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Section A :
Physique théorique.

The irreducible ray representations of the full inhomogeneous Galilei group

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ABSTRACT. — *Out of the discrete symmetries space-inversion, time-inversion, space-time-inversion and the universal covering group of the inhomogeneous Galilei group a covering group of the full inhomogeneous Galilei group is constructed. The continuous multipliers on this covering group are calculated, and all those continuous irreducible ray representations are constructed, the restrictions of which to the unit component of the group have a nontrivial multiplier or are of class II in the nomenclature of Inönü and Wigner* [6]. *The direct product of two irreducible multiplier representations of the covering group is decomposed into its irreducible components. Finally, physical conclusions are drawn.*

RÉSUMÉ. — *Nous construisons un groupe de couverture du groupe Galiléen inhomogène complet engendré par le groupe de couverture universel du groupe Galiléen inhomogène et les symétries discrètes de l'inversion d'espace, de temps, et d'espace-temps. Les multiplicateurs continus sur ce groupe de couverture sont calculés, et toutes ces représentations projectives irréductibles sont construites, dont les restrictions à la composante connexe de l'unité du groupe sont équipées avec un multiplicateur non trivial, ou sont de classe II d'après la désignation de Inönü et Wigner* [6]. *Le produit direct de deux représentations irréductibles du groupe de couverture est décomposé en ses composantes irréductibles. Enfin, nous examinons les conséquences pour la théorie quantique non relativiste.*

INTRODUCTION

Though one knows about sixty years the Poincaré group to be the kinematical symmetry group of physics, interest in the Galilei group has remained for mainly two reasons: In a Galilean theory, first, the interpretation of observables and results is much more evident than in a relativistic theory, and, second, interactions can be included in a straightforward way. Furthermore, by a detailed study of Galilean invariance it is seen, that a host of effects usually considered to be relativistic, really is common to both relativistic and non-relativistic quantum theory: for example, the spin of particles, the different properties of spin for particles with mass and those without mass, and the gyromagnetic factor of the electron [*9*].

In sec. 1 we give a description of the full inhomogeneous Galilei group FIGG and construct a covering group $\widetilde{\mathrm{FIGG}}$, the components of which all are simply connected. We prove, that every continuous ray representation of $\widetilde{\mathrm{FIGG}}$ is generated by a continuous multiplier representation, and calculate all the continuous multipliers. Using a method of Mutze [*10*], in sec. 2 we calculate up to equivalence all those continuous irreducible ray representations of $\widetilde{\mathrm{FIGG}}$, which might describe non-relativistic particles. The direct product of multiplier representations of $\widetilde{\mathrm{FIGG}}$ is decomposed in sec. 3. In sec. 4, the implications for non-relativistic quantum theory are discussed and compared to those resulting from Poincaré invariance for relativistic quantum theory.

1. The full inhomogeneous Galilei group FIGG.

Algebraic properties. We denote the elements of FIGG by $(\mathrm{D}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_\mathrm{S}, \varepsilon_\mathrm{T})$, where $\mathrm{D} \in \mathrm{SO}(3)$, $(\boldsymbol{v}, \boldsymbol{a}, \overset{0}{a}) \in \mathbb{R}^7$, and $(\varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}) \in \mathrm{Z}_2 \times \mathrm{Z}_2$, $\mathrm{Z}_2 := \{+, -\}$. FIGG is a subgroup of the real affine group in four dimensions defined by

$$(1) \qquad (\mathrm{D}\, \boldsymbol{v}, ; \boldsymbol{a}, \overset{0}{a}; \varepsilon_\mathrm{S}, \varepsilon_\mathrm{T})(\boldsymbol{x}, t) := (\varepsilon_\mathrm{S}\mathrm{D}\boldsymbol{x} + \varepsilon_\mathrm{T}\boldsymbol{v}t + \boldsymbol{a}, \varepsilon_\mathrm{T}t + \overset{0}{a}).$$

FIGG can be considered as an algebraic subgroup of GL(5, $\mathbb{R}$):

$$(2) \qquad (\mathrm{D}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}) = \begin{bmatrix} \varepsilon_\mathrm{S}\mathrm{D} & \varepsilon_\mathrm{T}\boldsymbol{v} & \boldsymbol{a} \\ \mathbf{0} & \varepsilon_\mathrm{T}1 & \overset{0}{a} \\ \mathbf{0} & 0 & 1 \end{bmatrix};$$

this subgroup leaves invariant the hyperplanes perpendicular to (0 0 0 0 1). In the following we shortly write $(\mathrm{D}, \mathbf{v}, \boldsymbol{a}, \overset{0}{a})$ for $(\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; +, +)$, I for the identity $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; +, +)$, $\mathrm{I_S}$ for the space-inversion $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; -, +)$, $\mathrm{I_T}$ for the time-inversion $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; +, -)$, $\mathrm{I_{ST}}$ for the space-time-inversion $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; -, -)$. The group law is

$$
\begin{aligned}
&(\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_S, \varepsilon_T)(\mathrm{D}', \mathbf{v}'; \boldsymbol{a}', \overset{0}{a}'; \varepsilon_S', \varepsilon_T') := \\
&\quad = \big(\mathrm{DD}', \mathbf{v} + \varepsilon_S\varepsilon_T\mathrm{D}\mathbf{v}'; \boldsymbol{a} + \varepsilon_S\mathrm{D}\boldsymbol{a}' + \varepsilon_T\overset{0}{a}'\mathbf{v}, \overset{0}{a} + \varepsilon_T\overset{0}{a}'; \varepsilon_S\varepsilon_S', \varepsilon_T\varepsilon_T'\big), \\
(3)\quad &(\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_S, \varepsilon_T)^{-1} = \\
&\quad = (\mathrm{D}^{-1}, -\varepsilon_S\varepsilon_T\mathrm{D}^{-1}\mathbf{v}; -\varepsilon_S\mathrm{D}^{-1}(\boldsymbol{a} - \overset{0}{a}\mathbf{v}), -\varepsilon_T\overset{0}{a}; \varepsilon_S, \varepsilon_T).
\end{aligned}
$$

So we have $\mathrm{I_S}^2 = \mathrm{I_T}^2 = \mathrm{I_{ST}}^2 = \mathrm{I}$, $\mathrm{I_S I_T} = \mathrm{I_T I_S} = \mathrm{I_{ST}}$, and

$$
(4)\quad (\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; -, +) = (\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S}, \ (\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; +, -) = (\mathrm{D}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_T}.
$$

Topological properties. FIGG is an algebraic and hence closed subgroup of GL(5, $\mathbb{R}$) and therefore a Lie group in the induced topology. It is homeomorphic to $\mathrm{SO}(3) \times \mathbb{R}^7 \times Z_2 \times Z_2$, where Z_2 bears the discrete topology; from this immediately follows, that FIGG consists of four two-fold connected components. Let IGG be the unit component of FIGG, and IGG_i the subgroup generated by IGG and I_i, $i = \mathrm{S}, \mathrm{T}, \mathrm{ST}$.

Now let $\widetilde{\mathrm{FIGG}} := \mathrm{SU}(2) \times \mathbb{R}^7 \times Z_2 \times Z_2$ be the Lie group defined by the group law

$$
\begin{aligned}
&(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_S, \varepsilon_T)(\mathrm{U}', \mathbf{v}'; \boldsymbol{a}', \overset{0}{a}'; \varepsilon_S', \varepsilon_T') := \\
&\quad = \big(\mathrm{UU}', \mathbf{v} + \varepsilon_S\varepsilon_T\mathrm{D(U)}\mathbf{v}'; \boldsymbol{a} + \varepsilon_S\mathrm{D(U)}\boldsymbol{a}' + \varepsilon_T\overset{0}{a}'\mathbf{v}, \overset{0}{a} + \varepsilon_T\overset{0}{a}'; \varepsilon_S\varepsilon_S', \varepsilon_T\varepsilon_T'\big), \\
(5)\quad &(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_S, \varepsilon_T)^{-1} = \\
&\quad = (\mathrm{U}^{-1}, -\varepsilon_S\varepsilon_T\mathrm{D(U)}^{-1}\mathbf{v}; -\varepsilon_S\mathrm{D(U)}^{-1}(\boldsymbol{a} - \overset{0}{a}\mathbf{v}), -\varepsilon_T\overset{0}{a}; \varepsilon_S, \varepsilon_T)
\end{aligned}
$$

where D: $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is the canonical homomorphism. Writing again $(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})$ for $(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; +, +)$, I for $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; +, +)$, $\mathrm{I_S}$ for $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; -, +)$, $\mathrm{I_T}$ for $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; +, -)$, and $\mathrm{I_{ST}}$ for $(\mathbf{1}, \mathbf{0}; \mathbf{0}, 0; -, -)$, we obtain

$$
(6)\quad
\begin{aligned}
&(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; -, +) = (\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S}, \ \mathrm{I_S}(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S} = (\mathrm{U}, -\mathbf{v}; -\boldsymbol{a}, \overset{0}{a}), \\
&(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a}; +, -) = (\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_T}, \ \mathrm{I_T}(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_T} = (\mathrm{U}, -\mathbf{v}; \boldsymbol{a}, -\overset{0}{a}).
\end{aligned}
$$

$I_S^2 = I_T^2 = I_{ST}^2 = I$, $I_S I_T = I_T I_S = I_{ST}$. $\widetilde{\text{FIGG}}$ like FIGG consists of four components, but each component is simply connected. Again let $\widetilde{\text{IGG}}$ be the unit component, and $\widetilde{\text{IGG}}_i$ the subgroup generated by $\widetilde{\text{IGG}}$ and I_i, $i = $ S, T, ST. The mapping

$$(7) \quad k : \widetilde{\text{FIGG}} \to \text{FIGG},\ k(\mathrm{U}, \mathbf{v} ; \boldsymbol{a}, \overset{0}{a} ; \varepsilon_S, \varepsilon_T) := (\mathrm{D}(\mathrm{U}), \mathbf{v} ; \boldsymbol{a}, \overset{0}{a} ; \varepsilon_S, \varepsilon_T)$$

being an analytic homomorphism, and its restriction to $\widetilde{\text{IGG}}$, the universal covering group of IGG, being the canonical homomorphism, $\widetilde{\text{FIGG}}$ is a covering group of FIGG.

The multipliers of FIGG. Suppose R to be a continuous ray representation of FIGG, then $\mathrm{R} \circ k$ is a continuous ray representation of $\widetilde{\text{FIGG}}$, and there exists one and only one $m \in \mathbb{R}$ such, that the multipliers of $\mathrm{R} \circ k|_{\widetilde{\text{IGG}}}$ can be chosen as [*2*]

$$(8) \quad \exp(im\,\tau),\ \tau(\mathrm{U}, \mathbf{v} ; \boldsymbol{a}, \overset{0}{a} ; \mathrm{U}', \mathbf{v}' ; \boldsymbol{a}', \overset{0}{a}{}') := \mathbf{v}.\mathrm{D}(\mathrm{U})\boldsymbol{a}' + \overset{0}{a}{}'\mathbf{v}^2/2.$$

(9) **Theorem:** Let R be a continuous ray representation of $\widetilde{\text{FIGG}}$ such, that the multipliers of $\mathrm{R}|_{\widetilde{\text{IGG}}}$ are nontrivial. Then the space-inversion is represented by an unitary operator ray and the time-inversion is represented by an anti-unitary operator ray.

