ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 13, nº 2 (1970), p. 103-127
<http://www.numdam.org/item?id=AIHPA_1970__13_2_103_0>

Ann. Inst. Henri Poincaré,
Vol. XIII, n° 2, 1970, p. 103-127.

Section A :
Physique théorique.

A class of solvable Lie groups and their relation to the canonical formalism

by

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ABSTRACT. — Some facts concerning symplectic vector spaces, their automorphism groups Spl($2n$, $\mathbb{R}$, E), and derivation Lie algebras spl($2n$, $\mathbb{R}$, E) are given. For every element R of these Lie algebras a solvable Lie group exp($\mathbb{R}$R) $\times$ E $\times$ $\mathbb{R}$ is constructed, which is nilpotent iff R is nilpotent. We calculate the Lie algebras $\mathbb{R}$R $\oplus$ E $\oplus$ $\mathbb{R}$ of these groups, all of which contain the Heisenberg Lie algebra. Automorphism groups and derivation Lie algebras of $\mathbb{R}$R $\oplus$ E $\oplus$ $\mathbb{R}$, and faithful finite dimensional representations of them together with the corresponding representations of exp($\mathbb{R}$R) $\times$ E $\times$ $\mathbb{R}$ are given. In Part II a modification weyl(E, σ) of the universal enveloping algebra of the Heisenberg Lie algebra is defined. We realize the Lie algebras $\mathbb{R}$R $\oplus$ E $\oplus$ $\mathbb{R}$ in this algebra. Finally some automorphisms and derivations of $\mathbb{R}$R $\oplus$ E $\oplus$ $\mathbb{R}$ are constructed by means of the adjoint representation of weyl(E, σ). Attention is given to the case of the harmonic oscillator and especially to the free nonrelativistic particle whose group is nilpotent.

RÉSUMÉ. — Quelques qualités concernant des espaces vectoriels symplectiques, leurs groupes d'automorphismes Spl($2n$, $\mathbb{R}$, E) et leurs algèbres de Lie des dérivations sont discutés. Pour chaque élément R d'une telle algèbre de Lie, on construit un groupe de Lie solvable, exp($\mathbb{R}$R) $\times$ E $\times$ $\mathbb{R}$, qui est nilpotent si et seulement si R est nilpotent. On calcule les algèbres de Lie $\mathbb{R}$R $\oplus$ E $\oplus$ $\mathbb{R}$ de ces groupes qui contiennent tous l'algèbre de Lie d'Heisenberg. On donne les groupes d'automorphismes et les algèbres de

Lie des dérivations de $\mathbb{R}R \oplus E \oplus \mathbb{R}$, ainsi que des représentations finies fidèles ensemble avec les représentations correspondantes de $\exp(\mathbb{R}R) \times E \times \mathbb{R}$. Dans la 2e partie, on définit une modification weyl(E, σ) de l'algèbre universelle enveloppante appartenant à l'algèbre de Lie d'Heisenberg. Dans cette algèbre, nous réalisons les algèbres de Lie $\mathbb{R}R \oplus E \oplus \mathbb{R}$.

Finalement, on construit quelques automorphismes et dérivations de $\mathbb{R}R \oplus E \oplus \mathbb{R}$ avec l'aide de la représentation adjointe de weyl(E, σ). On observe l'oscillateur harmonique et plus spécialement la particule libre non relativiste (dont le groupe est nilpotent).

INTRODUCTION

In § I-1 we collect some facts on the symplectic matrix group Spl($2n$, $\mathbb{R}$, E) and its Lie algebra spl($2n$, $\mathbb{R}$, E). In § II-8 we give an isomorphism R between spl($2n$, $\mathbb{R}$, E) and the Lie algebra of all bilinear polynomials of the position and momentum operators q_i and p^i. With the help of this isomorphism, we define for every Hamilton operator which is bilinear in the q_i and p^i a $2n + 2$-dimensional solvable group, each containing the Heisenberg group as a subgroup. Their Lie algebras are isomorphic to the Lie algebras formed by the identity element, the linear combinations of the q_i and p^i, and the chosen Hamilton operator in the infinite dimensional associative algebra weyl(E, σ), which is a certain modification of the universal enveloping algebra of the (Heisenberg) Lie algebra of the canonical commutation relations.

These solvable Lie groups were suggested (see [*1*]) as « spectrum generating » groups of the chosen Hamilton operator, i. e. their inequivalent irreducible unitary representations should label the physical states of the Hamilton operator, excluding spin. We find the hard work of classifying the unitary representations easier for the above solvable groups than for the corresponding « invariance » groups, since their dimensions are in general smaller and their algebraic structures easier to handle. Besides this, in general it is not known which of the local but not global isomorphic covering groups is the invariance group; for instance, it is hard to say which covering group fo the infinitely connected group U(n, $\mathbb{C}$) is the invariance group of the n dimensional harmonic oscillator.

There is another advantage in using these solvable groups instead of invariance groups: having classified their representations, we will have no trouble with the construction of position operators since they are included from the beginning.

We denote the direct vector space sum of two vector spaces A and B by $A \oplus B$, the direct Lie algebra sum of two Lie algebras A and B by $A \diamondplus B$, their semidirect sum, B being the ideal, by $A \mathbin{+\!\!>} B$, and the semidirect product of two groups A and B, B being the normal subgroup, by $A \circledS B$, their direct product by $A \otimes B$.

PART I

A CLASS OF SOLVABLE LIE GROUPS

§ I-1: Symplectic Vector Spaces and the Symplectic Group.

A pair (E, σ) of a real vector space E (in the following we consider only finite dimensional ones) and a nondegenerate antisymmettic bilinear form σ: $E \times E \to \mathbb{R}$ is called a *symplectic vector space*. Between the (antisymmetric) bilinear forms σ and the (antisymmetric) matrices A exists the bijection

$$\sigma(x, y) = \vec{\xi}^{T} A \vec{\eta} \qquad \text{for all } x, y \in E$$

where $x = \sum_{=1}^{n} \xi^i e_i$ with $\xi^i \in \mathbb{R}$ and $e_i \in E$ is the general element of E, $\vec{\xi}^T$ being the row vector $(\xi^1, \ldots, \xi^n)$ of x and $\vec{\xi}$ the corresponding column vector. The bilinear form σ is nondegenerate iff $\det(A) \neq 0$. Because of $\det(A) = (-1)^{\dim(E)} \det(A)$ this is possible only for evendimensional E. We write $\dim(E) = : 2n$.

(1) LEMMA [2; p. 10]. — In (E, σ) we can introduce a basis $e_1, \ldots, e_n$ $f^1, \ldots, f^n$ so that for all $i, k = 1, \ldots, n$

$$\sigma(e_i, e_k) = \sigma(f^i, f^k) = 0 \quad , \quad \sigma(e_i, f^k) = -\sigma(f^k, e_i) = \delta_i^k.$$

In the following the elements of (E, σ) are written $x = \Sigma(\xi^i e_i + \xi_i f^i)$, and the basis of E is chosen such that $\sigma(x, y) = \vec{\xi}^T \mathfrak{J} \vec{\eta}$, where $\mathfrak{J}$ is the $2n \times 2n$ matrix $\begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}$. The automorphism group of (E, σ) is called the *symplectic group* Spl $(2n, \mathbb{R}, E)$. It consists of all invertible $2n \times 2n$ matrices which invariant the bilinear form σ:

(2) $$\mathrm{Spl}\,(2n, \mathbb{R}, E) = \{S \in \mathrm{Gl}\,(2n, \mathbb{R}, E)/\sigma(Sx, Sy) = \sigma(x, y) \quad \text{for all } x, y \in E\}.$$

The defining condition is in matrix form $S^T \mathfrak{J} S = \mathfrak{J}$. The (abstract) symplectic group is a $n(2n + 1)$-dimensional, noncompact, simple, infinitely connected, connected Lie group; with the help of the exponential mapping we get its Lie algebra which is the derivation Lie algebra of (E, σ)

(3) $$\mathrm{spl}\,(2n, \mathbb{R}, E) = \{R \in \mathrm{gl}\,(2n, \mathbb{R}, E)/\sigma(Rx, y) + \sigma(x, Ry) = 0 \quad \forall x, y \in E\}.$$

Herein the defining relation is in matrix form $R^T \mathfrak{J} + \mathfrak{J} R = 0$.

Every matrix of Spl $(2n, \mathbb{R}, E)$ is multiplicatively generated by $\mathfrak{J}$ and symplectic matrices of the type $\begin{pmatrix} \mathrm{id}_n & B \\ 0 & \mathrm{id}_n \end{pmatrix}$, where B is a symmetric $n \times n$ matrix [3; p. 140]. From this follows $\det(S) = +1$ for all $S \in \mathrm{Spl}\,(2n, \mathbb{R}, E)$ and center $(\mathrm{Spl}\,(2n, \mathbb{R}, E)) = \{\pm \mathrm{id}_{2n}\}$. The set of all matrices $\begin{pmatrix} U & V \\ -V & U \end{pmatrix}$, where U and V are real $n \times n$ matrices with $UV^T = VU^T$ and $U^T U + V^T V = \mathrm{id}_n$, is a subgroup of Spl $(2n, \mathbb{R}, E)$. The correspondence to the unitary matrix group in n dimensions is given by

(4) $$U + iV \mapsto \begin{pmatrix} U & V \\ -V & U \end{pmatrix} =: \mathcal{U}$$

