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Section A :
Physique théorique.

A model of the gravitational radiation of solids

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ABSTRACT. — The quantized model of a simple cubic lattice serves to study the gravitational radiation resulting from lattice vibrations. It is shown that the series expansion in terms of mass multipole moments has to be replaced by the retarded quadrupole moment if the extensions of the emitting body exceed the wavelength of the radiation. The gravitational analog of the Brillouin effect is analyzed. The results are obtained in a general form not depending explicitly on the crystal model. The intensities are too low for detection at the present stage.

I. INTRODUCTION

A first possible experimental success in the detection of gravitational waves has been announced very recently [*6*]; this experiment which seems to indicate the existence of long wavelength radiation from unidentified stellar sources, is based on a coincidence of signals at distant points. An involved statistical analysis is required to eliminate disturbing factors. We may thus finally be supplied by this method with the long desired proof of the existence of gravitational radiation but we can only expect real progress if either nature bestows upon us a suitable uniform polarized source of radiation or if we succeed to create one through our own efforts.

The significance of the subject suggests that all possible physical processes should be explored; the nature and magnitude of their interaction with the gravitational field and the widest range of possible experimental applicability should be determined. Only a limited amount of work has hitherto been done in this direction. Some of its main aspects are the following:

The gravitational radiation of binary stars [7] and neutron stars [8] is of higher power in the long wavelength domain than their total electromagnetic radiation, but the sources are probably all too remote for observation. Other stellar and planetary sources indicate no better results [9]. The theory of the interaction of gravitational waves with classical elastic bodies was developed by J. Weber [4]. He introduced forced oscillations to obtain coherent emission of a body that is larger than the acoustical wavelength. The thermal radiation of an elastic body was considered by Mironowsky [5]. The quantum features of the elastic body were partly taken into account in this work. B. de Witt and G. Papini (and somewhat earlier, one of the present authors in an unpublished series of seminar lectures) [10] [11] [12] began to analyze the interaction of a superconductor with gravitational fields. Halpern and Laurent discussed the radiation of microscopic systems as molecules, atoms and nuclei [1]. A first attempt was made in this paper to obtain an enhanced rate by stimulated emission. Kopvilem and Negibarov suggested enhancement of the emission by the creation of a superradiant state [13]. Several authors considered the classical gravitational synchrotron radiation. The most up-to-date work on this subject containing all references is probably to be published by I. Khriplovitch. All the above mentioned investigations are based on the linear approximation, which is considered to be a good approximation to the field equations of general relativity. There have been numerous attempts to quantize the gravitational field and to investigate its role in elementary particle physics. These are, however, too remote from our present topic to be considered here. Even the aforementioned publications are quoted only to demonstrate some of the principal features and are far from complete.

A detailed application of the theory of solids to the process of emission of gravitational radiation has not been carried out. The interaction of electromagnetic radiation with solids proves the existence of optical transitions and the Brillouin effect. In both cases an extended crystal contributes coherently to the emission. The gravitational analog of these processes were not considered in the mentioned works of Weber and of Mironowsky. It appears to us that Mironowsky did not take the conservation of crystal momentum (which applies even to a continuum as the limiting case of a crystal lattice) into account. His result in this case should be modified.

We have considered in the present work a simple model of a crystal lattice and investigated its gravitational radiation. The lattice is quantized and the gravitational field remains classical. The quantum of action is occasionally introduced for the energy flux of the gravitational field by dividing the latter by $h\nu$. The pattern follows largely the work by Halpern and Laurent [*1*]. Since one is dealing here with emitting bodies of extensions that are large compared to the wavelength, the multipole approximation cannot be applied. The main result of section II of the present work is that the series expansion in multipole moments has to be replaced by the retarded quadrupole moment if the emitting body is not smaller than the wavelength. Section III presents the crystal model, and section IV its quantization and interaction with the gravitational field. These sections have been expounded in some more detail for readers with limited knowledge of solid state physics. The results appear in a form that show that they apply in greater generality than for the model considered. Discussion of the results and conclusions are presented in section VI. We used, if not otherwise mentioned, units in which $\hbar = c = 1$ and length, time and energy^{-1} are measured in centimetres. The flat space metric is of the signature -2.

II. THE GRAVITATIONAL RADIATION OF AN EXTENDED BODY

The gravitational radiation emitted by the quantum transitions of solids is treated in the semiclassical, linear approximation. The procedure is analogous to that of the transitions of microscopic systems which has been discussed in a previous work [*1*]. The basic facts only are reviewed here and reference is made to section II and the Appendix of the quoted work for more details.

The classical gravitational field is expressed in terms of

$$\mathfrak{g}^{ik} = (-g)^{1/2} g^{ik} \tag{1}$$

where $g^{ik}(x)$ is the contravariant metric tensor and

$$g = \det.(g_{ik}).$$

A coordinate system is adopted for which

$$\mathfrak{g}^{ik}{}_{,k} = 0 \tag{1a}$$

The linear approximation of Einstein's equations relates the $\mathfrak{g}^{ik}$ to the energy-momentum (e. m.) density of matter $T^{ik}(x)$:

$$\Box \mathfrak{g}^{ik} = 16\pi G T^{ik} \tag{2}$$
$$\mathfrak{g}^{ik}{}_{,k} = 0$$

$G = 2{,}6.10^{-66}$ cm^2 is Newton's gravitational constant ($\hbar = c = 1$)

$$\Box = \eta^{ik} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k}$$

The general solutions of eqs. (2) is:

$$\mathfrak{g}^{ik}(x) = \mathfrak{g}^{ik}_{(0)}(x) + 4G \int \frac{[T^{ik}(x')]_{\text{ret}}}{r} d^3x' \quad , \quad r = |\bar{x} - \bar{x}'| \tag{2a}$$

We deal in the following exclusively with the time dependent gravitational field of the source T^{ik} so that we choose $\mathfrak{g}^{ik}_{(0)} = \eta^{ik}$. η^{ik} is the flat space metric tensor with the signature minus two.

The density of the energy flux of the gravitational radiation is expressed in the present approximation in terms of the entities:

$$\begin{aligned}(-g)^{1/2} t_0^\alpha = \frac{1}{64\pi G} \{ & \mathfrak{g}^{rs}{}_{,n} \mathfrak{g}^{uv}{}_{,0} (2\eta^{n\alpha} \eta_{ru} \eta_{sv} - \eta^{n\alpha} \eta_{rs} \eta_{uv} - 4\delta^\alpha_r \delta^n_u \eta_{sv}) \\ & - 2\mathfrak{g}^{\alpha u}{}_{,sm} (\mathfrak{g}^{sm} - \eta^{sm}) \eta_{u0} - 2\mathfrak{g}^{\alpha u}{}_{,s} \mathfrak{g}^{vk}{}_{,m} \eta_{uv} \eta_{k0} \eta^{sm} \\ & - 2\mathfrak{g}^{mu}{}_{,s} \mathfrak{g}^{\alpha s}{}_{,m} \eta_{u0} + 2\mathfrak{g}^{mu}{}_{,s} \mathfrak{g}^{vk}{}_{,m} \eta^{\alpha s} \eta_{uv} \eta_{k0} \}\end{aligned} \tag{3}$$

Greek letters are summed here from one to three and Latin letters from zero to three.