Proof : According to [*2*] we can choose a continuous representative U for $\mathrm{R}|_{\widetilde{\text{IGG}}}$ with the multiplier $\exp(im\,\tau)$, $m \neq 0$. Now it holds:

$$\mathrm{R}(\mathrm{I}_i)\mathrm{R}(g)\mathrm{R}(\mathrm{I}_i)^{-1} = \mathrm{R}(\mathrm{I}_i g \mathrm{I}_i)$$

for all $g \in \widetilde{\text{IGG}}$, $i = $ S, T, ST, and from this follows:

$$\exp(im\,\tau(\mathrm{I}_i g \mathrm{I}_i, \mathrm{I}_i g' \mathrm{I}_i)) = \delta_i(\exp(mi\,\tau(g, g')))$$

with $\delta_i(z) = z$ iff $\mathrm{R}(\mathrm{I}_i)$ is unitary and $\delta_i(z) = z^*$ iff $\mathrm{R}(\mathrm{I}_i)$ is anti-unitary. As we have for $i = $ S: $\tau(\mathrm{I_S} g \mathrm{I_S}, \mathrm{I_S} g' \mathrm{I_S}) = \tau(g, g')$, and for $i = $ T:

$$\tau(\mathrm{I_T} g \mathrm{I_T}, \mathrm{I_T} g' \mathrm{I_T}) = -\,\tau(g, g'),$$

$\mathrm{R}(\mathrm{I_S})$ has to be unitary and $\mathrm{R}(\mathrm{I_T})$ has to be anti-unitary. ■

Corollary: For any continuous ray representation of FIGG, the multipliers of which are nontrivial on any neighbourhood of the unit, space-inversion is an unitary operator ray and time-inversion is an anti-unitary operator ray.

For all ray representations $R \circ k$ of $\widetilde{\mathrm{FIGG}}$ we have, of course,

$$R \circ k(\pm \mathbf{1}, \mathbf{0}; \mathbf{0}, 0) = \mathrm{id}.$$

On the other hand, let R be some continuous ray representation of $\widetilde{\mathrm{FIGG}}$ with $R(\pm \mathbf{1}, \mathbf{0}; \mathbf{0}, 0) = \mathrm{id}$; then, just as in the case of connected Lie groups, $R \circ q$ is a continuous ray representation of FIGG for any choice of the section q: $\mathrm{FIGG} \to \widetilde{\mathrm{FIGG}}$, $k \circ q = \mathrm{id}_{\mathrm{FIGG}}$. Therefore instead of studying the ray representations of FIGG we can study the ray representations of $\widetilde{\mathrm{FIGG}}$ which represent the kernel of k trivially.

(10) **Theorem:** Every continuous ray representation of $\widetilde{\mathrm{FIGG}}$ is generated by a continuous multiplier representation of $\widetilde{\mathrm{FIGG}}$.

Proof: Let d: $\widetilde{\mathrm{FIGG}} \to \widetilde{\mathrm{FIGG}}$, $d(g) := \mathrm{I}$, $d(g\mathrm{I}_i) := \mathrm{I}_i$, $i = \mathrm{S, T, ST}$, for all $g \in \widetilde{\mathrm{IGG}}$, and U a representative of a continuous ray representation of $\widetilde{\mathrm{FIGG}}$ so, that $\mathrm{U}\,|_{\widetilde{\mathrm{IGG}}}$ is continuous and the multiplier on $\widetilde{\mathrm{IGG}}$ is some $\exp(im\,\tau)$, $\mathrm{U}(g\mathrm{I}_i) := \mathrm{U}(g)\mathrm{U}(\mathrm{I}_i)$ for all $i = \mathrm{S, T, ST}$, and $g \in \widetilde{\mathrm{IGG}}$. Let f: $\widetilde{\mathrm{FIGG}} \to \mathbb{C}_1 := \{ z \in \mathbb{C} : |z| = 1\}$, be defined by $f(g) := 1$,

$$f(g\mathrm{I}_i)\mathrm{U}(\mathrm{I}_i g \mathrm{I}_i) := \mathrm{U}(\mathrm{I}_i)\mathrm{U}(g)\mathrm{U}(\mathrm{I}_i)^{-1}$$

for all $g \in \widetilde{\mathrm{IGG}}$, $i = \mathrm{S, T, ST}$; then

$$f(g)\mathrm{U}(d(g)g) = f(g)\mathrm{U}(d(g)gd(g)^2) = \mathrm{U}(d(g))\mathrm{U}(gd(g))\mathrm{U}(d(g))^{-1}$$

for all $g \in \widetilde{\mathrm{FIGG}}$, and so f is continuous because d and $\mathrm{U}\,|_{\widetilde{\mathrm{IGG}}}$ are continuous. We have:

$$\begin{aligned}
\mathrm{U}(g\mathrm{I}_i)\mathrm{U}(g'\mathrm{I}_{i'}) := {} & \mathrm{U}(g)\mathrm{U}(\mathrm{I}_i)\mathrm{U}(g')\mathrm{U}(\mathrm{I}_{i'}) \\
= {} & \mathrm{U}(g)\mathrm{U}(\mathrm{I}_i)\mathrm{U}(g')\mathrm{U}(\mathrm{I}_i)^{-1}\mathrm{U}(\mathrm{I}_i)\mathrm{U}(\mathrm{I}_{i'}) \\
= {} & f(g'\mathrm{I}_i)\exp(im\,\tau(g, \mathrm{I}_i g'\mathrm{I}_i))w(\mathrm{I}_i, \mathrm{I}_{i'})\mathrm{U}(g\mathrm{I}_i g'\mathrm{I}_i)\mathrm{U}(\mathrm{I}_i\mathrm{I}_{i'}) \\
= {} & f(g'\mathrm{I}_i)\exp(im\,\tau(g, \mathrm{I}_i g'\mathrm{I}_i))w(\mathrm{I}_i, \mathrm{I}_{i'})\mathrm{U}(g\mathrm{I}_i g'\mathrm{I}_{i'})
\end{aligned}$$

where w is the multiplier defined on $\{\mathrm{I}, \mathrm{I}_\mathrm{S}, \mathrm{I}_\mathrm{T}, \mathrm{I}_{\mathrm{ST}}\}$. So our continuous multiplier is for $(g, g') \in \widetilde{\mathrm{FIGG}} \times \widetilde{\mathrm{FIGG}}$:

(11) $\lambda(g, g') = f(g'd(g'g))w(d(g), d(g'))\exp(im\,\tau(gd(g), d(g)g'd(g'g)))$;

the continuous multiplier representation is U,

$$\mathrm{U}(g) := \mathrm{U}(gd(g))\mathrm{U}(d(g)). \qquad \blacksquare$$

Moreover, for the continuous function f in (11) we have:

(12) $$f(gg'd) = \exp(im\,\tau(dgd, dg'd))\delta_d(\exp(-\,im\,\tau(g, g')))f(gd)f(g'd),$$
$$f(gdd') = f(d'gd'd)\delta_d(f(gd')),$$

for all $(g, g') \in \widetilde{\mathrm{IGG}} \times \widetilde{\mathrm{IGG}}$, $(d, d') \in \{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\} \times \{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\}$, where δ is a homomorphism of the subgroup $\{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\}$ into the operators $\{\mathrm{id}, \mathrm{K}\}$ on the complex numbers, $\mathrm{K}(z) := z^*$.

(13) **Theorem:** Any multiplier of a continuous representation of $\widetilde{\mathrm{FIGG}}$ with unitary space-inversion and anti-unitary time-inversion is equivalent to one of the multipliers $w_m^{\chi\varepsilon}$, $m \in \mathbb{R}$, $\chi, \varepsilon = \pm 1$:

$$w_m^{\chi\varepsilon}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}; \mathrm{U}', \boldsymbol{v}'; \boldsymbol{a}', \overset{0}{a}'; \varepsilon_\mathrm{S}', \varepsilon_\mathrm{T}') :$$
$$= w^{\chi\varepsilon}(\varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}; \varepsilon_\mathrm{S}', \varepsilon_\mathrm{T}') \exp\left(im\left(\varepsilon_\mathrm{S}\boldsymbol{v}\cdot\mathrm{D}(\mathrm{U})\boldsymbol{a}' + \varepsilon_\mathrm{T}\overset{0}{a}'\boldsymbol{v}^2/2\right)\right),$$
$$w^{\chi\varepsilon}(\varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}; \varepsilon_\mathrm{S}', \varepsilon_\mathrm{T}') := \left(\varepsilon_\mathrm{S}'^{(1-\varepsilon_\mathrm{T})/2}\right)^{(1-\chi)/2}\left(\varepsilon_\mathrm{T}'^{(1-\varepsilon_\mathrm{T})/2}\right)^{(1-\varepsilon)/2}.$$

$w_m^{\chi\varepsilon}$ is equivalent to $w_{m'}^{\chi'\varepsilon'}$ iff $m = m'$, $\chi = \chi'$, $\varepsilon = \varepsilon'$.

For all continuous representations of $\widetilde{\mathrm{FIGG}}$ with anti-unitary space-inversion or unitary time-inversion the multipliers can be chosen as class functions relative to $\widetilde{\mathrm{IGG}}$.

Proof: From (12) we have for $m \in \mathbb{R}$, because for $m \neq 0$ $\mathrm{I_S}$ has to be represented unitarily and $\mathrm{I_T}$ anti-unitarily: $f(gg'd) = f(gd)f(g'd)$ for all $g, g' \in \widetilde{\mathrm{IGG}}$ and $d = \mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}$. So f is a one-dimensional continuous vector representation of $\widetilde{\mathrm{IGG}}$ for fixed d. All such representations are of the form [4 ; p. 64]: $(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) \to \exp(ie\overset{0}{a})$, and so we have $f((\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})d) = \exp(ie_d\overset{0}{a})$ with $e_\mathrm{I} = 0$. Using again (12) we get for ray representations with

a) unitary space-inversion, unitary time-inversion:

$$1 = f(\mathrm{I_S}g\mathrm{I_S}\mathrm{I_S})f(g\mathrm{I_S}) = \exp(ie_\mathrm{S}\overset{0}{a} + ie_\mathrm{S}\overset{0}{a}) \Rightarrow e_\mathrm{S} = 0;\ 1 = f(g\mathrm{I_S}) =$$
$$= f(g\mathrm{I_T}\mathrm{I_{ST}}) = \exp(-\,ie_\mathrm{T}\overset{0}{a} + ie_\mathrm{ST}\overset{0}{a}) \Rightarrow e_\mathrm{ST} = e_\mathrm{T} =: e$$
$$\Rightarrow f(gd) = r(g'd)r(gd')/r(g'dgd'),\ r(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}; \varepsilon_\mathrm{S}, \varepsilon_\mathrm{T}) := \exp(ie\overset{0}{a}/2);$$

b) unitary space-inversion, anti-unitary time-inversion:

$$1 = f(g\mathrm{I_S I_S}) = f(\mathrm{I_S} g \mathrm{I_S I_S}) f(g\mathrm{I_S}) = \exp(ie_{\mathrm{S}}\overset{0}{a} + ie_{\mathrm{S}}\overset{0}{a}) \Rightarrow e_{\mathrm{S}} = 0\ ;$$
$$1 = f(g\mathrm{I_T I_T}) = f(\mathrm{I_T} g \mathrm{I_T I_T}) f(g\mathrm{I_T}) = \exp(-\, ie_{\mathrm{T}}\overset{0}{a} - ie_{\mathrm{T}}\overset{0}{a}) \Rightarrow e_{\mathrm{T}} = 0\ ;$$

in the same way we find: $e_{\mathrm{ST}} = 0$ and so $f \equiv 1$;

c) anti-unitary space-inversion, unitary time-inversion:

$$1 = f(g\mathrm{I_{ST} I_{ST}}) = f(\mathrm{I_{ST}} g \mathrm{I_{ST} I_{ST}}) f(g\mathrm{I_{ST}}) = \exp(-\, ie_{\mathrm{ST}}\overset{0}{a} - ie_{\mathrm{ST}}\overset{0}{a}) \Rightarrow e_{\mathrm{ST}} = 0$$
$$1 = f(g\mathrm{I_S I_T}) = f(\mathrm{I_T} g \mathrm{I_T I_S}) f(g\mathrm{I_T}) = \exp(-\, ie_{\mathrm{S}}\overset{0}{a} - ie_{\mathrm{T}}\overset{0}{a}) \Rightarrow e_{\mathrm{T}} = -\, e_{\mathrm{S}} = : e\ ;$$
$$f(gd) = r(g'd)\delta_d(r(gd'))/r(g'dgd'),\ r(\mathrm{U}, \boldsymbol{v}\ ;\ \boldsymbol{a}, \overset{0}{a}\ ;\ \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}}) : = \exp(ie\overset{0}{a}/2)\ ;$$

d) anti-unitary space-inversion, anti-unitary time-inversion:

$$1 = f(g\mathrm{I_T I_T}) = f(\mathrm{I_T} g \mathrm{I_T I_T}) f(g\mathrm{I_T}) = \exp(-\, ie_{\mathrm{T}}\overset{0}{a} - ei_{\mathrm{T}}\overset{0}{a}) \Rightarrow e_{\mathrm{T}} = 0\ ;$$
$$1 = f(g\mathrm{I_{ST} I_S}) = f(\mathrm{I_S} g \mathrm{I_S I_{ST}}) f(g\mathrm{I_S}) = \exp(ie_{\mathrm{ST}}\overset{0}{a} + ie_{\mathrm{S}}\overset{0}{a}) \Rightarrow e_{\mathrm{ST}} = -\, e_{\mathrm{S}} = : e$$

and so $f(gd) = r(g'd)\delta_d(r(gd'))/r(g'dgd'),\ r(\mathrm{U}, \boldsymbol{v}\ ;\ \boldsymbol{a}, \overset{0}{a}\ ;\ \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}}) = \exp(ie\overset{0}{a}/2)$.