[4; p. 350]. Here $(U + iV)$ is unitary iff $\mathcal{U} \in \mathrm{Spl}\,(2n, \mathbb{R}, E)$. It is easy to see that this correspondence is a Lie group isomorphism; we call this $2n$-dimensional representation $\mathcal{U}$ of the unitary group in n dimensions $U(n, \mathbb{R}, E)$. Similar results hold for the Lie algebras: the matrices $\begin{pmatrix} L & K \\ -K & L \end{pmatrix}$, with L a real antisymmetric, K a real symmetric $n \times n$ matrix, form a Lie algebra $u(n, \mathbb{R}, E)$, the correspondence to the unitary matrix Lie algebra in n dimensions being given by 4). The maximal compact subgroup of Spl $(2n, \mathbb{R}, E)$ is $U(n, \mathbb{R}, E)$. This follows from [4; Lemma 4.3, p. 345] and

(5) $$U(n, \mathbb{R}, E) = \mathrm{Spl}\,(2n, \mathbb{R}, E) \cap SO(2n, \mathbb{R}, E).$$

Every $R \in \mathrm{spl}\,(2n, \mathbb{R}, E)$ can be decomposed uniquely

$$\begin{aligned} R &= \frac{1}{2}(R + \mathfrak{J} R \mathfrak{J}^T) + \frac{1}{2}(R - \mathfrak{J} R \mathfrak{J}^T) \\ &= \begin{pmatrix} L & K \\ -K & L \end{pmatrix} + \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \end{aligned}$$

where K, A and B are symmetric $n \times n$ matrices, and L is antisymmetric. The first part is in $u(n, \mathbb{R}, \mathrm{E})$, the second not. According to this decomposition we have $\mathrm{spl}(2n, \mathbb{R}, \mathrm{E}) = u(n, \mathbb{R}, \mathrm{E}) \oplus \mathfrak{p}$ where the vector space $\mathfrak{p}$ is the intersection of $\mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$ with the (Jordan algebra of) symmetric matrices in $2n$ dimensions. It is not a Lie algebra but a so-called Lie triple system [*4*; p. 189, *5*; p. 78]. Actually it is the eigenspace of eigenvalue -1 of the involutive automorphism $\mathrm{R} \mapsto \mathfrak{J}\mathrm{R}\mathfrak{J}^{\mathrm{T}}$ of $\mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$. Since for $\mathrm{R} \in \mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$ we have $(\mathrm{R}\mathfrak{J})^{\mathrm{T}} = \mathrm{R}\mathfrak{J}$, the mapping $\mathrm{R} \mapsto \mathrm{R}\mathfrak{J}$ defines a bijection of $\mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$ onto the $n(2n+1)$-dimensional Jordan algebra of symmetric $2n \times 2n$ matrices. By means of this bijection Williamson [*6*; p. 911] has proved that every $\mathrm{S} \in \mathrm{Spl}(2n, \mathbb{R}, \mathrm{E})$ can be written uniquely in the form $\mathrm{S} = \exp(\alpha \mathrm{R}) \exp(\beta \mathrm{R}')$ with $\mathrm{R}, \mathrm{R}' \in \mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$ and $\alpha, \beta \in \mathbb{R}$. A one parameter subgroup $\exp(\alpha \mathrm{R})$ of $\mathrm{Spl}(2n, \mathbb{R}, \mathrm{E})$ is compact iff $\mathrm{R} \in u(n, \mathbb{R}, \mathrm{E})$. The one parameter subgroup

$$\exp(\alpha \mathfrak{J}) = (\cos \alpha)\, \mathrm{id}_{2n} + (\sin \alpha) \mathfrak{J} \tag{6}$$

is isomorphic as a Lie group to the one-dimensional torus in the usual normtopology on $\mathrm{U}(2n, \mathrm{E})$, given by

$$\| \mathrm{A} \| := (\dim(\mathrm{E}))^{-\frac{1}{2}} \sqrt{\mathrm{Spur}(\mathrm{A}^{\mathrm{T}}\mathrm{A})} \quad \text{for all } \mathrm{A} \in \mathrm{U}(2n, \mathrm{E}).$$

§ I-2: The Oscillator Group and the Heisenberg Group.

Let $e^{\alpha \mathrm{R}}$ with $\alpha \in \mathbb{R}$ and $\mathrm{R} \in \mathrm{gl}(2n, \mathbb{R}, \mathrm{E})$ be a one parameter subgroup of $\mathrm{Gl}(2n, \mathbb{R}, \mathrm{E})$. The topological manifold $e^{\mathbb{R}\mathrm{R}} \times \mathrm{E} \times \mathbb{R}$ becomes a Lie group if we define

$$(e^{\alpha \mathrm{R}}, x, \beta)(e^{\alpha' \mathrm{R}}, y, \beta) := (e^{(\alpha+\alpha')\mathrm{R}}, x + e^{\alpha \mathrm{R}} y, \sigma(x, e^{\alpha \mathrm{R}} y) + \beta + \beta') \tag{7}$$

and iff $\mathrm{R} \in \mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$. The identity element is $(\mathrm{id}_{2n}, 0, 0)$, the inverse of $(e^{\alpha \mathrm{R}}, x, \beta)$ is $(e^{-\alpha \mathrm{R}}, -e^{-\alpha \mathrm{R}} x, -\beta)$. The Lie group given by $\mathrm{R} = \mathfrak{J}$ is called *oscillator group* $\mathrm{Osz}(2n)$. Its dimension is $2n + 2$. We have the subgroups $(e^{\mathbb{R}\mathrm{R}}, 0, 0) \cong e^{\mathbb{R}\mathrm{R}}$, and $(e^{\mathbb{R}\mathrm{R}}, 0, 0)(\mathrm{id}_{2n}, 0, \mathbb{R}) \cong e^{\mathbb{R}\mathrm{R}} \otimes \mathbb{R}$, and the normal subgroups $(\mathrm{id}_{2n}, 0, \mathbb{R}) \cong \mathbb{R}$ (the center), and $(\mathrm{id}_{2n}, \mathrm{E}, \mathbb{R})$. The latter we call *Heisenberg group* $\mathrm{Heis}(2n)$. It is a $2n + 1$-dimensional Lie group on the manifold $\mathrm{E} \times \mathbb{R}$, connected and simply connected with composition

$$(x, \beta)(y, \beta') := (x + y, \sigma(x, y) + \beta + \beta'). \tag{8}$$

Every group $e^{\mathbb{R}\mathrm{R}} \times \mathrm{E} \times \mathbb{R}$ is the semidirect product of $e^{\mathbb{R}\mathrm{R}}$ and $\mathrm{Heis}(2n)$.

In the product topology on $e^{\mathbb{R}R} \times E \times \mathbb{R}$, given by the usual topology on Gl $(2n, \mathbb{R}, E)$, the normtopology $\| x \| := (\sigma(\mathfrak{J}x, x))^{1/2}$ on E, and the usual topology on $\mathbb{R}$, the groups $e^{\mathbb{R}R} \times E \times \mathbb{R}$ and Heis $(2n)$ are noncompact and connected. The *commutant* of two elements is

$$(9) \quad (e^{\alpha R}, x, \beta)(e^{\alpha' R}, y, \beta')(e^{\alpha R}, x, \beta)^{-1}(e^{\alpha' R}, y, \beta')^{-1}$$
$$= (\mathrm{id}_{2n}, (\mathrm{id}_{2n} - e^{\alpha' R})x - (\mathrm{id}_{2n} - e^{\alpha R})y, \sigma(x, e^{\alpha R}y) + \sigma(e^{\alpha' R}x, y)$$
$$- \sigma(x, y) + \sigma(e^{\alpha' R}x, e^{\alpha R}y) + \sigma(e^{\alpha' R}x, x) + \sigma(y, e^{\alpha R}y)).$$

Calculating the successive commutants we see that $e^{\mathbb{R}R} \times E \times \mathbb{R}$ is solvable, and even nilpotent iff R is nilpotent. Heis $(2n)$ is nilpotent but not commutative.

The manifold $\mathbb{R} \times E \times \mathbb{R}$ can be made a Lie group if we define

$$(10) \quad (\alpha, x, \beta)(\alpha', y, \beta') := (\alpha + \alpha', x + e^{\alpha R}y, \sigma(x, e^{\alpha R}y) + \beta + \beta')$$

and iff $R \in \mathrm{spl}(2n, \mathbb{R}, E)$. These groups have the same algebraic properties as the corresponding groups above, but are simply connected for all R. Since they have the same Lie algebras (see below) they are the universal covering groups of the original ones. For $R = \mathfrak{J}$ we get an infinite covering of the infinitely connected Osz$(2n)$. If the projection of R onto $u(n, \mathbb{R}, E)$ vanishes the groups are homeomorphic [*4*; lemma 4.3, p. 345].

On the manifold $e^{\mathbb{R}R} \times E$ we define a Lie group by

$$(11) \quad (e^{\alpha R}, x)(e^{\alpha' R}, y) := ((e^{(\alpha+\alpha')R}, x + e^{\alpha R}y) \qquad \forall R \in \mathrm{gl}\,(2n, \mathbb{R}, E)$$

with $(e^{\alpha R}, x)^{-1} = (e^{-\alpha R}, -e^{-\alpha R}x)$ and identity element $(\mathrm{id}_{2n}, 0)$. If E denotes the commutative additive group of elements of E, we have for this group $e^{\mathbb{R}R} \otimes E$. Its center is $\{(\mathrm{id}_{2n}, 0)\}$; it is solvable and for nilpotent R even nilpotent.

(12) THEOREM. — For $R \in \mathrm{spl}(2n, \mathbb{R}, E)$ the group $e^{\mathbb{R}R} \times E \times \mathbb{R}$ is the central extension of $e^{\mathbb{R}R} \otimes E$ by $(\mathrm{id}_{2n}, 0, \mathbb{R})$, i. e. the sequence

$$\{(\mathrm{id}_{2n}, 0, 0)\} \to (\mathrm{id}_{2n}, 0, \mathbb{R}) \to e^{\mathbb{R}R} \times E \times \mathbb{R} \xrightarrow{\varphi} e^{\mathbb{R}R} \otimes E \to \{(\mathrm{id}_{2n}, 0)\}$$

is exact, the homomorphism φ being the restriction.