The above expressions are obtained from the exact equations of general relativity by neglecting higher powers of $(\mathfrak{g}^{ik} - \eta^{ik})$ [*2*] (*). The first three terms enclosed by the bracket stem from the canonical part of the e. m. density of the gravitational field and the last four terms from the spin part of the same entity [*2*]. The question of the validity of the present approximation is discussed in some more detail in ref. [*1*]. The gravitational fields considered here are so weak that one may expect that t_0^α indicates the right

(*) The last proofs of reference [*2*] were unfortunately not supplied to the author so that it contains a large number of misprints.

order of magnitude of the energy flux. The relation (1*a*) allows us to express g^{0k} in terms of the $g^{\alpha\lambda}$ ($\alpha, \lambda = 1, 2, 3$). The spatial components $T^{\alpha\lambda}(x)$ of the momentum density of the emitting solid, from which $g^{\alpha\lambda}$ can be determined, are in general not well known; they depend on the nature of the forces acting between the lattice constituents. Reference [*1*] shows how the integral over $T^{\alpha\lambda}(x)$ can be transformed in good approximation into integrals over $T^{00}(x)$ if the extensions of the source are small compared to the wavelength emitted and its motion is nonrelativistic. The latter condition is even fulfilled in the present case; however, the source cannot be considered small compared to the wavelength, as in the multipole expansion. The extensions of the radiating solid may exceed considerably the wavelength of the gravitational radiation. In this case, the method has to be generalized.

Neglecting the interaction with gravitation, the e. m. density of the source of the gravitational field is conserved:

$$T^{ik}{}_{,k} = 0 \text{ therefore } T^{k\alpha}{}_{,\alpha} = -T^{k0}{}_{,0} \text{ and } T^{\alpha\lambda}{}_{,\alpha\lambda} = -T^{\lambda 0}{}_{,\lambda 0} = T^{00}{}_{,00} \quad (2b)$$

The retarded tensor components are :

$$T^{ik}_{R}(x') = T^{ik}(x'_0 - r, \bar{x}') \quad (4)$$

where r is the distance from $\bar{x}'$ to the observation point $\bar{x}$. Therefore:

$$\frac{d}{dx'_\alpha}(T^{ik}_{R}(x')) = T^{ik}(x'_0 - r, \bar{x}')_{,\alpha} - T^{ik}(x'_0 - r, \bar{x}')_{,0}\frac{x'_\alpha - x_\alpha}{r} \quad (4a)$$

The derivatives w. r. t. x_α in the first term on the r. h. side of eq. (4*a*) are assumed in this context to act only on the spatial part $\bar{x}'$ and not on r in the term $x_0 - r$ which stems from the retardation.

Forming the divergence:

$$\frac{d}{dx'_\alpha} T^{i\alpha}_{R}(x') = T^{i\alpha}_{R,\alpha} - T^{i\alpha}_{R,0}\frac{(\bar{x}' - \bar{x})_\alpha}{r} = -(T^{i0}(x'_0 - r, \bar{x}') + T^{in}(x'_0 - r, \bar{x}'))_{,0} \quad (4b)$$

where the index n in T^{in} denotes that component of the direction pointing from $\bar{x}$ to $\bar{x}'$. Accordingly:

$$\frac{d^2}{dx'_\alpha dx'_\lambda} T^{\alpha\lambda}_{R}(x') = (T^{00}_{R} + 2T^{0n}_{R} + T^{nn}_{R})_{,00} \quad (4c)$$

The integrands of the following integrals are divergence expressions so that they can be transformed into surface terms which vanish in case of a spatially restricted source:

$$\text{I.}\qquad \int \frac{d}{dx'_\alpha}\left(\frac{T_R^{k\alpha}}{r}\right)d^3x' = -\int\left(\frac{T_{R,0}^{kn}}{r} + \frac{T_{R,0}^{k0}}{r} + \frac{T_R^{kn}}{r^2}\right)d^3x'$$

$$\text{II.}\qquad \int \frac{d}{dx'_\alpha}\left(x'_\lambda \frac{T_R^{k\alpha}}{r}\right)d^3x' = \int\left\{\frac{T_R^{k\lambda}}{r} - x'_\lambda\left(\frac{T_{R,0}^{kn}}{r} + \frac{T_{R,0}^{k0}}{r} + \frac{T_R^{kn}}{r^2}\right)\right\}d^3x'$$

$$\text{III.}\qquad \int \frac{d}{dx'_\mu}\left(x'_\alpha x'_\lambda \frac{T_R^{k\mu}}{r}\right)d^3x' = \int\left(T_R^{k\alpha}\frac{x'_\lambda}{r} + T_R^{k\lambda}\frac{x'_\alpha}{r}\right)d^3x'$$
$$- \int x'_\alpha x'_\lambda\left(\frac{T_{R,0}^{kn}}{r} + \frac{T_{R,0}^{k0}}{r} + \frac{T_R^{kn}}{r^2}\right)d^3x' \qquad (4d)$$

$$\text{IV.}\qquad \int \frac{d^2}{dx'_\alpha dx'_\mu}\left(\frac{T_R^{\alpha\mu}}{r}\right)d^3x' = \int\left(2\frac{T_{R,00}^{0n}}{r} + \frac{T_{R,00}^{00}}{r} + 2\frac{T_{R,0}^{0n}}{r^2} + \frac{T_{R,00}^{nn}}{r}\right)d^3x'$$
$$+ \int\left(2\frac{T_{R,0}^{nn}}{r^2} + 3\frac{T_R^{nn}}{r^3} - \frac{T_{R\lambda}^{\lambda}}{r^3}\right)d^3x'$$

