ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 12, nº 4 (1970), p. 323-328
<http://www.numdam.org/item?id=AIHPA_1970__12_4_323_0>

Ann. Inst. Henri Poincaré,
Vol. XII, n° 4, 1970, p. 323-328.

Section A :
Physique théorique.

Monotone convergence in partially ordered vector spaces

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ABSTRACT. — In the operational approach to the theory of statistical physical systems, the set of states may be regarded as a cone in a real vector space. The existence of a strength functional on the set of states provides the cone with a base which in turn endows the vector space with a semi-norm. The assumption that countable mixtures of states exist is equivalent to the assumption that bounded monotone increasing sequences in the cone have unique limits in the cone. Under this assumption it is shown that the semi-norm is in fact a norm with respect to which the vector space is complete thus justifying an assumption made by Davies and Lewis.

RÉSUMÉ. — *La convergence filtrante croissante dans des espaces vectoriels partiellement ordonnés.* — Dans l'approche opérationnelle à la théorie des systèmes physiques statistiques, on peut considérer l'ensemble des états comme un cône dans un espace vectoriel réel. L'existence d'une forme qui mesure la force d'un état fournit le cône d'une base qui, à son tour, donne à l'espace vectoriel une semi-norme. Le postulat qu'il existe des mélanges dénombrables d'états équivaut à l'hypothèse que chaque suite bornée filtrante croissante dans le cône a une seule limite dans celui-ci.

Partant de cette hypothèse on démontre que la semi-norme est en fait une norme par rapport à laquelle l'espace vectoriel est complet, celui-ci justifie une hypothèse de Davies et de Lewis.

1. INTRODUCTION

In the operational approach to the theory of statistical physical systems inaugurated by Haag and Kastler [5] and formulated by Davies and Lewis [1], the states of the system are represented by elements of a generating cone K for a partially ordered vector space V. Further, it is supposed that there exists an affine functional e on K such that $e(0) = 0$ which measures the strength of any state. e clearly extends to a strictly positive linear functional on V. It follows from results of Ellis [2] that the set

$$\{ f : f \in \mathrm{K}, e(f) = 1 \}$$

forms a base B for K in the sense that each element g of K, $g \neq 0$, has a unique representation $g = \alpha f$ with $\alpha > 0$, $f \in \mathrm{B}$. B is defined to be the set normalised states of the system. The Minkowski functional corresponding to the convex hull S of $\mathrm{B} \cup -\mathrm{B}$ defines a semi-norm $\| . \|_{\mathrm{B}}$ on V. For $f \in \mathrm{V}$,

$$\| f \|_{\mathrm{B}} = \inf \{ \lambda > 0 : f \in \lambda \mathrm{S} \}.$$

Clearly, for $f \in \mathrm{K}$, $\| f \|_{\mathrm{B}} = e(f)$. Without further discussion Davies and Lewis supposed that $\| . \|_{\mathrm{B}}$ was in fact a norm on V with respect to which V is complete. Moreover, they also supposed that K was closed in the norm topology of V. Notice that, following Ellis, when $\| . \|_{\mathrm{B}}$ is a norm for V, (V, B) is said to be a base norm space, and this is the case if and only if S is linearly bounded. When (V, B) is a base norm space, it is clear that S contains the open unit ball and is contained in the closed unit ball though S coincides with the closed unit ball when S is linearly compact. The purpose of this note is to examine more closely the assumptions of Davies and Lewis. The question which is answered is the following. What conditions can be placed on the set of states K in order that the assumptions of Davies and Lewis hold, and have these conditions any physical justification.