So all the multipliers of continuous ray representations of $\widetilde{\mathrm{FIGG}}$ are equivalent to those of the form $\lambda(gd, g'd') = w(d, d') \exp(im\tau(g, dg'd))$ and so for $m = 0$ they can be chosen as class functions relative to $\widetilde{\mathrm{IGG}}$. For $m \neq 0$, according to (9) $\mathrm{I_S}$ is represented unitarily and $\mathrm{I_T}$ is represented anti-unitarily, and so w is of the form $w^{\chi\varepsilon}$ [3 ; p. 169]. ■ Let R be a continuous ray representation of FIGG ; then $\mathrm{R} \circ k$, k as in (7), is a continuous ray representation of $\widetilde{\mathrm{FIGG}}$, and we can choose a continuous representative $\mathrm{R}e$ of $\mathrm{R} \circ k$, that has a multiplier as defined in (13). So for some section q: FIGG $\to \widetilde{\mathrm{FIGG}}$ we get a multiplier representation $\mathrm{R}e \circ q$ of FIGG, that is a representative of R. However, the choice of such a section q defines an unique section U: SO(3) $\to$ SU(2) and v. v. so, that $q(\mathrm{D}, \boldsymbol{v}\ ;\ \boldsymbol{a}, \overset{0}{a}\ ;\ \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}}) = (\mathrm{U(D)}, \boldsymbol{v}\ ;\ \boldsymbol{a}, \overset{0}{a}\ ;\ \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}})$. Now, $\mathrm{R}e(-\mathbf{1}) = \pm \mathbf{1}$, and for $\mathrm{R}e(-\mathbf{1}) = +\mathbf{1}$ we get: $\mathrm{R}e \circ q(g)\, \mathrm{R}e \circ q(g') = \lambda(q(g), q(g'))\mathrm{R}e \circ q(gg')$, where λ is the multiplier of $\mathrm{R}e$. For $\mathrm{R}e(-\mathbf{1}) = -\mathbf{1}$ we get: $\mathrm{R}e \circ q(g)\, \mathrm{R}e \circ q(g') = \zeta(\mathrm{D}, \mathrm{D}')\lambda(q(g), q(g'))\mathrm{R}e \circ q(gg')$ with $g = (\mathrm{D}, \boldsymbol{v}\ ;\ \boldsymbol{a}, \overset{0}{a}\ ;\ \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}})$, $g' = (\mathrm{D}', \boldsymbol{v}'\ ;\ \boldsymbol{a}', \overset{0}{a}{}'\ ;\ \varepsilon_{\mathrm{S}}', \varepsilon_{\mathrm{T}}')$, where

(14) $\zeta(\mathrm{D}, \mathrm{D}') : = 1$ iff $\mathrm{U(D)U(D')} = \mathrm{U(DD')}$, $\zeta(\mathrm{D}, \mathrm{D}') : = -\, 1$ else;

as is shown in [5], ζ is a nontrivial multiplier on SO(3).

Corollary: The multipliers of continuous ray representations of FIGG are equivalent to such of the form λ_i,

$$\lambda_i(\mathrm{D}, \mathbf{v}\,;\, \boldsymbol{a}, \overset{0}{a}\,;\, \varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}};\, \mathrm{D}', \mathbf{v}'\,;\, \boldsymbol{a}', \overset{0}{a}{}'\,;\, \varepsilon_{\mathrm{S}}', \varepsilon_{\mathrm{T}}'):=$$
$$= \zeta(\mathrm{D}, \mathrm{D}')^{2i} w(\varepsilon_{\mathrm{S}}, \varepsilon_{\mathrm{T}}\,;\, \varepsilon_{\mathrm{S}}', \varepsilon_{\mathrm{T}}') \exp\left(im(\varepsilon_{\mathrm{S}}\mathbf{v}\cdot\mathrm{D}\boldsymbol{a}' + \varepsilon_{\mathrm{T}}\overset{0}{a}{}'\mathbf{v}^2/2)\right),$$

with $m \in \mathbb{R}$, $i = 0, 1/2$, where w is a multiplier on $\{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\}$; w for $m \neq 0$ is equivalent to a $w^{\chi\varepsilon}$ as defined in (13).

2. Some irreducible continuous ray representations of FIGG.

The irreducible ray representations of IGG have been studied for trivial multipliers by Inönü and Wigner [*6*] and for nontrivial multipliers by Levy-Leblond [*8*] and by Brennich [*4*]. As we need them for the construction of the ray representations of FIGG, we give a complete list of these ray representations. For mathematical convenience they are defined on $\widetilde{\mathrm{IGG}}$. From now on, by Hilbert space we mean a separable Hilbert space, and by representations we mean such on a separable Hilbert space.

(15) **Theorem** [4 ; p. 67]: All continuous ray representations of $\widetilde{\mathrm{IGG}}$ are of type I ; every continuous irreducible ray representation of $\widetilde{\mathrm{IGG}}$ is unitary equivalent to one of the following ray representations:

I $\quad [s, \mathbf{0}; \boldsymbol{p}, e, m]$, $s = 0, 1/2, 1, \ldots, \boldsymbol{p} \in \mathbb{R}^3, e \in \mathbb{R}, m \in \mathbb{R}\backslash\{0\}$;

representative $\underline{(s, \mathbf{0}; \boldsymbol{p}, e, m)}$ $(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\mathbf{x}):=$

$$= \exp\left(i(e - \boldsymbol{p}^2/2m + \mathbf{x}^2/2m)\overset{0}{a} - i\mathbf{x}\cdot\boldsymbol{a}\right)\mathrm{D}^{(\mathrm{S})}(\mathrm{U})\mathrm{F}(\mathrm{D}(\mathrm{U})^{-1}(\mathbf{x} - m\mathbf{v})),$$

where $\mathrm{F} \in \mathrm{L}_2(\mathbb{R}^3, \mathbb{C}^{2s+1})$ and $\mathrm{D}^{(s)}$ is the well known irreducible unitary representation of SU(2);

II $\quad [s, \mathbf{0}; \boldsymbol{p}, e, 0]$, $s = 0, \pm 1/2, \pm 1, \ldots, \boldsymbol{p} \in \mathbb{R}^3\backslash\{\mathbf{0}\}, e \in \mathbb{R}$;

representative $\underline{(s, \mathbf{0}; \boldsymbol{p}, e, 0)}$ $(\mathrm{U}, \mathbf{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\mathbf{x}, t):=$

$$= \exp\left(is\,\chi(\boldsymbol{p}, \mathbf{x}; \mathrm{U}) + i(e + t)\overset{0}{a} - i\mathbf{x}\cdot\boldsymbol{a}\right)\mathrm{F}(\mathrm{D}(\mathrm{U})^{-1}\mathbf{x}, t - \mathbf{x}\cdot\mathbf{v}),$$

where $\mathrm{F} \in \mathrm{L}_2(\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = |\boldsymbol{p}|\} \times \mathbb{R}, \mathbb{C})$ and χ is defined in (16);

III $[s, \boldsymbol{k}; \boldsymbol{p}, e, 0], s = 0, 1/2, \boldsymbol{k} \in \mathbb{R}^3\backslash\{\mathbf{0}\}, \boldsymbol{p} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$ with $\boldsymbol{k}\cdot\boldsymbol{p} = 0$, $e \in \mathbb{R}$;

representative $\underline{(s, \boldsymbol{k}; \boldsymbol{p}, e, 0)}\,(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{y}, \boldsymbol{x}, t) : =$

$$= \varepsilon(\boldsymbol{k}, \boldsymbol{y}; \chi(\boldsymbol{p}, \boldsymbol{x}; \mathrm{U})\boldsymbol{p}/|\,\boldsymbol{p}\,|)^{2s} \exp\,(i(e + t)\overset{0}{a} - i\boldsymbol{x}\cdot\boldsymbol{a})$$
$$\times \exp\,(i(\boldsymbol{x} \times \boldsymbol{v})\cdot(\boldsymbol{p} \times \boldsymbol{y})/p^2 + i(\boldsymbol{p}\cdot\boldsymbol{x} \times \boldsymbol{v})(\boldsymbol{p}\cdot\boldsymbol{x} \times \boldsymbol{y})/\boldsymbol{p}^2(\boldsymbol{p}^2 + \boldsymbol{p}\cdot\boldsymbol{x}))$$
$$\times\ \mathrm{F}(\mathrm{D}(\mathrm{U}(\chi(\boldsymbol{p}, \boldsymbol{x}; \mathrm{U})\boldsymbol{p}/|\,\boldsymbol{p}\,|))^{-1}\boldsymbol{y}, \mathrm{D}(\mathrm{U})^{-1}\boldsymbol{x}, t - \boldsymbol{x}\cdot\boldsymbol{v}),$$

where

$\mathrm{F} \in \mathrm{L}_2(\{\boldsymbol{y} \in \mathbb{R}^3 : |\,\boldsymbol{y}\,| = |\,\boldsymbol{k}\,|, \boldsymbol{y}\cdot\boldsymbol{p} = 0\} \times \{\boldsymbol{x} \in \mathbb{R}^3 : |\,\boldsymbol{x}\,| = |\,\boldsymbol{p}\,|\} \times \mathbb{R}, \mathbb{C})$

and χ and ε are defined in (16) ;

IV $[s, \mathbf{0}; \mathbf{0}, e, 0], s = 0, 1/2, 1, \ldots, e \in \mathbb{R}$;

representative $\underline{(s, \mathbf{0}; \mathbf{0}, e, 0)}\,(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) : = \exp\,(ie\overset{0}{a})\mathrm{D}^{(s)}(\mathrm{U})$;