$$\text{V.}\qquad \int \frac{d^2}{dx'_\alpha dx'_\mu}\left(x'_\lambda \frac{T_R^{\alpha\mu}}{r}\right)d^3x' = \int\left\{x'_\lambda\left(2\frac{T_{R,00}^{0n}}{r} + \frac{T_{R,00}^{00}}{r} + \frac{T_{R,00}^{nn}}{r} + 2\frac{T_{R,0}^{0n}}{r^2}\right.\right.$$
$$\left.\left. + 2\frac{T_{R,0}^{nn}}{r^2} + 3\frac{T_R^{nn}}{r^3} - \frac{T_{R\alpha}^{\alpha}}{r^3}\right) - 2\left(\frac{T_{R,0}^{\lambda n}}{r} + \frac{T_{R,0}^{\lambda 0}}{r} + \frac{T_R^{\lambda n}}{r^2}\right)\right\}d^3x'$$

$$\text{VI.}\qquad \int \frac{d^2}{dx'_\alpha dx'_\mu}\left(x'_\lambda x'_\gamma \frac{T_R^{\alpha\mu}}{r}\right)d^3x' = \int\left\{x'_\lambda x'_\gamma\left(2\frac{T_{R,00}^{0n}}{r} + \frac{T_{R,00}^{00}}{r} + \frac{T_{R,00}^{nn}}{r} + 2\frac{T_{R,0}^{0n}}{r^2}\right.\right.$$
$$\left. + 2\frac{T_{R,0}^{nn}}{r^2} + 3\frac{T_R^{nn}}{r^3} - \frac{T_{R\alpha}^{\alpha}}{r^3}\right) + 2\frac{T_R^{\lambda\gamma}}{r} - 2x'_\lambda\left(\frac{T_{R,0}^{\gamma n}}{r}\right.$$
$$\left.\left. + \frac{T_{R,0}^{\gamma 0}}{r} + \frac{T_R^{\gamma n}}{r^2}\right) - 2x'_\gamma\left(\frac{T_{R,0}^{\lambda n}}{r} + \frac{T_{R,0}^{\lambda 0}}{r} + \frac{T_R^{\lambda n}}{r^2}\right)\right\}d^3x'$$

The summation convention for Greek letters extends from one to three. The derivatives on the l. h. side of eqs. I-IV have been written as total derivatives although they refer only to one component of $\bar{x}$; this serves to indicate that each derivative includes even the space dependent retardation term: $x_0 - r$; this distinguishes the derivatives on the l. h. s. from the derivatives which in eq. (4a) were denoted by a comma. Considering the case of a spatially restricted source one can make use of the fact that the

r. h. sides of eqs. (4*d* II and III) vanish to bring the r. h. side of eq. (4*d* VI) into the form:

$$\int\left\{-2\frac{\mathrm{T}_{\mathrm{R}}^{\lambda\gamma}}{r}+x'_\lambda x'_\gamma\left(2\frac{\mathrm{T}_{\mathrm{R},00}^{0n}}{r}+\frac{\mathrm{T}_{\mathrm{R},00}^{00}}{r}+\frac{\mathrm{T}_{\mathrm{R},00}^{nn}}{r}\right.\right.$$
$$\left.\left.+2\frac{\mathrm{T}_{\mathrm{R},0}^{0n}}{r^2}+2\frac{\mathrm{T}_{\mathrm{R},0}^{nn}}{r^2}+3\frac{\mathrm{T}_{\mathrm{R}}^{nn}}{r^3}-\frac{\mathrm{T}_{\mathrm{R}\alpha}^{\alpha}}{r^3}\right)\right\}d^3x' \quad (4e)$$

or :

$$\int\left\{-\left(x'_\lambda\frac{\mathrm{T}_{\mathrm{R}}^{\gamma n}}{r^2}+x'_\gamma\frac{\mathrm{T}_{\mathrm{R}}^{\lambda n}}{r^2}\right)+x'_\lambda x'_\gamma\left(\frac{\mathrm{T}_{\mathrm{R},0}^{0n}}{r^2}+\frac{\mathrm{T}_{\mathrm{R},0}^{nn}}{r^2}+3\frac{\mathrm{T}_{\mathrm{R}}^{nn}}{r^3}-\frac{\mathrm{T}_{\mathrm{R}\alpha}^{\alpha}}{r^3}\right)\right\}d^3x'$$

even these integrals vanish in case of a localized source.

We shall specialize to the case when $r = |\bar{x}' - \bar{x}|$ is much larger than both, the wavelength of the gravitational radiation as well as the extension of the source. Terms of higher powers of r^{-1} can thus be neglected. The motion of the source is nonrelativistic so that the rest mass of the molecules contributes mainly to T^{00} and the energy density of the fields acting between these molecules can here be neglected compared to it. The contribution from terms of the form $\mathrm{T}^{00}{}_{,00}$ are then in general much greater than from terms of the form $\mathrm{T}^{0n}{}_{,00}$ and $\mathrm{T}^{nn}{}_{,00}$. The first forms in a way the analog of magnetic moments whereas the last constitute some higher moments that have no analog in electromagnetic theory [*1*]. Neglecting of all these smaller terms allows us to transform the radiation field of eq. (2*a*) with the help of eq. (2*b*) and eq. (4*d*, *e*) into:

$$g^{\alpha\mu}(x) = \eta^{\alpha\mu} + 2\mathrm{G}\int x'_\alpha x'_\mu \frac{\mathrm{T}_{\mathrm{R},00}^{00}(x')}{r}\,d^3x' \quad (2c)$$

$\mathrm{T}_{\mathrm{R}}^{00}$ denotes here again the value of $\mathrm{T}^{00}(x')$ retarded w. r. t. the point $\bar{x}$.

One arrives thus at the conclusion that the quadrupole approximation can be replaced by the retarded quadrupole approximation if one deals with an extended body.

III. DETAILS OF THE CRYSTAL MODEL

We consider here the gravitational radiation resulting from the vibrations of crystal lattices. We shall see that it suffices to consider just one such model because the results are of a form that is quite general. The nuclei of the lattice constituents provide the principal contribution to the energy density T^{00}. We assume the validity of the adiabatic approximation for the electron states. The electrons in this approximation follow the motion of the center of gravity of the molecule or atom adiabatically. The validity of this assumption depends on a large ratio of the magnitude of the energy level differences of the electronic states to those of the states of the lattice [3]. The adiabatic approximation results in the existence of a potential energy which governs the motion of the lattice constituents. This potential energy is a function of the position of all the heavy particles of the lattice. Restriction is made here to the case of a Bravais lattice with a single atom as the basis. We assume the existence of periodic boundary conditions with N atoms per periodicity interval. The potential energy $V(\bar{x}_1 \ldots \bar{x}_N)$ can be expanded in powers of the deviations $\bar{u}(n)$ from the mean positions $\bar{R}_n$ of the lattice constituents. Thus for the nth particle:

$$\bar{x}(n) = \bar{R}_n + \bar{u}(n) \tag{1}$$

and the potential energy is:

$$\begin{aligned} V(\bar{x}_1 \ldots \bar{x}_N) = V(\bar{R}_1 \ldots \bar{R}_N) &+ \sum_n V_{n\alpha}(\bar{R}_1 \ldots \bar{R}_n) u^\alpha(n) \\ &+ \frac{1}{2!} \sum_{n,m} V_{n\alpha m\mu}(\bar{R}_1 \ldots \bar{R}_N) u^\alpha(n) u^\mu(n) \\ &+ \frac{1}{3!} \sum_{n,m,r} V_{n\alpha m\mu r\lambda}(\bar{R}_1 \ldots \bar{R}_N) u^\alpha(n) u^\mu(m) u^\lambda(r) + \ldots \end{aligned}$$

(The summation convention for Greek indices extends again from one to three.)