In previous axiomatic theories of quantum systems (see Mackey [6]), one assumption which has been made is that it is possible to form countable mixtures of states. Whilst this assumption is of a mathematical rather than a physical nature it is consistent with the assumption that probabilities can take real rather than rational values. In the approach described above this assumption reduces to the following. Let $\{f_n\}$ be a countable collection of elements of K such that for $m \geqslant n, f_m - f_n \in K$. Alternatively $\{f_n\}$ may be described as a monotone increasing sequence of elements of K. Suppose that for each f_n, $e(f_n) \leqslant k$. Then, there exists uniquely $f \in K$ such that $f - f_n \in K$ and $\lim e(f_n) = e(f)$. Equivalently one can postulate that every bounded monotone increasing sequence in K has a unique limit in K. Under this assumption it will be shown that (V, B) is a complete base norm space thus justifying one of the assumptions of Davies and Lewis.

A precise statement of the main theorem is the following.

THEOREM. — *Let* K *be a generating cone for the real vector space* V *and let* B *be a base for* K. *Suppose that for each monotone increasing sequence* $\{f_n\} \in K$, *such that* $e(f_n) \leqslant k$, *some finite* k, *there exists uniquely* $f \in K$ *with* $f - f_n \in K$ *and* $\lim e(f_n) = e(f)$. *Then the mapping* $\|\,.\,\|_B$ *defined for* $g \in V$ *by*

$$\| g \|_B = \inf \{ \lambda > 0 : g \in \lambda S \}$$

where $S = \text{conv}(B \cup -B)$, *is a norm for* V *with respect to which* V *is complete.*

An alternative point of view to that adopted here may be taken. One can suppose that the basic object is not K but B, the normalised states of the system. Then the assumption made in the theorem above is equivalent to one which supposes that countable convex combinations of elements of B exist and lie in B. A complete theory of such « σ-convex sets » can be developed. For details the reader is referred to Gerzon [*4*].

The theorem above does not provide a complete answer to the question posed earlier, since it need not follow that the K is closed for the norm $\|\,.\,\|_B$. However, a result of Ellis [*3*] shows that the norm closure $\overline{K}$ of K is a cone in V having the base $\overline{B}$. Moreover, $(V, \overline{B})$ is a complete base norm space with norm coinciding with the norm in (V, B). It follows therefore, that the assumptions of Davies and Lewis are valid providing one is prepared to add some possibly non-physical states to K in order to form $\overline{K}$.

2. PROOF OF THEOREM

$\|\,.\,\|_B$ is a semi-norm on V and so in order to show that $\|\,.\,\|_B$ is a norm on V it remains to prove that for $f \in V$, $\| f \|_B = 0$ implies that $f = 0$. Suppose $f \in V$, $\| f \|_B = 0$, $f \neq 0$. Then, for $n = 0, 1, 2, \ldots,$

$f \in 2^{-n}S$ which implies that there exist g_n, $h_n \in B$, $t_n \in [0, 1]$ such that

$$f = 2^{-n}(t_n g_n - (1 - t_n)h_n).$$

Hence,

$$e(f) = 2^{-n}(2t_n - 1)$$

since $e(g) = 1$ for $g \in B$, which implies that

$$| e(f) | < 2^{-n}.$$

Since this result holds for all non-negative integers n, it follows that $e(f) = 0$ and hence that $t_n = \frac{1}{2}$. Hence,

$$2f = 2^{-n}(g_n - h_n).$$

Let

$$a_N = \sum_{n=1}^{N} 2^{-n} g_n.$$

Then, $\{ a_N \}$ is monotone increasing and lim $e(a_N) = 1$. It follows that there exists uniquely $a \in K$ such that $e(a) = 1, a \geqslant a_N$. Hence $a \in B$ and

$$\begin{aligned} a &= a - a_N + \sum_{n=1}^{N} 2^{-n}(g_n - h_n) + \sum_{n=1}^{N} 2^{-n}h_n \\ &= (a - a_N) + 2Nf + \sum_{n=1}^{N} 2^{-n}h_n \end{aligned}$$