V $[s, \boldsymbol{k}; \mathbf{0}, e, 0], s = 0, \pm 1/2, \pm 1, \ldots, \boldsymbol{k} \in \mathbb{R}^3\backslash\{\mathbf{0}\}, e \in \mathbb{R}$;

representative $\underline{(s, \boldsymbol{k}; \mathbf{0}, e, 0)}\,(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{x}) : =$

$$= \exp\,(is\chi(\boldsymbol{k}, \boldsymbol{x}; \mathrm{U}) + ie\overset{0}{a} + i\boldsymbol{x}\cdot\boldsymbol{v})\mathrm{F}(\mathrm{D}(\mathrm{U})^{-1}\boldsymbol{x}),$$

where $\mathrm{F} \in \mathrm{L}_2(\{\boldsymbol{x} \in \mathbb{R}^3 : |\,\boldsymbol{x}\,| = |\,\boldsymbol{k}\,|, \mathbb{C})$ and χ is defined in (16). If two irreducible ray representations of $\widetilde{\mathrm{IGG}}$ are equivalent, denoted by $\sim$, they necessarily belong to the same case; then

$$\begin{aligned}
&[s, \mathbf{0}; \boldsymbol{p}, e, m] \sim [s', \mathbf{0}; \boldsymbol{p}', e', m'] &&\Leftrightarrow\ s = s',\ |\,m\,| = |\,m'\,|;\\
&[s, \mathbf{0}; \boldsymbol{p}, e, 0] \sim [s', \mathbf{0}; \boldsymbol{p}', e', 0] &&\Leftrightarrow\ s = s',\ |\,\boldsymbol{p}\,| = |\,\boldsymbol{p}'\,|;\\
&[s, \boldsymbol{k}; \boldsymbol{p}, e, 0] \sim [s', \boldsymbol{k}'; \boldsymbol{p}', e', 0] &&\Leftrightarrow\ s = s',\ |\,\boldsymbol{k}\,| = |\,\boldsymbol{k}'\,|,\ |\,\boldsymbol{p}\,| = |\,\boldsymbol{p}'\,|;\\
&[s, \mathbf{0}; \mathbf{0}, e, 0] \sim [s', \mathbf{0}; \mathbf{0}, e', 0] &&\Leftrightarrow\ s = s';\\
&[s, \boldsymbol{k}; \mathbf{0}, e, 0] \sim [s', \boldsymbol{k}'; \mathbf{0}, e', 0] &&\Leftrightarrow\ s = s',\ |\,\boldsymbol{k}\,| = |\,\boldsymbol{k}'\,|.
\end{aligned}$$

In theorem (15) χ and ε are defined as

$\chi : \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^6 : |\,\boldsymbol{x}\,| = |\,\boldsymbol{y}\,| \neq 0\} \times \mathrm{SU}(2) \to \mathbb{R}$ with

$$\exp\left(i\chi\left(\boldsymbol{x}, \boldsymbol{y}\,;\, \cos\frac{t}{2}\mathbf{1} - i\sin\frac{t}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\Big/2\right) : =$$

$$(16)\qquad = \frac{(\boldsymbol{x}^2 + \boldsymbol{x}\cdot\boldsymbol{y})\cos\frac{t}{2} + \boldsymbol{n}\cdot(\boldsymbol{x}\times\boldsymbol{y} + i(\boldsymbol{x}+\boldsymbol{y})\,|\,\boldsymbol{x}\,|\,)\sin\frac{t}{2}}{\sqrt{\left((\boldsymbol{x}^2 + \boldsymbol{x}\cdot\boldsymbol{y})\cos\frac{t}{2} + \boldsymbol{n}\cdot(\boldsymbol{x}\times\boldsymbol{y})\sin\frac{t}{2}\right)^2 + (\boldsymbol{n}\cdot(\boldsymbol{x}+\boldsymbol{y}))^2\boldsymbol{x}^2\sin^2\frac{t}{2}}}$$

$$\varepsilon : \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^9 : |\,\boldsymbol{x}\,| = |\,\boldsymbol{y}\,| \neq 0, \boldsymbol{x}\cdot\boldsymbol{z} = \boldsymbol{y}\cdot\boldsymbol{z} = 0\} \to \{+1, -1\},$$
$$\varepsilon(\boldsymbol{x}, \boldsymbol{y}\,;\, \boldsymbol{z}) : = \mathrm{sign}\,(\cos\,(\arccos\,(\boldsymbol{x}\cdot\boldsymbol{y}/\boldsymbol{x}^2) + |\,\boldsymbol{z}\,|\,\mathrm{sign}\,(\boldsymbol{z}\cdot\boldsymbol{x}\times\boldsymbol{y}))),$$

where arc cos is chosen in the interval $[0, \pi]$.

The ray representations $[s, \mathbf{0}; \boldsymbol{p}, e, m]$, $m \neq 0$, and possibly $[s, \mathbf{0}; \boldsymbol{p}, e, 0]$ describe non-relativistic particles [8] [4]. In the following these ray representations of $\widetilde{\text{IGG}}$ are continued to ray representations of $\widetilde{\text{FIGG}}$. For this purpose we introduce the somewhat shorter notation: $[m, s; e]$ for $[s, \mathbf{0}; \mathbf{0}, e, m]$, $(m, s; e)$ for $\underline{(s, \mathbf{0}; \mathbf{0}, e, m)}$, $m \neq 0$, and $[0, s; p]$ for $[s, \mathbf{0}; 0, 0, p, 0, 0]$, $(0, s; p)$ for $\underline{(s, \mathbf{0}; 0, 0, p, 0, 0)}$, $p > 0$. We further note, that $(m, s; e)$ is a multiplier representation of $\widetilde{\text{IGG}}$ with the multiplier $\exp(im\tau)$ from (8), and that $(m, s; e)$ and $(m', s'; e')$ are unitary equivalent iff $m = m'$, $s = s'$, and $e = e'$.

A theorem on the ray representation of discrete symmetries. Let S be a continuous ray representation of a topological groupe G. Then the unitary subgroup of S, i. e. the subset of elements of G represented by an unitary operator ray, is a closed normal subgroup of index 1 or 2. This follows immediately from the fact, that card $[G/f^{-1}(N')] \leqslant \text{card}[G'/N']$ for any group homorphism $f\colon G \to G'$ and any normal subgroup N' of G', and that the set of unitary operator rays is a closed normal subgroup of index 2 of the group of all operator rays [10; p. 16].

(17) **Theorem** [10; p. 21]: Let G be a topological group with a closed normal subgroup N of index 2 and $d \in G\backslash N$. Then for any continuous irreducible unitary ray representation S of N with the representative U, the multiplier of which is w, we have:

If S satisfies the condition:

(U_s) there exists an unitary operator ray V with

$$V^2 = S(d^2) \text{ and } \bigwedge_{g \in N} S(g) = VS(d^{-1}gd)V^{-1}$$

the mapping $\bar{S}$, $\bigwedge_{g \in N} (\bar{S}(g) := S(g), \bar{S}(gd) := S(g)V)$

defines a continuous irreducible unitary ray representation of G. For an equivalent ray representation S' of N, V' may be chosen so, that $\bar{S}'$ is equivalent to $\bar{S}$.

If w satisfies the condition :

$$(U_d) \quad \bigwedge_{g \in N} \bigvee_{r(g) \in C_1} \Big(r(d^2) = 1,\ w(g, d^2)w(d^2, d^{-2}gd^2)^{-1} = r(g)r(d^{-1}gd),$$
$$\bigwedge_{g' \in N} w(g, g')w(d^{-1}gd, d^{-1}g'd)^{-1} = r(g)r(g')/r(gg')\Big)$$

the mapping $\bar{S}$,

$$\bigwedge_{g\in N} \bar{S}(g) := \left[\begin{bmatrix} U(g) & 0 \\ 0 & r(g)U(d^{-1}gd) \end{bmatrix}\right], \qquad \bar{S}(gd) := \bar{S}(g)\left[\begin{bmatrix} 0 & U(d^2) \\ 1 & 0 \end{bmatrix}\right]$$

defines an unitary ray representation of G that is irreducible iff no operator ray V satisfying (U_s) exists. The choice of U is of no importance, as a different U gives the same $\bar{S}$ if we choose r accordingly. If U is continuous, $\bar{S}$ is continuous iff r is continuous. For an equivalent ray representation S′ of N, U′ and r' can be chosen so, that $\overline{S'}$ is equivalent to $\bar{S}$.

Furthermore, any irreducible continuous unitary ray representation of G is unitary equivalent to a ray representation as defined after (U_s) and (U_d).

If S satisfies the condition:

(A_s) there exists an anti-unitary operator ray W with

$$W^2 = S(d^2) \text{ and } \bigwedge_{g\in N} S(g) = WS(d^{-1}gd)W^{-1}$$

the mapping $\bar{S}$, $\bigwedge_{g\in N} (\bar{S}(g) := S(g), \bar{S}(gd) := S(g)W)$

is a continuous irreducible ray representation of G with unitary subgroup N. For an equivalent ray representation S′ of N, W′ may be chosen so, that $\overline{S'}$ is equivalent to $\bar{S}$.

If S satisfies (A_s) and A is a representative of W, for every homomorphism χ: N $\to$ $\mathbb{C}_1$ such, that

$\bigwedge_{g\in N} \chi(g) := \chi(d^{-1}gd)$, the mapping $\bar{S}$,

$$\bigwedge_{g\in N} \bar{S}(g) := \left[\begin{bmatrix} U(g) & 0 \\ 0 & \chi(g)U(g) \end{bmatrix}\right], \qquad \bar{S}(gd) := \bar{S}(g)\left[\begin{bmatrix} 0 & A \\ \varepsilon\sqrt{\chi(d^2)}A & 0 \end{bmatrix}\right],$$

$\varepsilon \in \{+1, -1\}$, is a ray representation of G with unitary subgroup N, that is irreducible iff there is no unitary operator D satisfying

$$\text{(R)} \qquad \bigwedge_{g\in N} U(g) = \chi(g)DU(g)D^{-1}, \qquad A = \varepsilon\sqrt{\chi(d^2)}DAD.$$

$\bar{S}$ does not depend on the choice of U, and if U is continuous, $\bar{S}$ is continuous iff χ is continuous. For an equivalent ray representation S′ of N, U′, χ', and ε' can be chosen so, that $\overline{S'}$ is equivalent to $\bar{S}$.

If w satisfies the condition:

$$(A_d)\quad \bigwedge_{g\in N}\bigvee_{r(g)\in C_1}(w(g,d^2)/w(d^2,d^{-2}gd^2) = r(g)/r(d^{-1}gd),$$
$$\bigwedge_{g'\in N} w(g,g')w(d^{-1}gd,d^{-1}g'd) = r(g)r(g')/r(gg'))$$

the mapping $\bar{S}$ for $K = K^+ = K^{-1}$ anti-linear and $\varepsilon \in \{+1, -1\}$

$$\bigwedge_{g\in N}\bar{S}(g) := \left[\begin{bmatrix} U(g) & 0 \\ 0 & r(g)KU(d^{-1}gd)K \end{bmatrix}\right],$$
$$\bigwedge_{g\in N}\bar{S}(gd) := \bar{S}(g)\left[\begin{bmatrix} 0 & \varepsilon r(d^2)^{-1/2}U(d^2)K \\ K & 0 \end{bmatrix}\right]$$

is a ray representation of G with unitary subgroup N, that is irreducible if (A_s) is not satisfied and else is equivalent to a ray representation as defined after (A_s). The choice of U is of no importance, as a different U gives the same $\bar{S}$ if we choose r accordingly. If U is continuous, $\bar{S}$ is continuous iff r is continuous. For an equivalent ray representation S′ of N, U′, r', K′, and ε' can be chosen so, that $\bar{S'}$ is equivalent to $\bar{S}$.

Furthermore, any irreducible continuous ray representation of G with unitary subgroup N is unitary equivalent to a ray representation as defined after (A_s) and (A_d).

The physical irreducible ray representations of FIGG. We call « *physical* » those continuous ray representations of FIGG, in which space-inversion is represented unitarily, time-inversion is represented anti-unitarily and the restriction of which to IGG decomposes into ray representations of case I and II. As the case I ray representations of $\widetilde{\text{IGG}}$ have nontrivial multipliers, while the case II ray representations of $\widetilde{\text{IGG}}$ have trivial multipliers, in such a decomposition only either case I or either case II ray representations can occur.