The anharmonic terms of powers higher than the second in u give rise to the phonon-phonon interaction as well as to thermal expansion, etc, We shall later take into account in a phenomenologic way some of the effects which these higher terms produce. Here we shall treat the excitations of the model of a simple cubic lattice in the harmonic approximation. Forces of short range acting between the heavy particles are assumed so that the contributions to V stem mainly from the interaction of neighbouring particles. We include the interaction of nearest, second- and third nearest neighbours (these include all the neighbouring lattice points along the edges of the cubes as well as along their face and body diagonals). The rest of the terms of the potential energy are neglected because of the short range of the forces. We consider saturated units in order to avoid as far as possible long range electromagnetic fields which compete with the gravitational fields. A crystal of the Van der Waals type may be closest to our model. The restriction to the harmonic approximation excludes such effects as thermal expansion; one can in this approximation consider the mean position $\bar{x} = \bar{R}$ of the lattice constituents as the equilibrium position. The potential energy at this position $V(\bar{R}_1 \ldots \bar{R}_N)$ is then a minimum so that

$$V_{n\alpha}(\bar{R}_1 \ldots \bar{R}_N) = (\partial V/\partial x(n)_\alpha)_{\bar{x}(n)=\bar{R}_n} = 0 \tag{1a}$$

The potential energy then is:

$$V(\bar{x}_1 \ldots \bar{x}_N) = \frac{1}{2}\sum_{n,m} V_{n\alpha m\mu}(\bar{R}_1 \ldots \bar{R}_N)u^\alpha(n)u^\mu(m) \tag{1b}$$

Assuming central forces between the lattice points, the periodicity of the lattice allows us to write this in the form:

$$V(\bar{x}_1 \ldots \bar{x}_N) = \frac{1}{2}\sum_{n,m} U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)u^\alpha(n)u^\mu(m) \tag{1c}$$

$$U_{\alpha\mu}(\bar{R}_m - \bar{R}_u) = -F_n(n)$$

where $F_\mu(n)$ is the μ-component of the force acting on the nth particle as a result of a unit displacement from equilibrium position of the mth particle in the α-direction. The potential energy is not altered by displacements of the crystal as a whole so that:

$$U_{\alpha\mu}(\bar{R}_n - \bar{R}_m) = U_{\mu\alpha}(\bar{R}_m - \bar{R}_n) \quad \text{and} \quad \sum_m U_{\alpha\mu}(\bar{R}_n - \bar{R}_m) = 0 \tag{1d}$$

The crystal assumes a state of equilibrium even if it is subjected to a uniform strain. The displacement of the nth lattice point in such a state is proportional to a component of R_n. The equilibrium condition is thus:

$$\sum_n U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)R_n^\alpha = 0 \tag{1e}$$

The invariance of the potential energy w. r. t. rigid rotations of the crystal gives rise to additional conditions between coefficients of different powers of the displacements. To fulfill these rigorously requires terms beyond the harmonic approximation. The conditions for terms up to the second power are in our case:

$$\sum_m U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)\bar{R}_m^\lambda = \sum_m U_{\lambda\mu}(\bar{R}_n - \bar{R}_m)R_m^\alpha \tag{1f}$$

$U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)$ depends only on the difference of the vectors $\bar{R}$ which determine the mean position of the particles forming the lattice; its value is due to the periodicity of the crystal determined by m-n alone.

We denote by $\overset{\mu}{t}$ ($\mu = 1, 2, 3$) three mutually perpendicular vectors each of which is parallel and equal in length to one of the edges of an elementary cube of the lattice. The only coefficients $U_{\alpha\lambda}(\bar{R}_n - \bar{R}_m)$ that do not vanish are:

$$U_{\alpha\alpha'}\left(c\overset{\mu}{t}\right) = -\gamma\delta_{\alpha\alpha'}\delta_{\mu\alpha}$$

$$U_{\alpha\alpha'}\left(c_1\overset{\mu}{t} + c_2\overset{\nu}{t}\right) = \phi[\delta_{\alpha\alpha'}(\delta_{\alpha\mu} + \delta_{\alpha\nu}) + (\delta_{\alpha\mu}\delta_{\alpha'\nu} + \delta_{\alpha\nu}\delta_{\alpha'\mu})c_1c_2]\ (\mu \neq \nu) \quad (2)$$

$$U_{\alpha\alpha'}\left(c_1\overset{1}{t} + c_2\overset{2}{t} + c_3\overset{3}{t}\right) = \psi[\delta_{\alpha\alpha'} + (1 - \delta_{\alpha\alpha'})c_\alpha c_{\alpha'}]$$

$$U_{\alpha\alpha'}(0) = \delta_{\alpha\alpha'}(2\gamma - 8\phi - 8\psi) \geq 0 \qquad \text{(no summation)}$$

No summation is to be performed here over double indices. c, c_i can each assume the values ± 1. The short range of the forces requires that neighbours along the edges of the cube interact much more strongly than neighbours along a face- or body diagonal so that:

$$\gamma > \phi > \psi. \qquad (2a)$$

We shall use the expressions:

$$Y_{\alpha\mu}(\bar{k}) = \sum_n U_{\alpha\mu}(\bar{R}n)e^{-i\bar{k}.\bar{R}_n} \qquad (2b)$$

their values can be expressed in terms of the constants γ, ϕ, ψ and the lattice constant a:

$$\begin{aligned} Y_{\alpha\alpha'}(\bar{k}) = \delta_{\alpha\alpha'} \Big\{ & 4\gamma \sin^2\left(\frac{1}{2}ak_\alpha\right) - 4\phi\Big[\sin^2\frac{1}{2}a(k_\alpha + k_\mu) + \sin^2\frac{1}{2}a(k_\alpha - k_\mu) \\ & + \sin^2\frac{1}{2}a(k_\alpha + k_\lambda) + \sin^2\frac{1}{2}a(k_\alpha - k_\lambda)\Big] - 4\psi\Big[\sin^2\frac{1}{2}a(k_\alpha + k_\mu + k_\lambda) \\ & + \sin^2\frac{1}{2}a(k_\alpha - k_\mu + k_\lambda) + \sin^2\frac{1}{2}a(k_\alpha + k_\mu - k_\lambda) + \sin^2\frac{1}{2}a(-k_\alpha + k_\mu + k_\lambda)\Big]\Big\} \\ & + (1 - \delta_{\alpha\alpha'})\{-4\phi \sin k_\alpha \sin k_{\alpha'} - 8\psi \sin k_\alpha \sin k_{\alpha'} \cos k_\mu\} \\ & = \delta_{\alpha\alpha'}e_\alpha(\bar{k}) + (1 - \delta_{\alpha\alpha'})f_{\alpha\alpha'}(\bar{k}) \end{aligned} \qquad (2c)$$

here: $\alpha \neq \mu \neq \lambda \neq \alpha$.