For the notation see [7]. The proof is straightforward. Without difficulty we can define a class of locally isomorphis Lie groups on $e^{\mathbb{R}R} \times E \times$ Tor if we substitute the torus $e^{i\mathbb{R}}$ for $\mathbb{R}$. In the same way we get bigger Lie groups by inserting the whole group Spl$(2n, \mathbb{R}, E)$ instead of $e^{\mathbb{R}R}$, which remain solvable if we restrict Spl$(2n, \mathbb{R}, E)$ to a commutative subgroup.

§ I-3: The Lie Algebras of Osz(2n) and Heis(2n).

(13) PROPOSITION. — The Lie algebra of the group $e^{\mathbb{R}R} \times E \times \mathbb{R}$ is the vector space $\mathbb{R}R \oplus E \oplus \mathbb{R}$ together with the Lie bracket

$$[(\alpha R, x, \beta), (\alpha' R, y, \beta')]_- = \left(0, \frac{1}{2}(\alpha' R x - \alpha R y), 2\sigma(x, y)\right).$$

Proof. — The Lie algebra is the tangential space of $e^{\mathbb{R}R} \times e \times \mathbb{R}$ in $(\mathrm{id}_{2n}, 0, 0)$ [4; p. 88*ff*]. We calculate its elements with the help of the two one parameter subgroups θ, θ': $\mathbb{R} \to e^{\mathbb{R}R} \times E \times \mathbb{R}$ defined by θ: $\mu \mapsto (e^{\mu\alpha R}, 0, 0)$ and θ': $\mu \mapsto (\mathrm{id}_{2n}, \mu x, \mu\beta)$. From

$$\frac{d}{d\mu}(e^{\mu\alpha R}, 0, 0)\,|_{\mu=0} = (\alpha R, 0, 0)$$

$$\frac{d}{d\mu}(\mathrm{id}_{2n}, \mu x, \mu\beta)\,|_{\mu=0} = (0, x, \beta)$$

follows the first statement. To get the Lie brackets consider the commutant of the two one parameter subgroups θ and θ', which is $(\mathrm{id}_{2n}, (\mathrm{id}_{2n} - e^{\alpha R})y, \sigma(y, e^{\alpha R}y))$ from (9). From the curve segment

$$\mu \mapsto \left(\mathrm{id}_{2n}, (\mathrm{id}_{2n} - e^{\sqrt{\mu}\alpha R})\sqrt{\mu}y, \mu\sigma(y, e^{\sqrt{\mu}\alpha R}y)\right) \qquad \mu \geqslant 0$$

we get [4; p. 97] the tangent vector

$$\begin{aligned}[(\alpha R, 0, 0), (0, y, \beta)]_- :&= \frac{d}{d\mu}\left(\mathrm{id}_{2n}, (\mathrm{id}_{2n} - e^{\sqrt{\mu}\alpha R})\sqrt{\mu}y, \mu\sigma(y, e^{\sqrt{\mu}\alpha R}y)\right)|_{\mu=0} \\ &= \left(0, -\frac{1}{2}\alpha R y, 0\right).\end{aligned}$$

By the same line of reasoning we get from the commutant of two different elements of θ' the element $(\mathrm{id}_{2n}, 0, 2\sigma(x,y))$, and from this the tangent vector

(14) $$[(0, x, \beta), (0, y, \beta')]_- = (0, 0, 2\sigma(x, y)).$$

These Lie brackets are just those of the Heisenberg subgroup. The isomorphic Lie algebras of the groups $\mathbb{R} \times E \times \mathbb{R}$ are found in a similar way. We call the Lie algebra $\mathbb{R}\mathfrak{J} \oplus E \oplus \mathbb{R}$ *oscillator Lie algebra* osz(2n) [8] and the subalgebra $E \oplus \mathbb{R}$ *Heisenberg Lie algebra* heis(2n). The algebraic facts of § I-2 immediately carry over to the Lie algebras; especially

(15) $$\mathrm{osz}\,(2n) = \mathbb{R}\mathfrak{J} \mathbin{+\!\!\!\!\supset} \mathrm{heis}\,(2n).$$

Inserting the basis from lemma 1) in the Lie algebra $\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R}$, writing $-2(\mathrm{R}, 0, 0) =: \mathrm{H_R}$, $(0, e_i, 0) =: \sqrt{2}q_i$, $(0, f^i, 0) =: \sqrt{2}p^i$ and $2(0, 0, 1) =: c$ we get for the elements

$$\xi_\mathrm{R}\mathrm{H_R} + \sum_{i=1}^{n}(\xi^i q_i + \xi_i p^i) + \xi_0 c =: \xi_\mathrm{R}\mathrm{H_R} + \underline{x}$$

with $\xi_\mathrm{R}, \xi_0 \in \mathbb{R}$ the general Lie bracket relations

$$(16) \qquad [\xi_\mathrm{R}\mathrm{H_R} + \underline{x}, \eta_\mathrm{R}\mathrm{H_R} + \underline{y}]_- = \xi_\mathrm{R}\mathrm{R}y - \eta_\mathrm{R}\mathrm{R}x + \sigma(x, y)c,$$

which specializes in the case of osz$(2n)$ to

$$(17) \qquad [\mathrm{H_R}, q_i]_- = -p^i, \quad [\mathrm{H_J}, p^i]_- = q_i, \quad [q_i, p^k]_- = \delta_i^{\,k} c$$

(rest zero). The Lie algebra of the group (11) is calculated by the same way. Its Lie bracket relations are

$$(18) \qquad [\xi_\mathrm{R}\mathrm{H_R} + x, \eta_\mathrm{R}\mathrm{H_R} + y]_- = \xi_\mathrm{R}\mathrm{R}y - \eta_\mathrm{R}\mathrm{R}x.$$

It will be shown in § I-6 that this Lie algebra $\mathbb{R}\mathrm{R} \mathbin{+\!\!\!>} \mathrm{E}$ is just the « adjoint » Lie algebra of $\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R}$.

§ I-4: Automorphisms and Derivations of Heis(2n).

Let G be a $2n \times 2n$ matrix, $\alpha \in \mathbb{R}$, b^T a $2n$ row vector and a another $2n$ column vector. Then the matrix $\begin{pmatrix} \mathrm{G} & \alpha \\ b^\mathrm{T} & a \end{pmatrix} =: \mathrm{A}$ is an automorphism of heis$(2n)$ iff $\det(\mathrm{A}) \neq 0$ and the defining relation for automorphisms

$$(19) \qquad \mathrm{A}([\underline{x}, \underline{y}]_-) = [\mathrm{A}\underline{x}, \mathrm{A}\underline{y}]_-$$

holds. From this we get the automorphism group of heis$(2n)$

$$(20) \quad \mathrm{Aut}(\mathrm{heis}(2n)) = \{\mathrm{A} \in \mathrm{Gl}(2n+1, \mathbb{R}, \mathrm{E} \oplus \mathbb{R})/a = 0, 0 \neq \alpha \in \mathbb{R},\ b \text{ an arbitrary } 2n \text{ vector}, \sigma(\mathrm{G}x, \mathrm{G}y) = \alpha\sigma(x, y)\}.$$

With the matrix $\Sigma_1 := \begin{pmatrix} 0 & \mathrm{id}_n \\ \mathrm{id}_n & 0 \end{pmatrix}$ the matrix A becomes

$$(21) \qquad \begin{pmatrix} \sqrt{|\alpha|}\left\{\dfrac{(1+\mathrm{sign}\,\alpha)}{2}\mathrm{id}_{2n} + \dfrac{(1-\mathrm{sign}\,\alpha)}{2}\Sigma_1\right\}\mathrm{S} & 0 \\ b^\mathrm{T} & \alpha \end{pmatrix}$$

with $S \in \mathrm{Spl}(2n, \mathbb{R}, E)$. Here we used that because of $\Sigma_1{}^T\mathfrak{J}\Sigma_1 = -\mathfrak{J}$ every *antisymplectic* matrix F (i. e. $F^T\mathfrak{J}F = -\mathfrak{J}$) can be written $\Sigma_1 S$, where now $S \in \mathrm{Spl}(2n; \mathbb{R}, E)$. Proof ; Given $F^T\mathfrak{J}F = -\mathfrak{J}$; it follows

$$(\Sigma_1 F)^T\mathfrak{J}(\Sigma_1 F) = -F^T\mathfrak{J}F = \mathfrak{J},$$

that is $\Sigma_1 F =: S \in \mathrm{Spl}(2n, \mathbb{R}, E)$ and $F = \Sigma_1{}^{-1}S = \Sigma_1 S$. We have $\det(A) = |\alpha|^{n+1}$. Aut (heis $(2n)$) is a $n(2n+1) + 2n + 1$-dimensional matrix group which decomposes in two nonconnected pieces. We write its general element $A(S, b, |\alpha|, \operatorname{sign}\alpha)$. For the identity component we have

(22) $A(S, b, |\alpha|, 1)\, A(S', b', |\alpha'|, 1) = A(SS', S'^T b + |\alpha| b', |\alpha\alpha'|, 1)$,

the inverse is in the both components respectively

$$(23)\qquad \begin{aligned} &A\left(S^{-1}, -|\alpha|^{-3/2}(S^{-1})^T b, \frac{1}{|\alpha|}, 1\right)\\ &A\left(\Sigma_1 S^{-1}\Sigma_1, |\alpha|^{-3/2}\Sigma_1 S^{-1T} b, \frac{1}{|\alpha|}, -1\right). \end{aligned}$$

We get the following subgroups of the identity component

(24) $\{A(S, 0, 1, 1)\} = \mathrm{Spl}(2n, \mathbb{R}, E)$

(25) $\{A(\mathrm{id}_{2n}, 0, |\alpha|, 1)\} =: \mathrm{Dil}(2n)$ *dilatations on* E

(26) $\{A(\mathrm{id}_{2n}, b, 1, 1)\} =: \mathcal{F}(2n)$

(27) $\{A(S, b, 1, 1)\} = \mathcal{F}(2n) \otimes \mathrm{Spl}(2n, \mathbb{R}, E)$

(28) $\{A(S, 0, |\alpha|, 1)\} = \mathrm{Dil}(2n) \otimes \mathrm{Spl}(2n, \mathbb{R}, E)$

(29) $\{A(\mathrm{id}_{2n}, b, |\alpha|, 1)\} = \mathrm{Dil}(2n) \otimes \mathcal{F}(2n)$.

