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Section A :
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Continuous bases for unitary irreducible representations of SU(1, 1)

by

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ABSTRACT. — The UIR's of SU(1, 1) are studied in two different continuous bases obtained by diagonalizing a noncompact generator belonging to the hyperbolic and parabolic class, respectively. The space of differentiable vectors of a UIR and its dual containing the « generalized eigenvectors » of the noncompact generators are described in the discrete and continuous bases. The matrix elements of finite transformations and generators are given.

§ 1. INTRODUCTION

The noncompact group SU(1, 1) of all matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1,$$

is of great interest in physics as it is homomorphic to the subgroup SO(2, 1) of the Lorentz group that leaves a spacelike vector invariant, and in mathematics as the simplest noncompact semi-simple group. Bargmann [*1*] found all unitary irreducible representations (UIR's) of SU(1, 1) and the corresponding matrix elements in the discrete basis where the compact generator is diagonal.

Mukunda [*2*] and Barut and Phillips [*3*] investigated the case where a noncompact generator is diagonal. The basis vectors are then labelled by a continuous index, in other words, a noncompact generator has a continuous spectrum. The results, however, were incomplete. We wish to give a reasonably complete and rigorous discussion of this problem. The layout of the paper is as follows. After a brief review of the properties of SU(1, 1) and its UIR's we give a rigorous definition of the « generalized eigenvectors » of a noncompact generator using the concept of « Gelfand triplet ». Then the components of these generalized eigenvectors in the discrete basis are calculated departing from a difference equation. The matrix elements of the finite transformations and the generators in the continuous bases are given. Finally the relation of this work with ref. [2] is discussed.

§ 2. THE GROUP SU(1, 1)

A. Subgroups

SU(1, 1) has three classes of conjugate one-parameter subgroups. These can be represented by the following specimens

$$
\begin{aligned}
k(\theta) &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} && \text{elliptic class} \\
a_1(s) &= \begin{pmatrix} \cosh\frac{s}{2} & -i\sinh\frac{s}{2} \\ i\sinh\frac{s}{2} & \cosh\frac{s}{2} \end{pmatrix} && \text{hyperbolic class} \\
a_2(t) &= \begin{pmatrix} \cosh\frac{t}{2} & -\sinh\frac{t}{2} \\ -\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix} && \text{hyperbolic class} \\
n(\xi) &= \begin{pmatrix} 1-\frac{i\xi}{2} & -\frac{i\xi}{2} \\ \frac{i\xi}{2} & 1+\frac{i\xi}{2} \end{pmatrix} && \text{parabolic class.}
\end{aligned}
\tag{2.1}
$$

The elliptic subgroups are compact, the hyperbolic and parabolic subgroups are noncompact.

An arbitrary element g can be parametrized in various ways e. g.

$$
\begin{aligned}
g &= k(\theta)\cdot a_2(t)\cdot k(\psi) \\
g &= k(\theta)\cdot a_1(s)\cdot a_2(t) \\
g &= k(\theta)\cdot a_2(t)\cdot n(\xi)
\end{aligned}
\tag{2.2}
$$

B. Lie algebra

As a set of linearly independent elements of the Lie algebra of SU(1, 1) we can choose i times the generators of the subgroups $k(\theta)$, $a_1(s)$ and $a_2(t)$, denoted by J_0, J_1 and J_2, respectively, hence $k(\theta) = \exp(-i\theta J_0)$, etc. (The same notation will be used for the generators of the representations.) Their commutation relations are

$$[J_0, J_1] = iJ_2, \quad [J_0, J_2] = -iJ_1, \quad [J_1, J_2] = -iJ_0 \tag{2.3}$$

$n(\xi)$ is generated by

$$K_+ = J_0 + J_1 \tag{2.4}$$

which satisfies

$$[K_+, J_2] = -iK_+ \tag{2.5}$$

The ladder operators

$$J_\pm = J_1 \pm iJ_2 \tag{2.6}$$

satisfy the relations

$$[J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = -2J_0 \tag{2.7}$$

The Casimir invariant is given by

$$C_2 = J_0^2 - J_1^2 - J_2^2 \tag{2.8}$$

C. UIR's

The UIR's can be grouped into three classes according to the spectrum of C_2 and J_0. In all cases we can choose a standard basis $\{|j, m\rangle\}$ where

$$\begin{aligned}
\langle j, m | j, m' \rangle &= \delta_{mm'} \\
C_2 |j, m\rangle &= j(j+1) |j, m\rangle \\
J_0 |j, m\rangle &= m |j, m\rangle \\
J_+ |j, m\rangle &= [(m+j+1)(m-j)]^{1/2} |j, m+1\rangle \\
J_- |j, m\rangle &= [(m+j)(m-j-1)]^{1/2} |j, m-1\rangle
\end{aligned} \tag{2.9}$$

The three series of UIR's are

1° The continuous principal series.

$$j = -\frac{1}{2} + is, \qquad 0 < s < \infty$$

$$m = 0, \pm 1, \pm 2, \ldots \quad \text{or} \quad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots;$$

notation C_j^δ, where $\delta = 0$ and 1, respectively.

2° The supplementary series.

$$-\frac{1}{2} < j < 0 \qquad m = 0, \pm 1, \pm 2, \dots;$$

notation E_j.

3° The discrete principal series.

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, \dots$$
$$m = -j, -j+1, -j+2, \dots \quad \text{or} \quad m = j, j-1, \dots;$$

notation D_j^+ and D_j^-, respectively.

There exists an outer automorphism of the Lie algebra $J_i \to J_i'$ where

$$(J_0', J_1', J_2') = (-J_0, -J_1, J_2) \tag{2.10}$$

In the case of the series C_j^δ and E_j this automorphism can be realized by $J_i' = PJ_iP^{-1}$ where

$$P\,|j, m\rangle = e^{i\pi m}\,|j, -m\rangle \tag{2.11}$$

Hence $P^2 = 1$ and the eigenvalues of P are ± 1. As $[P, J_2] = 0$, P and J_2 can be diagonalized simultaneously.

§ 3. GENERALIZED EIGENVECTORS OF NONCOMPACT GENERATORS

A. We start from a UIR $U(G) = \{ U(g) ; g \in SU(1, 1) \}$, defined in a standard basis.

The Hilbert space of the representation is then

$$\mathscr{H} = \{ x = \Sigma x_m\,|j, m\rangle; \quad \|x\|^2 = \Sigma\,|x_m|^2 < \infty \} \tag{3.1}$$

All summations go over the spectrum of J_0.

Introduce the space of « rapidly decreasing sequences »

$$\mathscr{D} = \{ x = \Sigma x_m\,|j, m\rangle; \quad \lim_{|m|\to\infty} m^n x_m = 0, \quad \text{all } n \}, \tag{3.2}$$

and the space of « slowly increasing sequences »

$$\mathscr{D}' = \{ x' \sim \Sigma x_m'\,|j, m\rangle; \quad \lim_{|m|\to\infty} x_m'/m^N = 0, \quad \text{some } N = N(x') \}. \tag{3.3}$$

Evidently we have $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$, with $\mathscr{D}$ dense in $\mathscr{H}$, and for $x \in \mathscr{D}$, $x' \in \mathscr{D}'$ we can define a generalized « scalar product »

$$(x', x) = \Sigma x_m'^* x_m; \quad (x, x') = (x', x)^*.$$

We shall sometimes also use the Dirac bra and ket notation: $\langle x' \mid x \rangle$ and $\langle x \mid x' \rangle$.

A topology is defined on $\mathscr{D}$ by the set of norms $\{ p_n \}_0^\infty$, where

$$p_n(x) = [\Sigma(m^2 + 1)^n \mid x_m \mid^2]^{1/2}$$

$\mathscr{D}$ and $\mathscr{D}'$ have the following properties:

1. $\mathscr{D}$ is the set of differentiable vectors of U(G). $\mathscr{D}$ is thus invariant under all J_i and $U(g)$, and these operators are continuous in $\mathscr{D}$. All J_i are essentially self-adjoint on $\mathscr{D}$.

2. $\mathscr{D}$ is a nuclear Frechet space, i. e. a complete, metrizable, nuclear space. This implies that $\mathscr{D}$ is a Montel space and reflexive.

3. $\mathscr{D}'$ is the dual of $\mathscr{D}$, and $\mathscr{H}$ — hence also $\mathscr{D}$ — is dense in $\mathscr{D}'$ (in the weak or strong topology on $\mathscr{D}'$: as $\mathscr{D}'$ is the dual of a Montel space, it is a Montel space, and the weak and strong topologies coincide on bounded sets, especially on convergent sequences; see [*4*], IV.3.4, p. 90). Any operator A in $\mathscr{H}$, which has an adjoint A^+ leaving $\mathscr{D}$ invariant and continuous in $\mathscr{D}$—as e. g. J_i and $U(g)$—can be extended to an operator A' in $\mathscr{D}'$ by $(A'y', x) = (y', A^+x)$, $x \in \mathscr{D}$, $y' \in \mathscr{D}'$. A' is continuous in $\mathscr{D}'$.

4. *Nuclear spectral theorem* (see e. g. [*5*] or [*6*]).

If A is a self-adjoint operator, leaving $\mathscr{D}$ invariant and continuous in $\mathscr{D}$, then A has a complete set of generalized eigen-vectors in $\mathscr{D}'$, i. e. there is a set $\{ \mid \lambda, i \rangle : \lambda \in \mathrm{Sp}A,\ i = 1, 2, \ldots, n_\lambda \} \subset \mathscr{D}'$, where SpA is the spectrum of A, and $n_\lambda \leq \infty$ is the multiplicity of SpA at the point λ, and a measure μ on SpA, so that

$$A' \mid \lambda, i \rangle = \lambda \mid \lambda, i \rangle, \quad \lambda \in \mathrm{Sp}A, \quad i = 1, 2, \ldots, n_\lambda, \tag{3.5}$$

and for any $x, y \in \mathscr{D}$ the completeness relation

$$(x, y) = \int \sum_{i=1}^{n_\lambda} \langle x \mid \lambda, i \rangle \langle \lambda, i \mid y \rangle \, d\mu(\lambda) \tag{3.6}$$

is fulfilled.

B. We now proceed to the proofs of 1.-3.

Proof of 1. — We recall the definition of a differentiable vector of a unitary representation $g \to U(g)$ of a Lie group G in a Hilbert space $\mathscr{H}$:

$x \in \mathscr{H}$ is differentiable (analytic) if the mapping $G \ni g \to U(g)x \in \mathscr{H}$ is infinitely differentiable (analytic) in g.

