now $\sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n$ is analytic on $]-t_x, t_x[$, therefore ([1], 9.3.5) $(\alpha_t(x))'$ is absolutely and uniformly converging on the same interval

$$(\alpha_t(x))' = \sum_{n=1}^{\infty} \frac{D^n(x)}{(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{D^{n+1}(x)}{n!} t^n$$

We write $z_j = \sum_{n=0}^{j} \frac{D^{n+1}(x)}{n!} t^n$, then $(z_j)$ is a Cauchy sequence for $|| \cdot ||$ and:

$$z_j - z_k = \sum_{n=k+1}^{j} \frac{D^{n+1}(x)}{n!} t^n = D(y_j - y_k).$$

So $D(y_j - y_k)$ converges to 0 as $j$ and $k$ go to infinity. Let

$$z = \lim_{j \to \infty} D(y)_j.$$
Now, \( y = \lim_{j, \infty} y_j \) As \( D \) is closed, \( z = D(y) \)

\[
D(y) = \sum_{n=1}^{\infty} \frac{D^n(D(x))}{n!} t^n, \quad \lim_{j, \infty} D(y) = \alpha_t(D(x)).
\]

hence

(2.2) \[
D(\alpha_t(x)) = \alpha_t(D(x))
\]

and consequently

\[
\alpha_t(\alpha_t(x)) = \lim_{t, \infty} \sum_{k=0}^{\infty} \frac{D^k}{k!} \left( \sum_{n=0}^{\infty} \frac{D^n(x)}{n!} t^n \right) t^{rk} = \lim_{t, \infty} \sum_{n=0}^{\infty} \sum_{k=0}^{l} \frac{D^{k+n}(x)}{k! n!} t^n t^{rk}
\]

which is absolutely converging as \( l \) goes to infinity, so we can rearrange the terms:

\[
\sum_{m=0}^{\infty} \frac{D^m(x)}{m!} (t + t')^m = \alpha_{t+t'}(x).
\]

Through elementary calculations, taking advantage of the absolutely convergence of the series and of the continuity of * one gets:

(2.3) \[
\alpha_t(\lambda x + \lambda y) = \lambda \alpha_t(x) + \mu \alpha_t(y)
\]

(2.4) \[
\alpha_t(xy) = \alpha_t(x) \alpha_t(y)
\]

(2.5) \[
\alpha_t(x^*) = (\alpha_t(x))^*
\]

for \( t \in \mathbb{R} \) sufficiently small.

Moreover \( \forall t \in \mathbb{R}, \exists m \in \mathbb{N}, |t| < mt \); we write

\[
\alpha_t(x) = \left[ \frac{\alpha_t}{m} \right]^m (x)
\]

\( \alpha_t \) is now well defined for all \( t \in \mathbb{R} \) on \( \mathcal{A}^{(a)} \) and fulfils (2.1) and (2.2) for every \( \chi \) in \( \mathcal{A}^{(a)} \).

\( \alpha_t \) is a *-algebra isomorphism applying \( \mathcal{A}^{(a)} \) into \( \mathcal{A}^{(a)} \) and \( \forall \chi \in \mathcal{A}^{(a)} \), \( t \to \alpha_t(\chi) \) is an analytic function. We shall extend \( \alpha_t \) to \( \mathcal{A} \). We can assume that \( \mathcal{A} \) has a unit element, for, if not, we can define \( D \) on \( \mathcal{A} = C \times \mathcal{A} \), the algebra obtained from \( \mathcal{A} \) by adjunction of a unit element,

\[
D(\lambda, x) = (0, D(x)).
\]

Moreover, we can assume that \( e \in \mathcal{A}^{(1)} \); because if not one settles: \( D(e) = 0. \)
Note that $\alpha_t(e) = e$ because $D(e) = 0$. If $y = \alpha_0(y)$ is invertible, there exists a neighbourhood of 0 such that $\alpha_t(y)$ is invertible. Now if $t \to \alpha_t(y)$ is analytic, then $t \to (\alpha_t(y))^{-1}$ is also analytic. We can put $\alpha_t(y^{-1}) = (\alpha_t(y))^{-1}$ so $y \in \mathcal{A}^{(a)} \Rightarrow y^{-1} \in \mathcal{A}^{(a)}$ for $x \in \mathcal{A}^{(a)}$; $\lambda \in \mathbb{C}$.

$x - \lambda e$ invertible $\Rightarrow \exists y$ and $(x - \lambda e)y = e$ 
$\Rightarrow \alpha_t$ is well defined on $y$ and $[\alpha_t(x) - \lambda e] \alpha_t(y) = e$
therefore $(\alpha_t(x) - \lambda e)$ is invertible; hence $\text{Spec}' \alpha_t(x) \subset \text{Spec}' x$.

