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# On the theory of unitary representations of the SL(2, C) group

by

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ABSTRACT. — The canonical basis of the unitary representations of the group SL(2, C) are constructed in the explicit form on the base of homogeneous functions. The matrix elements for finite transformations are found. The maximal degenerate principal series of SL(*n*, C) is also considered.

## § 1. INTRODUCTION

In a series of papers (Barut, Budini and Fronsdal [*1*], Dothan, Gell-Mann and Ne'eman [*2*], Fronsdal [*3*], Delbourgo, Salam and Strathdee [*4*], Ruhl [*5*], Michel [*6*], Todorov [*7*] and Nguyen van Hieu [*8*]) the possibility of using the unitary representations of non-compact groups to classify the elementary particles was discussed. It was shown that in the symmetry theory with the group G

$$G = \mathfrak{T} \,\boxed{\sim}\, S, \quad S \supset SL(2, C)$$

which is the semi-direct product of the Poincaré group $\mathfrak{T}$ and some internal symmetry non-compact group S containing some SL(2, C) subgroup there exists no contradiction with the unitary condition for S-matrix [*8*], and in this theory we can introduce the field operators in such a manner that the free field operators obey the normal commutation of anticommutation relations (with the normal connection between spins and statistics).

Before to study the experimental consequences of this new symmetry theory we must solve some mathematical problems:

1. To study the irreducible unitary representations of the internal symmetry non-compact group S and the splitting of these representations into direct sum of the irreducible finite-dimensional representations of the maximal compact subgroup of S.
2. To calculate for the unitary representations the matrix elements of the finite transformations of the group S which correspond to the pure Lorentz transformations.

These problems were considered in all above mentioned papers, some partial results were obtained there, but none of them was solved finally. In the present work we study the irreducible unitary representations of the SL(2, C) group and calculate the matrix elements of the finite transformations for these representations. The method developped here can be also generalized to study the group SL($n$, C) and SU($p$, $q$). We note that the theory of unitary representations of these groups was developped in the work of Gelfand and Neimark [*9*]. The Gelfand-Neimark theory is a rigorous one from the mathematical point of view. However, the method used by Gelfand and Neimark is not convenient for the physical applications. Here we apply an other method based on using the homogeneous functions to realize the irreducible unitary representations of non-compact groups. This method is very convenient for the applications to physics. The possibility of using the homogeneous functions to study the representations of non-compact groups was discussed in references [*10*] [*3*] [*5*] [*11*]. Within this method we can obtain the Gelfand-Neimark results in a very simple manner.

## § 2. UNITARY REPRESENTATIONS OF THE SL(2, C) group

SL(2, C) is the group of all $2 \times 2$ complex matrices with determinant equal to 1. We now realize the representations of this group in the Hilbert space of the functions $f(z_1, z_2)$ depending on two complex variables $z_1$ and $z_2$. For every matrix $g \in \mathrm{SL}(2, \mathrm{C})$ we define a corresponding operator $\mathrm{T}_g$ in the given Hilbert space of functions $f(z_1, z_2)$:

$$g \to \mathrm{T}_g \tag{2.1}$$

$$\mathrm{T}_g f(z_1, z_2) = f(z_1', z_2') \; ; \qquad z_a' = z_b g_{ba} \,|\, a, b = 1, 2 \,|$$

It is not difficult to prove that

$$T_{g_1}T_{g_2} = T_{g_1 g_2}$$

Therefore the corresponding $g \to T_g$ is a representation of the group SL(2, C). We define now the scalar product in the Hilbert space of functions $f(z_1, z_2)$:

$$\langle f_1, f_2 \rangle = \left(\frac{i}{2}\right)^2 \int f_1(z_1, z_2)\overline{f_2(z_1, z_2)}\,.\,dz_1 d\bar{z}_1 . dz_2 d\bar{z}_2 \tag{2.2}$$

In this case our Hilbert space consists of all square integrable functions of two complex variables. It is not difficult to prove that with this scalar product (2.2) the operators $T_g$ are unitary:

$$\langle T_g f_1, T_g f_2 \rangle = \langle f_1, f_2 \rangle$$

Thus we obtain an unitary representation of the SL(2, C) group, which is not yet an irreducible one, however. In order to get the irreducible representations we use the homogeneous functions. A function $f(z_1, z_2)$ is called a homogeneous function of degree $(\lambda_1, \lambda_2)$ where $\lambda_1$ and $\lambda_2$ are complex numbers, if for any complex number $\sigma \neq 0$ we have

$$f(\sigma z_1, \sigma z_2) = \sigma^{\lambda_1}\bar{\sigma}^{\lambda_2} f(z_1, z_2) \tag{2.3}$$

This definition makes sense only if the difference $\lambda_1 - \lambda_2$ is an integer number. From this definition and from (2.1) it follows that if $f(z_1, z_2)$ is a homogeneous function of degree $(\lambda_1, \lambda_2)$ then $T_g f(z_1, z_2)$ is also a homogeneous function with the same degree. Thus the Hilbert space $D_\lambda$ of homogeneous functions of some degree $\lambda = (\lambda_1, \lambda_2)$ realizes a representation of the group SL(2, C).