It is not difficult to show (see [7] for details) that if $J_1, \ldots, J_n$ form a basis of self-adjoint generators of the representation U(G), an alternative definition of the set $\mathscr{D}_G$ of differentiable vectors is the following:

$\mathscr{D}_G$ is the largest subset of $\mathscr{H}$ such that

$$\left.\begin{array}{ll} a. & \mathscr{D}_G \subset \bigcap_1^n \mathscr{D}(J_i) \\ b. & J_i \mathscr{D}_G \subset \mathscr{D}_G, \quad i = 1, \ldots, n. \end{array}\right\} \tag{3.7}$$

Here

$$\mathscr{D}(J_i) = \{ x ; \lim_{t \to 0} [U(e^{-itJ_i})x - x]/t = -iJ_i x \text{ exists} \} \tag{3.8}$$

is the domain of definition of J_i.

$\mathscr{D}$ is by definition nothing but the set $\mathscr{D}_K$ of differentiable vectors of the maximal compact subgroup $K = \{ e^{-itJ_0} \}$ of SU(1, 1): evidently

$$\frac{d^n}{dt^n}(e^{-itJ_0}x) = \Sigma(-im)^n e^{-imt} x_m \,|\, j, m \rangle.$$

Clearly $\mathscr{D}_K \supset \mathscr{D}_G$.

It is shown in Bargmann's paper ([1], p. 602) that the basis vectors $\{ |j, m\rangle \}$ belong to the domain of definition of J_1 and J_2. It follows from this fact and (2.9) that J_0, J_1 and J_2 are defined in $\mathscr{D}$ and leave $\mathscr{D}$ invariant. Thus conditions *a.* and *b.* above are satisfied, and it follows that

$$\mathscr{D} = \mathscr{D}_K = \mathscr{D}_G.$$

An alternative way of showing this is to use the characterization in Nelson's paper ([8], p. 592, proof of theorem 3) of $\mathscr{D}_G$ as $\bigcap_1^\infty \mathscr{D}(\bar{\Delta}^n)$, where $\bar{\Delta}$ is the self-adjoint closure of $-\sum_0^2 J_i^2$ (our notation differs from that of Nelson). Correspondingly we have $\mathscr{D}_K = \bigcap_1^\infty \mathscr{D}(\bar{J}_0^{2n})$. But as

$$C_2 = J_0^2 - J_1^2 - J_2^2 = j(j+1)I,$$

I = identity operator in $\mathscr{H}$, we have $\bar{\Delta} = -2\overline{J_0^2} + j(j+1)I$, so that $\mathscr{D}_K = \mathscr{D}_G$.

The invariance of $\mathscr{D}$ under all U(g) follows from the fact that $\mathscr{D}$ is the set of differentiable vectors.

Using (2.9) one can easily derive

$$p_n(J_i x) \leq C(n) p_{n+1}(x), \quad \text{some suitable} \quad C(n).$$

This shows that $\{ J_i \}$ are continuous in $\mathscr{D}$.

To show that $U(g)$ is continuous in $\mathscr{D}$, we observe that $p_{2n}(x) = \| (J_0^2 + 1)^n x \|$. As $\overline{J_0^2} = -\frac{1}{2}\overline{\Delta} - \frac{1}{2} j(j+1) I$, an equivalent set of norms is obtained from

$$p'_{2n}(x) = \| (-\overline{\Delta} + 1)^n x \|.$$

Now $p'_{2n}(U(g)x) = \| (-U(g^{-1})\overline{\Delta} U(g) + 1)^n x \|$.

$U(g^{-1})\overline{\Delta} U(g)$ is evidently a quadratic expression in $\{ J_i \}$. Lemma 6.3, p. 588, in [8] then gives

$$p'_{2n}(U(g)x) \leq C'(n) \cdot p'_{2n}(x).$$

That all J_i are essentially self-adjoint on $\mathscr{D}$ follows from Segal's result [9] that J_i is essentially self-adjoint on the Garding subspace, which is contained in $\mathscr{D}$ (cf. also [10], p. 371, IV).

Proof of 2. — As $\mathscr{D}$ has a countable set of norms, it is metrizable. Completeness and nuclearity follows e. g. from the criteria given in [11], 6.1, p. 87-88. Now every nuclear Frechet space is a Montel space: from [11], 0.5.7, p. 7 and 4.4.7, p. 73 follows that a closed bounded set in $\mathscr{D}$ is compact, and as $\mathscr{D}$ is a Frechet space, it is barreled, and hence a Montel space. Finally a Montel space is reflexive (See [4], III.1.1, p. 2 and IV.3.4, p. 89-90, for the last three statements.)

Proof of 3. — It is immediately realized that $\mathscr{D}'$, as defined by (3.3), is just the space of continuous linear functionals on $\mathscr{D}$. It is also easy to see that $\mathscr{H}$ is dense in $\mathscr{D}'$ in the weak topology on $\mathscr{D}'$, and hence also in the strong topology.

The definition of A' shows that A' is weakly continuous in $\mathscr{D}'$. From [4], IV.4.2, p. 103 follows that A' is continuous also in the strong topology on $\mathscr{D}'$.

C. We thus have a Gelfand triplet $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$, where $\mathscr{D}$ is a complete nuclear space, dense in $\mathscr{H}$, and $\mathscr{H}$ is dense in $\mathscr{D}'$, the dual of $\mathscr{D}$. The infinitesimal generators $\{ J_i \}$ and finite group transformations $\{ U(g) \}$ can be extended to continuous operators in $\mathscr{D}'$; actually this extension is nothing but closure by continuity of continuous operators defined originally on the dense subspace $\mathscr{D}$ of $\mathscr{D}'$. In $\mathscr{D}'$ we have, apart from the same freedom of operating with $\{ J_i \}$ and $\{ U(g) \}$ as in $\mathscr{D}$, the further advantage that the eigenvalue equations of the operators $\{ J_i \}$ have, in the sense of (3.6), a complete set of solutions.

We also make some short remarks on some other subspaces of $\mathscr{H}$, namely the space of « finite sequences »

$$\mathscr{D}_C = \{ x = \Sigma x_m | j, m \rangle ; \quad x_m = 0, \quad | m | > M(x) \} \tag{3.9}$$

and the space of analytic vectors

$$\mathscr{A} = \{ x = \Sigma x_m | j, m \rangle ; \quad p_n(x) \leq C(x) \cdot [a(x)]^n | n! \}. \tag{3.10}$$

Obviously $\mathscr{D}_C \subset \mathscr{A} \subset \mathscr{D} \subset \mathscr{H}$, and $\mathscr{D}_C$ is dense in $\mathscr{H}$. With a suitable topology on $\mathscr{D}_C$ its dual $\mathscr{D}'_C$ is the space of all sequences without any limitation in growth as $| m | \to \infty$.

$\mathscr{A}$ is by definition the set of analytic vectors of the subgroup $K = \{ e^{-itJ_0} \}$. However, one can easily show that for $x \in \mathscr{D}$ we have

$$\| J_i x \| \leq C(\| J_0 x \| + \| x \|),$$

where $C = \max (1, \sqrt{| j(j+1) |})$, and also

$$[J_{i_1}, [J_{i_2}, \ldots, [J_{i_n}, J_0], \ldots] = \varepsilon J_i, \; | \varepsilon | = 1.$$

It then follows from [*8*], Cor. 3.2, p. 577, that $\mathscr{A}$ is actually the set of analytic vectors of U(G).

$\mathscr{A}$ is invariant under $\{ J_i \}$ and $\{ U(g) \}$, whereas $\mathscr{D}_C$ is invariant under $\{ J_i \}$ but not $\{ U(g) \}$. One can show that all J_i are essentially self-adjoint on $\mathscr{D}_C$.

In [*7*] it is shown that all the properties of $\mathscr{D}$ and $\mathscr{D}'$ given in 1.3. remain true for a UIR U(G) of an arbitrary semi-simple Lie group G with a finite centre, provided we define $\mathscr{D}$ as $\mathscr{D}_K$, the set of differentiable vectors of the representation U(K) of a maximal compact subgroup K of G. The topology on $\mathscr{D}$ is defined by the set of norms $\{ p_n \}$, $p_n(x) = \| (-\Delta_K + 1)^{n/2} x \|$, where $-\Delta_K$ is the Casimir operator of U(K) ($= J_0^2$ for $G = SU(1, 1)$). One can also prove that $\mathscr{A} = \mathscr{A}_K$, the set of analytic vectors of U(G) is the same as the set of analytic vectors of U(K).

§ 4. DIAGONALIZATION OF A GENERATOR OF HYPERBOLIC CLASS

In this paragraph we will calculate the sequence $\{ x'_m \}$ corresponding to a generalized eigenvector of the noncompact generator J_2. Denote the eigenvectors by $| j, \lambda \rangle$:

$$\begin{aligned} C_2 | j, \lambda \rangle &= j(j+1) | j, \lambda \rangle \\ J_2 | j, \lambda \rangle &= \lambda | j, \lambda \rangle \end{aligned} \tag{4.1}$$

For the series C_j^δ and E_j the spectrum of J_2 is the real line with multiplicity two. In this case the basis can be chosen such that P is diagonal (P was defined by (2.11)):

$$P\,|\,j, \lambda \pm \rangle = \pm\,|\,j, \lambda \pm \rangle \tag{4.2}$$

We will also use the notation $|\,j, \lambda(-)^\sigma \rangle$ where $\sigma = 0,1$.

For D_j the spectrum is the real line.