The matrix $Y_{\alpha\alpha'}(\bar{k})$ assumes thus the form:

$$(Y) = \begin{pmatrix} e_1\,, & f_{12}, & f_{13} \\ f_{12}, & e_2\,, & f_{23} \\ f_{13}, & f_{23}, & e_3 \end{pmatrix} \tag{2c'}$$

The Hamiltonian of our system is:

$$H = \frac{1}{2}\sum_{n,\alpha} \frac{1}{m} p_\alpha(n)^2 + \frac{1}{2}\sum_m U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)u^\alpha(n)u^\mu(m) \tag{3}$$

The equations of motion are:

$$m\ddot{u}^\alpha(n) = -\sum_m U_{\alpha\mu}(R_n - R_m)u^\mu(m) \tag{3a}$$

The angular frequencies of the normal modes of the system are determined by the equations:

$$m\omega^2 u^\alpha(n) = \sum_m U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)u^\mu(m) \tag{3b}$$

The periodicity of the lattice implies that the solutions of these equations are of the form:

$$u^\alpha(n) = y^\alpha(\bar{k})e^{i\bar{k}.\bar{R}_n} \tag{3c}$$

(see ref. [3]) consequently:

$$m\omega^2 y^\alpha(\bar{k})e^{i\bar{k}.\bar{R}_n} = \sum_m U_{\alpha\mu}(\bar{R}_n - \bar{R}_m)y^\mu(\bar{k})e^{i\bar{k}.\bar{R}_m} \tag{3d}$$

expressing $U_{\alpha\mu}$ in terms of $Y_{\alpha\mu}$ eq. (2*b*) this becomes:

$$m\omega^2 y^\alpha(\bar{k}) = Y_{\alpha\mu}(\bar{k})y^\mu(\bar{k}) \tag{3e}$$

Here

$$Y_{\alpha\mu}(-\bar{k}) = Y^*_{\alpha\mu}(\bar{k}) \tag{3f}$$

to obtain real values for $U_{\alpha\mu}$.

m is the mass of the atom or molecule that forms the lattice.

The eigenvalues $m\omega^2$ are found by solving the equation:

$$\det.(Y - m\omega^2 I) = 0 \qquad \text{with} \qquad I_{\alpha\mu} = \delta_{\alpha\mu} \tag{3g}$$

The resulting equation is of the third degree and the solution is a function of the six parameters $e_\alpha(\bar{k}), f_{\mu\lambda}(\bar{k})$. We consider here the particular case of lattice waves propagating in the direction of one of the body diagonals of the elementary cube. The three coefficients $e_\alpha(\bar{k})$ are in this case equal to one value $e(\bar{k})$ and in the same way all the $f_{\mu\lambda}(\bar{k})$ are equal to one value $f(\bar{k})$. There are two mutually perpendicular eigenvectors:

$$\bar{\xi}_1 = \left(\frac{1}{2}\right)^{\frac{1}{2}}(1, -1, 0), \qquad \bar{\xi}_2 = \left(\frac{2}{3}\right)^{\frac{1}{2}}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

with the same eigenvalue:

$$m\omega_{1,2}^2 = e(\bar{k}) - f(\bar{k})$$

and one eigenvector parallel to $\bar{k}$:

$$\bar{\xi}_3 = \left(\frac{1}{3}\right)^{\frac{1}{2}}(1, 1, 1)$$

with the eigenvalue:

$$m\omega_3^2 = e(\bar{k}) + 2f(\bar{k})$$

IV. QUANTIZATION OF THE LATTICE AND THE EMISSION OF GRAVITATIONAL RADIATION

The solutions of the equations (III.3*a*, *e*) constitute lattice waves. The eigenfrequencies are expressed as functions of the wavelength $\lambda = 2\pi/k$ by inserting the expressions of eq. (III.2*c*) for $e(k)$ and $f(k)$. Neglecting for the moment the smaller constants ϕ and ψ one finds:

$$\omega(\lambda) \approx \omega_0 \mid \sin(a\pi/\lambda) \mid \quad , \quad \omega_0 = (4\gamma/m)^{1/2} \tag{1}$$

In solids $\omega(\lambda)$ assumes values for which the phase velocity is smaller by a factor of about 10^{-5} than the velocity of light. The gravitational waves propagate even in the crystal with a velocity that is practically equal to that of light because their interaction with matter is extremely weak. The phase velocity of electromagnetic waves in the crystal is c/n where n is the index of refraction. The large ratio of gravitational to elastic wavelengths prevents an elastic body which vibrates with one given frequency from coherent emission of gravitational radiation if its extensions exceed the order of magnitude of the acoustic wavelength. J. Weber was led by this fact to introduce forced oscillations of his classical elastic body so that its vibrations remain in phase with the gravitational radiation [4]. The difficulty was however apparently not considered by Mironowsky in his semi-classical treatment of the thermal gravitational radiation of an elastic body [5]. A crystal lattice composed of more than one constituent may be capable of optical vibrations that have a phase velocity equal to that of gravitational waves. Optical lattice vibrations of higher frequency and the correct phase velocity are however difficult to excite; electromagnetic waves which may serve for the excitation have themselves been shown to propagate with a phase velocity that is too small. The superposition of two lattice waves of acoustical character with somewhat differing frequencies offers another possibility of generating gravitational radiation. The radiation emitted according to quantum theory is in general not of the same frequency as the vibrations of the source; its frequency is the difference of the frequencies of two different states of the source. The differing assumptions about the process of emission of radiation in classical and quantum theory must yield compatible results in the description of slow vibrations of a macroscopic source. According to the classical theory the position of every small part of the source is known as a function of time. This situation can only be approximated in the quantum theory by superposition of states of different frequencies of vibration. The matrix elements of transition of these different states also superimpose thus approximating the classical result. One has in addition to this case in quantum theory (at least within the harmonic approximation) the case of excitation of lattice waves of a sharply defined frequency. Such vibrations may be excited for example with the help of a laser. We consider here the superposition of two such lattice waves of frequencies ω and ω' and wave vectors $\bar{p}$ and $\bar{p}'$ so that $\bar{p} + \bar{p}' = \bar{k}$ and $\omega + \omega' = \omega''$ are equal to the wave vector and the frequency of a gravitational wave propagating in the z-direction. The quantum theory predicts a transition where one quantum of each of the two states of vibration disappears and gravitational radiation of the frequency ω'' is produced.

