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**Possible derivation
of some $SO(p, q)$ group representations
by means of a canonical realization
of the $SO(p, q)$ lie algebra**

by

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ABSTRACT. — Starting from the usual realization of the inhomogeneous $SO(p-1, q-1)$ algebra in terms of quantum mechanical operators, we get a realization of the $SO(p, q)$ algebra by introducing supplementary variables and arbitrary parameters. In this way we show how we can derive some representations of the $SO(p, q)$ group ($pq \neq 0$).

SOMMAIRE. — En partant de la réalisation habituelle de l'algèbre de Lie du groupe inhomogène $SO(p-1, q-1)$ écrite en termes d'opérateurs de la mécanique quantique, nous obtenons une réalisation de l'algèbre de Lie des groupes de rotation non compacts $SO(p, q)$ par addition de constantes arbitraires et de variables canoniques supplémentaires. Nous montrons alors comment on peut déduire de cette façon certaines représentations des groupes $SO(p, q)$ ($pq \neq 0$).

I. — INTRODUCTION

In this note, we investigate a realization of the Lie algebra of the non-compact rotation groups $SO(p, n-p)$ ($n > p, p \neq 0$) in terms of quantum mechanical operators. Writing the elements of the inhomogeneous

$SO(p-1, n-p-1)$ subalgebra as functions of the canonical variables x_i and p_i ($i = 1, 2, \dots, n-2$), we could express all the $SO(p, n-p)$ generators in terms of the x_i 's and p_i 's. But in this way we derive only some representations of $SO(p, n-p)$. Indeed, if we want to get all the unitary irreducible ones, we know [1] that the $SO(p, n-p)$ Lie algebra must be expressed with

$$(1) \quad N = 1/2 \{ n(n-1)/2 - [n/2] \}$$

pairs of conjugate variables and $[n/2]$ arbitrary parameters [2] so that in the general case we have to use, in addition to the x_i 's and p_i 's new canonical variables [3].

In fact, we show in Section II that we have to consider two cases, one when $n \leq 4$, the other when $n > 4$. In this later case, we derive the general expression of the $SO(p, n-p)$ generators and deduce in Section III the most degenerate series of representations.

In Section IV, we investigate the case $n \leq 4$ and show as an example how we can derive the representations of the Lorentz group.

II. — THE REALIZATION OF THE $SO(p, n-p)$ ALGEBRA ($n > 4$)

Let us consider the $SO(p, n-p)$ group for which the generators $M_{\mu\nu}$ satisfy the following commutation rules:

$$(2) \quad [M_{\mu\nu}, M_{\rho\sigma}] = i \{ g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho} \}$$

where $\mu, \nu, \rho, \sigma = 1, 2, \dots, p, p+1, \dots, n$, and $g_{\mu\mu} = 1$ if $\mu \leq p$, $g_{\mu\mu} = -1$ if $\mu \geq p+1$, $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

We know that the generalized Wigner rotations [5] [6] leaving invariant a given light-like vector, generate a subgroup of $SO(p, n-p)$, isomorphic to the inhomogeneous $SO(p-1, n-p-1)$ group. The generators of this subgroup expressed in terms of the $M_{\mu\nu}$ are chosen to be [7]

$$(3) \quad \Gamma_i^- = M_{ip} - M_{in} \quad \text{and} \quad M_{ij}$$

with [8] $i, j = 1, 2, \dots, p-1, p+1, \dots, n-1$.

We can enlarge this group by adding a supplementary generator namely the dilatation operator D and get in this way the similitude group which

we define as the group leaving invariant *the direction* of a given light-like vector. It is easily seen that the dilatation operator D is

$$(4) \quad D = M_{pn}$$

Moreover, we put

$$(5) \quad \Gamma_i^+ = M_{ip} + M_{in}$$

which can be interpreted as the « special conformal transformations » acting in an euclidian space of signature $(p-1, n-p-1)$. The commutation rules between the generators (3) (4) (5) are

$$(6) \quad (a) \quad [D, M_{ij}] = 0 \quad (b) \quad [D, \Gamma_i^\pm] = \mp i\Gamma_i^\pm$$

$$(7) \quad [M_{ij}, M_{kl}] = i \{ g_{il}M_{jk} + g_{jk}M_{il} - g_{ik}M_{jl} - g_{jl}M_{ik} \}$$

$$(8) \quad [M_{ij}, \Gamma_i^\pm] = i(g_{jk}\Gamma_i^\pm - g_{ik}\Gamma_j^\pm)$$

$$(9) \quad [\Gamma_i^\pm, \Gamma_j^\pm] = 0$$

$$(10) \quad [\Gamma_i^-, \Gamma_j^+] = -2i(M_{ij} + g_{ij}D)$$

Let us choose now the usual canonical realization of the inhomogeneous $SO(p-1, n-p-1)$ subalgebra, namely

$$(11) \quad M_{ij} = x_i p_j - x_j p_i$$

$$(12) \quad \Gamma_i^- = p_i$$

with

$$[x_i, p_j] = -ig_{ij}$$

Following the comment of Section I, we are led to introduce in addition to the $(n-2)$ pairs of variables x_i, p_i