Note that the elements of $\mathcal{F}(2n)$ do not act as translations on E, as would do the matrices $\begin{pmatrix} \mathrm{id}_{2n} & a \\ 0 & 1 \end{pmatrix}$ for nonvanishing a. We write the discrete group $\{\mathrm{id}_{2n}, \Sigma_1\} =: \mathbb{Z}_2$. Then we have

(30) $\mathrm{Aut}(\mathrm{heis}(2n)) = \mathbb{Z}_2 \otimes (\mathcal{F}(2n) \otimes (\mathrm{Spl}(2n, \mathbb{R}, E) \otimes \mathrm{Dil}(2n)))$

where $\mathcal{F}(2n)$ is the group of inner automorphisms. $\mathbb{Z}_2$ interchanges the basiselements q_i and p^i of E. Therefore it is a good candidate for the Legendre transformations known from classical mechanics. $\mathrm{Spl}(2n, \mathbb{R}, E)$ can be interpreted as the group of canonical transformations of a linear phase space E.

$D \in \mathrm{gl}(2n+1, \mathbb{R}, E \oplus \mathbb{R})$ is a *derivation* of heis$(2n)$ iff

(31) $$D([\underline{x}, \underline{y}]_-) = [D\underline{x}, \underline{y}]_- + [\underline{x}, D\underline{y}]_-.$$

The derivations of heis$(2n)$ form a Lie algebra of linear transformations which is given by the set of matrices $\begin{pmatrix} N & 0 \\ p^T & \beta \end{pmatrix}$, where p is an arbitrary $2n$ vector, $\beta \in \mathbb{R}$ and N is a $2n \times 2n$ matrix subject to the condition

$$\sigma(Nx, y) + \sigma(x, Ny) = \beta\sigma(x, y).$$

Therefore

$$\text{(32)} \quad \begin{aligned} \operatorname{Der}(\operatorname{heis}(2n)) &= \left\{ \begin{pmatrix} V + \frac{1}{2}\beta \operatorname{id}_{2n} & 0 \\ p^T & \beta \end{pmatrix} \middle/ \begin{matrix} p \text{ an arbitrary } 2n \text{ vector,} \\ V \in \operatorname{spl}(2n, \mathbb{R}, E), \beta \in \mathbb{R}. \end{matrix} \right\} \\ &= A(2n) \Leftarrow\!\!\!+ (\operatorname{spl}(2n, \mathbb{R}, E) \oplus \operatorname{dil}(2n)), \end{aligned}$$

where $A(2n)$ is the Lie algebra of $\mathcal{F}(2n)$ and dil $(2n)$ that of Dil $(2n)$.

§ I-5: Automorphisms and Derivations of the Oscillator Lie Algebras.

Given $0 \neq R \in \operatorname{spl}(2n, \mathbb{R}, E)$, the *centralizer of* R *in* spl $(2n, \mathbb{R}, E)$ is

$$\text{(33)} \qquad \operatorname{zent}(R) := \{ V \in \operatorname{spl}(2n, \mathbb{R}, E) / [V, R]_- = 0\}$$

and the *centralizer of* R *in* Spl $(2n, \mathbb{R}, E) \cup \Sigma_1$ Spl $(2n, \mathbb{R}, E)$ is

$$\text{(34)} \qquad \operatorname{Zent}(R) := \{ G \in \operatorname{Spl}(2n, \mathbb{R}, E) \cup \Sigma_1 \operatorname{Spl}(2n, \mathbb{R}, E) / GRG^{-1} = R\}.$$

Let $\mathfrak{Z}_K$ denote the vector space of all $V \in \operatorname{spl}(2n, \mathbb{R}, E)$ subject to

$$[V, R]_- = \nu_V R \qquad \nu_V \in \mathbb{R}$$

where $\nu_V = 0$ only for $V = 0$, and $\mathfrak{Z}_R$ the manifold of all

$$G \in \operatorname{Spl}(2n, \mathbb{R}, E) \cup \Sigma_1 \operatorname{Spl}(2n, \mathbb{R}, E)$$

with

$$GRG^{-1} = \delta_G R \qquad 0 \neq \delta_G \in \mathbb{R}$$

where $\delta_G = 1$ only for $G = \operatorname{id}_{2n}$. Then we have

(35) Lemma. — The set of elements $V \in \operatorname{spl}(2n, \mathbb{R}, E)$ with

$$\text{(36)} \qquad [V, R]_- = \nu_V R \qquad \nu_V \in \mathbb{R}$$

is just the matrix Lie algebra zent $(R) \oplus \mathfrak{Z}_R$.

The set of elements $G \in \mathrm{Spl}(2n, \mathbb{R}, E) \cup \Sigma_1 \mathrm{Spl}(2n, \mathbb{R}, E)$ with

$$GRG^{-1} = \delta_G R \qquad 0 \neq \delta_G \in \mathbb{R} \tag{37}$$

is just the matrix group $\mathrm{Zent}(R) \times \mathfrak{Z}_R$.

In this Lie algebra (resp. Lie group) zent(R) (resp. Zent(R)) is an ideal (resp. normal subgroup).

Proof. — Every element V with (36) can be decomposed uniquely into $V = V_0 + V_v$ where now $V_0 \in \mathrm{zent}(R)$ and $V \in \mathfrak{Z}_R$. By definition from $V \in \mathrm{zent}(R) \cap \mathfrak{Z}_R$ follows $V = 0$, i. e. the vector space sum of zent(R) and $\mathfrak{Z}_R$ is direct. From the Jacobi identity follows that zent (R) is an ideal. The proof for the group theoretical statement is similar. ■

For the physical interesting R the vector space $\mathfrak{Z}_R$ is one-dimensional or zero. Then the matrix Lie algebra of the $V \in \mathrm{spl}(2n, \mathbb{R}, E)$ with (36) is $\mathrm{zent}(R) \mathbin{\diamondsuit\!\!\!\!+} \mathfrak{Z}_R$ or only zent(R). The corresponding facts hold for the groups.

(38) THEOREM. — Given $0 \neq R \in \mathrm{spl}(2n, \mathbb{R}, E)$, $R^2 \neq 0$, an element of $\mathrm{Gl}(2n + 2, \mathbb{R}, \mathbb{R} \oplus E \oplus \mathbb{R})$ is in $\mathrm{Aut}(\mathbb{R}R \times E \times \mathbb{R})$ iff it has the form

$$\begin{pmatrix} \delta & 0 & 0 \\ -\dfrac{1}{\alpha} GR\mathfrak{J}b & G & 0 \\ \tau & b^T & \alpha \end{pmatrix} \qquad \begin{array}{l} 0 \neq \delta \in \mathbb{R} \\ 0 \neq \alpha \in \mathbb{R} \end{array}$$

where we have $GRG^{-1} = \delta R$, i. e. $G \in \mathrm{Zent}(R) \times \mathfrak{Z}_R$, and

$$G = \sqrt{|\alpha|} \left\{ \frac{(1 + \mathrm{sign}\,\alpha)}{2} \mathrm{id}_{2n} + \frac{(1 - \mathrm{sign}\,\alpha)}{2} \Sigma_1 \right\} S, \tag{39}$$

$S \in \mathrm{Spl}(2n, \mathbb{R}, E)$.

Proof. — We insert the matrix $\begin{pmatrix} \delta & d^T & \gamma \\ c & G & a \\ \tau & b^T & \alpha \end{pmatrix}$ into the automorphism condition (19) of the Lie algebra relations (16). This gives for $\xi_R \neq 0$, $y \neq 0$, rest zero

(a) $\sigma(\mathfrak{J}d, Ry) = 0$ for all $y \in E$

(b) $GRy = \delta RGy - \sigma(\mathfrak{J}d, y)Rc$ for all $y \in E$

(c) $\sigma(\mathfrak{J}b, Ry) = \sigma(c, Gy)$ for all $y \in E$, for $x \neq 0$, $y \neq 0$, rest zero

(d) $\gamma = 0$

(e) $\sigma(\mathfrak{J}d, x)\mathrm{RG}y - \sigma(\mathfrak{J}d, y)\mathrm{RG}x = \sigma(x, y)a$ for all $x, y \in \mathrm{E}$

(f) $\sigma(\mathrm{G}x, \mathrm{G}y) = \alpha\sigma(x, y)$ for all $x, y \in \mathrm{E}$,

and for $x \neq 0$, $\eta_0 \neq 0$, rest zero

(g) $\sigma(\mathrm{G}x, a) = 0$ for all $x \in \mathrm{E}$.