A. Construction of the eigenvectors

From § 3 we know that

$$|\,j, \lambda \rangle \sim \sum_m A_m(j, \lambda)\,|\,j, m \rangle \tag{4.3}$$

where $A_m(j, \lambda) \equiv \langle j, m\,|\,j, \lambda \rangle \equiv x'_m$ is « slowly increasing » as a function of m. Equations (2.9) and (4.1) give:

$$\lambda A_m = \frac{i}{2}\{\,[(m + j + 1)(m - j)]^{\frac{1}{2}}A_{m+1} - [(m + j)(m - j - 1)]^{\frac{1}{2}}A_{m-1}\,\} \tag{4.4}$$

Introduce B_m by

$$A_m(j, \lambda) = N(j, \lambda)[\Gamma(m - j)/\Gamma(m + j + 1)]^{\frac{1}{2}}B_m(j, \lambda) \tag{4.5}$$

where N is a normalization constant and the square root is defined to be equal to $[\Gamma(m - j)\Gamma(m + j + 1)]^{\frac{1}{2}}/\Gamma(m + j + 1)$ except when $m - j$ = negative integer (i. e. for D_j^-). With this definition we have

$$[\Gamma(m - j)/\Gamma(m + j + 1)]^{\frac{1}{2}} = e^{-i\pi m}[\Gamma(-m - j)/\Gamma(-m + j + 1)]^{\frac{1}{2}} \tag{4.6}$$

The right hand side is taken as a definition of the root for the series D_j^-. B_m satisfies the following difference equation:

$$(j - m)B_{m+1} + (j + m)B_{m-1} = 2i\lambda B_m \tag{4.7}$$

Note that if $B_m(j, \lambda)$ satisfies the equation, so does $B_{-m}(j, \lambda)$ and $e^{i\pi m}B_m(j, -\lambda)$. This equation is solved by the method of Laplace [12]. With the « Ansatz »

$$B_m(j, \lambda) = \int_C t^{m-1}v(t)dt \tag{4.8}$$

(4.7) is replaced by the differential equation

$$(t^2 - 1)\cdot\frac{dv}{dt} + \{\,(j + 1)\frac{t^2 + 1}{t} - 2i\lambda\,\}\,v = 0$$

with the solution

$$v(t) = t^{j+1}(t-1)^{-j-1+i\lambda}(t+1)^{-j-1-i\lambda}$$

Hence

$$B_m(j, \lambda) = \int_C t^{m+j}(t-1)^{-j-1+i\lambda}(t+1)^{-j-1-i\lambda}dt \tag{4.9}$$

The contour C must be such that

$$I_C t^{m+j}(t-1)^{-j+i\lambda}(t+1)^{-j-i\lambda} = 0$$

This condition is satisfied if either

a) C is a contour between $t = -1$ and $t = 1$ or

b) C is a closed contour along which the integrand is single-valued.

The difference equation (4.7) obviously has two linearly independent solutions except when the sequence terminates. This happens when $j - m$ or $j + m$ equals zero for some m, which is the case for the discrete series. For the series C_j^δ and E_j it is possible to choose two different paths C_1 and C_2 connecting $t = -1$ and $t = 1$, which give two linearly independent solutions for B_m:

C_1 = semicircle in Im $t > 0$, center $t = 0$

C_2 = semicircle in Im $t < 0$, center $t = 0$.

In the discrete case B_m must satisfy:

$B_m = 0$ when $m \leq j$ for D_j^+

$B_m = 0$ when $m \geq -j$ for D_j^-.

(Note that the square root in (4.5) equals zero when $j < m < -j$, hence $A_m = 0$ when $m < -j$ and $m > j$, respectively).

Only one solution exists in each case.

D_j^+: C_3 = the circle $|t| = R > 1$ with the t-plane cut from -1 to 1.

D_j^-: C_4 = the circle $|t| = R < 1$ with the cuts $(-\infty, -1)$ and $(1, \infty)$.

We will not give the details of the calculations, but restrict ourselves to some remarks.

The normalization constant $N(j, \lambda)$ is determined up to a phase by the condition:

$$\sum_m \langle j, \lambda \pm | j, m \rangle \langle j, m | j, \lambda' \pm \rangle = \delta(\lambda - \lambda') \tag{4.10}$$

The left hand side is calculated by inserting the integral representations for B_m given below and interchanging the order of summation and integration. The phase of $N(j, \lambda)$ will be chosen such that the matrix elements calculated in § 6 have similar forms for the three series of representations.

Note that the difference equation has solutions for every complex λ. These solutions actually give admissible vectors in $\mathscr{D}'$ (see section 4.C). The usual argument that a self-adjoint operator has only real eigenvalues does not apply to this case. The reason is that we study the extension of J_2, self-adjoint in $\mathscr{H}$, to a larger space $\mathscr{D}'$ which is not a Hilbert space (cf. § II and the conclusion of [2]). The set $|j, \lambda \pm\rangle$ with real λ, however, forms a complete set in the sense of the nuclear spectral theorem (see (4.18)).

Now we give the coefficients $\langle j, m | j, \lambda \pm \rangle$ for the different series of UIR's

1. C_j^δ

Including an m-independent factor in $N(j, \lambda)$ we obtain from (4.9) and equation 2.1. (10) of [13] the two solutions corresponding to the paths C_1 and C_2:

$$\begin{aligned} B_m^{1,2}(j, \lambda) &= \frac{1}{2}\int_0^\pi d\varphi e^{\pm im\varphi}\left(\cos\frac{\varphi}{2}\right)^{-j-1-i\lambda}\left(\sin\frac{\varphi}{2}\right)^{-j-1-i\lambda} \\ &= \frac{\Gamma(-j-i\lambda)\Gamma(-j+i\lambda)}{\Gamma(-2j)} e^{\mp\frac{i\pi}{2}(j-i\lambda)} \\ &\quad \cdot {}_2F_1(-m-j, -j+i\lambda; -2j; 2\mp i0) \end{aligned} \tag{4.11}$$

The corresponding eigenvectors do not satisfy (4.2). This equation implies that

$$A^{\pm}_{-m}(j, \lambda) = \pm e^{i\pi m} A^{\pm}_m(j, \lambda)$$

(4.5) gives $B^{\pm}_{-m} = \pm B^{\pm}_m$. But $B^1_{-m} = B^2_m$.
Hence (4.2) is satisfied if we choose

$$B^{\pm}_m = B^1_m \pm B^2_m \tag{4.12}$$

The procedure outlined above gives $N = (2\pi)^{-1}$. The result can be reformulated using equations 2.9 (11), (21) and (29) of [13]. Finally:

$$\begin{aligned} &\langle j, m | j, \lambda(-)^\sigma \rangle \\ &= \frac{1}{2}\left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)}\right]^{\frac{1}{2}} \frac{\Gamma(-j-i\lambda)}{i^\sigma \sin\frac{\pi}{2}(-j+\sigma+i\lambda)} \left\{\frac{{}_2F_1(m-j, -j-i\lambda; m+1-i\lambda; -1)}{\Gamma(-m-j)\Gamma(m+1-i\lambda)}\right. \\ &\left. + (-1)^\sigma \frac{{}_2F_1(-m-j, -j-i\lambda; -m+1-i\lambda; -1)}{\Gamma(m-j)\Gamma(-m+1-i\lambda)}\right\} \end{aligned} \tag{4.13}$$

2. E_j

The results are identical up to (4.12) but the normalization is different for A_m^+ and A_m^- in this case. With a certain choice of phase we have

$$N^{(-)^\sigma} = (2\pi)^{-1}\Gamma\left(\frac{j+1+\sigma-i\lambda}{2}\right)/\Gamma\left(\frac{-j+\sigma-i\lambda}{2}\right) \quad (4.14)$$

$$\langle j, m|j, \lambda\pm\rangle = N^{\pm}[\Gamma(m-j)/\Gamma(m+j+1)]^{\frac{1}{2}}B_m^{\pm}(j, \lambda) \quad (4.15)$$

where $B_m^{\pm}$ can be read off from (4.13).

3. $D_j^{\pm}$

In the same way we have

$$B_m^3(j, \lambda) = \frac{1}{\pi}\sin\pi(-j+i\lambda)\int_{-1}^{1} dt\, t^{m+j}(1+t)^{-j-1-i\lambda}(1-t)^{-j-1+i\lambda}$$
$$= \frac{2^{-2j-1}\Gamma(-j-i\lambda)}{\Gamma(m-j)\Gamma(-m+1-i\lambda)}\cdot {}_2F_1(-m-j, -j-i\lambda; -m+1-i\lambda; -1) \quad (4.16)$$

and

$$N(j, \lambda) = (2\pi^2)^{-\frac{1}{2}}\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda}{2}\right)/(j+1-i\lambda)_{-2j-1}$$

Hence

$$\langle j, m|j, \lambda\rangle = \frac{2^{-2j-1}}{\sqrt{2\pi^2}}\cdot\left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)}\right]^{\frac{1}{2}}\frac{\Gamma(j+1-i\lambda)\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda}{2}\right)}{\Gamma(m-j)\Gamma(-m+1-i\lambda)}$$
$$\cdot {}_2F_1(-m-j, -j-i\lambda; -m+1-i\lambda; -1) \quad (4.17)$$

For the series D_j^- the normalization can be chosen such that

$$\langle j, m|j, \lambda\rangle_{D^-} = e^{-i\pi m}\langle j, -m|j, \lambda\rangle_{D^+}.$$

Thus we need not treat this case separately.

4. Completeness relations.

The relations

$$C_j^\delta \text{ and } E_j: \quad \sum_{+,-}\int_{-\infty}^{\infty} d\lambda\,\langle j, m|j, \lambda\pm\rangle\langle j, \lambda\pm|j, m'\rangle = \delta_{mm'}$$

$$D_j^{\pm}: \quad \int_{-\infty}^{\infty} d\lambda\,\langle j, m|j, \lambda\rangle\langle j, \lambda|j, m'\rangle = \delta_{mm'}$$

are proved by inserting the integral representations for B_m and interchanging the order of integration.

B. Analyticity of the coefficients $\langle j, m \,|\, j, \lambda \rangle$

1. C_j^δ

From (4.11) and (4.12) it is easily shown that

$$\begin{aligned}
B_0^+ &= \frac{1}{\Gamma(-j)}\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j-i\lambda}{2}\right)\\
B_1^- &= \frac{2i}{\Gamma(-j+1)}\Gamma\left(\frac{-j+1+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda}{2}\right)\\
B_{1/2}^+ &= \frac{1}{\Gamma\left(-j+\frac{1}{2}\right)}\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda}{2}\right)\\
B_{1/2}^- &= \frac{i}{\Gamma\left(-j+\frac{1}{2}\right)}\Gamma\left(\frac{-j+1+i\lambda}{2}\right)\Gamma\left(\frac{-j-i\lambda}{2}\right)
\end{aligned}$$

But from the difference equation (4.7) it is evident that B_m^+/B_0^+ and B_m^-/B_1^- in the integral case and $B_m^\pm/B_{1/2}^\pm$ in the half-integral case are polynomials in λ of degree m, $m-1$ and $m-\frac{1}{2}$, respectively.

Hence

$$\langle j, m \,|\, j, \lambda\pm \rangle = S^\pm(j,\lambda)\cdot R_m^\pm(j,\lambda) \tag{4.19}$$

where

$$S^{(-)^\sigma} = \Gamma\left(\frac{-j+\sigma+i\lambda}{2}\right)\Gamma\left(\frac{-j+\sigma+(-1)^\sigma\cdot\delta-i\lambda}{2}\right)$$

and $R_m^\pm$ are polynomials in λ.