(18) **Theorem:** Any continuous irreducible ray representation of $\widetilde{\text{FIGG}}$, the restriction of which to $\widetilde{\text{IGG}}$ has a nontrivial multiplier, is equivalent to one of the ray representations $[m, s; e]^{\chi\varepsilon}$, $m \in \mathbb{R}\backslash\{0\}$, $s = 0, 1/2, 1, \ldots$, $e \in \mathbb{R}$, χ, $\varepsilon \in \{+, -\}$, which are defined by their representatives

R: $= (m, s; e)_\eta^{+\varepsilon}$, $\eta = \pm$, and R: $= (m, s; e)^{-\varepsilon}$, respectively, as following:

$$\mathrm{R}(gd) := \mathrm{R}(g)\mathrm{R}(d) \text{ for all } g \in \widetilde{\mathrm{IGG}},\ d \in \{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\},$$

$$\mathrm{R}(\mathrm{I_{ST}}) := \mathrm{R}(\mathrm{I_S})\mathrm{R}(\mathrm{I_T}) \text{ and with } \mathrm{F}, \mathrm{G} \in \mathrm{L}_2(\mathbb{R}^3, \mathbb{C}^{2s+1}):$$

$$(m, s; e)_\eta^{++}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{x}) := (m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{x}),$$

$$(m, s; e)_\eta^{++}(\mathrm{I_S})\mathrm{F}(\boldsymbol{x}) := \eta \mathrm{F}(-\boldsymbol{x}),$$

$$(m, s; e)_\eta^{++}(\mathrm{I_T})\mathrm{F}(\boldsymbol{x}) := \mathrm{D}^{(s)}(i\sigma_2)\mathrm{F}(-\boldsymbol{x})^*;$$

$$(m, s; e)_\eta^{+-}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \begin{bmatrix}(m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}\\ (m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{G}\end{bmatrix}(\boldsymbol{x}),$$

$$(m, s; e)_\eta^{+-}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \eta\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(-\boldsymbol{x}),$$

$$(m, s; e)_\eta^{+-}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \begin{bmatrix}-\mathrm{D}^{(s)}(i\sigma_2)\mathrm{G}(-\boldsymbol{x})^*\\ \mathrm{D}^{(s)}(i\sigma_2)\mathrm{F}(-\boldsymbol{x})^*\end{bmatrix};$$

$$(m, s; e)^{-\varepsilon}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \begin{bmatrix}(m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}\\ (m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{G}\end{bmatrix}(\boldsymbol{x}),$$

$$(m, s; e)^{-\varepsilon}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \begin{bmatrix}\mathrm{F}\\ -\mathrm{G}\end{bmatrix}(-\boldsymbol{x}),$$

$$(m, s; e)^{-\varepsilon}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}) := \begin{bmatrix}\varepsilon\mathrm{D}^{(s)}(i\sigma_2)\mathrm{G}(-\boldsymbol{x})^*\\ \mathrm{D}^{(s)}(i\sigma_2)\mathrm{F}(-\boldsymbol{x})^*\end{bmatrix}.$$

The multiplier of $(m, s; e)_\eta^{+\varepsilon}$ is $w_m^{+,(-)^{2s}\varepsilon}$, the multiplier of $(m, s; e)^{-\varepsilon}$ is $w_m^{-,(-)^{2s}\varepsilon}$, $w_m^{\chi\varepsilon}$ as defined in (13). Two of the multiplier representations defined above are unitary equivalent iff they are identical, while $[m, s; e]^{\chi\varepsilon}$ and $[m', s'; e']^{\chi'\varepsilon'}$ are (unitary or anti-unitary) equivalent iff $|m| = |m'|$, $s = s'$, $\chi = \chi'$, $\varepsilon = \varepsilon'$.

Proof: As $\widetilde{\mathrm{FIGG}}$ is a Lie group, the unitary subgroup of any ray representation contains $\widetilde{\mathrm{IGG}}$; furthermore according to (9), space-inversion must be represented unitarily and time-inversion anti-unitarily. So the unitary subgroup of ray representations of $\widetilde{\mathrm{FIGG}}$, the multipliers of which restricted to $\widetilde{\mathrm{IGG}}$ are nontrivial, is $\widetilde{\mathrm{IGG}}_{\mathrm{S}}$. Because of this we first have to construct all irreducible unitary continuous ray representations of $\widetilde{\mathrm{IGG}}_{\mathrm{S}}$, the multipliers of which restricted to $\widetilde{\mathrm{IGG}}$ are nontrivial; this is done by means of (15) and (17). Now for $(m, s; e)$, $m \neq 0$, $\mathrm{P_S F}(\boldsymbol{x}) := \mathrm{F}(-\boldsymbol{x})$ satisfies $\mathrm{P_S}^2 = \mathbf{1}$, $\mathrm{P_S}(m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{P_S} = (m, s; e)(\mathrm{U}, -\boldsymbol{v}; -\boldsymbol{a}, \overset{0}{a})$, and so for $[m, s; e]$, $[\mathrm{P_S}]$ is the only operator ray satisfying condition $(\mathrm{U_S})$. So every continuous irreducible unitary ray representation of $\widetilde{\mathrm{IGG}}_{\mathrm{S}}$, the multipliers of which restricted to $\widetilde{\mathrm{IGG}}$ are nontrivial, is unitary equivalent to one of the kind RR, $\mathrm{RR}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) := [m, s; e](\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})$, $\mathrm{RR}((\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S}) := [m, s; e](\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})[\mathrm{P_S}]$.

Defining $\mathrm{T_S F}(\boldsymbol{x}) := \mathrm{D}^{(s)}(i\sigma_2)\mathrm{F}(-\boldsymbol{x})^*$, we see that $\mathrm{T_S}$ is an anti-unitary operator and $\mathrm{T_S}^2 = (-)^{2s}\mathbf{1}$, $\mathrm{P_S T_S} = \mathrm{T_S P_S}$. So $[\mathrm{T_S}]$ satisfies $(\mathrm{A_S})$ for the ray representations RR, and is the only operator ray satisfying $(\mathrm{A_S})$; thus we immediately get $[m, s; e]^{++}$ and its representatives. There remain those ray representations of $\widetilde{\mathrm{FIGG}}$, the restrictions of which to $\widetilde{\mathrm{IGG}}$ decompose into direct sum representations. We start from the continuous multiplier representations $(m, s; e)_\eta$ of $\widetilde{\mathrm{IGG}}_{\mathrm{S}}$, $(m, s; e)_\eta(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) := (m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})$, $(m, s; e)_\eta((\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S}) := \eta(m, s; e)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{P_S}$; we have to consider the continuous homomorphisms $\chi\colon \widetilde{\mathrm{IGG}}_{\mathrm{S}} \to \mathbb{C}_1$ with $\chi(g) = \chi(\mathrm{I_T} g \mathrm{I_T})$ for all $g \in \widetilde{\mathrm{IGG}}_{\mathrm{S}}$. For the restrictions of these homomorphisms to $\widetilde{\mathrm{IGG}}$ we have [4 ; p. 64]: $\chi(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) = \exp(ie\overset{0}{a})$, $e \in \mathbb{R}$, and so with $\mathrm{I_T}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_T} = (\mathrm{U}, -\boldsymbol{v}; \boldsymbol{a}, -\overset{0}{a})$: $\chi(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) = 1$; there remains $\chi(\mathrm{I_S}) = \pm$. With this we get for $\chi(\mathrm{I_S}) = 1$ the representations $(m, s; e)_\eta^{+\varepsilon}$, and for $\chi(\mathrm{I_S}) = -1$ the representations $(m, s; e)^{-\varepsilon}(\eta = +)$. The irreducibility of these representations follows from the fact, that $(m, s; e)$ is an irreducible multiplier representation of $\widetilde{\mathrm{IGG}}$. On the other hand it is easily seen that the ray representations constructed with $\chi \equiv 1$ and $\varepsilon = 1$ as after (17 $\mathrm{A_S}$) are reducible. The calculation of the multipliers and the establishment of the equivalence is a straightforward task, which we refrain from performing. ■

As is easily seen we have defined an abundance of representatives: $(m, s; e)_\eta^{+\varepsilon}$ and $(m, s; e')_{-\eta}^{+\varepsilon}$ differ only by a phase function and have the same multipliers, i. e. this phase function defines an one-dimensional vector representation of $\widetilde{\mathrm{FIGG}}$.

(19) **Theorem:** Any continuous irreducible ray representation of $\widetilde{\mathrm{FIGG}}$ with unitary space-inversion and anti-unitary time-inversion, the restriction of which to $\widetilde{\mathrm{IGG}}$ contains a case II representation, is equivalent to one of the ray representations $[0, s; p]^{\chi\varepsilon}$, $s = 0, \pm 1/2, \pm 1 \ldots, p \in (0, \infty)$, $\chi = \pm$, $\varepsilon = \pm$, which are defined by their representatives R: $= (0, 0; p)_\eta^{+\varepsilon}$, $\eta = \pm$, $(0, 0; p)^{-\varepsilon}$, or $(0, s; p)^{\chi\varepsilon}$, $s \neq 0$, respectively as following :

$\mathrm{R}(gd) := \mathrm{R}(g)\mathrm{R}(d)$ for all $g \in \widetilde{\mathrm{IGG}}$, $d \in \{\mathrm{I}, \mathrm{I_S}, \mathrm{I_T}, \mathrm{I_{ST}}\}$, $\mathrm{R}(\mathrm{I_{ST}}) :=$
$= \mathrm{R}(\mathrm{I_S})\mathrm{R}(\mathrm{I_T})$ and with $\mathrm{F}, \mathrm{G}, \mathrm{F}_\pm, \mathrm{G}_\pm \in \mathrm{L}_2(\{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = p\} \times \mathbb{R}, \mathbb{C})$:

$$(0, 0; p)_\eta^{++}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{x}, t) := (0, 0; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\boldsymbol{x}, t),$$
$$(0, 0; p)_\eta^{++}(\mathrm{I_S})\mathrm{F}(\boldsymbol{x}, t) := \eta\mathrm{F}(-\boldsymbol{x}, t),$$
$$(0, 0; p)_\eta^{++}(\mathrm{I_T})\mathrm{F}(\boldsymbol{x}, t) := \mathrm{F}(-\boldsymbol{x}, t)^* ;$$

$$(0, 0; p)_\eta^{+-}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}(0, 0; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}\\ (0, 0; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{G}\end{bmatrix}(\boldsymbol{x}, t),$$
$$(0, 0; p)_\eta^{+-}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \eta\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(-\boldsymbol{x}, t),$$
$$(0, 0; p)_\eta^{+-}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}-\mathrm{G}(-\boldsymbol{x}, t)^*\\ \mathrm{F}(-\boldsymbol{x}, t)^*\end{bmatrix};$$

$$(0, 0; p)^{-\varepsilon}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}(0, 0; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}\\ (0, 0; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{G}\end{bmatrix}(\boldsymbol{x}, t),$$
$$(0, 0; p)^{-\varepsilon}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}\mathrm{F}\\ -\mathrm{G}\end{bmatrix}(-\boldsymbol{x}, t),$$
$$(0, 0; p)^{-\varepsilon}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}\varepsilon\mathrm{G}(-\boldsymbol{x}, t)^*\\ \mathrm{F}(-\boldsymbol{x}, t)^*\end{bmatrix};$$

with $\varphi(\boldsymbol{x}) := (x_1 + ix_2)/|x_1 + ix_2|$:

$$(0, s; p)^{\chi+}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}(0, s; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}_+\\ (0, -s; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{F}_-\end{bmatrix}(\boldsymbol{x}, t),$$
$$(0, s; p)^{\chi+}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}\varphi(-\boldsymbol{x})^{2s}\mathrm{F}_-(-\boldsymbol{x}, t)\\ \varphi(\boldsymbol{x})^{-2s}\mathrm{F}_+(-\boldsymbol{x}, t)\end{bmatrix},$$
$$(0, s; p)^{\chi+}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\end{bmatrix}(\boldsymbol{x}, t) := \begin{bmatrix}\varphi(\boldsymbol{x})^{2s}\mathrm{F}_+(-\boldsymbol{x}, t)^*\\ \chi\varphi(\boldsymbol{x})^{-2s}\mathrm{F}_-(-\boldsymbol{x}, t)^*\end{bmatrix};$$