From (f) we have $\det(\mathrm{G}) = \alpha^n$; since every automorphism must be invertible from (g) it follows that $a = 0$ iff G invertible. Let G be not invertible: i. e. $a \neq 0$ and $\alpha = 0$. From (e) we have for $x = \mathrm{R}z$ and $y = \mathrm{R}v$ with the help of (a) for all $v, z \in \mathrm{E}$: $0 = \sigma(\mathrm{R}z, \mathrm{R}v)a = -\sigma(z, \mathrm{R}^2 v)a$, i. e. $\mathrm{R}^2 = 0$, which is not true. So G must be invertible. From (b) we get with the help of (a) for $y = \mathrm{R}z$ and all $z \in \mathrm{E}$: $0 \neq \mathrm{GR}^2 z = \delta\mathrm{RGR}z$, i. e. $\delta \neq 0$ and $\mathrm{RGR} \neq 0$. For $x = \mathrm{R}z$ we get therefore from (e) for all $y, z \in \mathrm{E}$, $\sigma(\mathfrak{J}d, y)\mathrm{RGR} = 0$, i. e. $d = 0$. The result for the vector c follows from (c) and (f), and the rest of the theorem from lemma 35) and (21) *ff.* ∎

(40) THEOREM. — Given $0 \neq \mathrm{R} \in \mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$, $\mathrm{R}^2 \neq 0$, an element of $\mathrm{gl}\,(2n + 2, \mathbb{R}, \mathbb{R} \oplus \mathrm{E} \oplus \mathbb{R})$ is in $\mathrm{der}(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$ iff it has the form

$$\begin{pmatrix} v & 0 & 0 \\ -\mathrm{R}\mathfrak{J}p & \mathrm{V} + \frac{1}{2}\beta\, id_{2n} & 0 \\ \rho & p^{\mathrm{T}} & \beta \end{pmatrix}$$

where $[\mathrm{V}, \mathrm{R}]_- = v\mathrm{R}$, i. e. $\mathrm{V} \in \mathrm{zent}\,(\mathrm{R}) \oplus \mathfrak{Z}_{\mathrm{R}} \subset \mathrm{spl}\,(2n, \mathbb{R}, \mathrm{E})$.

The proof is similar to the proof above. For the special case $\mathrm{R} = \mathfrak{J}$ (the *harmonic oscillator*) we get

(41) COROLLARY. — $\mathrm{Aut}(\mathrm{osz}(2n))$ is given by all matrices of the form

$$\begin{pmatrix} \operatorname{sign}\alpha & 0 & 0 \\ \frac{1}{\alpha}\mathrm{G}b & \mathrm{G} & 0 \\ \tau & b^{\mathrm{T}} & \alpha \end{pmatrix} \qquad 0 \neq \alpha \in \mathbb{R}$$

where G is subject to (39) and $\mathrm{S} \in \mathrm{Zent}\,(\mathfrak{J}) = \mathrm{U}(n, \mathbb{R}, \mathrm{E})$.

der(osz(2n)) is given by all matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ p & \mathrm{V} + \frac{1}{2}\beta\,\mathrm{id}_{2n} & 0 \\ \rho & p^{\mathrm{T}} & \beta \end{pmatrix} \qquad \mathrm{V} \in u(n, \mathbb{R}, \mathrm{E})$$

Proof. — From $[\mathrm{V}, \mathfrak{I}]_- = \nu\mathfrak{I}$ and $\mathrm{V}^{\mathrm{T}}\mathfrak{I} + \mathfrak{I}\mathrm{V} = 0$ follows $\mathrm{V} + \mathrm{V}^{\mathrm{T}} = \nu\,\mathrm{id}_{2n}$. Since $\mathrm{id}_{2n} \notin \mathrm{spl}(2n, \mathbb{R}, \mathrm{E})$ we have $\nu = 0$. For the group we have $\mathrm{S}^{\mathrm{T}}\mathrm{S} = \delta\,(\mathrm{sign}\,\alpha)\,\mathrm{id}_{2n}$, from which we have $\delta = \mathrm{sign}\,\alpha$, since $\mathrm{S}^{\mathrm{T}}\mathrm{S} \in \mathrm{Spl}\,(2n, \mathbb{R}, \mathrm{E})$ is positive definite. The rest follows from (5). ■

§ I-6: « Self »-representations and adjoint representations.

The invertible matrices ($\vec{\xi}$ being the column vector corresponding to x)

$$(42) \qquad \begin{pmatrix} 1 & 0 & 0 \\ \vec{\xi} & \mathrm{S} & 0 \\ \beta & \vec{\xi}^{\mathrm{T}}\mathfrak{I}\mathrm{S} & 1 \end{pmatrix} =: \mathrm{M}(\mathrm{S}, x, \beta)$$

are the elements of the group Spl $(2n, \mathbb{R}, \mathrm{E}) \otimes$ Heis $(2n)$ with the composition law

$$(43) \quad \mathrm{M}(\mathrm{S}, x, \beta)\,\mathrm{M}(\mathrm{U}, y, \beta') = \mathrm{M}(\mathrm{SU}, x + \mathrm{S}y, \sigma(x, \mathrm{S}y) + \beta + \beta')$$

iff $\mathrm{S}, \mathrm{U} \in \mathrm{Spl}\,(2n, \mathbb{R}, \mathrm{E})$. For $\mathrm{S} = e^{\alpha\mathrm{R}}$ and $\mathrm{U} = e^{\alpha'\mathrm{R}}$ we get a $2n + 2$-dimensional faithful representation of $e^{\mathbb{R}\mathrm{R}} \times \mathrm{E} \times \mathbb{R}$, for $\mathrm{R} = 0$ of Heis $(2n)$. The lie algebra of this matrix group is given by the matrices

$$(44) \qquad \begin{pmatrix} 0 & 0 & 0 \\ \vec{\xi} & \mathrm{V} & 0 \\ \beta & \vec{\xi}^{\mathrm{T}}\mathfrak{I} & 0 \end{pmatrix} =: \mathrm{N}(\mathrm{V}, x, \beta)$$

which have the commutation relations

$$(45) \quad [\mathrm{N}(\mathrm{V}, x, \beta), \mathrm{N}(\mathrm{Z}, y, \beta')]_- = \mathrm{N}([\mathrm{V}, \mathrm{Z}]_-, \mathrm{V}y - \mathrm{Z}x, 2\sigma(x, y))$$

iff $\mathrm{V}, \mathrm{Z} \in \mathrm{spl}\,(2n, \mathbb{R}, \mathrm{E})$. For $\mathrm{V} = \xi_{\mathrm{R}}\mathrm{R}$ and $\mathrm{Z} = \eta_{\mathrm{R}}\mathrm{R}$ we have the commutation relations of proposition 13). So (44) gives faithful representations of the Lie algebras $\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R}$, for $\mathrm{R} = \mathfrak{I}$ of osz $(2n)$, and for $\mathrm{R} = 0$ of heis $(2n)$.

The *adjoint representation* ad: $\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R} \to \mathrm{inder}(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$ onto the *inner derivations* ad $(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$, the kernel of which is the center $(0, 0, \mathbb{R})$, is given by

$$\xi_{\mathrm{R}}\mathrm{H}_{\mathrm{R}} + \underline{x} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{R}\vec{\xi} & \xi_{\mathrm{R}}\mathrm{R} & 0 \\ 0 & \vec{\xi}^{\mathrm{T}}\mathfrak{J} & 0 \end{pmatrix} \tag{46}$$

These matrices form a faithful $2n + 2$-dimensional representation of the Lie algebra (18); they are an ideal in $\mathrm{der}(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$. The exact sequence from theorem 12), read in the Lie algebraic form, is thus nothing else than the well known exact sequence of the adjoint algebra of a Lie algebra.

The adjoint representation Ad: $\exp(\mathbb{R}\mathrm{R}) \times \mathrm{E} \times \mathbb{R} \to \mathrm{Int}(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$ onto the *inner automorphisms* of $\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R}$, the kernel of which is the center $(0, 0, \mathbb{R})$, is given by

$$(e^{\alpha\mathrm{R}}, x, \beta) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -\mathrm{R}\vec{\xi} & e^{\alpha\mathrm{R}} & 0 \\ -\frac{1}{2}\sigma(x, \mathrm{R}x) & \vec{\xi}^{\mathrm{T}}\mathfrak{J}e^{\alpha\mathrm{R}} & 1 \end{pmatrix}. \tag{47}$$

These matrices form a faithful $2n + 2$-dimensional representation of the Lie group (11); they are a normal subgroup in $\mathrm{Aut}(\mathbb{R}\mathrm{R} \oplus \mathrm{E} \oplus \mathbb{R})$. Thus theorem 12) gives the exact sequence of the *adjoint group* $e^{\mathbb{R}\mathrm{R}} \otimes \mathrm{E}$ of $\exp(\mathbb{R}\mathrm{R}) \times \mathrm{E} \times \mathbb{R}$ [*4*; p. 116]. The restriction to heis $(2n)$ of the above matrices gives for $\alpha = 0$ the adjoint representation of heis $(2n)$ and Heis $(2n)$, which correspond to $\mathrm{A}(2n)$ in (32) and to $\mathcal{F}(2n)$ in (30). Since

$$\mathrm{spl}\,(2n, \mathbb{R}, \mathrm{E}) \subset \mathrm{sl}\,(2n, \mathbb{R}, \mathrm{E})$$

the determinant of the matrices (47) equals 1; therefore the groups $\exp(\mathbb{R}\mathrm{R}) \times \mathrm{E} \times \mathbb{R}$ are unimodular [*4*; p. 366].

PART II

THE WEYL ALGEBRA

§ II-7: Definition of the Weyl Algebra.

Let ten(heis($2n$)) be the tensor algebra of the vector space $\mathrm{E} \oplus \mathbb{R}c$ of heis($2n$), $\otimes$ the tensor multiplication, and $([a, b]_- - (a \otimes b - b \otimes a))$

the two-sided ideal of ten(heis(2n)), which is generated by all elements of the form $[a, b]_- - (a \otimes b - b \otimes a)$ with $a, b \in$ heis (2n) $\subset$ ten (heis (2n)). Then the infinite dimensional associative algebra

(48) $\quad u(\text{heis}\ (2n))\colon = \text{ten}\ (\text{heis}\ (2n))/(\sigma(x, y)c - (\underline{x} \otimes \underline{y} - \underline{y} \otimes \underline{x}))$

is called *universal enveloping algebra* of heis (2n). For the notion of universal enveloping algebras and the following statements see [*4*; p. 90], [*9*; p. 151], [*10*; p. 26] and [*11*; exposé n° 1]:

(49) LEMMA. — heis(2n) is imbedded injectively in u (heis (2n)).