The quantization of the lattice vibrations is achieved by replacing the classical entities $p_\alpha(n)$, $u_\alpha(n)$ by Hermitian operators satisfying the commutation relations [4]:

$$[p_\alpha(n), u_\mu(m)] = - i\delta_{\alpha\mu}\delta_{mn} \tag{2}$$

with any two p's and any two u's commuting. One introduces then as new variables the amplitudes of excitation of the normal modes of vibrations of given $\bar{k}$:

$$a_\lambda(\bar{k}) = (2\omega_\lambda(\bar{k}))^{-1/2} \sum_n \bar{\xi}_\lambda(\bar{k}).(\bar{p}(n) - im\omega_\lambda \bar{u}(n))e^{-i\bar{k}.\bar{R}_n} \qquad (2a)$$

which fulfill the commutation relations:

$$[a_\lambda(\bar{k}), a_\mu^+(\bar{k}')] = \delta_{\lambda\mu}\delta_{\bar{k}\bar{k}'} \qquad (2b)$$

with any two a's and any two a^+'s commuting. Then:

$$\begin{aligned}\bar{p}(n) &= \sum_{\bar{k},\lambda} \left(\frac{1}{2} m\omega_\lambda(\bar{k})\right)^{\frac{1}{2}} \{ a_\lambda(\bar{k})\xi_\lambda(\bar{k})e^{i\bar{k}.\bar{R}_n} + a_\lambda^+(\bar{k})\xi_\lambda(\bar{k})e^{-i\bar{k}.\bar{R}_n} \} \\ \bar{u}(n) &= \sum_{\bar{k},\lambda} (2m\omega_\lambda(\bar{k}))^{-\frac{1}{2}} \{ a_\lambda(\bar{k})\xi_\lambda(\bar{k})e^{i\bar{k}.\bar{R}_n} - a_\lambda^+(\bar{k})\xi_\lambda(\bar{k})e^{-i\bar{k}.\bar{R}_n} \}\end{aligned} \qquad (2c)$$

and the Hamiltonian (III.3) becomes:

$$\mathrm{H} = \sum_{\bar{k},\lambda} \omega_\lambda(\bar{k}) \left\{ a_\lambda^+(\bar{k})a_\lambda(\bar{k}) + \frac{1}{2} \right\} \qquad (2d)$$

a normalized state of n quanta (phonons) of polarization λ and momentum $\bar{p}$ as well as n' phonons of polarization λ' and momentum $\bar{p}'$ becomes:

$$(n\,!\,n'\,!)^{-\frac{1}{2}}(a_\lambda^+(\bar{p}))^n(a_{\lambda'}^+(\bar{p}'))^{n'} \mid 0 \rangle = \mid n(\lambda\bar{p})n'(\lambda'\bar{p}) \rangle \qquad (2d')$$

where $\mid 0 \rangle$ is the state of zero phonons.

The classical expression (see eq. II.2c):

$$\mathrm{T}^{00}_{\mathrm{R},00} x_\alpha x_\mu / r \qquad (2e)$$

is in our approximation replaced by the matrix element of the same expression formed out of the corresponding operators between the initial state $\mid n(\lambda, \bar{p})n'(\lambda'\bar{p}') \rangle$ and the final state $\mid (n - 1)(\lambda\bar{p})(n' - 1)(\lambda'\bar{p}') \rangle$. The precise values of $\mathrm{T}^{00}(x)$ depend on the character of the forces acting between the lattice constituents. The overwhelming contribution stems however from the rest masses; the contribution to (2e) due to the rest masses is:

$$m\left(\sum_n x_\alpha(n)x_\mu(n)/r\right)_{\mathrm{R},00} \qquad (2e')$$

One forms thus the matrix element:

$$\langle (n-1)(\lambda\bar{p})(n'-1)(\lambda'\bar{p}') \mid \sum_n u_\alpha(n) u_\mu(n)/r \mid n(\lambda,\bar{p}) n'(\lambda'\bar{p}') \rangle \tag{2f}$$

The vectors $\bar{x}(n)$ of eq. (2e′) are directed from the center of gravity of the crystal to the instantaneous position of the particle in the nth cell (*). One can replace $x_\alpha(n)$ by $u^\alpha(n)$ in eq. (2f) because only the displacements $\bar{u}$ contribute to the expression of eq. (2e′) after the time derivatives are taken.

The shape of the crystal is suitably chosen so that its extensions in one direction let us say the z-direction is much greater than in any direction perpendicular to it. The point P where the gravitational radiation should be observed is then chosen on an axis (z-axis) which is perpendicular to the smallest surface of the radiating body and traverses its center. The distance from the center of the crystal to the observation point P is large compared to its extensions; one can therefore replace in good approximation the factor $1/r$ of eqs. (II.2c and 2e, e′, f) by the constant distance R between P and the center of the crystal. The retardation of different points on one x-y plane of the crystal with respect to P differs the less the larger R is chosen. Points of the same x-y plane thus have due to our choice of the extensions and distances practically the same retardation. Substituting the expression of eq. (2c) for $\bar{u}$ the matrix element eq. (2f) becomes then:

$$\frac{1}{2} \mathrm{R}^{-1} (nn')^{\frac{1}{2}} (\omega(\bar{p})\omega'(\bar{p}'))^{-\frac{1}{2}} \xi^\alpha_\lambda \xi^\mu_{\lambda'} \sum_n \exp i \{ (\bar{p}+\bar{p}').\bar{\mathrm{R}}_n - (\omega+\omega')t \} \tag{2f′}$$

The momenta and frequencies have been chosen in such a way that:

$$|\bar{p}+\bar{p}'| = |\bar{k}| = \omega+\omega' = \omega'' \quad \text{and} \quad \bar{p}+\bar{p}' = \bar{k} \tag{2g}$$

falls in the z-direction. The crystal momenta $\bar{p}, \bar{p}'$ are almost antiparallel because their absolute values are so much greater than that of the wave vector of the gravitational radiation $\bar{k}$. The exponentials of eq. (2f′) is

$$\exp i \{ (\bar{p}+\bar{p}').\bar{\mathrm{R}}_n - \omega'' t \} = \exp - i\omega'' \{ (t - \mathrm{R}_{nz}) \} \tag{2g′}$$

(*) One may of course choose the origin of the coordinate system at a point other than the center of gravity but it can be shown that the difference does not contribute to the time derivative of the mass quadrupole moment.

