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Section A :
Physique théorique.

Relations between « Inner » and « Outer » multiplicities for the classical groups (*)

by

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ABSTRACT. — For the groups SU(n + 1), SO(2n + 1), Sp(2n), SO(2n) and G_2 two relations are derived between the (« inner ») multiplicities of the weights contained in irreducible representations forming a direct product and the (« outer ») multiplicity of an irreducible representation contained in the Clebsch-Gordan series. The second relation relates the multiplicity of one of the irreducible representations forming the direct product to the outer multiplicity of the Clebsch-Gordan series and is therefore well suited to calculate the outer multiplicity from the inner multiplicity. In the appendix an example is given for SU(3), SU(6) and SU(12) respectively.

1. Using Weyl's formula, the fact that a weight-diagram is left invariant under any operation S of the Weyl group W and the ortogonality of the trigonometric functions, relations can be derived between the inner multiplicities [*1*] $\gamma(\overline{m})$ of weights $\overline{m}$ of irreducible representations forming a direct product and the outer multiplicity [*1*] $\overline{\gamma}(m)$ of an irreducible representation D(m) with highest weight m contained in that Clebsch-Gordan series.

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The relations hold for all the classical groups $SU(n+1)$, $SO(2n+1)$, $Sp(2n)$, $SO(2n)$ as well as for the exceptional group G_2.

From the said it is clear that these relations can be used to calculate the Clebsch-Gordan series for these groups if the inner multiplicities of the irreducible representations are known. In fact it will turn out that for the second relation the inner multiplicities $\gamma(\overline{m})$ of only one of the two irreducible representations forming the direct product has to be known. The relations are analogous to Steinbergs formula [2], the second having the advantage that the summation involved is considerably simpler [3]. In the appendix this second relation is used to calculate examples for SU(3), SU(6) and SU(12).

2. Let $\chi(m)$ denote the character of the irreducible representation having as highest weight m, then, as is well known [4]

$$\chi(m)\chi(m') = \sum_{m''} \overline{\gamma}(m'')\chi(m''), \tag{1}$$

where $\overline{\gamma}(m'')$ is the outer multiplicity, i. e. $\overline{\gamma}(m'')$ is the number of $\chi(m'')$ on the right side of (1). The $\chi(m)$ is left invariant under the application of any $S \in W$, S being an element of the Weyl group W, i. e.,

$$S\chi(m) = \chi(m). \tag{2}$$

This since in

$$\chi(m) = \sum_{\overline{m}' \in D(m)} \gamma(\overline{m}')e^{i(\overline{m}',\varphi)} \tag{3}$$

and in

$$S\chi(m) \equiv \sum_{\overline{m}' \in D(m)} \gamma(S\overline{m}')e^{i(S\overline{m}',\varphi)} \quad , \quad \gamma(S\overline{m}') = \gamma(\overline{m}'), \tag{4}$$

for any given $S \in W$ the right sides are the same. The summation in eq. (3) and eq. (4) goes over all weights $\overline{m}'$ of the irreducible representation $D(m)$ with highest weight m. Therefore, using eq. (2), eq. (1) can be rewritten as

$$S\chi(m)S\chi(m') = \sum_{m''} \overline{\gamma}(m'')S\chi(m'') \quad , \quad S \in W. \tag{5}$$

Now Weyl's formula is [5]

$$\chi(m)\sum_{S \in W} \delta_s e^{i(SR_0,\varphi)} = \sum_{S \in W} \delta_s e^{i(S(m+R_0),\varphi)} \tag{6}$$

where R_0 is half the sum over the positive roots, $\delta_s = -1$ is $S \in W$ is a reflection, $\delta_s = 1$ otherwise; or equivalently, using eq. (2),

$$\sum_{S \in W} \delta_s S\chi(m) e^{i(SR_0,\varphi)} = \sum_{S \in W} \delta_s e^{i(S(m+R_0),\varphi)}. \tag{7}$$