(50) LEMMA. — We have inj (heis (2n)) $\cap \mathbb{R}1 = \{0\}$; here 1 is the identity element of u (heis (2n)), and inj (heis (2n)) is the isomorphic image of heis (2n) in u (heis (2n)).

(51) LEMMA. — A basis of u (heis (2n)) is given by the identity element and the standard monomials of the basis elements of inj (heis (2n)).

Because of lemma (49) we identify heis (2n) and inj (heis (2n)). Because of lemma (50) we cannot identify the element inj (c) of inj (heis (2n)) with the identity element of u (heis (2n)). But actually this always is done in physical applications, for instance in the Poisson bracket Lie algebra of position and momentum variables in classical mechanics, and the commutator Lie algebra of position and momentum operators in quantum mechanics. Therefore we consider instead of u (heis (2n)) a different noncommutative associative infinite dimensional algebra which identifies c and the identity element. Let $(c - 1)$ be the two-sided ideal of u (heis (2n)) which is generated by the elements $c - 1 \in u$ (heis (2n)). Then the algebra

(52) $\quad \text{weyl}\ (\mathrm{E}, \sigma)\colon = u\ (\text{heis}\ (2n))/(c - 1)$

is identical with the algebra

(53) $\quad \text{ten}\ (\mathrm{E})/(\sigma(x, y)1 - (x \otimes y - y \otimes x)),$

where ten (E) is the tensor algebra over the vector space E, 1 the identity element of ten(E), and $x, y \in \mathrm{E} \subset$ ten(E). We call this algebra *Weyl algebra of* (E, σ) [*12*; p. 148). heis (2n) is embedded injectively in weyl(E, σ). A basis of weyl(E, σ) is given by the standard monomials of the basis elements of E $\subset$ weyl(E, σ) and 1. One should compare (53) with the definition of the Clifford algebra over an orthogonal vector space [*12*; p 148], [*13*, p. 367].

For the following we need two other universal algebras. Let sym (heis $(2n)$) be the universal enveloping algebra of the trivial Lie algebra on the vector space $E \oplus \mathbb{R}c$ of heis $(2n)$, we get by demanding every Lie bracket to vanish, i. e. for $\underline{x}, \underline{y} \in$ heis $(2n) \subset$ ten (heis $(2n)$)

$$\text{(54)} \qquad \text{sym (heis } (2n)): = \text{ten (heis } (2n))/(\underline{x} \otimes \underline{y} - \underline{y} \otimes \underline{x}).$$

It is clear that the universal enveloping algebra of the commutative trivial Lie algebra on E, i. e.

$$\text{(55)} \qquad \text{sym (E)}: = \text{ten (E)}/(x \otimes y - y \otimes x)$$

is isomorphic as an associative algebra to the algebra

$$\text{(56)} \qquad \text{sym (heis } (2n))/(c - 1).$$

Let us denote the inclusion sym (E) $\to$ sym (heis $(2n)$) by j, the projection u(heis $(2n)$) $\to$ weyl (E, σ) by π and the vector space in u (heis $(2n)$) generated linearly by all monomials $\underline{x}_1\underline{x}_2\ldots\underline{x}_i$ (for all $\underline{x}_k \in$ heis $(2n) \subset u$ (heis $(2n)$)) by u_i(heis$(2n)$) with u_0 (heis$(2n)$): $= \mathbb{R}1$. The vector space

$$\text{(57)} \qquad W_i: = u_i \text{ (heis } (2n))/(c - 1)$$

is generated linearly by the monomials $x_1x_2 \ldots x_i$ with $x_k \in E \subset$ weyl (E, σ). We have $W_0 = \mathbb{R}1$. In the following we denote the product of X and Y in sym (heis $(2n)$) by X.Y, and by Λ' the mapping which we get by linear continuation from

$$\Lambda' : x_1x_2 \ldots x_k \mapsto \frac{1}{k!} \sum_{\tau \in \gamma_k} x_{\tau(1)}x_{\tau(2)} \ldots x_{\tau(k)}, \; x_i \in \text{heis } (2n)$$

(here γ_k denotes the symmetric permutation group of k objects).

(58) Lemma. — The mapping

$$\Lambda' : \text{sym (heis } (2n)) \to u \text{ (heis } (2n))$$

is a vector space isomorphism (i. e. bijective and linear).

This lemma is due to Harish-Chandra [*4*; p. 98, p. 392]. Lemma (51), the Birkhoff-Witt-theorem, shows that the total symmetrized standard monomials form a basis of the commutative algebra sym (heis $(2n)$), and from lemma (58) we have the same basis for u (heis $(2n)$). We define a mapping Λ: sym (E) $\to$ weyl(E, σ) by $\Lambda: = \pi \circ \Lambda' \circ j$, which is linear and bijective. It follows that the total symmetrized standard monomials of the basis elements of $E \in \text{weyl}(E, \sigma)$ and the identity element 1 form a basis of weyl(E, σ). Let us write in the following

$$(59) \qquad \Lambda x_1 \ldots x_i := \frac{1}{i\,!} \sum_{\tau \in \gamma_i} x_{\tau(1)} \ldots x_{\tau(i)} \quad x_k \in E \subset \text{weyl}\,(E, \sigma).$$

The decomposition weyl(E, σ) $= \mathbb{R}1 + W_1 + W_2 + \ldots$ of weyl(E, σ) is not direct, since the vector spaces W_i are not disjoint for different indexes; for instance the element $xy - yx$ is in $\mathbb{R}1$ and in W_2. We get a direct decomposition of weyl(E, σ) if we consider the vector spaces ΛW_i, defined as linearly generated by the symmetrized standard monomials of degree i

$$(60) \qquad \text{weyl}\,(E, \sigma) = \mathbb{R}1 \oplus E \oplus \Lambda W_2 \oplus \Lambda W_3 \oplus \ldots,$$

instead of the W_i. To prove this we apply the linear transformation $\text{ad}\,(p^1)^{i_1} \ldots \text{ad}\,(p^n)^{i_n}\, \text{ad}\,(q_1)^{k_1} \ldots \text{ad}\,(q_n)^{k_n}$ of weyl(E, σ) on the equation

$$\xi_0 1 + \xi^1 q_1 + \ldots + \xi_n p^n + \ldots + \xi \Lambda q_1^{\,i_1} \ldots q_n^{\,i_1} (p^1)^{k_1} \ldots (p^n)^{k_n} = 0,$$

showing that $\xi = 0$. Continuing this process causes all coefficients to vanish, which is equivalent to the directness of the decomposition (60). Even in this decomposition weyl(E, σ) is not graded, i. e.

$$(61) \qquad \Lambda W_i \Lambda W_k \not\subset \Lambda W_{i+k};$$

for instance we have $\Lambda x \Lambda y = \Lambda xy + \frac{1}{2}\sigma(x, y)1$. Instead we have a filtration on weyl(E, σ) from which we can construct a graduation by standard procedure [*9*] and [*11*; exposé n° 1]; but since the resulting algebra is commutative (in fact it is isomorphic to sym (E), cf. [*11*]), it does not seem to be interesting for physical applications. It is easy to prove the formula

$$(62) \qquad \Lambda x_1 \ldots x_r x_{r+1} = \frac{1}{(r+1)\,!} \sum_{\tau \in \gamma_{r+1}} (\Lambda x_{\tau(1)} \ldots x_{\tau(r)}) x_{\tau(r+1)}$$

with the help of which we get by induction

$$(63) \qquad [\Lambda x_1 \ldots x_r, y]_- = \Lambda [x_1 \ldots x_r, y]_- \qquad x_k, y \in E \subset \text{weyl}\,(E, \sigma).$$

From this we have

$$(64) \qquad [\Lambda W_r, E]_- \subset \Lambda W_{r-1};$$

from this and (60) follows that the center of weyl(E, σ) is trivial [*12*, p. 148].

§ II-8: Some Lie Algebras in weyl(E, σ).

We now look for Lie algebras which are formed by linear and bilinear polynomials of elements $x_k \in \mathrm{E} \subset \mathrm{weyl}(\mathrm{E}, \sigma)$. We have for $\mathrm{W}_1 \oplus \Lambda \mathrm{W}_2$

(65) $[\Lambda xy, \Lambda vz]_- = \sigma(x, v)\Lambda yz + \sigma(x, z)\Lambda yv + \sigma(y, v)\Lambda xz + \sigma(y, z)\Lambda xv$

(66) $[\Lambda xy, z]_- = \sigma(x, z)y + \sigma(y, z)x.$

For $x = y$ and $v = z$ the first equation reduces to

$$[xx, zz]_- = 4\sigma(x, z)\Lambda xz, \tag{67}$$

which is equivalent to the original, since by linearizing it twice (i. e. by substituting $x \mapsto x + y$ and $z \mapsto v + z$) we regain the original one. (For commuting x and y we can drop Λ in Λxy).

(68) Lemma. — The *normalizer of* E *in* weyl(E, σ) (i. e. the set

$$\{ \mathrm{Y} \in \mathrm{weyl}\,(\mathrm{E}, \sigma)/[\mathrm{Y}, x]_- \in \mathrm{E} \qquad \text{for all } x \in \mathrm{E}\})$$

equals $\Lambda \mathrm{W}_2$.