On the other hand, for an hermitian element $y$ of $\mathcal{A}$:

$$||y|| = \sup_{\zeta \in \text{Spec}' y} |\zeta|$$

([I], 15.4.14.1); hence:

$$||\alpha_t(x)||^2 = ||\alpha_t(x^*x)|| = \sup_{\zeta \in \text{Spec}' \alpha_t(x^*x)} |\zeta| \leq \sup_{\zeta \in \text{Spec}' x^*x} |\zeta| = ||x^*x|| = ||x||^2$$

and finally $||\alpha_t(x^*x)|| = ||x||$ on $\mathcal{A}^{(a)}$. We extend $\alpha_t$ to $\mathcal{A}$ (2.1) to (2.5) still hold $\forall x \in \mathcal{A}$, $\exists (y_n)_n$, $y_n \in \mathcal{A}^{(a)}$ and $x = \lim_n y_n$. Therefore

$$\lim_n ||\alpha_t(x) - \alpha_t(y_n)|| = 0.$$ 

$t \to \alpha_t(x)$ is continuous as a uniform limit of continuous functions. So that the one-parameter unitary group $\{\alpha_t | t \in \mathbb{R}\}$ is strongly continuous.

Comment. — We get an extension to C*-algebras of the work of E. Nelson on Hilbert spaces ([5]).

3. CONVERSE PROPOSITION

We give a new proof of the result of Kastler-Pool-Poulsen [4], which improves some one of I. Guelfand [3].

Let $\mathcal{E}$ be a Banach space, $\{\alpha_t\}_{t \in \mathbb{R}}$ a strongly continuous one-parameter group of uniformly bounded linear operators, i.e.

$$\exists M > 0 \forall t \in \mathbb{R} \quad ||\alpha_t|| \leq M$$

$\forall x \in \mathcal{E}$, $\forall \rho \in L^1_c(\mathbb{R})$; let $\alpha(\rho)x = \int_{-\infty}^{+\infty} \alpha_t(x)\rho(t)dt$, which exists in the Bochner’s sense since $||\alpha_t(x)\rho(t)|| \leq M ||x|| ||\rho(t)||$ and one has that:

$$t \to ||\alpha_t(x)\rho(t)|| \in L^1_c(\mathbb{R}).$$

PROPOSITION. — $\mathcal{E}^{(e)}$ ($= \{x \in \mathcal{E} | t \in \mathbb{R} \to \alpha_t(x)$ is entire $\}$) is dense in $\mathcal{E}$.
Proof. — Let $\rho$ be a function in $C^0$ so that $\hat{\rho} \in D$. Then $\rho \in \mathcal{S}$, and $\rho \in \mathcal{L}^{1}(\mathbb{R})$. Moreover, suppose that $\int_{-\infty}^{+\infty} \rho(t)dt = 1$. We notice that

\[ \forall \varepsilon > 0, \exists \eta > 0 \text{ so that } \int_{-\infty}^{-\eta} |\rho(t)| dt \leq \varepsilon \text{ and } \int_{\eta}^{+\infty} |\rho(t)| dt \leq \varepsilon. \]

Now if $\rho_n(t) = \eta \rho(nt)$, $\int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)| dt \leq \varepsilon$ and $\int_{\eta}^{+\infty} |\rho_n(t)| dt \leq \varepsilon$

\[ \| \alpha(\rho_n)x - x \| = \left\| \int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt - \int_{-\infty}^{+\infty} \rho_n(t)xdt \right\| \]

\[ \leq \int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)| \| \alpha_t(x) - x \| dt + \int_{\frac{\eta}{n}}^{+\infty} \ldots + \int_{\frac{\eta}{n}}^{+\infty} \ldots \]

Now, $\forall \varepsilon > 0, \exists n_0$ such that $\forall n \geq n_0$ and $|t| \leq \frac{\eta}{n} \Rightarrow \| \alpha_t(x) - x \| \leq \varepsilon$. On the other hand $\| \alpha_t(x) \| \leq M \| x \|$, hence

\[ \| \alpha(\rho_n)x - x \| \leq [2(M + 1) \| x \| + 1]\varepsilon \text{ and } x = \lim_n \alpha(\rho_n)x. \]

We prove that $\alpha(\rho_n)x \in \mathcal{S}^{(e)} \forall x \in \mathcal{S}$. Indeed:

\[ \alpha_t(\alpha(\rho_n)x) = \alpha_t\left(\int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt\right) = \int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(\alpha_t(x))dt \]

\[ = \int_{-\infty}^{+\infty} \rho_n(t-r)\alpha_r(x)dt \]

\[ = (\rho_n * h)(r) \]

where $h(r) = \alpha_r(x)$.

Now, $h$ being continuous and bounded, $\rho_n * h \in \mathcal{S}$ and $\hat{\rho_n * h} = \hat{\rho_n} \hat{h}$ is a distribution (cf. [6]) with compact support, hence due to the Paley-Wiener theorem $\rho_n * h$ is an entire function.

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