Eq. (5) holds for any $S \in W$, therefore it is also true that

$$\sum_{S \in W} \delta_s S\chi(m) S\chi(m') e^{i(SR_0,\varphi)} = \sum_{S \in W} \sum_{m''} \delta_s \bar{\gamma}(m'') S\chi(m'') e^{i(SR_0,\varphi)}, \tag{8}$$

or equivalently

$$\sum_{S \in W} \delta_s \sum_{\bar{m} \in D(m)} \sum_{\bar{m}' \in D(m')} \gamma(\bar{m})\gamma(\bar{m}') e^{i(S(\bar{m}+\bar{m}'+R_0),\varphi)} = \sum_{m''} \bar{\gamma}(m'') \sum_{S \in W} \delta_s e^{i(S(m''+R_0),\varphi)}, \tag{9}$$

using eq. (3) and eq. (7).

For SU$(n+1)$ the components m_i of any weight

$$m = (m_1, m_2, \ldots, m_{n+1}), \quad \sum_{i=1}^{n+1} m_i = 0,$$

are of the form integer/$n+1$, for SO$(2n+1)$, Sp$(2n)$ and SO$(2n)$ the components of a weight m are eigher all integers or all half-integers, and if $m' + m + R_0$ is a weight having integer (half-integer) components, so are $m'' + R_0$ and all weights $\bar{m}' + \bar{m} + R_0$, and for G_2 the components of a weight

$$m = (m_1, m_2, m_3), \quad \sum_{i=1}^{3} m_i = 0,$$

are integers.

Therefore, if for the SU$(n+1)$ groups the ortogonality relations

$$\frac{1}{2(n+1)\pi} \int_0^{2(n+1)\pi} e^{i\frac{1}{n+1} n_k \varphi_k} e^{-i\frac{1}{n+1} n'_k \varphi_k} d\phi_k = \delta_{n_k, n'_k} \tag{10}$$

(no summation over k ; n_k, n'_k integers), for each component $n_k/n+1$ of the weight m are used, and for SO$(2n+1)$, Sp$(2n)$, SO$(2n)$

$$\frac{1}{4\pi} \int_0^{4\pi} e^{im_k \varphi_k} e^{-im'_k \varphi_k} d\phi_k = \delta_{m_k, m'_k}, \tag{11}$$

(m_k, m'_k both integer or half-integer), in the same sense, and for G_2 eq. (10) with $n = 1$, then one obtains from (9) for

$$D(m) \otimes D(m') = \sum_{m''} \bar{\gamma}(m'')D(m'')$$

the relation

$$\bar{\gamma}(m'') = \sum_{S \in W} \sum_{\bar{m} \in D(m)} \sum_{\bar{m}' \in D(m')} \delta_s \gamma(\bar{m})\gamma(\bar{m}')\,\delta_{S(\bar{m}'+\bar{m}+R_0),m''+R_0}, \tag{12}$$

where $\delta_{x,y}$ is the Kroneckersymbol. It should be noted that $\delta_{s(\bar{m}'+\bar{m}+R_0),m''+R_0}$ restricts the sum in such a way that for each $S \in W$ at most over the weights of one of the representations $D(m)$ or $D(m')$ is summed. Eq. (12) is the first relation connecting the inner multiplicities to the outer multiplicity.

3. Formula (12) could be used to calculate the Clebsch-Gordan series from known inner multiplicities. However, in spite of the fact that the Kroneckersymbol reduces the summations, in general still too many terms contribute to allow the formula to be simple. So, another relation between inner and outer multiplicities will be derived, in which even for groups of high rank only a few terms of the sums will contribute and thus make this relation useful for calculating $\bar{\gamma}(m'')$. Again starting from eq. (1) and using eq. (2) one can write

$$\chi(m)S\chi(m') = \sum_{m''} \bar{\gamma}(m'')S\chi(m'') \quad , \quad S \in W, \tag{13}$$

and analogous to eq. (8) one obtains

$$\sum_{S \in W} \delta_s \chi(m)S\chi(m')e^{i(SR_0,\varphi)} = \sum_{S \in W}\sum_{m''} \delta_s \bar{\gamma}(m'')S\chi(m'')e^{i(SR_0,\varphi)}. \tag{14}$$