2. E_j

The only difference from the preceding case comes from the normalization factor. (4.19) is valid with

$$S^{(-)^\sigma}(j,\lambda) = \Gamma\left(\frac{-j+\sigma+i\lambda}{2}\right)\Gamma\left(\frac{j+1+\sigma-i\lambda}{2}\right) \tag{4.20}$$

3. D_j^+

From (4.16) follows

$$B_{-j}^3 = \frac{2^{-2j-1}\Gamma(-j-i\lambda)}{\Gamma(-2j)\Gamma(j+1-i\lambda)}$$

But B_m^3/B_{-j}^3 is a polynomial of degree $m+j$ in λ. Including the normalization factor we find:

$$\langle j, m \,|\, j, \lambda \rangle = S(j,\lambda)\cdot R_m(j,\lambda)$$

where
$$S(j, \lambda) = \Gamma\left(\frac{-j + i\lambda}{2}\right)\Gamma\left(\frac{-j + 1 - i\lambda}{2}\right) \tag{4.21}$$

4. Note that $\langle j, \lambda\pm \mid j, m \rangle \equiv \langle j, m \mid j, \lambda\pm \rangle^*$ are antiholomorphic functions of λ (apart from poles) and that $\langle j, \lambda\pm \mid J_2 \mid j, m\rangle = \lambda^* \langle j, \lambda\pm \mid j, m\rangle$.

Therefore it is more convenient to use the functions $\langle j, \lambda^*\pm \mid j, m \rangle$ instead. Then the coefficients $\varphi^{\pm}(\lambda) \equiv \langle j, \lambda^*\pm \mid \varphi \rangle$ for $\mid \varphi \rangle \in \mathscr{D}$ and the matrix elements $\langle j, \lambda^*\pm \mid U(g) \mid j, \lambda'\pm \rangle$ will turn out to be meromorphic functions of λ (λ and λ') (see § 6 and § 7). The completeness relation is not affected as it contains an integration over real λ only.

C. Asymptotic formulas

The behaviour of $\langle j, m \mid j, \lambda\pm \rangle$ when $\mathrm{Re}\,\lambda \to \pm\infty$ follows from the results of the preceding section.

C_j^δ and E_j ($\delta=0$ for E_j): $|\langle j, m|j, \lambda\pm \rangle| \simeq C(j)|\mathrm{Re}\,\lambda|^{\mp\frac{1}{2}(1-\delta)}e^{-\frac{\pi}{2}|\mathrm{Re}\,\lambda|}|R_m^{\pm}(j, \lambda)|$

$$D_j^+ : \quad |\langle j, m \mid j, \lambda \rangle| \simeq C(j) \mid \mathrm{Re}\,\lambda \mid^{-j-\frac{1}{2}} e^{-\frac{\pi}{2}|\mathrm{Re}\,\lambda|} \mid R_m(j, \lambda) \mid \tag{4.22}$$

Hence $\langle j, m \mid j, \lambda\pm \rangle$ are square-integrable over any line $\mathrm{Im}\,\lambda = \text{constant}$ that does not pass through a pole.

The asymptotic expansion when $m \to \pm\infty$ can be derived from equations (18) and (19) of [*14*]. The leading term is of the form

$$|\langle j, m \mid j, \lambda\pm \rangle| \simeq C^{\pm}(j, \lambda)\cdot |m|^{|\mathrm{Im}\,\lambda| - \frac{1}{2}} \tag{4.23}$$

i. e. the coefficients are at most slowly increasing for any complex λ.

It is necessary to have an upper bound for $\langle j, m \mid j, \lambda\pm \rangle$ for all m and λ (excluding the poles in the λ-plane). The following rough upper limit for the polynomials $R_m^{\pm}$ is easily derived from a suitable integral formula (e. g. (A.7) of [*15*]).

$$|R_m^{\pm}(j, \lambda)| \leq C(j)\cdot e^{\alpha|\lambda|}(|m| + 1)^{|\mathrm{Im}\,\lambda|+1} \tag{4.24}$$

for some constants C and α.

§ 5. DIAGONALIZATION OF A GENERATOR OF PARABOLIC CLASS

The generalized eigenvectors of the generator $K_+ = J_0 + J_1$ are denoted by $|j, \eta\rangle$:

$$K_+ \mid j, \eta \rangle = \eta \mid j, \eta \rangle \tag{5.1}$$

We will find that the spectrum of K_+ is the real line for the continuous series C_j^δ and E_j, but only the positive real line for D_j^+ and the negative real line for D_j^-.

A. Construction of the eigenvectors

The « Ansatz »

$$| j, \eta \rangle \sim \sum_m A_m(j, \eta) \, | j, m \rangle$$

$$A_m = N(j, \eta)[\Gamma(m - j)/\Gamma(m + j + 1)]^{\frac{1}{2}} B_m(j, \eta) \tag{5.2}$$

and equations (2.9) and (5.1) give the difference equation:

$$2(m - \eta)B_m + (m - j)B_{m+1} + (m + j)B_{m-1} = 0 \tag{5.3}$$

Using the same method as in § 4 we arrive at the solution (including a change of variable)

$$B_m(j, \eta) = \int_C t^{-m+j}(1 - t)^{m+j} e^{2\eta t} dt \tag{5.4}$$

with the subsidiary condition on the contour C:

$$I_C t^{-m+j-2}(1 - t)^{m+j} e^{2\eta t} = 0$$

Exactly as in the hyperbolic case the difference equation has two linearly independent solutions for the continuous series, but only one for the discrete series. The eigenvectors of K_+ in $\mathscr{D}'$ are determined by the further restriction that the sequence $\{ A_m \}$ must be slowly increasing. This condition can only be satisfied for real η. The normalization is determined by

$$\langle j, \eta \, | \, j, \eta' \rangle = \sum_m \langle j, \eta \, | \, j, m \rangle \langle j, m \, | \, j, \eta' \rangle = \delta(\eta - \eta') \tag{5.5}$$

For the continuous series the contours C_1 and C_2 give linearly independent solutions.

$$C_1: \quad t = \frac{1}{2}(1 + i\xi) \quad -\infty < \xi < \infty$$

$$C_2: \quad (1+, 0+, 1-, 0-)$$

where we have used the notation of [13], § 1.6 and the t-plane is cut from $-\infty$ to 0 and from 1 to $+\infty$.

From equation 6.11 (1) of [13] follows that

$$B_m^{(2)}(j, \eta) = - \frac{4\pi^2 \cdot e^{i2\pi j}\Phi(-m+j+1, 2j+2; 2\eta)}{\Gamma(m-j)\Gamma(-m-j)\Gamma(2j+2)} \tag{5.6}$$

(where Φ is a confluent hypergeometric function). From the asymptotic expansion of Φ when $m \to \pm\infty$ (see next section) it is obvious that the solution given by $B^{(2)}$ does not belong to $\mathscr{D}'$.

When $\eta > 0$

$$\int_{C_1} dt \ldots = \int_{-\infty}^{(0+)} dt$$

and when $\eta < 0$

$$\int_{C_1} dt \ldots = \int_{\infty}^{(1-)} dt \ldots$$

From these relations and equation 6.11 (9) of [13] we find for $\eta > 0$:

$$B_m^{(1)} = \frac{2\pi i}{\Gamma(m-j)} \Psi(-m+j+1, 2j+2; 2\eta)$$

and for $\eta < 0$:

$$B_m^{(1)} = \frac{2\pi i}{\Gamma(-m-j)} e^{2\eta}\Psi(m+j+1, 2j+2; -2\eta) \tag{5.7}$$

This solution is well-behaved when $m \to \pm\infty$.

In the case of the discrete series the correct choice is:

$$C_3 = (0^+) \quad \text{for} \quad D_j^+, \qquad C_4 = (1^+) \quad \text{for} \quad D_j^-$$

(Then $B_m = 0$ for $m \leq j$ and $m \geq -j$, respectively).

Note that if $B_m(j, \eta)$ is a solution of the difference equation, so is

$$B'_m(j, \eta) = B_{-m}(j, -\eta).$$

Hence a solution for D_j^+ immediately gives a solution for D_j^-, and the latter case does not need a separate treatment.

Obviously $\int_{-\infty}^{(0+)} dt \ldots = \int_{C_3} dt \ldots$ when $\eta > 0$ and $m - j =$ integer.

Hence

$$B_m^{(3)} = \frac{2\pi i}{\Gamma(m-j)} \Psi(-m+j+1, 2j+2, 2\eta) \tag{5.8}$$

Analytic continuation gives the same formula for $\eta < 0$. From the asymptotic expansion of Ψ follows that the corresponding eigenvector belongs to $\mathscr{D}'$ only if $\eta > 0$.

The normalization constants are calculated from (5.5) in the same way as in the hyperbolic case.

It is possible to choose

$$N(j, \eta) = \frac{-i}{\sqrt{2\pi^2}} |2\eta|^{j+\frac{1}{2}} e^{-\eta} \tag{5.9}$$

for all series. Then the final result is:

$$\langle j, m | j, \eta \rangle = \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)}\right]^{\frac{1}{2}} \frac{|\eta|^{-1/2}}{\Gamma(\varepsilon m - j)} W_{\varepsilon m, j+\frac{1}{2}}(2|\eta|) \tag{5.10}$$

where $\varepsilon = \text{sign}\ \eta$ and W is the Whittaker function defined in [*13*], 6.9. (2).

The completeness relations for $\langle j, m | j, \eta \rangle$ are easily checked:

$$\int_{-\infty}^{\infty} \langle j, m | j, \eta \rangle \langle j, \eta | j, m' \rangle \, d\eta = \delta_{mm'} \quad \text{for} \quad C_j^{\delta} \quad \text{and} \quad E_j$$

$$\int_{0}^{\infty} \langle j, m | j, \eta \rangle \langle j, \eta | j, m' \rangle \, d\eta = \delta_{mm'} \quad \text{for} \quad D_j^{+} \tag{5.11}$$

B. Asymptotic behaviour

1. $m \to \pm \infty$

From (5.6) and equation 6.13 (12) in [*13*] we find for $\eta > 0$:

$$B_m^{(2)} \simeq C(j, \eta) \cdot \sin \pi(-m-j) \cdot m^{j+\frac{3}{4}} \cdot \cos\left[\sqrt{8m\eta} - \pi\left(j + \frac{3}{4}\right)\right] \qquad (m \to +\infty)$$

$$B_m^{(2)} \simeq C(j, \eta) \cdot \sin \pi(m-j) \cdot (-m)^{j+\frac{3}{4}} \cosh\left[\sqrt{8|m|\eta} + i\pi\left(j + \frac{3}{4}\right)\right] \quad (m \to -\infty)$$

(When $\eta < 0$ the two formulas are interchanged).