$$(0,\, s\,;\, p)^{\chi-}(\mathrm{U},\, \boldsymbol{v}\,;\, \boldsymbol{a},\, \overset{0}{a})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\\ \mathrm{G}_+\\ \mathrm{G}_-\end{bmatrix}(\mathbf{x},\, t) := \begin{bmatrix}(0,\, s\,;\, p)(\mathrm{U},\, \boldsymbol{v}\,;\, \boldsymbol{a},\, \overset{0}{a})\mathrm{F}_+\\ (0,\, -s\,;\, p)(\mathrm{U},\, \boldsymbol{v}\,;\, \boldsymbol{a},\, \overset{0}{a})\mathrm{F}_-\\ (0,\, s\,;\, p)(\mathrm{U},\, \boldsymbol{v}\,;\, \boldsymbol{a},\, \overset{0}{a})\mathrm{G}_+\\ (0,\, -s;\, p)(\mathrm{U},\, \boldsymbol{v}\,;\, \boldsymbol{a},\, \overset{0}{a})\mathrm{G}_-\end{bmatrix}(\mathbf{x},\, t),$$

$$(0,\, s\,;\, p)^{\chi-}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\\ \mathrm{G}_+\\ \mathrm{G}_-\end{bmatrix}(\mathbf{x},\, t) := \begin{bmatrix}\varphi(-\mathbf{x})^{2s}\mathrm{F}_-(-\mathbf{x},\, t)\\ \varphi(\mathbf{x})^{-2s}\mathrm{F}_+(-\mathbf{x},\, t)\\ \chi\varphi(-\mathbf{x})^{2s}\mathrm{G}_-(-\mathbf{x},\, t)\\ \chi\varphi(\mathbf{x})^{-2s}\mathrm{G}_+(-\mathbf{x},\, t)\end{bmatrix},$$

$$(0,\, s\,;\, p)^{\chi-}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}_+\\ \mathrm{F}_-\\ \mathrm{G}_+\\ \mathrm{G}_-\end{bmatrix}(\mathbf{x},\, t) := \begin{bmatrix}-\varphi(\mathbf{x})^{2s}\mathrm{G}_+(-\mathbf{x},\, t)^*\\ -\varphi(\mathbf{x})^{-2s}\mathrm{G}_-(-\mathbf{x},\, t)^*\\ \varphi(\mathbf{x})^{2s}\mathrm{F}_+(-\mathbf{x},\, t)^*\\ \varphi(\mathbf{x})^{-2s}\mathrm{F}_-(-\mathbf{x},\, t)^*\end{bmatrix}.$$

The multiplier of $(0,\, 0;\, p)_\eta^{+\varepsilon}$ is $w_0^{+\varepsilon}$, that of $(0,\, 0;\, p)^{-\varepsilon}$ is $w_0^{-\varepsilon}$, and that of $(0,\, s;\, p)^{\chi\varepsilon}$ is $w_0^{\chi,(-)^{2s}\varepsilon}$, $w_0^{\chi\varepsilon}$ as defined in (13). $(0,\, 0;\, p)_\eta^{+\varepsilon}$ and $(0,\, 0;\, p')_{\eta'}^{+\varepsilon'}$ are unitary equivalent iff $p = p'$, $\varepsilon = \varepsilon'$; $\eta = \eta'$; $(0,\, 0;\, p)^{-\varepsilon}$ and $(0,\, 0;\, p')^{-\varepsilon'}$ are unitary equivalent iff $p = p'$; $\varepsilon = \varepsilon'$; $(0,\, s;\, p)^{\chi\varepsilon}$ and $(0,\, s';\, p')^{\chi'\varepsilon'}$ are unitary equivalent iff $|\, s\,| = |\, s'\,|$, $p = p'$, $\chi = \chi'$, $\varepsilon = \varepsilon'$. $(0,\, 0;\, p)_\eta^{+\varepsilon}$ and $(0,\, 0;\, p')^{-\varepsilon'}$ are inequivalent, $(0,\, 0;\, p)_\eta^{+\varepsilon}$ and $(0,\, s;\, p')^{\chi\varepsilon'}$ are inequivalent for all $s \neq 0$, and $(0,\, 0;\, p)^{-\varepsilon}$ and $(0,\, s,\, p')^{\chi\varepsilon'}$ are inequivalent for all $s \neq 0$. The same equivalence relations are valied for the ray representations $[0, 0; p]^{\chi\varepsilon}$ and $[0, s; p]^{\chi'\varepsilon'}$, $s \neq 0$, except for the condition $\eta = \eta'$, as $(0, 0; p)_+^{+\varepsilon}$ and $(0,\, 0;\, p)_-^{+\varepsilon}$ define the same ray representation $[0,\, 0;\, p]^{+\varepsilon}$ of $\widetilde{\mathrm{FIGG}}$.

Proof: The unitary subgroup must be $\widetilde{\mathrm{IGG}}_\mathrm{S}$, as the space-inversion is to be represented unitarily and the time-inversion is to be represented anti-unitarily. Defining $\mathrm{P}_s\mathrm{F}(\mathbf{x},\, t) := \varphi(\mathbf{x})^{-2s}\mathrm{F}(-\mathbf{x},\, t)$ we get: $\mathrm{P}_s(0,\, s;\, p)(\mathrm{U},\, \boldsymbol{v};\, \boldsymbol{a},\, \overset{0}{a})\mathrm{P}_s^{-1} = (0,\, -s;\, p)(\mathrm{U},\, -\boldsymbol{v};\, -\boldsymbol{a},\, \overset{0}{a})$, and as $[0,\, s;\, p]$ and $[0,\, -s,\, p]$ are identical for $s = 0$ and inequivalent for $s \neq 0$, we get the following continuous irreducible unitary ray representations RR of $\widetilde{\mathrm{IGG}}_\mathrm{S}$, the restrictions of which to $\widetilde{\mathrm{IGG}}$ contain a case II ray representation: $s = 0$: $\mathrm{RR}(\mathrm{U},\, \boldsymbol{v};\, \boldsymbol{a},\, \overset{0}{a}) := [0,\, 0;\, p](\mathrm{U},\, \boldsymbol{v};\, \boldsymbol{a},\, \overset{0}{a})$, $\mathrm{RR}(\mathrm{I_S}) := [\mathrm{P_0}]$; $s \neq 0$: Choosing the representatives $(0,\, s;\, p)$ for $[0,\, s;\, p]$, we have to look for the continuous functions r: $\widetilde{\mathrm{IGG}} \to \mathbb{C}_1$ with $r(g)r(\mathrm{I_S}g\mathrm{I_S}) = 1$ and $r(g)r(g') = r(gg')$, as the multiplier of $(0,\, s;\, p)$ is 1; froom the second condition follows [4 ; p. 64]: $r(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) = \exp(ie\overset{0}{a})$, $e \in \mathbb{R}$, and from the first with $\mathrm{I_S}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a})\mathrm{I_S} = (\mathrm{U}, -\boldsymbol{v}; -\boldsymbol{a}, \overset{0}{a})$: $r \equiv 1$. So we get for the ray representations:

$$\mathrm{RR}(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) := \left[\begin{bmatrix} (0, s; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) & 0 \\ 0 & (0, s; p)(\mathrm{U}, -\boldsymbol{v}; -\boldsymbol{a}, \overset{0}{a}) \end{bmatrix}\right],$$

$$\mathrm{RR}(\mathrm{I_S}) := \left[\begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}\right].$$

By performing the unitary transformation

$$\mathrm{RR} \to \mathrm{RR}', \quad \mathrm{RR}'(g) := \ddot{\mathrm{A}}\mathrm{RR}(g)\ddot{\mathrm{A}}^{-1}, \quad \ddot{\mathrm{A}} := \left[\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathrm{P}_s \end{bmatrix}\right],$$

we obtain

$$\mathrm{RR}'(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) = \left[\begin{bmatrix} (0, s; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) & 0 \\ 0 & (0, -s; p)(\mathrm{U}, \boldsymbol{v}; \boldsymbol{a}, \overset{0}{a}) \end{bmatrix}\right].$$

$$\mathrm{RR}'(\mathrm{I_S}) := \left[\begin{bmatrix} 0 & \mathrm{P}_s^{-1} \\ \mathrm{P}_s & 0 \end{bmatrix}\right].$$

Defining $\mathrm{T}_s\mathrm{F}(\boldsymbol{x}, t) := \varphi(\boldsymbol{x})^{2s}\mathrm{F}(-\boldsymbol{x}, t)^*$, we have

$$\mathrm{P}_s^{-1} = (-)^{2s}\mathrm{P}_{-s}, \quad \mathrm{T}_s = (-)^{2s}\mathrm{T}_s^{-1}, \quad \mathrm{P}_s\mathrm{T}_s = \mathrm{T}_{-s}\mathrm{P}_s = \mathrm{T}_s\mathrm{P}_{-s} = \mathrm{P}_{-s}\mathrm{T}_{-s},$$

and one verifies $[\mathrm{T}_0]$ to be the only anti-unitary operator ray satisfying (17 A_s) for $s = 0$, and the only anti-unitary operator rays satisfying (17 A_s) for $s \neq 0$ to be $\left[\begin{bmatrix} \mathrm{T}_s & 0 \\ 0 & \chi\mathrm{T}_{-s} \end{bmatrix}\right]$. So for $s = 0$ we get the representations $(0, 0; p)_\eta^{+\varepsilon}$ and $(0, 0; p)^{-\varepsilon}$ as in the case $m \neq 0$; for $s \neq 0$ we get the representations $(0, s; p)^{\chi+}$ using $\left[\begin{bmatrix} \mathrm{T}_s & 0 \\ 0 & \chi\mathrm{T}_{-s} \end{bmatrix}\right]$ and not doubling the representations RR′, and $(0, s; p)^{\chi-}$, doubling once more, and noting that ε must be chosen -1 for these representation to be irreducible, and that the doubled representations constructed with $\left[\begin{bmatrix} \mathrm{T}_s & 0 \\ 0 & \mathrm{T}_{-s} \end{bmatrix}\right]$ and $\left[\begin{bmatrix} \mathrm{T}_s & 0 \\ 0 & -\mathrm{T}_{-s} \end{bmatrix}\right]$ are equivalent. The statements about the multipliers and about equivalence follow by a straightforward calculation. ■

In contrast to the case $m \neq 0$, for $m = 0$ there also exist ray representations of $\widetilde{\mathrm{FIGG}}$, which represent $\mathrm{I_S}$ anti-unitarily and $\mathrm{I_T}$ unitarily. They can be found with the same methods ; however, we refrain from doing so.

3. The decomposition of the direct product of irreducible multiplier representations of $\widetilde{\text{FIGG}}$.

The decomposition of the direct product of irreducible multiplier representations of $\widetilde{\text{IGG}}$. — Here some results have been obtained by Levy-Leblond [8] and by Brennich [4].