Then using Weyl's formula, i. e., eq. (7), it follows

$$\chi(m)\sum_{S \in W} \delta_s e^{i(S(m'+R_0),\varphi)} = \sum_{m''} \bar{\gamma}(m'')\sum_{S \in W} \delta_s e^{i(S(m''+R_0),\varphi)}. \tag{15}$$

Eq. (15) is the formula used by the graphical method [6] [7] [8] for decomposition of direct products. If eq. (15) is rewritten as

$$\sum_{S\in W}\sum_{\overline{m}\in D(m)} \delta_s \gamma(\overline{m}) e^{i(\overline{m}+S(m'+R_0),\varphi)} = \sum_{m''}\sum_{S\in W} \delta_s \overline{\gamma}(m'') e^{i(S(m''+R_0),\varphi)} \tag{16}$$

and again the orthogonality relations, eq. (10) and eq. (11), are used, one obtains for

$$D(m) \otimes D(m') = \sum_{m''} \overline{\gamma}(m'') D(m'')$$

the relation

$$\overline{\gamma}(\overline{\overline{m}}' + m') = \sum_{S\in W}\sum_{\overline{m}\in D(m)} \delta_s \gamma(\overline{m}) \delta_{\overline{m}+S(m'+R_0),\overline{\overline{m}}'+m'+R_0} \quad , \quad \overline{\overline{m}}' \in D(m), \tag{17}$$

where $\delta_{x,y}$ is again the Kronecker-symbol. Therefore

$$D(m) \otimes D(m')$$
$$= \sum_{\overline{\overline{m}}'\in D(m)}\sum_{S\in W}\sum_{\overline{m}\in D(m)} \delta_s \gamma(\overline{m}) \delta_{\overline{m}+S(m'+R_0),\overline{\overline{m}}'+m'+R_0} D(m' + \overline{\overline{m}}') \tag{18}$$

with $D(m' + \overline{\overline{m}}') = 0$ if $m' + \overline{\overline{m}}'$ is not a dominant weight (the $\overline{\gamma}(m' + \overline{\overline{m}}')$ for non dominant weights is in general not zero). In eq. (17) at most the sum over W contributes and out of this sum only those elements $S \in W$ for which the $\overline{m}$ defined through the Kronecker-symbol is a weight of $D(m)$. If in $D(m) \otimes D(m')$ the $D(m)$ is taken to be the « smaller » irreducible representation (in the sense that the length of the vector m is smaller than the length of the vector m'), then in general only a few (*) $S \in W$ (with respect to the order of W) contribute to eq. (17) since for most of the $S \in W$ there exists no weight $\overline{m} \in D(m)$ such that

$$\overline{m} + S(m' + R_0) = \overline{\overline{m}}' + m' + R_0. \tag{19}$$

4. The multiplicity $\overline{\gamma}(m' + \overline{\overline{m}}')$ of an irreducible representation $D(m' + \overline{\overline{m}}')$

(*) In fact it can be seen that for the complete Clebsch-Gordan series the number of γ's with $S \neq 1$ equal to the number of weights $m' + \overline{\overline{m}}'$, $\overline{\overline{m}}' \in D(m)$, which lie outside the fundamental domain.

in the decomposition of $D(m) \otimes D(m')$ is therefore obtained as follows: for given m, m' and $\bar{\bar{m}}'$ the relation

$$\bar{\bar{m}}' + m' + R_0 - S(m' + R_0) = \bar{m} \tag{20}$$

is formed. Is for a given $S \in W$ the weight $\bar{m} \in D(m)$, then $\gamma(\bar{m})$ contributes with sign δ_s. Is $\bar{m} \notin D(m)$, then this $S \in W$ does not contribute.