This means that $B_m^{(2)}$ increases exponentially in one direction. Consequently $B_m^{(2)}$ does not give a vector in $\mathscr{D}'$.

From (5.8) and [*13*], 6.13 (9) follows

$$B_m^{(3)} \simeq C(j, \eta) m^{j+\frac{1}{4}} \cos\left[\sqrt{8m\pi} - m\pi + \frac{\pi}{4}\right] \quad (m \to \infty)$$

When $\eta < 0$ the last factor is exponentially increasing, that is $\{ A_m \} \notin \mathscr{D}'$ for $\eta < 0$.

In the same way we obtain from (5.10):

$$\langle j, m | j, \eta \rangle \simeq \sqrt{\frac{2}{\pi}} (2m\eta)^{-1/4} \cos\left[\sqrt{8m\eta} - m\pi + \frac{\pi}{4}\right] \quad (m \to \infty, \ \eta > 0) \tag{5.12}$$

$$\langle j, m | j, \eta \rangle \simeq \sqrt{\frac{2}{\pi}} (2 | m | \eta)^{-1/4} \sin \pi(- | m | - j)$$
$$\cdot \exp [- \sqrt{8 | m | \eta} + i | m | \pi] \qquad (m \to -\infty,\ \eta > 0) \quad (5.13)$$

$\eta < 0$: $\langle j, m | j, \eta \rangle = e^{-i\pi m} \langle j, -m | j, -\eta \rangle$.

When $\operatorname{Im} \eta \neq 0$ these expressions will not give vectors in $\mathscr{D}'$.

2. $\eta \to \pm\infty$

[13], 6.13. (1) gives

$$\langle j, m | j, \eta \rangle \simeq \left[\frac{\Gamma(m - j)}{\Gamma(m + j + 1)} \right]^{\frac{1}{2}} \frac{\sqrt{2} | 2\eta |^{\varepsilon m - \frac{1}{2}}}{\Gamma(\varepsilon m - j)} e^{-|\eta|} \qquad (5.14)$$

3. $\eta \to 0$

Use equation 6.8 (2) of [13]. The result depends on the type of representation:

$$C_j^\delta: \quad \langle j, m | j, \eta \rangle \simeq \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)} \right]^{\frac{1}{2}} \cdot \frac{\sqrt{2}}{\Gamma(\varepsilon m - j)}$$
$$\left\{ \frac{\Gamma(-2j-1)}{\Gamma(-\varepsilon m - j)} | 2\eta |^{j+\frac{1}{2}} + \frac{\Gamma(2j+1)}{\Gamma(-\varepsilon m + j + 1)} | 2\eta |^{-j-\frac{1}{2}} \right\} \qquad (5.15)$$

$$E_j: \quad \langle j, m | j, \eta \rangle \simeq \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)} \right]^{\frac{1}{2}} \cdot \frac{\sqrt{2}}{\Gamma(\varepsilon m - j)} \cdot \frac{\Gamma(2j+1)}{\Gamma(-\varepsilon m + j + 1)} \cdot | 2\eta |^{-j-\frac{1}{2}} \qquad (5.16)$$

$$D_j^+: \quad \langle j, m | j, \eta \rangle \simeq \left[\frac{\Gamma(m-j)}{\Gamma(m+j+1)} \right]^{\frac{1}{2}} \cdot \frac{\sqrt{2}}{\Gamma(-2j)} \cdot | 2\eta |^{-j-\frac{1}{2}} \qquad (5.17)$$

C. Transformation between the bases $| j, \lambda \rangle$ and $| j, \eta \rangle$

As the generalized eigenvectors $| j, \lambda \rangle$ and $| j, \eta \rangle$ of J_2 and K_+, respectively, both belong to $\mathscr{D}'$, their « scalar product » may be undefined. That is, we cannot expect the sum $\sum_m \langle j, \lambda^* | j, m \rangle \langle j, m | j, \eta \rangle$ to be convergent from general considerations. Hence we must consider $\langle j, \lambda^* | j, \eta \rangle$ as a generalized function which satisfies

$$\varphi^\pm(\lambda) \equiv \langle j, \lambda^* \pm | \varphi \rangle = \int_{-\infty}^{\infty} \langle j, \lambda^* \pm | j, \eta \rangle \varphi(\eta) d\eta$$
$$\varphi(\eta) \equiv \langle j, \eta | \varphi \rangle = \sum_{+,-} \int_{-\infty}^{\infty} \langle j, \eta | j, \lambda \pm \rangle \varphi^\pm(\lambda) d\lambda \qquad (5.18)$$

for all $| \varphi \rangle \in \mathscr{D}$.

Straightforward calculations, however, yield an expression for $\langle j, \lambda^* | j, \eta \rangle$ which is a well-behaved function.

The correctness of (5.18) can be checked by noting that with the formulas given below

$$\langle j, \lambda^* \pm | j, m \rangle = \int_{-\infty}^{\infty} \langle j, \lambda^* \pm | j, \eta \rangle \langle j, \eta | j, m \rangle \, d\eta$$

and the inverse relation are satisfied.

For a general $| \varphi \rangle = \Sigma \varphi_m | j, m \rangle \in \mathscr{D}$ the uniform convergence and fast decrease in λ and η proved in § 7 implies that (5.18) is valid.

1. C_j^δ and E_j

$$\langle j, \lambda^*(-)^\sigma | j, \eta \rangle = (-i\varepsilon)^\sigma \frac{2^{-i\lambda}}{\sqrt{4\pi}} \frac{\Gamma\left(\dfrac{-j^* + \sigma - i\lambda}{2}\right)}{\Gamma\left(\dfrac{-j + \sigma + i\lambda}{2}\right)} | \eta |^{-\frac{1}{2}+i\lambda} \tag{5.19}$$

($j^* = -j - 1$ for C_j^δ, $j^* = j$ for E_j).

2. D_j^+

$$\langle j, \lambda^* | j, \eta \rangle = \frac{2^{-i\lambda}}{\sqrt{2\pi}} \cdot \frac{\Gamma\left(\dfrac{-j - i\lambda}{2}\right)}{\Gamma\left(\dfrac{-j + i\lambda}{2}\right)} \eta^{-\frac{1}{2}+i\lambda} \tag{5.20}$$

§ 6. MATRIX ELEMENTS OF FINITE TRANSFORMATIONS

In this paragraph we give the matrix elements of the finite transformations $\exp(-i\theta J_0)$, $\exp(-itJ_2)$ and $\exp(-i\xi K_+)$ in the continuous bases.

The interpretation of a matrix element like

$$\langle j, \lambda^* | \exp(-i\theta J_0) | j, \lambda' \rangle = \sum_m \langle j, \lambda^* | j, m \rangle \exp(-im\theta) \langle j, m | j, \lambda' \rangle$$

is parallel to that given for $\langle j, \lambda^* | j, \eta \rangle$ in § 5, C and the value is calculated by the methods used in § 4 (For the noncompact transformations the summation is replaced by integration). The calculations are valid for real λ and λ' but obviously the result can be continued to a meromorphic function of λ and λ'.

When performing integrations of the type

$$\langle j, \lambda^* \mid \exp(-i\theta J_0) \mid \varphi \rangle = \int_{-\infty}^{\infty} d\lambda' \langle j, \lambda^* \mid \exp(-i\theta J_0) \mid j, \lambda' \rangle \varphi(\lambda')$$

(where $|\varphi\rangle \in \mathscr{D}$, $\varphi(\lambda) = \langle j, \lambda^* \mid \varphi \rangle$)
let λ be real as well as λ' and interprete the factors $\Gamma(\pm i(\lambda - \lambda'))$ (see below) as $\Gamma(\pm i(\lambda - \lambda') + \varepsilon)$ with $\varepsilon > 0$. Then the integral is well-defined and the result can actually be continued to a meromorphic function of λ (see § 7).

A. Matrix elements of $\exp(-i\xi K_+)$ in the $|j, \lambda\rangle$ basis

Define $d^j_{\lambda(-)^\sigma, \lambda'(-)^{\sigma'}}(\xi) = \langle j, \lambda^*(-)^\sigma \mid \exp(-i\xi K_+) \mid j, \lambda'(-)^{\sigma'} \rangle$.
$\Delta\lambda = \lambda - \lambda'$.

1. C_j^δ and E_j

$$d^j_{\lambda(-)^\sigma, \lambda'(-)^{\sigma'}}(\xi) = \frac{1}{2\pi} \frac{\Gamma\left(\dfrac{-j^* + \sigma - i\lambda}{2}\right)\Gamma\left(\dfrac{-j + \sigma' + i\lambda'}{2}\right)}{\Gamma\left(\dfrac{-j + \sigma + i\lambda}{2}\right)\Gamma\left(\dfrac{-j^* + \sigma' - i\lambda'}{2}\right)} \Gamma(i\Delta\lambda) \cdot \cos\frac{\pi}{2}(i\Delta\lambda + \sigma - \sigma') \, |2\xi|^{-i\Delta\lambda} (\operatorname{sign}\xi)^{\sigma - \sigma'} \tag{6.1}$$

2. D_j^+

$$d^j_{\lambda\lambda'}(\xi) = \frac{1}{2\pi} \frac{\Gamma\left(\dfrac{-j - i\lambda}{2}\right)\Gamma\left(\dfrac{-j + i\lambda'}{2}\right)}{\Gamma\left(\dfrac{-j + i\lambda}{2}\right)\Gamma\left(\dfrac{-j - i\lambda'}{2}\right)} \Gamma(i\Delta\lambda) |2\xi|^{-i\Delta\lambda} \cdot \exp\left[\frac{\pi}{2}\Delta\lambda \operatorname{sign}\xi\right] \tag{6.2}$$