Theorem [8 ; p. 786]: For $m + m' \neq 0 \neq mm'$, $\sigma := \operatorname{sign}\left(\frac{mm'}{m + m'}\right)$, we have

$$(20)\quad (m, s; e)\otimes(m', s'; e') \cong \bigoplus_{L=0}^{\infty} \bigoplus_{S=|s-s'|}^{s+s'} \bigoplus_{J=|L-S|}^{L+S} \int_0^{\infty} \oplus dE(m+m', J; e+e'+\sigma E)$$

Theorem [4 ; p. 75]: For $m \neq 0$, $j := 0$ iff $s + s'$ integer, and $j := 1/2$ else, we have:

$$(21)\quad (m, s ; e) \otimes (-m, s' ; e') \cong$$
$$\cong (2s + 1)(2s' + 1) \int_0^{\infty} \oplus\, dk \int_0^{\infty} dp\underline{(j, 0, k, 0 ; 0, 0, p, e + e', 0)}.$$

Theorem: For $m \neq 0$ and $p > 0$ we have

$$(22)\quad (m, s ; e) \otimes (0, s' ; p) \cong \bigoplus_{L=0}^{\infty} \bigoplus_{J=||s'|+L-s|}^{|s'|+L+s} \int_{-\infty}^{+\infty} \oplus\, dE(m, J ; E)$$

Proof : Let us shortly write R for $(m, s; e) \otimes (0, s'; p)$; then

$$R(U, \mathbf{v} ; \boldsymbol{a}, \overset{0}{a})F(\mathbf{x}, \mathbf{y}, t) = \exp\,(i(e + t + \mathbf{x}^2/2m)\overset{0}{a} - i(\mathbf{x} + \mathbf{y})\cdot\boldsymbol{a}$$
$$+ is'\chi(00p, \mathbf{y} ; U)) \times D^{(s)}(U)F(D(U)^{-1}(\mathbf{x} - m\mathbf{v}), D(U)^{-1}\mathbf{y}, t - \mathbf{y}\cdot\mathbf{v}),$$

where $\mathbf{x}$ varies over $\mathbb{R}^3$, $\mathbf{y}$ varies over the sphere of radius p, and t varies over $\mathbb{R}$. With the unitary operator A_1,

$$A_1F(\mathbf{X}, \mathbf{y} ; t) := F(\mathbf{X} - \mathbf{y}, \mathbf{y}, t + (\mathbf{X} - \mathbf{y})\cdot\mathbf{y}/m),$$

we get :

$$A_1R(U, \mathbf{v} ; \boldsymbol{a}, \overset{0}{a})A_1{}^{-1}F(\mathbf{X}, \mathbf{y} ; t) =$$
$$= \exp\,(i(e - p^2/2m + t + \mathbf{X}^2/2m)\overset{0}{a} - i\mathbf{X}\cdot\boldsymbol{a} + is'\chi(00p, \mathbf{y} ; U))$$
$$\times D^{(s)}(U)F(D(U)^{-1}(\mathbf{X} - m\mathbf{v}), D(U)^{-1}\mathbf{y}, t),$$

or, defining

$$\mathrm{D}_t(\mathrm{U}, \boldsymbol{v} ; \boldsymbol{a}, \overset{0}{a})\mathrm{F}(\mathbf{X}, \mathbf{y}) := \exp(i(t + \mathbf{X}^2/2m)\overset{0}{a} - i\mathbf{X}\cdot\boldsymbol{a} + is'\chi(00p, \mathbf{y} ; \mathrm{U})) \times \mathrm{D}^{(s)}(\mathrm{U})\mathrm{F}(\mathrm{D}(\mathrm{U})^{-1}(\mathbf{X} - m\boldsymbol{v}), \mathrm{D}(\mathrm{U})^{-1}\mathbf{y}),$$

$$\mathrm{A}_1\mathrm{R}\mathrm{A}_1{}^{-1} = \int_{-\infty}^{+\infty} \oplus\, dt\mathrm{D}_t.$$

Consider the continuous representation of SU(2) defined by

$$\mathbf{U}^s\mathrm{F}(\boldsymbol{x}) := \exp(is\chi(001, \boldsymbol{x}; \mathrm{U}))\mathrm{F}(\mathrm{D}(\mathrm{U})^{-1}\boldsymbol{x}),$$

where $\boldsymbol{x} \in \mathrm{S}_2$ and $\mathrm{F} \in \mathrm{L}_2(\mathrm{S}_2, \mathbb{C})$. It leaves invariant the subspaces spanned by the functions

$$\mathrm{D}^{(j)}(\mathrm{U}_{\boldsymbol{x}}{}^{-1})_{-s,m}, m = -j, \ldots, j, \quad \text{for all} \quad j = |s| + \mathrm{L}, \mathrm{L} = 0, 1, 2, \ldots,$$

where $\mathrm{U}_{\boldsymbol{x}}$ is defined in [4 ; p. 62] and has the properties:

$$\mathrm{D}(\mathrm{U}_{\boldsymbol{x}})(0\,0\,1) = \boldsymbol{x}, \mathrm{U}_{\boldsymbol{x}}{}^{-1}\mathrm{U}\mathrm{U}_{\mathrm{D}(\mathrm{U})^{-1}\boldsymbol{x}} = \mathrm{U}(\chi(0\,0\,1, \boldsymbol{x}; \mathrm{U})(0\,0\,1))$$

for all $\mathrm{U} \in \mathrm{SU}(2)$. These subspaces are irreducible, and using the complete decomposability of all continuous vector representations of SU(2), it can be shown [11 ; p. 53], that these functions form an orthogonal base in $\mathrm{L}_2(\mathrm{S}_2, \mathbb{C})$: they are a generalisation of the spherical harmonics. So we have for this representation: $\mathbf{U}^s = \bigoplus_{\mathrm{L}=0}^{\infty} \mathrm{D}^{(|s|+\mathrm{L})}$. With this the theorem follows immediately. ■

The decomposition of the direct product of irreducible multiplier representations of $\widetilde{\text{FIGG}}$.

Theorem: For $m + m' \neq 0 \neq mm'$, $\sigma := \operatorname{sign}\left(\dfrac{mm'}{m + m'}\right)$, we have:

(23) $$(m, s ; e)_\eta^{++} \otimes (m', s' ; e')_{\eta'}^{+\varepsilon} \cong \bigoplus_{\mathrm{L}=0}^{\infty} \bigoplus_{\mathrm{S}=|s-s'|}^{s+s'} \bigoplus_{\mathrm{J}=|\mathrm{L}-\mathrm{S}|}^{\mathrm{L}+\mathrm{S}} \int_0^{\infty} \oplus\, d\mathrm{E}(m + m', \mathrm{J} ; e + e' + \sigma\mathrm{E})_{\eta\eta'(-)^{\mathrm{L}}}^{+\varepsilon} ;$$

$$(m, s ; e)_\eta^{++} \otimes (m', s' ; e)^{-\varepsilon} \cong \bigoplus_{\mathrm{L}=0}^{\infty} \bigoplus_{\mathrm{S}=|s-s'|}^{s+s'} \bigoplus_{\mathrm{J}=|\mathrm{L}-\mathrm{S}|}^{\mathrm{L}+\mathrm{S}} \int_0^{\infty} \oplus\, d\mathrm{E}(m + m', \mathrm{J} ; e + e' + \sigma\mathrm{E})^{-\varepsilon} ;$$

$$(m, s ; e)_\eta^{+-} \otimes (m', s' ; e')_{\eta'}^{+-} \cong 4\cdot \bigoplus_{\mathrm{L}=0}^{\infty} \bigoplus_{\mathrm{S}=|s-s'|}^{s+s'} \bigoplus_{\mathrm{J}=|\mathrm{L}-\mathrm{S}|}^{\mathrm{L}+\mathrm{S}} \int_0^{\infty} \oplus\, d\mathrm{E}(m + m', \mathrm{J} ; e + e' + \sigma\mathrm{E})_{\eta\eta'(-)^{\mathrm{L}}}^{++} ;$$

$$(m, s\,; e)_\eta^{+-} \otimes (m', s'\,; e')^{-\varepsilon} \cong$$
$$= 2.\bigoplus_{L=0}^{\infty}\ \bigoplus_{S=|s-s'|}^{s+s'}\ \bigoplus_{J=|L-S|}^{L+S} \int_0^\infty \oplus\, dE(m+m', J\,; e+e'+\sigma E)^{-,-\varepsilon}\,;$$

$$(m, s\,; e)^{-+} \otimes (m', s'\,; e')^{-+} \cong$$
$$\cong 2.\bigoplus_{L=0}^{\infty}\ \bigoplus_{J=|s-s'|}^{s+s'}\ \bigoplus_{S=|L-S|}^{L+S} \int_0^\infty \oplus\, dE\left\{(m+m', J\,; e+e'+\sigma E)_+^{++}\right.$$
$$\left.\oplus\,(m+m', J\,; e+e'+\sigma E)_-^{++}\right\};$$

$$(m, s\,; e)^{-+} \otimes (m', e'\,; e')^{--} \cong$$
$$\cong \bigoplus_{L=0}^{\infty}\ \bigoplus_{S=|s-s'|}^{s+s'}\ \bigoplus_{J=|L-S|}^{L+S} \int_0^\infty \oplus\, dE\left\{(m+m', J\,; e+e'+\sigma E)_+^{+-}\right.$$
$$\left.\oplus\,(m+m', J\,; e+e'+\sigma E)_-^{+-}\right\};$$

$$(m, s\,; e)^{--} \otimes (m', s'\,; e')^{--} \cong$$
$$= 2.\bigoplus_{L=0}^{\infty}\ \bigoplus_{S=|s-s'|}^{s+s'}\ \bigoplus_{J=|L-S|}^{L+S} \int_0^\infty \oplus\, dE\left\{(m+m', J\,; e+e'+\sigma E)_+^{++}\right.$$
$$\left.\oplus\,(m+m', J\,; e+e'+\sigma E)_-^{++}\right\}.$$

Proof: First we consider $(m, s; e) \otimes (m', s'; e')$. By a transformation to the center of mass system:

$$\ddot{A}F(\mathbf{X}, \mathbf{x}) := F\left(\frac{m}{m+m'}\mathbf{X} + \mathbf{x},\ \frac{m'}{m+m'}\mathbf{X} - \mathbf{x}\right),$$

$$\ddot{A}^{-1}F(\mathbf{x}, \mathbf{y}) = F\left(\mathbf{x} + \mathbf{y},\ \frac{mm'}{m+m'}(\mathbf{x}/m - \mathbf{y}/m')\right),$$

we get

$$\ddot{A}(m, s\,; e) \otimes (m', s'\,; e')(U, \mathbf{v}\,; \boldsymbol{a}, \overset{0}{a})\ddot{A}^{-1}F(\mathbf{X}, \mathbf{x}) =$$
$$= \exp\left(i(e + e' + \mathbf{x}^2/2\mu + \mathbf{X}^2/2M)\overset{0}{a}\right)$$
$$\times \exp(-i\mathbf{X}\cdot\boldsymbol{a})D^{(s)}(U) \otimes D^{(s')}(U)F(D(U)^{-1}(\mathbf{X} - M\mathbf{v}), D(U)^{-1}\mathbf{x})$$

with $\mu := \dfrac{mm'}{m+m'}$, $M := m + m'$. So we obtain the decomposition of $(m, s; e) \otimes (m', s'; e')$ by expanding $F(\mathbf{X}, \mathbf{x})$ into an integral over the modulus of $\mathbf{x}$, and then by expanding the functions $F(\mathbf{X}, \mathbf{x})$ with fixed modulus of $\mathbf{x}$ into spherical harmonics. To space-inversion and time-inversion happen the following:

$$\ddot{A}(m, s\,; e)_\eta^{++} \otimes (m', s'\,; e')_{\eta'}^{++}(I_S)\ddot{A}^{-1}F(\mathbf{X}, \mathbf{x}) = \eta\eta' F(-\mathbf{X}, -\mathbf{x}),$$

and using $Y_l^m(-\mathbf{x}) = (-)^l Y_l^m(\mathbf{x})$, we obtain the decomposition of

$(m, s\,; e)_\eta^{++} \otimes (m', s'\,; e')_{\eta'}^{++}(\mathrm{I_S})$.