The proof follows from (60), (63), (64) and (66). We have

$$\Lambda xy = \sum_{i,k} \left\{ \frac{1}{2}(\xi^i\eta^k + \xi^k\eta^i)q_iq_k + \frac{1}{2}(\xi_i\eta_k + \xi_k\eta_i)p^ip^k + \frac{1}{2}(\xi^i\eta_k + \xi_k\eta^i)\Lambda q_ip^k \right\}$$

which suggests to introduce in $\Lambda \mathrm{W}_2$ the $n(2n + 1)$ basis elements

$$\Lambda q_iq_k = \Lambda q_kq_i, \qquad \Lambda p^ip^k = \Lambda p^kp^i, \qquad \Lambda q_ip^k \neq \Lambda q_kp_i. \tag{69}$$

They have the commutation relations [*14*; p. 1203]

$$\begin{aligned}
&[\Lambda q_iq_k, \Lambda q_lq_m]_- = [\Lambda p^ip^k, \Lambda p^lp^m]_- = 0\\
&[\Lambda q_ip^k, \Lambda q_lp^m]_- = \delta_i^m\Lambda q_lp^k - \delta_l^k\Lambda q_ip^m\\
&[\Lambda q_iq_k, \Lambda p^lp^m]_- = \delta_i^m\Lambda q_kp^l + \delta_i^l\Lambda q_kp^m + \delta_k^l\Lambda q_ip^m + \delta_k^m\Lambda q_ip^l\\
&[\Lambda q_iq_k, \Lambda q_lp^m]_- = \delta_i^m\Lambda q_kq_l + \delta_k^m\Lambda q_iq_l\\
&[\Lambda p^ip^k, \Lambda q_lp^m]_- = -\delta_l^i\Lambda p^kp^m - \delta_l^k\Lambda p^ip^m.
\end{aligned} \tag{70}$$

We prove that the elements of $\Lambda \mathrm{W}_2$ span the Lie algebra spl($2n$, ℝ). First note that from the Jacobi identity in weyl(E, σ) we have

(71) $[\mathrm{ad}\,(\Lambda xy), \mathrm{ad}\,(\Lambda vz)]_-$
$= \sigma(x, v)\,\mathrm{ad}\,(\Lambda yz) + \sigma(x, z)\,\mathrm{ad}\,(\Lambda yv) + \sigma(y, v)\,\mathrm{ad}\,(\Lambda xz) + \sigma(y, z)\,\mathrm{ad}\,(\Lambda vx)$

i. e. the $2n \times 2n$ matrices ad $(\Lambda xy)\,|_{\mathrm{E}}$ (ad restricted to E) form a Lie algebra, which is a subalgebra of spl $(2n, \mathbb{R}, \mathrm{E})$ because of

$$\sigma(\mathrm{ad}\,(\Lambda xy)v, z) + \sigma(v, \mathrm{ad}\,(\Lambda xy)z) = 0. \tag{72}$$

It is straightforward to prove that the kernel of the Lie algebra homomorphism $\mathrm{R}\colon \Lambda\mathrm{W}_2 \to \mathrm{spl}\,(2n, \mathbb{R}, \mathrm{E})$ defined by

$$\mathrm{Z} \mapsto \mathrm{ad}\,(\mathrm{Z})\,|_{\mathrm{E}} =: \mathrm{R}(\mathrm{Z}) \qquad \mathrm{Z} \in \Lambda\mathrm{W}_2 \tag{73}$$

is zero. From a dimensional arguement now follows that this homomorphism is a Lie algebra isomorphism. It follows that the Lie algebra $\mathbb{R}1 \oplus \mathrm{W}_1 \oplus \Lambda\mathrm{W}_2$ is isomorphic to heis$(2n) \leftarrow\!\!\!\!+$ spl$(2n, \mathbb{R})$. Choosing now a special element of $\Lambda\mathrm{W}_2$, say H_{R}, we get the commutation relations (16) of $\mathbb{R}\mathrm{R} \,+\!\!\!\!\rightarrow$ heis$(2n)$ where now

$$\mathrm{R} = \mathrm{R}(\mathrm{H}_{\mathrm{R}}) = \mathrm{ad}\,(\mathrm{H}_{\mathrm{R}})\,|_{\mathrm{E}} \qquad \mathrm{H}_{\mathrm{R}} \in \Lambda\mathrm{W}_2. \tag{74}$$

This is for instance for the *Hamilton operator* of the *harmonic oscillator*

$$\mathfrak{J} = \mathrm{R}\Big(-\frac{1}{2}\sum_i (q_i q_i + p^i p^i)\Big) = \mathrm{ad}\Big(-\frac{1}{2}\sum_i (q_i q_i + p^i p^i)\Big)\Big|_{\mathrm{E}}. \tag{75}$$

In the following we list some subalgebras of $\Lambda\mathrm{W}_2$ and their images under the action of the isomorphism R:

The n^2 elements $\mathrm{N}_i^k := \Lambda q_i p^k$ form a subalgebra of $\Lambda\mathrm{W}_2$ with

$$[\mathrm{N}_i^k, \mathrm{N}_l^m]_- = \delta_i^m \mathrm{N}_l^k + \delta_l^k \mathrm{N}_i^m \tag{76}$$

which under the action of R is gl$(n, \mathbb{R}, \mathrm{E})$ given by

$$\mathrm{R}(\mathrm{N}) = \begin{pmatrix} -\mathrm{G} & 0 \\ 0 & \mathrm{G}^{\mathrm{T}} \end{pmatrix} \quad \mathrm{G} \text{ arbitrary } n \times n \text{ matrix,} \tag{77}$$

and, using the matrix exponential mapping, we get

$$\exp\,(\mathrm{R}(\mathrm{N})) = \begin{pmatrix} (e^{\mathrm{G}})^{-1} & 0 \\ 0 & (e^{\mathrm{G}})^{\mathrm{T}} \end{pmatrix} \in \mathrm{Spl}\,(2n, \mathbb{R}, \mathrm{E}) \tag{78}$$

where N is some linear combination of the N_i^k.

The $\frac{1}{2}\,n(n-1)$ elements $\mathrm{M}_{ik} := \Lambda q_i p^k - \Lambda q_k p^i$ form a subalgebra of this Lie algebra with

$$[\mathrm{M}_{ik}, \mathrm{M}_{lm}]_- = \delta_i^l \mathrm{M}_{km} + \delta_i^m \mathrm{M}_{lk} + \delta_k^l \mathrm{M}_{im} + \delta_m^k \mathrm{M}_{il} \tag{79}$$

which under the action of R is isomorphic to so(n, $\mathbb{R}$, E) given by the above matrices (77) for antisymmetric G. The corresponding matrix Lie group is generated by the matrices (78) for orthogonal e^{G}. The $\frac{1}{2}n(n+1)$ elements $H_{ik}: = \Lambda(q_i q_k + p^i p^k)$ form together with the M_{ik} another Lie algebra in ΛW_2 with

$$\begin{aligned} [H_{ik}, H_{lm}]_- &= \delta_i^l M_{km} + \delta_i^m M_{lk} + \delta_k^l M_{im} + \delta_m^k M_{il} \\ [H_{ik}, M_{lm}]_- &= \delta_k^m H_{il} + \delta_i^m H_{kl} - \delta_k^l H_{im} - \delta_i^l H_{km}. \end{aligned} \tag{80}$$

Under the action of R this Lie algebra is isomorphic to $u(n, \mathbb{R}, E)$:

$$R(M) = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \qquad R(H) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \tag{81}$$

with L a real antisymmetric and K a real symmetric $n \times n$ matrix (see § I-1), and M resp. H a linear combination of the M_{ik} resp. the H_{ik}. The corresponding matrix group U(n, $\mathbb{R}$, E) is given by the matrices (4).

§ II-9: Realization of some Derivations and Automorphisms.

We want to identify derivations, resp. automorphisms, with elements of the form $\mathrm{ad}(Z)\,|_{\mathrm{heis}(2n)}$, resp. $\exp(\mathrm{ad}(Z)\,|_{\mathrm{heis}(2n)}$ for suitable $Z \in \mathrm{weyl}(E, \sigma)$.

Besides the inner derivations of heis($2n$), which are given by $\mathrm{ad}\,(E)\,|_{\mathrm{heis}(2n)}$ we have the derivations $\mathrm{ad}\,(\Lambda W_2)\,|_{\mathrm{heis}(2n)}$ of heis($2n$):

$$\mathrm{ad}\,(\Lambda xy)[\underline{x}, \underline{y}]_- = [\mathrm{ad}\,(\Lambda xy)\underline{v}, \underline{z}]_- + [\underline{v}, \mathrm{ad}\,(\Lambda xy)\underline{z}]_- \tag{82}$$

(note (66)). In the notation of (32)

$$\begin{pmatrix} V & 0 \\ p^{T} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{ad}\,(Z)\,|_E & 0 \\ \mathrm{ad}\,(Jp)\,|_E & 0 \end{pmatrix} \qquad p \in E,\ Z \in \Lambda W_2. \tag{83}$$

From (64) now follows that no elements of $\mathrm{ad}\,(\mathrm{weyl}(E, \sigma))\,|_{\mathrm{heis}(2n)}$ can be identified with derivations given by $\beta \neq 0$ in (32). The corresponding facts hold for automorphisms given by $\alpha \neq 1$ in (21) and the special automorphism $A(\mathrm{id}_{2n}, 0, 1, -1)$.

From (64) follows that there is no representation by elements of $\mathrm{ad}\,(\mathrm{weyl}(E, \sigma))\,|_{\mathbb{R}\mathbb{R} \oplus E \oplus \mathbb{R}}$ of those derivations of $\mathbb{R}\mathbb{R} \oplus E \oplus \mathbb{R}$ which are given by nonvanishing ρ and β in (40). The remaining derivations in (40) are given by the elements of ad (E) $|_{\mathbb{R}R \oplus E \oplus \mathbb{R}}$ and by those elements $Z \in \Lambda W_2$ which obey the conditions

$$\text{(84)} \qquad \mathrm{ad}\,(Z)H_R = \nu H_R, \qquad \mathrm{ad}\,(Z)\,|_E = V.$$

The corresponding automorphisms of the Lie algebras $\mathbb{R}R \oplus E \oplus \mathbb{R}$, see (38), are now generated by the matrix exponential mapping of these derivations.