B. Matrix elements of $\exp(-i\theta J_0)$ in the $|j, \lambda\rangle$ basis

Define $d^j_{\lambda(-)^\sigma, \lambda'(-)^{\sigma'}}(\theta) = \langle j, \lambda^*(-)^\sigma \mid \exp(-i\theta J_0) \mid j, \lambda'(-)^{\sigma'} \rangle$

$$f_1(\theta) = \left(\cos\frac{\theta}{2}\right)^{-2j-2} \left|2\tan\frac{\theta}{2}\right|^{-i\Delta\lambda} \cdot {}_2F_1\left(j+1-i\lambda, j+1+i\lambda'; 1-i\Delta\lambda; -\tan^2\frac{\theta}{2}\right)$$

$$f_2(\theta) = \left(\cos\frac{\theta}{2}\right)^{-2j-2} \left|2\tan\frac{\theta}{2}\right|^{i\Delta\lambda} \cdot {}_2F_1\left(j+1+i\lambda, j+1-i\lambda'; 1+i\Delta\lambda; -\tan^2\frac{\theta}{2}\right)$$

where $-\pi < \theta \leq \pi$

1. C_j^δ and E_j ($\delta = 0$ for E_j)

$$d^j_{\lambda(-)^\sigma,\lambda'(-)^{\sigma'}}(\theta) = \frac{1}{2\pi}\left\{\frac{\Gamma\left(\frac{-j^*+\sigma-i\lambda}{2}\right)\Gamma\left(\frac{-j+\sigma'+i\lambda'}{2}\right)}{\Gamma\left(\frac{-j+\sigma+i\lambda}{2}\right)\Gamma\left(\frac{-j^*+\sigma'-i\lambda'}{2}\right)}\cdot\Gamma(i\Delta\lambda)f_1(\theta)\right.$$
$$+(-1)^\delta\frac{\Gamma\left(\frac{j+1+\delta+(-1)^\delta\sigma+i\lambda}{2}\right)\Gamma\left(\frac{j^*+1+\delta+(-1)^\delta\sigma'-i\lambda'}{2}\right)}{\Gamma\left(\frac{j^*+1+\delta+(-1)^\delta\sigma-i\lambda}{2}\right)\Gamma\left(\frac{j+1+\delta+(-1)^\delta\sigma'+i\lambda'}{2}\right)}$$
$$\left.\cdot\Gamma(-i\Delta\lambda)f_2(\theta)\right\}\cos\frac{\pi}{2}(i\Delta\lambda+\sigma-\sigma')(\text{sign }\theta)^{\sigma-\sigma'} \quad (6.3)$$

2. D_j^+

$$d^j_{\lambda,\lambda'}(\theta) = \frac{1}{2\pi}\left\{\frac{\Gamma\left(\frac{-j-i\lambda}{2}\right)\Gamma\left(\frac{-j+i\lambda'}{2}\right)}{\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j-i\lambda'}{2}\right)}\Gamma(i\Delta\lambda)\cdot\exp\left[\frac{\pi}{2}\Delta\lambda\text{ sign }\theta\right]f_1(\theta)\right.$$
$$\left.+\frac{\Gamma\left(\frac{-j+1+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda'}{2}\right)}{\Gamma\left(\frac{-j+1-i\lambda}{2}\right)\Gamma\left(\frac{-j+1+i\lambda'}{2}\right)}\Gamma(-i\Delta\lambda)\exp\left[-\frac{\pi}{2}\Delta\lambda\text{ sign }\theta\right]f_2(\theta)\right\} \quad (6.4)$$

C. **Matrix elements of** $\exp(-i\theta J_0)$ **in the** $|j,\eta\rangle$ **basis**

Define $d^j_{\eta\eta'}(\theta) = \langle j,\eta\,|\exp(-i\theta J_0)|\,j,\eta'\rangle$.
$\varepsilon = \text{sign }\eta$, K_ν = modified Bessel function.

1. C_j^δ and E_j ($\delta = 0$ for E_j)

$$d^j_{\eta\eta'}(\theta) = \frac{\exp\left[i(\eta+\eta')\cot\text{an}\frac{\theta}{2}\right]}{\pi\left|\sin\frac{\theta}{2}\right|}$$
$$\left\{\exp[i\pi(\varepsilon-\varepsilon')(2j+1)\text{ sign }\theta]K_{-2j-1}\left[\exp\left(\frac{i\pi}{4}(\varepsilon+\varepsilon')\text{ sign }\theta\right)\frac{2|\eta\eta'|^{\frac{1}{2}}}{\left|\sin\frac{\theta}{2}\right|}\right]\right.$$
$$+(-1)^\delta\exp[-i\pi(\varepsilon-\varepsilon')(2j+1)\text{ sign }\theta]$$
$$\left.K_{-2j-1}\left[\exp\left(-\frac{i\pi}{4}(\varepsilon+\varepsilon')\text{ sign }\theta\right)\frac{2|\eta\eta'|^{\frac{1}{2}}}{\left|\sin\frac{\theta}{2}\right|}\right]\right\} \quad (6.5)$$

2. D_j^+

$$d^j_{\eta\eta'}(\theta) = \frac{\exp\left[i\pi j \operatorname{sign}\theta + i(\eta+\eta')\operatorname{cotan}\frac{\theta}{2}\right]}{\left|\sin\frac{\theta}{2}\right|}\cdot J_{-2j-1}\left(\frac{2\,|\eta\eta'|^{\frac{1}{2}}}{\left|\sin\frac{\theta}{2}\right|}\right) \tag{6.6}$$

(This formula can be derived from (6.5) putting $\varepsilon = \varepsilon'$, and using the fact that j is integer of half-integer).

D. **Matrix elements of** $\exp(-itJ_2)$ **in the** $|j, \eta\rangle$ **basis**

For all series we have

$$d_{\eta\eta'}(t) \equiv \langle j, \eta\,|\exp(-itJ_2)\,|\,j, \eta'\rangle = \left|\frac{\eta'}{\eta}\right|^{\frac{1}{2}}\delta(\eta' - e^t\eta) \tag{6.7}$$

§ 7. GENERATORS AND DIFFERENTIABLE VECTORS IN A CONTINUOUS BASIS

From § 3 we know that $\mathscr{D}$ is the maximal invariant common domain of the generators in $\mathscr{H}$. Hence $\langle j, \lambda^* \pm |\,J_i\,|\,\varphi\rangle$ and $\langle j, \eta\,|\,J_i\,|\,\varphi\rangle$ exist for all $\varphi \in \mathscr{D}$. The explicit expressions for these matrix elements are calculated below, and at the same time we obtain alternative characterizations of the space $\mathscr{D}$ in terms of the functions

$$\varphi^{\pm}(\lambda) = \langle j, \lambda^* \pm |\,\varphi\rangle \quad \text{or} \quad \varphi(\eta) = \langle j, \eta\,|\,\varphi\rangle.$$

$\mathscr{D}'$ (which contains the generalized eigenvectors) is also an invariant common domain of the generators. The action of J_i on $|j, \lambda\pm\rangle$ and $|j, \eta\rangle$ is given in section C.

A. **The** $|j, \lambda\pm\rangle$ **basis**

1. Let $|\varphi\rangle = \Sigma\varphi_m\,|\,j, m\rangle \in \mathscr{D}$ (i. e. φ_m is a rapidly decreasing sequence). From § 4.B we have

$$\varphi^{\pm}(\lambda) = S^{\pm}(j, \lambda^*)^*\Sigma\varphi_m R_m^{\pm}(j, \lambda^*)^* \tag{7.1}$$

The rapid decrease of $\{\varphi_m\}$ and the upper limit on $|R_m^{\pm}|$ given by (4.24) imply that the sums $\Sigma\varphi_m R_m^{\pm *}$ converge uniformly on every compact subset of the λ-plane to holomorphic functions $\hat{\varphi}^{\pm}(\lambda)$. Hence $\varphi^{\pm}(\lambda)$ are of the form

$$\varphi^{\pm}(\lambda) = S^{\pm}(j, \lambda^*)^*\hat{\varphi}^{\pm}(\lambda) \tag{7.2}$$

Convergence in $\mathscr{D}$ means: $|\varphi_{(\nu)}\rangle \to |\varphi\rangle$ iff

$$p_n''(\varphi_{(\nu)} - \varphi) = \max_m \, [(m^2 + 1)^n \, | \varphi_{(\nu)m} - \varphi_m |^2]^{\frac{1}{2}} \to 0$$

for all n. It is evident that this condition and (4.24) give uniform convergence $\hat{\varphi}^{\pm}_{(\nu)}(\lambda) \to \hat{\varphi}^{\pm}(\lambda)$ in every compact subset of the λ-plane ($\{ p_n'' \}$ is equivalent to $\{ p_n \}$, (3.4)).

2. Equation (4.24) does not give any limit to the growth of $\varphi^{\pm}(\lambda)$ when $|\operatorname{Im}\lambda| \to \infty$. It is evident, however, that $\varphi^{\pm}(\lambda)$ are square integrable. In fact

$$\int_{-\infty}^{\infty} (|\varphi^{+}(\lambda)|^2 + |\varphi^{-}(\lambda)|^2) d\lambda = \Sigma \, |\varphi_m|^2 < \infty.$$

But $\mathscr{D}$ is invariant under J_2. Hence $\lambda^n \varphi^{\pm}(\lambda)$ must belong to $\mathscr{D}$ for all $n = 0, 1, \ldots$ This is possible only if $\varphi^{\pm}(\lambda)$ decrease faster than any inverse power of λ when $\lambda \to \pm\infty$. More generally, this is true for $\operatorname{Im}\lambda$ fixed, arbitrary, and $\operatorname{Re}\lambda \to \pm\infty$.

3. Consider a representation in the class C_j^0 and a vector $|\varphi\rangle \in \mathscr{D}$. The action of K_+ is given by:

$$\langle j, \lambda^* \pm | K_+ | \varphi \rangle = i\left[\frac{d}{d\xi}\langle j, \lambda^* \pm | e^{-i\xi K_+} | \varphi \rangle\right]_{\xi=0} \tag{7.3}$$

where

$$\langle j, \lambda^* \pm | e^{-i\xi K_+} | \varphi \rangle = \int_{-\infty}^{\infty} d\lambda' [d_{\lambda\pm,\lambda'+}(\xi)\varphi^{+}(\lambda') + d_{\lambda\pm,\lambda'-}(\xi)\varphi^{-}(\lambda')] \tag{7.4}$$

Calculate the component $\langle j, \lambda^* + | K | \varphi \rangle$.