The retarded expression obtained from (2g') is:

$$\exp - i\omega''(t - r + R_{nz} - R_{nz}) = \exp - i\omega''(t - r) \qquad (2g'')$$

The effect of the retardation is thus compensated by the factor $\exp i(\bar{p}+\bar{p}').\bar{R}$ so that crystal momentum is conserved and the whole crystal can contribute to the emission. This result is exact in the above approximation which is justified by the choice of the dimensions.

V. DETAILS ON THE EMISSION OF GRAVITATIONAL RADIATION

The energy flux density of the gravitational radiation is assumed to be equal to the $t^{\alpha}{}_{0}$-component of the energy-momentum pseudotensor in a coordinate system in which the De Donder condition (II.1a) holds [1]. Expressed in terms of

$$\varphi^{ik} = \mathfrak{g}^{ik} - \eta^{ik} \qquad (1)$$

this condition reads:

$$\varphi^{jk}{}_{,k} = 0 \qquad (1a)$$

The source of the gravitational fields is of the time dependence: $e^{-i\omega'' t}$ with $\omega'' = \omega + \omega'$ and thus because of $\varphi^{j\alpha}{}_{,\alpha} = -\varphi^{j0}{}_{,0}$:

$$\varphi^{j\alpha}{}_{,\alpha} = - i\omega''\varphi^{j0} \qquad \text{and} \qquad \varphi^{\gamma\alpha}{}_{,\gamma\alpha} = - \omega''^2\varphi^{00}{}_{,00} \qquad (1b)$$

Considering the harmonic time dependence and the retardation $\varphi^{\alpha\gamma}(x)$ is according to eq. (II.2c) an expression of the general form :

$$\varphi^{\alpha\gamma}(x) = \int \frac{Q^{\alpha\gamma}(\bar{x}')}{|\bar{x} - \bar{x}'|} e^{-i\omega''(t - |\bar{x}-\bar{x}'|)} d^3x' \qquad (2)$$

thus

$$\varphi^{\alpha\gamma}{}_{,\mu} = \int Q^{\alpha\gamma}(\bar{x}') \left\{ - \frac{(x - x')_\mu}{|\bar{x} - \bar{x}'|^3} + i\omega'' \frac{(x - x')_\mu}{|x - x'|^2} \right\} e^{-i\omega''(t-|x-x'|)} d^3x' \qquad (2a)$$

The part of $\varphi^{\alpha\gamma}{}_{,\mu}$ that survives at large distances from the source is proportional to $- n_\mu\varphi^{\alpha\gamma}{}_{,0}/r$ where $\bar{n} = (\bar{x}' - \bar{x})/|\bar{x} - \bar{x}'|$. The components of $\bar{n}$ do not change appreciably at different points of the source if the extensions of the source are small compared to the distance R. Specializing to the dimensions and position of the source quoted in section IV one sees that in case of an observation point near the z-axis n_3 is the only significant component of $\bar{n}$. Therefore due to eq. (1a) $\varphi^{\alpha 3}{}_{,3} \approx -\varphi^{\alpha 0}{}_{,0}$ and due to eq. (2a) even:

$$\varphi^{\alpha\gamma}{}_{,3} \approx \varphi^{\alpha\gamma}{}_{,0} = -i\omega''\varphi^{\alpha\gamma} \tag{2b}$$

so that:

$$\varphi^{\alpha 0} \approx -\varphi^{\alpha 3} \quad \text{and} \quad \varphi^{00} \approx -\varphi^{03} \approx \varphi^{33} \tag{2c}$$

and furthermore: $\varphi = \varphi^{n}{}_{n} \approx -(\varphi^{22} + \varphi^{11})$

$$\varphi^{ik}\varphi_{ik} \approx (\varphi^{22})^2 + (\varphi^{11})^2 + 2(\varphi^{12})^2 + 2(\varphi^{23})^2 + 2(\varphi^{31})^2 - 2(\varphi^{02})^2 - 2(\varphi^{01})^2$$

use has again been made of the fact that the distance to the observation point is large compared to the source.

The terms of t_0^3 that remain after cancellations are in this case:

$$t_0^3 \approx -\frac{1}{64\pi G}\left\{2\varphi^{ik,3}\varphi_{ik,0} - \varphi,^3\varphi_{,0}\right\} = \frac{\omega\pi^2}{64\pi G}\left\{(\varphi^{22} - \varphi^{11})^2 + (2\varphi^{12})^2\right\} \tag{3}$$

The real part of the φ^{ik} is to be inserted in this expression. Writing: $\varphi^{ik} = \tilde{\varphi}^{ik}e^{-i\omega''t}$ one obtains the time average of t_0^3:

$$\frac{\omega''^2}{128\pi G}\left\{-(\varphi^{22} - \varphi^{11})^*(\varphi^{22} - \varphi^{11}) + 2\varphi^{12*}\varphi^{12}\right\} \tag{3a}$$

$Q^{\alpha\gamma}$ of eq. (2) is then expressed in terms of the matrix elements of eq. (IV.2f) and the $\varphi^{\gamma\alpha}(x)$ resulting from eq. (II.2c, V.2) and these expressions are inserted into eq. (3a). The resulting flux density in the immediate neighbourhood of the z-axis is:

$$t_0^3(x) \approx G\frac{(\omega + \omega')^6 nn'}{32\pi R^2\omega'^2} \approx G\frac{2nn'\omega^4}{\pi R^2} \approx 2G\frac{EE'\omega^2}{\pi R^2} \quad \text{with } \begin{array}{l} E = n\omega \\ E' = n'\omega' \approx n'\omega \end{array} \tag{3b}$$

Use has been made of the fact that the two phonon frequencies ω and ω' are nearly of the same value and $\omega + \omega' = \omega''$ in order to obtain the second and third expression of eq. (3b) from the first.