§ II-10: The Special Case of the (Nonrelativistic) Free Particle.

We discuss the above results for the *free particle* whose *Hamilton operator* is given by

$$\text{(85)} \qquad H_F := \frac{1}{2}\sum p^i p^i, \qquad R_F := R(H_F) = \begin{pmatrix} 0 & -id_n \\ 0 & 0 \end{pmatrix},$$

$R_F^2 = 0$. Because of this nilpotency theorem (38) is not applicable. We show that for $n \geqslant 2$ theorem (38) holds in the same form, i. e. that the automorphism group of $\mathbb{R}R_F \rhd\!\!\!+ \mathrm{heis}(2n)$ is given by the matrices (38) with $0 \neq \delta \neq 1$ in general. It follows that the elements of

$$\mathrm{der}\,(\mathbb{R}R_F \rangle\, \mathrm{heis}\,(2n))$$

are given by the matrices (40) with $\nu \neq 0$ in general. First note that from (*b*) in the proof of theorem (38) we have for $R^2 = 0$ (multiplying by R from the left) $RGR = 0$. So from (*e*) we have for $x = Rz$ with the help of (*a*) $a = 0$. This is true for all R with $R^2 = 0$.

What remains to show is $d = 0$. From $R_F G R_F = 0$ and G invertible follows that G has the form $\begin{pmatrix} A & B \\ 0 & \frac{1}{\delta}A \end{pmatrix}$ with A invertible. It follows from (*e*) for $x = d$

$$(e') \qquad \sigma(\Im d, d)RGy = \sigma(\Im d, y)RGd \qquad \text{for all } x \in E.$$

Writing $y^T = (y_1{}^T, y_2{}^T)$ and the same for d where now y_1, y_2, d_1, d_2 are n column vectors, from (*a*) follows $d_1 = 0$. Denoting the scalar product in n dimensions by $\langle\, , \rangle$, we have from (e') since A is invertible for all n vectors y_2 the condition $\langle d_2, d_2 \rangle y_2 = \langle d_2, y_2 \rangle d_2$, from which for $n \geqslant 2$ follows $d_2 = 0$. To prove this we used two peculiarities of R_F, which will not hold in general for all R with $R^2 = 0$, namely $d_1 = 0$ and the special form of G above.

The set of elements of spl $(2n, \mathbb{R}, E)$ obeying the equation (36) for R_F is the set of matrices $\begin{pmatrix} A & B \\ 0 & -A^T \end{pmatrix}$ with $B = B^T$ and $A + A^T = \nu\, \mathrm{id}_n$. For $A - \frac{1}{2}\nu\, \mathrm{id}_n =: L$ we get $L = -L^T$. The $n^2 + 1$-dimensional Lie algebra $\mathrm{zent}(R_R) \oplus \mathbb{Z}R_F$ is given by the matrices $\begin{pmatrix} L & B \\ 0 & L \end{pmatrix} + \frac{1}{2}\nu \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & -\mathrm{id}_n \end{pmatrix}$. Contrary to the harmonic oscillator $(R = \mathfrak{J})$, where $v = 0$ from (41), for the free particle $\nu \neq 0$. It is straightforward to show that because of theorem (38) the automorphism group of $\mathbb{R}R_F \mathbin{+\!\!\!>} \mathrm{heis}(2n)$ is given by the set of matrices

$$\begin{pmatrix} (\mathrm{sign}\,\alpha)\delta & 0 & 0 \\ -\frac{1}{\alpha} GR\mathfrak{J}b & G & 0 \\ \tau & b^T & \alpha \end{pmatrix} \qquad \begin{matrix} 0 \neq \alpha \in \mathbb{R} \\ 0 < \delta \in \mathbb{R} \\ \tau \in \mathbb{R} \end{matrix} \tag{86}$$

where the $2n \times 2n$ matrix G has the form

$$G = \sqrt{|\alpha|} \left\{ \frac{1 + \mathrm{sign}\,\alpha}{2} \begin{pmatrix} \sqrt{\delta}\, A & B \\ 0 & \frac{1}{\sqrt{\delta}} A \end{pmatrix} + \frac{1 - \mathrm{sign}\,\alpha}{2} \begin{pmatrix} \sqrt{\delta}\, C & D \\ 0 & -\frac{1}{\sqrt{\delta}} C \end{pmatrix} \right\},$$

and the $n \times n$ matrices A, B, C, D are subject to the conditions $A^TA = \mathrm{id}_n$, $C^TC = \mathrm{id}_n$, A^TB and C^TD symmetric.

The one-dimensional Lie algebra $\frac{1}{2}\mathbb{R}\begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & -\mathrm{id}_n \end{pmatrix}$ is the image of $\frac{1}{2}\mathbb{R}\sum_{l=1}^{n} \Lambda q_l p^l$ under the isomorphism R; the $\frac{1}{2}n(n-1)$-dimensional Lie algebra of the matrices $\begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$ is the R-image of the so$(n, \mathbb{R})$ Lie algebra (78), and the commutative $\frac{1}{2}n(n+1)$-dimensional Lie algebra of the matrices $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is the R-image of the $\Lambda p^i p^k$.

Note that the kernel of the homomorphism

$$(\exp \mathbb{R}R_F) \times E \times \mathbb{R} \to \mathbb{R}R_F \times E \times \mathbb{R}$$

defined by $(\mathrm{id}_{2n} + \alpha R_F, x, \beta) \mapsto (\alpha, x, \beta)$ is $\{(\mathrm{id}_{2n}, 0, 0)\}$. Therefore the nilpotent group of the free particle, $\exp(\mathbb{R}R_F) \times E \times \mathbb{R}$, is simply connected and global isomorphic to the group (10), contrary to the solvable group $(\exp \mathbb{R}\mathfrak{J}) \times E \times \mathbb{R}$ of the harmonic oscillator, which is infinitely connected by (6).

The representations (42) and (47) of $\exp(\mathbb{R}R_F) \times E \times \mathbb{R}$ and $\exp(\mathbb{R}R_F) \otimes E$ may be extended to faithful $2n + 2$-dimensional representations of the *extended* and the *pure Galilei group* in n dimensions by substituting $\exp(\mathbb{R}R_F) \to O(n, \mathbb{R}, E) \otimes \exp(\mathbb{R}R_F)$.

CONCLUSION

For completeness one should also know the automorphism group and the derivation Lie algebra of those Lie algebras $\mathbb{R}R \oplus \mathrm{heis}(2n)$ which fulfill $R^2 = 0$, though there seem to be no more physical relevant Hamilton operators H which are subject to $R(H)^2 = 0$ (besides the free particle, which was treated seperately and will be treated elsewhere for the relativistic case). In addition it seems to be likely that the vector space $\mathfrak{z}_R$ is one-dimensional or zero. Then the set of solutions of equation (36) would reduce to zent $(R) \oplus \mathfrak{z}_R$ as it did in the special cases we calculated explicitely. It would also be desirable to have a necessary and sufficient condition for the vanishing of $\mathfrak{z}_R$, for instante « $\mathfrak{z}_R = \{0\}$ iff R is invertible » or « $z_R = \{0\}$ iff $R^2 \neq 0$ ».

We were able to identify (some) derivations and automorphisms with linear transformations realized by means of the adjoint representation of weyl(E, σ) and to identify (all) elements of the Lie algebras $\mathbb{R}R \oplus \mathrm{heis}(2n)$ with elements of weyl(E, σ). It should be possible to construct a « closure » $\overline{\mathrm{weyl}}(E, \sigma)$ of weyl(E, σ) by including « formal potence series » in such a way that even the Lie groups $\exp(\mathbb{R}R) \times E \times \mathbb{R}$ or at least some covering groups of them can be identified with some subsets of $\overline{\mathrm{weyl}}(E, \sigma)$ [*15*; p. 33]. In addition it should be possible to find in $\overline{\mathrm{weyl}}(E, \sigma)$ a twofold covering group of Spl $(2n, \mathbb{R}, E)$ the Lie algebra of which is given by ΛW_2 [*16*; p. 143]. The analogy to the case of the Clifford algebras and the covering groups Spin(n) of the (pseudo-) orthogonal groups is obvious [*17*; p. 64]. We remark that this analogy is very farreaching if one regards orthogonal vector spaces instead of symplectic ones, Jordan algebras instead of Lie algebras and so on.

By induction one proves that for $x, y, z_1, \ldots, z_r \in \mathrm{E}$

$$\text{(87)}\quad \operatorname{ad}(\Lambda xy)\Lambda z_1 \ldots z_r = \sum_{i=1}^{n} \{ \sigma(x, z_i)\Lambda y z_1 \ldots z_{i-1}z_{i+1} \ldots z_r + \sigma(y, z_i)\Lambda x z_1 \ldots z_{i-1}z_{i+1} \ldots z_r \}.$$

This together with the Jacobi identity shows that the restrictions of the linear transformations ad ( ) of the vector space weyl(E, σ) to the finite-dimensional vector spaces $\Lambda \mathrm{W}_r$ are representations of spl $(2n, \mathbb{R})$ which are monomorphic as is easily to check. The first one in W_0 is the trivial zero representation, the second one in E the self-representation discussed above and the third one in $\Lambda \mathrm{W}_2$ the adjoint representation of spl$(2n, \mathbb{R})$. A similar representation of the orthogonal Lie algebras in W_3 is well known [*16*; p. 151].

The author is indebted to Dr. H. D. Döbner for critical remarks. Some of the above results are already to be found in his work [*1*]. This work was supported in part by the Deutsche Forschungsgemeinschaft.

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(Manuscrit reçu le 10 octobre 1969).