From (2.10) follows $PK_+P^{-1} = -K_+$. But this implies that

$$\langle j, \lambda^* + | K_+ | \varphi \rangle$$

depends only on $\varphi^{-}(\lambda)$, i. e. only the second term in (7.4) contributes to (7.3) in this case. (6.1) and (4.19) give

$$\int_{-\infty}^{\infty} d\lambda' d_{\lambda+,\lambda'-}(\xi)\varphi^{-}(\lambda') \cdot$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} d\lambda' \frac{\Gamma\left(\frac{j+1-i\lambda}{2}\right)\Gamma\left(\frac{-j+1+i\lambda'}{2}\right)}{\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{j+2-i\lambda'}{2}\right)} \Gamma(i\Delta\lambda)$$

$$\sin\left(\frac{i\pi\Delta\lambda}{2}\right) | 2\xi |^{-i\Delta\lambda} \operatorname{sign} \xi \cdot \Gamma\left(\frac{j+2-i\lambda'}{2}\right)\Gamma\left(\frac{j+2+i\lambda'}{2}\right)\hat{\varphi}^{-}(\lambda')$$

Note that the singularities of $\varphi^-(\lambda')$ in the lower half-plane are cancelled by the zeros of $d_{\lambda+,\lambda'-}(\xi)$. Hence the integrand is holomorphic in the lower half-plane apart from the poles of $\Gamma(i\Delta\lambda)$. Now the integral can be written

$$\int_{\operatorname{Im}\lambda'=0} d\lambda' d_{\lambda+,\lambda'-}\varphi^-(\lambda') = -2\pi i \operatorname{Res}\,[d_{\lambda+,\lambda'-}\varphi^-(\lambda')]_{\lambda'=\lambda-i} + \int_{\operatorname{Im}\lambda'=-3/2} d\lambda' d_{\lambda+,\lambda'-}\varphi^-(\lambda')$$

Derivation with respect to ξ gives a constant from the pole term and from the second term an integral that contains the factor

$$|2\xi|^{-i\Delta\lambda-1} = |2\xi|^{\frac{1}{2}}\cdot e^{i\alpha(\lambda,\lambda')}$$

Thus the integral vanishes when $\xi \to 0$. A simple calculation yields

$$\langle j, \lambda^* + | K_+ | \varphi \rangle = i(-j+i\lambda)\varphi^-(\lambda - i) \tag{7.5}$$

In the same way we obtain:

$$\langle j, \lambda^* - | K_+ | \varphi \rangle = i(-j+i\lambda)\varphi^+(\lambda - i) \tag{7.6}$$

The action of $K_- = J_0 - J_1$ is easily derived from the relation

$$K_+K_- = C_2 + J_2^2 - iJ_2 \tag{7.7}$$

and (7.5), (7.6). The result is

$$\langle j, \lambda^* \pm | K_- | \varphi \rangle = i(j+i\lambda)\varphi^\mp(\lambda + i) \tag{7.8}$$

(7.5)-(7.8) are valid for representations in the class C_j^1 also.

The same method gives for the series E_j:

$$\begin{aligned}
\langle j, \lambda^* + | K_+ | \varphi \rangle &= i(-j+i\lambda)\varphi^-(\lambda - i)\\
\langle j, \lambda^* - | K_+ | \varphi \rangle &= i(j+1+i\lambda)\varphi^+(\lambda - i)\\
\langle j, \lambda^* + | K_- | \varphi \rangle &= i(-j-1+i\lambda)\varphi^-(\lambda + i)\\
\langle j, \lambda^* - | K_- | \varphi \rangle &= i(j+i\lambda)\varphi^+(\lambda + i)
\end{aligned} \tag{7.9}$$

and for D_j^+:

$$\begin{aligned}
\langle j, \lambda^* | K_+ | \varphi \rangle &= (-j-1+i\lambda)\frac{\Gamma\left(\frac{-j-i\lambda}{2}\right)\Gamma\left(\frac{-j-1+i\lambda}{2}\right)}{\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j-1-i\lambda}{2}\right)}\varphi(\lambda - i)\\
\langle j, \lambda^* | K_- | \varphi \rangle &= (-j-i\lambda)\frac{\Gamma\left(\frac{-j-i\lambda}{2}\right)\Gamma\left(\frac{-j+1+i\lambda}{2}\right)}{\Gamma\left(\frac{-j+i\lambda}{2}\right)\Gamma\left(\frac{-j+1-i\lambda}{2}\right)}\varphi(\lambda + i)
\end{aligned} \tag{7.10}$$

Corresponding expressions for the other generators are obtained through:

$$J_0 = \frac{1}{2}(K_+ + K_-) \qquad J_1 = \frac{1}{2}(K_+ - K_-)$$

4. From the formulas of the preceding section it is evident that the generators are defined for all pairs $\varphi^\pm(\lambda)$ that satisfy: (drop the $\pm$ for the discrete series)

a) $\varphi^\pm(\lambda) = S^\pm(j, \lambda^*)^*\hat{\varphi}^\pm(\lambda)$, where $\hat{\varphi}^\pm$ are holomorphic in the whole λ-plane.

b) $\varphi^\pm(\lambda) \to 0$ faster than any inverse power of λ when $\mathrm{Re}\,\lambda \to \pm\infty$ for $\mathrm{Im}\,\lambda$ fixed.

Furthermore, this set of functions is invariant under the generators. It was proved in 1. and 2. that if $|\varphi\rangle \in \mathscr{D}$, then $\varphi^\pm(\lambda)$ have the properties *a)* and *b)*. But $\mathscr{D}$ is maximal. Hence the set defined by *a)* and *b)* is identical with $\mathscr{D}$.

B. The $|j, \eta\rangle$ basis

1. Let $|\varphi\rangle = \Sigma\varphi_m |j, m\rangle \in \mathscr{D}$. From the integral formula (5.4) it is easily shown that

$$\left|\frac{d^k}{d\eta^k}\langle j, \eta | j, m\rangle\right| \leq \sum_{\nu=0}^{k} C_\nu(j, k)\frac{(|m| + 1)^{\nu+\frac{1}{2}-\mathrm{Re}\,j}}{|\eta|^{\nu+1}} \tag{7.11}$$

for all η, m when $C_\nu(j, k)$ are suitably chosen constants. Hence the sum

$$\Sigma\varphi_m \cdot \frac{d^k}{d\eta^k}\langle j, \eta | j, m\rangle$$

converges uniformly on every closed interval that does not contain $\eta = 0$, and the sum

$$\varphi(\eta) = \Sigma\varphi_m \langle j, \eta | j, m\rangle \tag{7.12}$$

is an infinitely differentiable function which can be differentiated term by term when $\eta \neq 0$. In the same way convergence in the topology of $\mathscr{D}$ implies uniform convergence of $\varphi(\eta)$ and all its derivatives when $\eta \neq 0$.

2. The argument of section A.2. can be repeated to show that $\varphi(\eta)$ decreases faster than any inverse power of η when $|\eta| \to \infty$. When $\eta \to 0$ the square integrability of $\varphi(\eta)$ means that $\varphi(\eta) \to \infty$ no faster than $|\eta|^{-\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$.

3. The transformation e^{-iJ_2t} has a very simple form in the $|j, \eta\rangle$ basis. From (6.7):

$$\langle j, \eta | e^{-iJ_2t} | \varphi \rangle = \int_{-\infty}^{\infty} d\eta' d_{\eta\eta'}(t)\varphi(\eta') = e^{t/2}\varphi(e^t\eta) \tag{7.13}$$

Differentiation gives $\left(\varphi'(\eta) \equiv \dfrac{d}{d\eta}\varphi(\eta)\right)$

$$\langle j, \eta | J_2 | \varphi \rangle = i\left[\frac{1}{2}\varphi(\eta) + \eta\varphi'(\eta)\right] \tag{7.14}$$

$\langle j, \eta | K_- | \varphi \rangle$ ($K_- = J_0 - J_1$) is obtained from (7.7):

$$\langle j, \eta | K_+K_- | \varphi \rangle = \eta \langle j, \eta | K_- | \varphi \rangle = \left(j + \frac{1}{2}\right)^2 \varphi - \eta\varphi' - \eta^2\varphi''$$

$$\langle j, \eta | K_- | \varphi \rangle = \frac{1}{\eta}\left(j + \frac{1}{2}\right)\varphi - \varphi' - \eta\varphi'' \tag{7.15}$$

It is easy to see from (5.15)-(5.17) that $\langle j, \eta | K_-^n | \varphi \rangle$ ($n = 0, 1, \ldots$) are well-behaved when $\eta \to 0$ for $|\varphi\rangle \in \mathscr{D}_C$ (= the set of finite linear combinations of $|j, m\rangle$). In order that this shall be true for arbitrary $|\varphi\rangle \in \mathscr{D}$ it is necessary that $\varphi(\eta)$ is of the following form in $\eta > 0$:

$$\begin{aligned} C_j^\delta \text{ and } E_j\colon \quad \varphi(\eta) &= \eta^{-j-\frac{1}{2}}f_1(\eta) + \eta^{j+\frac{1}{2}}f_2(\eta) \\ D_j^+\colon \quad \varphi(\eta) &= \eta^{-j-\frac{1}{2}}f(\eta) \end{aligned} \tag{7.16}$$

and in $\eta < 0$:

$$\varphi(\eta) = |\eta|^{-j-\frac{1}{2}}g_1(\eta) + |\eta|^{j+\frac{1}{2}}g_2(\eta)$$

and (for D_j^-)

$$\varphi(\eta) = |\eta|^{-j-\frac{1}{2}}g(\eta), \tag{7.17}$$

respectively, where $f_i(\eta)$ and $g_i(\eta)$ are ∞ differentiable, including $\eta = 0$.

5. Exactly as in section A.4. we arrive at a description of the set $\mathscr{D}$: $|\varphi\rangle \in \mathscr{D}$ iff

a) $\varphi(\eta)$ is ∞ differentiable for $\eta \neq 0$,
b) $\varphi(\eta)$ is rapidly decreasing when $|\eta| \to \infty$,
c) The behaviour when $\eta \to 0$ is given by (7.16) and (7.17).

C. Generators acting on the generalized eigenvectors

1. The $|j, \lambda\pm\rangle$ basis.

The fact that $\{J_i\}$ are continuous operators in $\mathscr{D}'$ justifies using the commutation relations without restrictions. Hence (2.5) implies

$$J_2K_+ |j, \lambda\pm\rangle = K_+(J_2 + i)|j, \lambda\pm\rangle = (\lambda + i)K_+ |j, \lambda\pm\rangle$$

With $PK_+P^{-1} = -K_+$ this gives

$$K_+ \mid j, \lambda \pm \rangle = C^{\pm} \mid j, \lambda + i, \mp \rangle$$

The constants cannot be determined from the commutation relations alone, as they depend on the choice of phase for the basis vectors. Use (7.5), (7.6) to find (for C_j^{δ})

$$K_+ \mid j, \lambda \pm \rangle = i(-j-1+i\lambda) \mid j, \lambda + i, \mp \rangle \tag{7.18}$$

In the same way

$$K_- \mid j, \lambda \pm \rangle = i(j+1+i\lambda) \mid j, \lambda - i, \mp \rangle \tag{7.19}$$

(and analogous formulas for the other series).