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APPENDIX

SU(3) : In order to show how eq. (17) works a simple and familiar example for $D(m) \otimes D(m')$ is calculated, namely $D(1, 0, -1) \otimes D(1, 0, -1)$. The dominant weights of $D(1, 0, -1)$ and their multiplicities are

$$\begin{array}{ll} (1, 0, -1) \;, & \text{multiplicity } 1 \\ (0, 0, 0) \;, & \text{multiplicity } 2. \end{array}$$

The complete weight diagram of $D(1, 0, -1)$ consists out of all weights obtained from the dominant ones by all possible permutations of their components. Then with

$$m' + R_0 = (2, 0, -2)$$

one obtains for

$\bar{\bar{m}}' = (-1, 0, 1)$:

$$\begin{aligned} \bar{m} &= \bar{\bar{m}}' + m' + R_0 - S(m' + R_0) \\ &= (1, 0, -1) + S(-2, 0, 2) \end{aligned}$$

and only for $S = 1$ the weight $\bar{m} \in D(m)$. Thus

$$\bar{\gamma}(0, 0, 0) = \gamma(-1, 0, 1) = 1.$$

For $\bar{\bar{m}}' = (0, -1, 1)$:

$\bar{m} = (2, -1, -1) + S(-2, 0, 2)$, and only the two permutations

$$\begin{array}{l} (-2, 0, 2) \\ (-2, 2, 0) \end{array}$$

result into weights

$$\bar{m} = (0, -1, 1), \quad (0, 1, -1) \in D(m) \text{ with } \delta_s = 1, -1,$$

respectively. Thus

$$\bar{\gamma}(1, -1, 0) = \gamma(0, -1, 1) - \gamma(0, 1, -1) = 0.$$

For $\bar{\bar{m}}' = (+1, -1, 0)$:

$$\bar{m} = (3, -1, -2) + S(-2, 0, 2),$$

with

$$\bar{m} = (1, -1, 0), \delta_s = 1,$$

and thus

$$\bar{\gamma}(2, -1, -1) = \gamma(1, -1, 0) = 1.$$

Similarly one obtains for

$$\begin{array}{ll} \bar{\bar{m}}' = (0, 1, -1) : & \bar{\gamma}(1, 1, -2) = \gamma(0, 1, -1) = 1, \\ \bar{\bar{m}}' = (1, 0, -1) : & \bar{\gamma}(2, 0, -2) = \gamma(1, 0, -1) = 1, \\ \bar{\bar{m}}' = (0, 0, 0) \quad : & \bar{\gamma}(1, 0, -1) = \gamma(0, 0, 0) \quad = 2, \\ \bar{\bar{m}}' = (-1, 1, 0) : & \bar{\gamma}(0, 1, -1) = \gamma(-1, 1, 0) - \gamma(1, -1, 0) = 0 \end{array}$$

Of course one does not have to calculate $\bar{\gamma}(0, 1, -1)$ and $\bar{\gamma}(1, -1, 0)$ since $(0, 1, -1)$

and $(1, -1, 0)$ are not highest weights of irreducible representations. $SU(6) : D(2, 0, 0, 0, 0, -2) \otimes D(2, 1, 0, 0, -1, -2)$: Some of the $\bar{\gamma}$ will be calculated.

Dominant weights of $D(2, 0, 0, 0, 0, -2)$ and multiplicity d :

$$\begin{array}{ll} (2, 0, 0, 0, 0, -2), & d = 1 \\ (1, 1, 0, 0, 0, -2), & d = 1 \\ (1, 1, 0, 0, -1, -1), & d = 1 \\ (2, 0, 0, 0, -1, -1), & d = 1 \\ (1, 0, 0, 0, 0, -1), & d = 5 \\ (0, 0, 0, 0, 0, 0) & d = 15. \end{array}$$

The whole weight diagram is again obtained by taking all weights resulting from all permutations of the components.

Then with

$$m' + R_0 = 1/2(9, 5, 1, -1, -5, -9)\,.$$

one obtains for

$$\bar{\bar{m}}' + m' + R_0 - S(m' + R_0) = \bar{m} \text{ with } \bar{m} \in D(m)$$

for $\bar{\bar{m}}' = (-1, 0, 0, 0, 0, 1)$:

$$\bar{\bar{m}}' + m' + R_0 - S(m' + R_0) = 1/2((7, 5, 1, -1, -5, -7) + S(-9, -5, -1, 1, 5, 9))$$