$$(13) \quad N_\xi = N - (n-2) = 1/2 \{ n(n-1)/2 - [n/2] - 2(n-2) \}$$

new pairs of canonical variables which we shall denote by ξ_μ and π_μ with $[\xi_\mu, \pi_\nu] = i\delta_{\mu\nu}$. The values of N and N_ξ [Formulas (1) and (13)] with respect to the type of non-compact rotation group are given in table I. Let us note that for $n \leq 4$ only, the number of variables x_i, p_i introduced in the generators (11) (12) of the inhomogeneous subgroup is sufficient to get all the representations of $SO(p, n-p)$. So, we have to consider as special

cases the groups for which $n \leq 4$, and we treat them in a complete way in Section IV.

Now, let us give the expression of the dilatation operator (4). From (6(a)) and (6(b)), we deduce that if $n \leq 4$, the only possible expression for D is $-\frac{1}{2} g_{ij}(x_i p_j + p_j x_i)$. In the general case $n > 4$, we shall add the term $1/2(\vec{\xi} \cdot \vec{\pi} + \vec{\pi} \cdot \vec{\xi})$ where $\xi_\mu \pi_\mu$ are conjugate variables with $\mu = 1, 2, \dots, N_\xi$. Finally, the expression of D is chosen to be:

$$(14) \quad D = 1/2 g_{ij}(x_i p_j + p_j x_i) + 1/2 (\vec{\xi} \cdot \vec{\pi} + \vec{\pi} \cdot \vec{\xi})$$

From (8), we see that the Γ_i^+ defined in (5) transform like a vector under the M_{ij} and then are of the type $S p_i + S' x_i$ where S and S' are scalar functions. It is easy to determine the expression of these two quantities with the aid of (6(b)) and (10). Finally, we are led to

$$(15) \quad \Gamma_i^+ = -1/2 [p_i, g_{jk} x_j x_k]_+ - [x_i, D]_+ - f(\vec{\xi}, \vec{\pi}) p_i$$

which satisfy (9), the only condition on $f(\vec{\xi}, \vec{\pi})$ being

$$(16) \quad [D, f(\vec{\xi}, \vec{\pi})] = -2if(\vec{\xi}, \vec{\pi})$$

So, the $[n/2]$ arbitrary parameters we have to introduce must be put in $f(\vec{\xi}, \vec{\pi})$. For instance, it can be shown that this function is of type $A \xi^2 + A' [\xi^2, \pi^2]_+$ for the SO(4, 1) and SO(3,2) groups [4], the vector ξ being one-dimensional and A, A' two arbitrary real parameters (see table I).

TABLE I

Groups	SO(2,1)	SO(3,1) SO(2,2)	SO(4,1) SO(3,2)	SO(4,2) SO(5,1)	SO(p, n-p)	
					n even > 4	n odd > 4
rank	1	2	2	3	$\frac{n}{2}$	$(n-1)/2$
N	1	2	4	6	$n(n-2)/4$	$[(n-1)/2]^2$
N_ξ	0	0	1	2	$(n-2)(n-4)/4$	$[(n-1)(n-5)/2] - 1$

III. — THE MOST DEGENERATE REPRESENTATIONS OF $SO(p, n - p)$ FOR $n > 4$

If we want to find the most degenerate series of representations of $SO(p, n - p)$, it is not useful in the general case to introduce the required number of variables of type ξ but only one and one arbitrary parameter. So, we suppose that in (14) and (15) we introduce only one pair of variables ξ, π and take $f(\xi, \pi)$ in the following form

$$(17) \quad f(\xi, \pi) = -A^2 \xi^2$$

which satisfy (16). A being an arbitrary real number.

Then, all the invariant operators are fixed except the second order Casimir operator:

$$(18) \quad c_2 = \frac{1}{2} g_{ij} g_{kl} M_{ik} M_{lj} - D^2 + \frac{1}{2} g_{ij} (\Gamma_i^+ \Gamma_j^- + \Gamma_j^- \Gamma_i^+)$$

which becomes

$$(19) \quad c_2 = -\frac{1}{4} (\xi \pi + \pi \xi)^2 + A^2 \xi^2 g_{ij} p_i p_j - \left(\frac{n-2}{2} \right)^2$$

By solving the eigenvalue problem, we are led to the following results: if $A \neq 0$, we get the discrete energy spectrum for which the eigen-functions are of type

$$(20) \quad \Phi(x_i, \xi) = e^{ig_{ij} p_i x_j} \xi^{-1/2} J_\nu(mA\xi)$$

corresponding to the eigen-values

$$(21) \quad c_2 = \nu^2 - \left(\frac{n-2}{2} \right)^2$$

$g_{ij} p_i p_j = m^2$ being the translational invariant operator of the inhomogeneous $SO(p-1, n-p-1)$ subgroup, and J_ν a Bessel function of which the order ν is an integer when n even and half integer when n is odd.