$$\ddot{\mathrm{A}}(m, s; e)_\eta^{++} \otimes (m', s'; e')_{\eta'}^{++}(\mathrm{I_T})\ddot{\mathrm{A}}^{-1}\mathrm{F}(\mathbf{X},\mathbf{x}) = \mathrm{D}^{(s)}\otimes\mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}(-\mathbf{X}, -\mathbf{x})^*,$$

and the decomposition of $(m, s; e)_\eta^{++} \otimes (m', s'; e')_{\eta'}^{++}(\mathrm{I_T})$ is obtained using $Y_l^m(\mathbf{x})^* = (-)^m Y_l^{-m}(\mathbf{x})$, from which follows:

$$Y_l^m(-\mathbf{x}) = \mathrm{D}^{(l)}(i\sigma_2)_{m'm} Y_l^{m'}(\mathbf{x}).$$

Noting, that for any linear operator B: $\mathrm{B} \oplus (-\mathrm{B})$ and $(-\mathrm{B}) \oplus \mathrm{B}$ are unitary equivalent, and that the phase of $(m, s; e)_\eta^{+\varepsilon}(\mathrm{I_T})$ and $(m, s; e)^{-\varepsilon}(\mathrm{I_T})$ is irrelevant, as for any anti-linear operator A: $w\mathrm{A} = w^{1/2}\mathrm{A}(w^{1/2})^{-1}$ for all $|\,w\,| = 1$, we have for the other cases, abbreviating

$$(\ddot{\mathrm{A}} \oplus \ddot{\mathrm{A}})(m, s; e)_\eta^{++} \otimes (m', s'; e')_{\eta'}^{+-}(\ddot{\mathrm{A}} \oplus \ddot{\mathrm{A}})^{-1} \text{ by R:}$$

$$\mathrm{R}(\mathrm{I_T})\begin{bmatrix}\mathrm{F}\\ \mathrm{G}\end{bmatrix}(\mathbf{X}, \mathbf{x}) = \begin{bmatrix}-\mathrm{D}^{(s)} \otimes \mathrm{D}^{(s')}(i\sigma_2)\mathrm{G}(-\mathbf{X}, -\mathbf{x})^*\\ \mathrm{D}^{(s)} \otimes \mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}(-\mathbf{X}, -\mathbf{x})^*\end{bmatrix},$$

and from this the decomposition of $(m, s; e)_\eta^{++} \otimes (m', s'; e')_{\eta'}^{+-}$ follows. The proof for $(m, s; e)_\eta^{++} \otimes (m', s'; e')^{-\varepsilon}$ follows along the same lines. Writing shortly R for $(m, s; e)_\eta^{+-} \otimes (m', s'; e')_{\eta'}^{+-}$, we have

$$\mathrm{R}(\mathrm{I_S})\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix}(\mathbf{x}, \mathbf{y}) = \eta\eta'\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix}(-\mathbf{x}, -\mathbf{y}).,$$

and an analogous formula for $\mathrm{R}(\mathrm{I_T})$. After transforming by $\ddot{\mathrm{A}}_4$,

$$\ddot{\mathrm{A}}_4\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix} := \begin{bmatrix}\ddot{\mathrm{A}}\mathrm{F}_{11} & \ddot{\mathrm{A}}\mathrm{F}_{12}\\ \ddot{\mathrm{A}}\mathrm{F}_{21} & \ddot{\mathrm{A}}\mathrm{F}_{22}\end{bmatrix},$$

$\mathrm{R}' := \ddot{\mathrm{A}}_4\mathrm{R}\ddot{\mathrm{A}}_4{}^{-1}$:

$$\mathrm{R}'(\mathrm{I_S})\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix}(\mathbf{X}, \mathbf{x}) = \eta\eta'\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix}(-\mathbf{X}, -\mathbf{x}),$$

$$\mathrm{R}'(\mathrm{I_T})\begin{bmatrix}\mathrm{F}_{11} & \mathrm{F}_{12}\\ \mathrm{F}_{21} & \mathrm{F}_{22}\end{bmatrix}(\mathbf{X}, \mathbf{x}) =$$

$$= \begin{bmatrix}\mathrm{D}^{(s)}\otimes\mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}_{22}(-\mathbf{X}, -\mathbf{x})^* & -\mathrm{D}^{(s)}\otimes\mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}_{21}(-\mathbf{X}, -\mathbf{x})^*\\ -\mathrm{D}^{(s)}\otimes\mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}_{12}(-\mathbf{X}, -\mathbf{x})^* & \mathrm{D}^{(s)}\otimes\mathrm{D}^{(s')}(i\sigma_2)\mathrm{F}_{11}(-\mathbf{X}, -\mathbf{x})^*\end{bmatrix}$$

So the decomposition of the 2×2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (b-c)/2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (b+c)/2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (a+d)/2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (a-d)/2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

defines the complete decomposition of $(m, s; e)_\eta^{+-} \otimes (m', s'; e')_{\eta'}^{+-}$. The rest follows along the same lines. ∎

4. Consequences for non-relativistic quantum theory.

Interpretation of the quantum numbers. According to [4 ; p. 70], $[m, s; e]$ for $m \neq 0$ is a ray representation of IGG describing a particle of mass $|m|$ and spin s. $[0, s; p]$ for $p > 0$ gives the non-relativistic description of particles of mass 0, spin $|s|$, helicity sign(s) for $s \neq 0$, and wave number p ; however, this description is rather insufficient, as there is no Doppler effect. As in the relativistic case, helicity is no more an invariant, if space-inversion symmetry is included: under space-inversion, the helicity changes sign.

The quantum numbers arising from discrete symmetries have not such a simple meaning. Let R be a representative of one of the ray representations $[m, s; e]^{\chi\varepsilon}$ as defined in (18) and (19); then $R(I_S)^2 = z\mathbf{1}$, $|z| = 1$, $R(I_T)^2 = \pm \mathbf{1}$, $R(I_{ST})^2 = \pm \mathbf{1}$, and $R(I_S)$ is unitary, $R(I_T)$ and $R(I_{ST})$ are anti-unitary. By a suitable choice of R we get $R(I_S)^2 = \mathbf{1}$, meaning that $R(I_S)$ is a self-adjoint operator and hence may be an observable; then we have independant of the choice of $R(I_T)$ and $R(I_{ST})$:

$$R(I_S)R(I_T) = \chi R(I_T)R(I_S),\ R(I_T)^2 = \varepsilon(-)^{2s}\mathbf{1}, \text{ and } R(I_{ST})^2 = \chi\varepsilon(-)^{2s}\mathbf{1}.$$

So the quantum number ε fixes (for a given spin) the square of $R(I_T)$, χ and ε fix the square of $R(I_{ST})$, and the quantum number χ determines wether $R(I_S)$ and $R(I_T)$ commute or anti-commute.

In the relativistic theory unitarity of the space-inversion and anti-unitarity of the time-inversion is assured by the physical demand, that the energy spectrum should be bounded to one side [3]. This demand has no consequence in the non-relativistic case, as for particles with non-zero mass it is satisfied automatically, while for massless particles it can not even be satisfied for IGG. In this case, for non-zero mass particles, one has to represent space-inversion unitarily and time-inversion anti-unitarily according to (9) for purely mathematical reasons ; for massless particles, however, one has to make an ad hoc assumption. In spite of these differences, many properties of the discrets symmetries are common to relativistic and non-relativistic theory.

Superselection rules. The existence of several non-equivalent multipliers of FIGG gives rise to a set of superselection rules.

a) Masses. In the Galilean quantum theory, mass is introduced via the multiplier [2 ; p. 41] ; to different masses there correspond inequivalent multipliers, and so there is a superselection rule.

b) Integer spin — non-integer spin. As every ray representation of FIGG defines a ray representation of its subgroups, the superselection rules for the subgroups also are valied for FIGG. As SO(3) has two inequivalent multipliers, the trivial one for integer spin, and ζ as defined in (14) for non-integer spin, we have a superselection rule.

c) Commuting — anti-commuting space- and time-inversion. For any physical ray representation of FIGG, the space-inversion can be represented by a self-adjoint operator ; then the representatives of space-inversion and of time-inversion commute or anti-commute, and as these two cases correspond to inequivalent multipliers, there again is a superselection rule.

d) Square of the time-inversion. For any representative R of any physical ray representation of FIGG, $R(I_T)^2$ is $+\mathbf{1}$ or $-\mathbf{1}$, independant of the choice of R, and of an equivalence transformation. These signs correspond to inequivalent multipliers, and there is a superselection rule.

All superselection rules except *a)* also are valied in the case of Poincaré invariance. Superselection rules provide an excellent means of testing a symmetry: a broken superselection rule has the immediate consequence, that the symmetry is broken. So, apart from the trouble with massless particles, alone the existence of nuclear reactions, that do not conserve the total mass, completely disproves Galilean invariance. A testing of the discrete symmetries by superselection rules is hampered by the fact, that usually only representations of the kind $[m, s; e]^{++}$ are assumed to be physical. Then the superselection rule *c)* is of no importance, and the superselection rule *d)* is a consequence of the superselection rule *b)*.

Parity breaking shows, that not all elementary particle can be described by ray representations of the kind $[m, s; e]^{++}$. Landau [7] suggested that not the parity P, but the product CP of parity and particle — anti-particle conjugation C, is a general symmetry operation. If we try to interpret CP as space-inversion, we get the following conditions: space-inversion exchanges particle and anti-particle states, time-inversion transforms particle states into particle states, anti-particle states into anti-particle states. In the context of ray representations this means, that particles must be described by representations of the kind $[m, s; e]^{\alpha+}$, as only these ray representations can be chosen to be diagonal for $g \in \mathrm{IGG_T}$. Then for $m \neq 0$, $[m, s; e]^{++}$ describes a particle equal to its anti-particle, while $[m, s; e]^{-+}$ describes a particle different from its anti-particle. However, assuming the photon to be described by $[0, 1; p]^{++}$, the proton by $[m_p, 1/2; e]^{-+}$, the electron by $[m_e, 1/2; e]^{-+}$, and the charged pion by $[m, 0; e]^{-+}$, we find, considering the decays of the « *stables* » enlisted in [*1*], that electrically neutral particles must be described by $[m, s; e]^{++}$, and electrically charged particles by $[m, s; e]^{-+}$. It might seem, that this contradiction can be solved by the following modification of Landau's proposition: space-inversion is represented by C_qP, where C_q is the electric charge conjugation. However, this would have the consequence, that the decay $n \to p^- + e^+ + \nu$ for inverted spatial impulses has the same cross section as the decay $n \to p^+ + e^- + \bar{\nu}$, which markedly contradicts experimental results.

From all this we conclude, that a consistent description of elementary particles is not possible in the context of ray representations of FIGG (and of IO(1, 3), too). This also can be predicted from the fact, that for charged particles there is a superselection rule between particle states and anti-particle states. For if there are superselection rules, the pure states of a physical system no more form the projective space $\mathrm{H}/\mathbb{C}$ of a separable Hilbert space H, but form the union $\bigcup_{i \in \mathrm{I}} \mathrm{H}_i/\mathbb{C}$ of such projective spaces, shortly called bundle space. The concept of ray automorphism has to be generalized to that of bundle automorphism, $b: \bigcup_{i \in \mathrm{I}} \mathrm{H}_i/\mathbb{C} \to \bigcup_{i \in \mathrm{I}} \mathrm{H}_i/\mathbb{C}$, where for all $i \in \mathrm{I}$ there is a $j \in \mathrm{I}$ so, that $b \mid \mathrm{H}_i/\mathbb{C}$ is a ray isomorphism from $\mathrm{H}_i/\mathbb{C}$ onto $\mathrm{H}_j/\mathbb{C}$. The set of these bundle automorphisms is a subgroup of the group of permutations of $\bigcup_{i \in \mathrm{I}} \mathrm{H}_i/\mathbb{C}$, and a bundle representation of a group G is a homomorphism of G into the bundle automorphisms of a bundle space (details of bundle representations can be found in [*10*]). Hence we finally conclude: **Elementary particles must be described by bundle representations.**

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