The solid angle Ω centered by the z-axis within which the gravitational energy flux density is of the same order of magnitude as that of eq. (3b) is determined by the relative phase of the product of the retardation factor $\exp i\omega''r$ and the phase factor of the gravitational wave: $\exp i\omega''(z - z')$. The wave vector $\bar{k}$ of the gravitational wave and the vector $\bar{n}$ (see eq. (2a)) form a nonvanishing angle if the observation point is sufficiently off the z-axis and this causes a phase difference of the two exponential factors of opposite sign. The phases do not cancel anymore in this case. The maximal phase difference that can be admitted without destructive interference from different sections of the crystal is about $\pi/4$. Thus $\omega''L(1 - \cos\psi)$, $\omega''L\,\frac{1}{2}\psi^2 \leq \pi/4$ where ψ denotes the angle formed by $\bar{k}$ and $-\bar{n}$ and L is the length of the crystal in the z-direction. The solid angle Ω is of the order of $\psi^2 \approx \frac{1}{2}\,\pi(\omega''L)$. The total rate of radiation emitted is thus of the order of $R^2\psi^2 t_0^3 \approx GEE'\omega/L$ or expressed in terms of the energy density $\rho = E/V$ (volume of crystal $V = qL$): $\approx G\rho\rho'\omega q^2 L$.

We have restricted hitherto our considerations to the harmonic approximation. Consideration of terms of the third and higher orders of $\bar{u}$ in the expansion of the potential energy (eq. (III.1)) takes account of the interaction between phonons which results in a limited lifetime of the phonons. The energy of each phonon is thus not precisely determined and there results a finite line width. The line width is in general large compared to the difference in frequency $\omega - \omega' \approx 10^{-5}\omega$ of the two colliding phonon beams which give rise to the gravitational radiation; one may therefore consider two colliding lattice waves of equal frequencies. Frequencies ω and ω' out of the whole domain of frequencies will add up to produce gravitational radiation. The resultant gravitational field will form a pulse of radiation of about equal lifetime as the phonons.

Equation ($3b'$) has been derived for a vanishing angle with respect to the z-axis. Consideration of the phase shift at a small non-vanishing angle may reduce the intensity up to one order of magnitude.

VI. DISCUSSION OF THE RESULTS

The gravitational radiation considered results from the anihilation of two acoustical phonons. The frequencies of the two phonons differ only by a fraction of the order of 10^{-5} and their wave vectors must be almost antiparallel in order to fulfill the law of conservation of crystal momentum. The emission of photons by the anihilation of two phonons is also possible. The difference $|\,\omega - \omega'\,|$ for a given $\omega'' = \omega + \omega'$ must be larger in this case by a factor equal to the inverse index of refraction than in the gravitational case. The most serious competition to the emission of gravitational radiation stems rather from the phonon-phonon interaction than from rivaling photon emission; the latter can be reduced by suitable choice of the material. The phonon-phonon interaction results in a limited phonon life time and by consequence in a frequency spread. This spread exceeds in general the required frequency difference of the two phonon frequencies; one has thus to deal with two phonon beams of the same frequency distribution and opposing directions of propagation. The frequency spread of the gravitational radiation will be of the same magnitude as that of the phonons and so is the pulselength of the gravitational wave resulting from the anihilation of two phonons; one can view the process as the anihilation of two particles of finite life time. The intensity resulting from two phonons of fixed frequencies and crystal momenta remains of the same order of magnitude within a solid angle of about $(\omega'' L)^{-1}$ where L is the length of the crystal contributing coherently to the emission. The rate of emission is proportional to each of the phonon occupation numbers of the two phonon states; it does not depend explicitly on the properties of the crystal. The relation (V.3*b*′) does not contain the mass of the lattice constituents and not even the mass of the whole crystal. The power emitted is thus at its best proportional to one quarter times the square of the kinetic energy of the crystal vibrations. This dependence seems to be in contrast to the relation usually given for the rate of emission of gravitational radiation of nonrelativistic systems; the latter depends on $GM^2L^4\omega^6$ where M is the rest mass of the system and L its maximal displacement from an equilibrium position. Agreement with the relation (V.3*b*′) is, however, obtained in the case of a harmonic motion by expressing a factor ω^4 in terms of the mass M and the centripetal force (which together determine ω). Generally one can show in the case of central forces decreasing with the distance that the second time derivative of the quadrupole moment of a mass point is at most of the same order of magnitude as its kinetic energy. The mass which determines the power of the gravitational radiation resulting from such a nonrelativistic motion is thus that of the kinetic energy of the system and not its rest mass.

The angular distribution of the radiation is peculiar; if the crystal momenta could be adjusted precisely enough emission should occur preponderantly into a small solid angle in one direction only. The order of magnitude of this solid angle is (λ/L) where λ is the wavelength of the gravitational radiation. A shapely focused beam of radiation can in any case be achieved by suitable choice of the crystal shape.

The smallness of the gravitational constant can only be compensated by high energy densities to improve the rate of emission. Expressed in terms of the energy density ρ of the crystal vibrations and the volume V of the crystal the number of quanta emitted per second is according to eq. (V.3*b*′) at most: $5.10^{-7}\rho^2V^2/(L)$; ρ has been expressed here already in Joules per cc. The crystal volume within which the oscillations can be excited is in general limited. The energy density of lattice vibrations of a given frequency can even for a short period not exceed 100 Joules per cc. One graviton per minute should therefore be an upper limit for the rate of emission of one crystal. Such an intensity is however as we shall see much too low for detection. A higher limit of the energy density ρ is obtained for other processes than simple lattice vibrations. Phonon assisted electron transitions allow a considerably higher energy to be stored in the crystal and may possibly also emit gravitational radiation coherently. The gravitational radiation of crystals remains of no practical use unless better methods for the detection are discovered. The relation (V.3*b*) shows that the transition probability is proportional to the phonon occupation numbers n, n'. The detection of one additional phonon becomes however very difficult if these phonon occupation numbers are already high. A crystal is thus not very suitable for the detection of gravitational radiation of high frequency. A different method of detection is the absorption by atomic or molecular quadrupole transitions; it was already mentioned briefly in ref. [*1*]. A large number of molecules which are capable of performing the quadrupole transition in question from their ground state are cooled down so that thermal excitation is practically excluded. Absorption of a graviton results in an excited state which can decay quickly to the ground state via two electric dipole transitions. One of the resulting two photons is likely to be trapped by the other molecules in the ground state whereas the second which is in general of a different frequency may be observed. The absorption cross section of one molecule for gravitational radiation of wavelength λ is $\lambda^2 R$ where R denotes the ratio of gravitational to electromagnetic transition probabilities [*1*]; it is for molecules at most $R = 10^{-33}$. The concentration of the radiation into a narrow beam might facilitate this method of detection somewhat. A volume V of matter consisting of such molecules would require a flux of $(\lambda^2 V)^{-1}.10^5$ gravitons/cm$^2\cdot$sec to achieve one absorption per day. We do not believe that any human effort could at present lead to the successful performance of an experiment of this kind but possibility of its realization may not be too remote.

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