If $A = 0$, the eigenfunctions Φ are of the form:

$$(22) \quad \Phi = e^{ig_{ij} p_i x_j} \xi^{\alpha-1/2}$$

where α is a number. The eigen-values of the Casimir operator (19) becomes:

$$(23) \quad c_2 = \alpha^2 - \left(\frac{n-2}{2} \right)^2$$

and then we get the continuous spectrum. We see that the case for which no variable of type ξ is introduced correspond to a particular value of (23), namely $c_2 = -[(n-2)/2]^2$.

Of course, these results are in accordance with those of Raczk Niedler Limic [9] and L. Castell [10] who obtains them in the case of the conformal group $SO(4,2)$ by an analogous way.

IV. — THE REALIZATION OF THE $SO(p, n-p)$ LIE ALGEBRA WHEN $n \leq 4$

The case of the $SO(2,1)$ group has been investigated in paper of ref [11] and we do not talk about.

So, let us consider the $SO(3,1)$ and $SO(2,2)$ groups for which we have to introduce two arbitrary parameters in their Lie algebra and no supplementary variable of type ξ (see table I). In the $SO(3,1)$ case, a simple calculus leads to the following expressions of D and $\Gamma_i^\pm$:

$$\begin{aligned}
 D &= \frac{1}{2} [x, p_x]_+ + \frac{1}{2} [y, p_y]_+ \\
 (24) \quad \Gamma_1^- &= p_x + \frac{Ax + By}{x^2 + y^2} \quad \Gamma_1^+ = \frac{1}{2} [p_x, x^2 + y^2]_+ - [x, D]_+ + Ax - By \\
 \Gamma_2^- &= p_y + \frac{Ay - Bx}{x^2 + y^2} \quad \Gamma_2^+ = \frac{1}{2} [p_y, x^2 + y^2]_+ - [y, D]_+ + Ay + Bx
 \end{aligned}$$

where A and B are real arbitrary parameters.

By analytic continuation, we get the $SO(2,2)$ case, namely

$$\begin{aligned}
 D &= \frac{1}{2} [x, p]_+ - \frac{1}{2} [x_0, p_0]_+ \\
 (25) \quad \Gamma_0^- &= p_0 + \frac{\alpha x_0 - \beta x}{x^2 - x_0^2} \quad \Gamma_0^+ = \frac{1}{2} [p_0, x^2 - x_0^2]_+ - [x_0, D]_+ + \alpha x_0 + \beta x \\
 \Gamma_1^- &= p + \frac{\alpha x - \beta x_0}{x^2 - x_0^2} \quad \Gamma_1^+ = \frac{1}{2} [p, x^2 - x_0^2]_+ - [x, D]_+ + \alpha x + \beta x_0
 \end{aligned}$$

α being real and β pure imaginary. The angular momentum J_3 is taken in its usual form.

The fundamental invariants take the values:

for $SO(3,1)$

$$(26) \quad \begin{aligned} F &= \vec{J}^2 - \vec{K}^2 = B^2 - A^2 - 1 \\ W &= \vec{J} \cdot \vec{K} = -AB \end{aligned}$$

for $SO(2,2)$

$$(27) \quad \begin{aligned} F &= \beta^2 - \alpha^2 - 1 \\ W &= -\alpha\beta \end{aligned}$$

The Hilbert space of representations will be given by solving the eigenvalue problem

$$(28) \quad \vec{\Gamma}^{-2} \Phi = c \Phi$$

$\vec{\Gamma}^{-2}$ being the translational invariant operator of the inhomogeneous subgroup. Thus, we shall get conditions on the parameters A , B and α , β .

For example, let us point out the possible derivation of the Lorentz group representations. In this case, the expression of $\vec{\Gamma}^{-2}$ becomes

$$(29) \quad \vec{\Gamma}^{-2} = p^2 + \frac{2A}{x} \frac{\vec{x} \cdot \vec{p}}{x} + \frac{A^2 + B^2}{x} - \frac{2B}{x} J_3$$

Solving the equation (28), we find the general solution

$$(30) \quad \Phi = e^{im\varphi} (r\sqrt{c})^{-iA} J_\nu(r\sqrt{c})$$

where (r, φ) are the polar coordinates in the plane. J_ν is a Bessel function of arbitrary real order ν and m the eigenvalues of the angular momentum ($m = 0, \pm 1, \pm 2, \dots$). The parameter B is related to the order ν by

$$(31) \quad \nu^2 = (m + B)^2$$

The constant A is left arbitrary. A more detailed discussion in which the construction of an orthonormal basis is considered, would give the usual values [12] of the invariants (26).

V. — CONCLUSION

We have shown that, starting with the usual canonical realization of the inhomogeneous $SO(p-1, n-p-1)$ group, we can get the $SO(p, n-p)$ Lie algebra with which it should be possible to reduce the irreducible unitary representations of the $SO(p, n-p)$ group with respect to the inhomogeneous subgroup ones. In particular, this is of physical interest for the conformal group $SO(4,2)$ since it contains as inhomogeneous subgroup the Poincaré group.

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