The constants can also be determined through the equality

$$\langle j, m \mid K_+ \mid j, \lambda \pm \rangle = m \langle j, m \mid j, \lambda \pm \rangle + \frac{1}{2}[(m+j+1)(m-j)]^{\frac{1}{2}} \langle j, m+1 \mid j, \lambda \pm \rangle$$
$$+ \frac{1}{2}[(m+j)(m-j-1)]^{\frac{1}{2}} \langle j, m-1 \mid j, \lambda \pm \rangle = C^{\pm} \langle j, m \mid j, \lambda + i, \pm \rangle$$

and the asymptotic formula for $\langle j, \lambda \pm \mid j, m \rangle$ when $m \to \infty$ by identifying the coefficients of the highest power of m on both sides (This is an alternative way to derive (7.5)-(7.10)).

It is easily checked that the vectors in (7.18) and (7.19) cannot be written as a (formal) integral over real λ

$$K_+ \mid j, \lambda \pm \rangle \sim \int_{-\infty}^{\infty} d\lambda' (K_+)_{\lambda'\pm, \lambda\pm} \mid j, \lambda' \pm \rangle.$$

Hence the « matrix elements » $\langle j, \lambda' \pm \mid K_+ \mid j, \lambda \pm \rangle$ cannot be given a sense even as distributions in λ and λ'.

It is possible to give a description of the vectors in $\mathscr{D}'$ in the $\mid j, \lambda \rangle$ basis by a method which is a generalization of [*16*], § 27.3 (we drop the index $\pm$ for the moment).

Introduce a topology in $\mathscr{D}$ by the set of norms

$$p_{\mu\nu}(\varphi) = \sup_{G_\mu} \mid (1 + \mid \mathrm{Re}\, \lambda \mid)^{\nu} \varphi(\lambda) \mid$$

where $G_\mu = \{ \lambda ; \mid \mathrm{Im}\, \lambda \mid < \mu, \mid \lambda - \lambda_i \mid > \mu^{-1}, \{ \lambda_i \} = \text{the poles of } S(j, \lambda^*)^* \}$.

It is easy to check that this topology is equivalent to that given by (3.4). Put $G = \Omega - (\{ \lambda_i \}, \infty)$ where Ω = the Riemann sphere. Then $\mathscr{D}$ = the set of functions which are holomorphic in G with simple poles in λ_i and rapidly decreasing when $\mid \mathrm{Re}\, \lambda \mid \to \infty$. Introduce the Banach space $B_\nu(\overline{G}_\mu)$

of functions holomorphic in G_μ with continuous boundary values on C_μ (= the boundary of G_μ) and the norm $\| \varphi \| = p_{\mu\nu}(\varphi)$.

If $| u \rangle \in \mathscr{D}'$ there exists μ, ν and M such that

$$| \langle u | \varphi \rangle | \leq M p_{\mu\nu}(\varphi), \quad \forall | \varphi \rangle \in \mathscr{D}.$$

Hahn-Banach's theorem implies that $| u \rangle$ can be extended to $B_\nu(\overline{G}_\mu)$ with the same upper bound M.

Now let λ_0 be any of the poles λ_i and $z \in \Omega - \overline{G}_\mu \equiv H_\mu$. Then

$$(\lambda_0 - \lambda)^{-\nu-1}(z - \lambda)^{-1} \in B_\nu(\overline{G}_\mu)$$

as a function of λ, hence $\tilde{u}_\nu(z) = \langle u | (\lambda_0 - \lambda)^{-\nu-1}(z - \lambda)^{-1} \rangle$ exists. It is easily proved that $\tilde{u}_\nu(z)$ is a holomorphic function in H_μ, bounded in $H_{\mu+1}$, and for $| \varphi \rangle \in \mathscr{D}$

$$\langle u | \varphi \rangle = \frac{1}{2\pi i} \int_{C_{\mu+1}} \tilde{u}_\nu(\lambda)(\lambda_0 - \lambda)^{\nu+1} \varphi(\lambda) d\lambda$$

[cf. [*16*], § 27.3, (8)] where the direction of $C_{\mu+1}$ is such that G_μ is to the left. On the other hand every $\tilde{v}(\lambda)$ which is holomorphic in H_μ for some μ and which does not grow faster than some power $| \lambda |^n$ when $| \lambda | \rightarrow \infty$ away from G_μ defines a continuous linear functional on $\mathscr{D}$ through

$$\langle v | \varphi \rangle = \frac{1}{2\pi i} \int_{C_{\mu+1}} \tilde{v}(\lambda)\varphi(\lambda)d\lambda$$

Hence we can write formally

$$| v \rangle \sim \frac{1}{2\pi i} \int_{C^*_{\mu+1}} \tilde{v}(\lambda^*)^* | j, \lambda \rangle \, d\lambda \tag{7.20}$$

The function $\tilde{v}(\lambda)$ is not unique. A closer investigation shows that if $\tilde{v}_1$ and $\tilde{v}_2$ satisfy the above conditions, they represent the same $v \in \mathscr{D}'$ if and only if $\tilde{v}_1 - \tilde{v}_2$ is a polynomial $P(\lambda)$ (of degree $\leq n$) in $| \operatorname{Im} \lambda | > \mu$ and $(\tilde{v}_1 - \tilde{v}_2)(\lambda_i) = P(\lambda_i)$ for all λ_i.

As a special case of (7.20) we have

$$K_+ | j, \lambda \pm \rangle \sim \frac{1}{2\pi i} \int_{C^*_\mu} d\lambda' \, \frac{i(-j + i\lambda')}{\lambda' - \lambda - i} | j, \lambda' \mp \rangle \tag{7.21}$$

where C^*_μ satisfies $\mu > | \operatorname{Im} \lambda | + 1$. C^*_μ can be deformed into a closed contour enclosing $\lambda + i$.

2. The $|j, \eta\rangle$ basis.

(7.14) can be written in the form

$$\langle j, \eta | J_2 | \varphi \rangle = -i \int_{-\infty}^{\infty} d\eta' [\eta' \delta'(n' - \eta) + \frac{1}{2} \delta(\eta' - \eta)] \varphi(\eta')$$

Hence

$$J_2 | j, \eta \rangle \sim i \int_{-\infty}^{\infty} d\eta' [\eta' \delta'(\eta' - \eta) + \frac{1}{2} \delta(\eta' - \eta)] | j, \eta' \rangle$$

In the same way from (7.15)

$$K_- | j, \eta \rangle \sim \int_{-\infty}^{\infty} d\eta' \left[\frac{1}{\eta'} \left(j + \frac{1}{2} \right)^2 \delta(\eta' - \eta) - \delta'(\eta' - \eta) - \eta' \delta''(\eta' - \eta) \right] | j, \eta' \rangle$$

In this case it is possible to define the matrix elements as generalized functions in η' (or η):

$$\langle j, \eta' | J_2 | j, \eta \rangle = i \left[\eta' \delta'(\eta' - \eta) + \frac{1}{2} \delta(\eta' - \eta) \right] \tag{7.22}$$

$$\langle j, \eta' | K_- | j, \eta \rangle = \frac{1}{\eta'} \left(j + \frac{1}{2} \right)^2 \delta(\eta' - \eta) - \eta' \delta''(\eta' - \eta) - \delta'(\eta' - \eta)$$

More generally, if $| u \rangle \in \mathscr{D}'$ then $\langle j, \eta | u \rangle$ is a tempered distribution everywhere except in $\eta = 0$ where further conditions must be imposed to match the behaviour (7.16) and (7.17).

§ 8. DISCUSSION

In order to see the relation between this paper and [2], consider a UIR of class C_j^0 in the $| j, \lambda \pm \rangle$ basis.

Define

$$\varphi^{1,2}(\lambda) = \frac{1}{\sqrt{2}} [\varphi^+(\lambda) \mp \varphi^-(\lambda)] \tag{8.1}$$

Then (7.5), (7.6) and (7.8) give

$$\begin{aligned} K_+: \quad & \varphi^{1,2}(\lambda) \to \mp i(-j + i\lambda) \cdot \varphi^{1,2}(\lambda - i) \\ K_-: \quad & \varphi^{1,2}(\lambda) \to \mp i(j + i\lambda) \cdot \varphi^{1,2}(\lambda + i) \end{aligned} \tag{8.2}$$

This is equation (3.33) of [2]. In the Fourier-transformed basis

$$\tilde{\varphi}^{\pm}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda q} \varphi^{\pm}(\lambda) d\lambda \tag{8.3}$$

the generators are given by

$$\begin{aligned} J_0: &\quad \tilde{\varphi}^{\pm}(q) \to -i\frac{d}{dq}\tilde{\varphi}^{\pm}(q) \\ K_+: &\quad \tilde{\varphi}^{\pm}(q) \to ie^{-q}\left(-j-1+\frac{d}{dq}\right)\tilde{\varphi}^{\mp}(q) \\ K_-: &\quad \tilde{\varphi}^{\pm}(q) \to ie^{q}\left(j+1+\frac{d}{dq}\right)\tilde{\varphi}^{\mp}(q) \end{aligned} \tag{8.4}$$

Using (8.1) we arrive at equation (3.10) of [*2*].

The difficulty of using the functions $\tilde{\varphi}^{1,2}(q)$ is, as Mukunda points out, that the conditions on the vectors in $\mathscr{D}$ involve relations between $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ at $q = \pm\infty$ which make it impossible to choose them independently. By using the combinations $\tilde{\varphi}^{\pm}$ these constraints are « diagonalized » i. e. the two components are independent (This amounts to choosing functions which are even and odd, respectively, in the variable φ used in equation (3.2) in [*2*]).

In terms of the functions $\varphi^{1,2}(\lambda)$ or $\varphi^{\pm}(\lambda)$ these conditions at $q = \pm\infty$ are directly related to the analyticity properties discussed in § 4.C and § 7.A. This is easily seen by inserting (7.2) in (8.3). Evidently the residues of the poles of $\varphi^1(\lambda)$ and $\varphi^2(\lambda)$ are related in a very inconvenient way. In view of these difficulties, it is probable that the discussion of § 7.A gives the simplest description of the differentiable vectors in the basis $|j, \lambda\rangle$.

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