$$\begin{array}{lll} \text{Then } \bar{m} & = (-1, 0, 0, 0, 0, 1), & \delta_s = 1 \\ & = (1, -2, 0, 0, 0, 1), & \delta_s = -1 \\ & = (-1, 0, 0, 0, 2, -1), & \delta_s = -1 \\ & = (-1, 0, 1, -1, 0, 1), & \delta_s = -1 \end{array}$$

and thus

$$\begin{aligned} \bar{\gamma}(1, 1, 0, 0, -1, -1) &= \gamma(-1, 0, 0, 0, 0, 1) \quad - \gamma(1, -2, 0, 0, 0, 1) \\ &\quad - \gamma(-1, 0, 0, 0, 2, -1) - \gamma(-1, 0, 1, -1, 0, 1) \\ &= 5 - 1 - 1 - 1 = 2 \end{aligned}$$

For $\bar{\bar{m}}' = (0, 0, 0, 0, 0, 0)$:

$$1/2((9, 5, 1, -1, -5, -9) + S(-9, -5, -1, 1, 5, 9)),$$

giving

$$\begin{aligned} \bar{\gamma}(2, 1, 0, 0, -1, -2) &= \gamma(0, 0, 0, 0, 0, 0) \quad - \gamma(0, 0, 1, -1, 0, 0) \\ &\quad - \gamma(2, -2, 0, 0, 0, 0) - \gamma(0, 0, 0, 0, 2, -2) \\ &\quad - \gamma(0, 2, -2, 0, 0, 0) - \gamma(0, 0, 0, 2, -2, 0) \\ &= 15 - 5 - 1 - 1 - 1 - 1 = 6. \end{aligned}$$

For $\bar{\bar{m}}' = (0, 1, 0, 0, -1, 0)$:

$$1/2((9, 7, 1, -1, -7, -9) + S(-9, -5, -1, 1, 5, 9)),$$

giving

$$\bar{\gamma}(2, 2, 0, 0, -2, -2) = 5 - 1 - 1 - 1 = 2.$$

For $\bar{\bar{m}}' = (-1, -1, 0, 0, 1, 1)$:

$$1/2((7, 3, 1, -1, -3, -7) + S(-9, -5, -1, 1, 5, 9)),$$

giving

$$\bar{\gamma}(-1, -1, 0, 0, 1, 1,) = \gamma(-1, -1, 0, 0, 1, 1,) = 1$$

etc.

SU(12) : The direct product

$$D(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1) \otimes D(2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3)$$

is considered. The dominant weights of $D(1, 0, \ldots, 0, -1)$ and their multiplicities d are

$$\begin{array}{ll} (1, 0, \ldots, 0, -1) \ , & d = 1 \\ (0, \ldots \qquad , 0) \ , & d = 11. \end{array}$$

$R_0 = 1/2\,(11, 9, 7, 5, 3, 1, -1, -3, -5, -7, -9, -11)$; proceeding as for SU(3) and SU(6) one obtains for example for $\overline{\overline{m}}' = (-1, 0, \ldots, 0, 1)$:

$$\bar{\gamma}(1, 1, 0, \ldots, -2) = \gamma(-1, 0, \ldots, 0, 1) = 1,$$

and for $\overline{\overline{m}}' = (0, \ldots, 0)$:

$$\begin{aligned} \bar{\gamma}(2, 1, 0, \ldots, 0, -3) &= \gamma(0, \ldots, 0) - \gamma(0, 0, 1, -1, 0, \ldots, 0) \\ &- \gamma(0, 0, 0, 1, -1, 0, \ldots, 0) - \gamma(0, 0, 0, 0, 1, -1, 0, \ldots, 0) \\ &- \gamma(0, 0, 0, 0, 0, 1, -1, 0, \ldots, 0) - \gamma(0, \ldots, 0, 1, -1, 0, 0, 0, 0) \\ &- \gamma(0, \ldots, 0, 1, -1, 0, 0, 0) - \gamma(0, \ldots, 0, 1, -1, 0, 0) \\ &- \gamma(0, \ldots, 0, 1, -1, 0) \\ &= 11 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 = 3. \end{aligned}$$

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