Hence, $a - 2Nf \in K$ and

$$e(a - 2Nf) = 1 - \sum_{n=1}^{N} 2^{-n} + \sum_{n=1}^{N} 2^{-n} = 1$$

which implies that $a - 2Nf \in B$. For $n \geqslant 2$, let

$$\begin{aligned} b_n &= (1 - n^{-1})a - 2nf \\ &= n^{-1}((n - 2)a + (a - 2n^2 f)) \in K \end{aligned}$$

since from above $a - 2Nf \in K$ for all non-negative integers N, Moreover,

$$b_{n+1} - b_n = (n(n + 1))^{-1}(a - 2n(n + 1)f) \in K$$

for the same reason. Hence $\{ b_n \}$ is a monotone increasing sequence in K and clearly $\lim e(b_n) = 1$. Hence, there exists uniquely $b \in \mathrm{K}$, $b \geqslant b_n$ with $e(b) = 1$. Further, for $n \geqslant 2$,

$$\begin{aligned} b_n &= (1 - n^{-1})a - 2nf \leqslant (1 - (n+1)^{-1})a - 2nf \\ &= b_{n+1} + 2f \\ &\leqslant b + 2f \end{aligned}$$

Moreover, $e(b + 2f) = 1$, and since the limit of the monotone increasing sequence $\{ b_n \}$ is unique it follows that $b = b + 2f$ and hence that $f = 0$. Hence we have arrived at a contradiction. Therefore $\| . \|_{\mathrm{B}}$ is a norm on V and (V, B) is a base norm space.

It remains to show that the space is complete. Let $\{ f_n \} \in \mathrm{V}$ be such that

$$\lim_{\mathrm{N} \to \infty} \sum_{n=1}^{\mathrm{N}} \| f_n \|_{\mathrm{B}} = k < \infty, \qquad \| f_n \|_{\mathrm{B}} \neq 0.$$

Then, for $\varepsilon > 0$, for $n = 1, 2, \ldots$ there exist g_n, $h_n \in \mathrm{K}$ such that

$$e(g_n) + e(h_n) < (1 + \varepsilon) \| f_n \|_{\mathrm{B}}$$

and $f_n = g_n - h_n$. This follows from the definition of $\| f_n \|_{\mathrm{B}}$. Let

$$a_{\mathrm{N}} = \sum_{n=1}^{\mathrm{N}} g_n, \qquad b_{\mathrm{N}} = \sum_{n=1}^{\mathrm{N}} h_n.$$

Then $\{ a_{\mathrm{N}} \}$, $\{ b_{\mathrm{N}} \}$ are monotone increasing sequences and

$$e(a_{\mathrm{N}}) < (1 + \varepsilon)k, \qquad e(b_{\mathrm{N}}) < (1 + \varepsilon)k.$$

Let a, b be the unique elements of K such that $e(a) = \lim e(a_{\mathrm{N}})$, $e(b) = \lim e(b_{\mathrm{N}})$, $a \geqslant a_{\mathrm{N}}$, $b \geqslant b_{\mathrm{N}}$, and let $g = a - b$. Then, a simple calculation shows that there exists an integer N′ such that for $\mathrm{N} \geqslant \mathrm{N}'$,

$$\left\| g - \sum_{n=1}^{\mathrm{N}} f_n \right\|_{\mathrm{B}} < \varepsilon.$$

Hence every absolutely convergent series in (V, B) has a limit in V which proves that (V, B) is complete. This completes the proof of the theorem.

RÉFÉRENCES

[*1*] E. B. Davies and J. T. Lewis, An operational approach to quantum probability *(to appear)*.

[*2*] A. J. Ellis, The duality of partially ordered normed vector spaces, *J. London Math. Soc.*, t. **39**, 1964, p. 730-744.

[*3*] A. J. Ellis, Linear operators in partially ordered normed vector spaces, *J. London Math. Soc.*, t. **41**, 1966, p. 323-332.

[*4*] M. A. Gerzon, Convex sets with infinite convex combinations *(under preparation)*.

[*5*] R. Haag and D. Kastler, An algebraic approach to quantum field theory, *J. Math. Phys.*, t. **5**, 1964, p. 846-861.

[*6*] G. W. Mackey, *Mathematical foundations of quantum mechanics*, New York, Benjamin, 1963.

(Manuscrit reçu le 20 octobre 1969).