For the homogeneous functions we cannot use the scalar product defined as in (2.2). Indeed, each function $f(z_1, z_2)$ from $D_\lambda$ is determined uniquely by a corresponding function of one variable $f(z) \equiv f(z, 1)$. Since

$$f(z_1, z_2) = z_2^{\lambda_1}\bar{z}_2^{\lambda_2} f\left(\frac{z_1}{z_2}, 1\right) = z_2^{\lambda_1}\bar{z}_2^{\lambda_2} f\left(\frac{z_1}{z_2}\right) \tag{2.4}$$

and the integral in the right-hand side of (2.2) can be written in the form of the product of two independent integrals

$$\int f_1(z_1, z_2)\overline{f_2(z_1, z_2)}\,.\,dz_1 d\bar{z}_1 . dz_2 d\bar{z}_2$$

$$= \int |z_2^{\lambda_1}\bar{z}_2^{\lambda_2}|^2 f_1\left(\frac{z_1}{z_2}\right)\overline{f_2\left(\frac{z_1}{z_2}\right)} . |z_2|^2 . d\left(\frac{z_1}{z_2}\right) . d\left(\overline{\frac{z_1}{z_2}}\right) . dz_2 d\bar{z}_2$$

$$= \int f_1(z)\overline{f_2(z)}\,dz d\bar{z} . \int |z_2^{\lambda_1+1}\bar{z}_2^{\lambda_2}|^2 dz_2 d\bar{z}_2,$$

It is not difficult to show that the seond integral tends to infinity.

Thus, in the Hilbert space $D_\lambda$ we must define the scalar product in another manner. Since the homogeneous functions $f(z_1, z_2)$ effectively depend only on one of variables then to define the scalar product we must use the complex path integral instead of the complex surface integral. Namely, we define the scalar product in the following manner:

$$\langle f_1, f_2 \rangle = \frac{i}{2} \int f_1(z_1, z_2)\overline{f_2(z_1, z_2)} d\omega_z, \tag{2.5}$$

where $d\omega_z$ is some measure invariant under the transformations

$$z_a \to z_a' = z_b g_{ba}, \qquad \det g = 1$$

It is easy to see that such a measure can be of the form

$$d\omega_z = (\overline{z_2} d\overline{z_1} - \overline{z_1} d\overline{z_2})(z_2 dz_1 - z_1 dz_2) \tag{2.6}$$

As in the case of (2.2) from the invariance of the measure it follows that all the operators $T_g$ are unitary with respect to the scalar product (2.5), and the representation of the SL(2, C) group in the Hilbert space $D_\lambda$ is an unitary one.

The norm of an element $f(z_1, z_2) \in D_\lambda$ is determined according to (2.5):

$$\|f\|^2 = \langle f | f \rangle = \frac{i}{2} \int |f(z_1, z_2)|^2 d\omega_z \tag{2.7}$$

Putting in the integral in the right-hand side of (2.7) $z_a = \sigma z_a'$ and using (2.3) we get

$$\|f\|^2 = |\sigma^{\lambda_1 + \overline{\lambda_2} + 2}|^2 . \|f\|^2 \tag{2.8}$$

for any complex $\sigma \neq 0$. This relation shows that $\lambda_1$ and $\lambda_2$ must satisfy the equation

$$\lambda_1 + \overline{\lambda_1} + 2 = 0 \tag{2.9}$$

whose solution is

$$\lambda_1 = \nu_0 + \frac{i\rho}{2} - 1$$

$$\lambda_2 = -\nu_0 + \frac{i\rho}{2} - 1,$$

where $\nu_0$ and $\rho$ are real numbers. Since $\lambda_1 - \lambda_2 = 2\nu_0$ must be an integer number, then $\nu_0$ is an integer or half-integer number. Thus we have obtained the unitary representations of the SL(2, C) group in the Hilbert space $D_\lambda$

of homogeneous functions of degrees $\lambda = \left(\nu_0 + \frac{i\rho}{2} - 1, -\nu_0 + \frac{i\rho}{2} - 1\right)$ where $\nu_0$ is any integer or half-integer number, $\rho$ is any real number. These representations will be denoted by $\mathfrak{C}_{\nu_0\rho}$. They are irreducible [10] and form the so called principal series. Together with this principal series there exists also the supplementary one [9] [10] [12] which can be considered in a similar manner. However, we do not study this series here.

Finally we note that from (2.6) and (2.9) it follows that for the representations of the principal series the scalar product defined by (2.5) is identical to the scalar product introduced by Gelfand and Neimark

$$\langle f_1, f_2 \rangle = \frac{i}{2} \int f_1(z)\overline{f_2(z)}dzd\bar{z} \tag{2.10}$$

## § 3. EQUIVALENT REPRESENTATIONS. SPLITTING OF THE UNITARY REPRESENTATIONS OF SL(2, C) GROUP INTO DIRECT SUMS OF THE REPRESENTATIONS OF SU(2) SUBGROUP

Let $\mathfrak{C}_{\nu_0\rho}$ (with operators $T_g$) and $\mathfrak{C}_{\nu_0'\rho'}$ (with operators $T_g'$) be two irreducible unitary representations of the group SL(2, C) which are realized in the Hilbert spaces of homogeneous functions

$$D_{\lambda \equiv \left(\nu_0 + \frac{i\rho}{2} - 1, -\nu_0 + \frac{i\rho}{2} - 1\right)} \text{ and } D_{\lambda' \equiv \left(\nu_0' + \frac{i\rho'}{2} - 1, -\nu_0' + \frac{i\rho'}{2} - 1\right)}$$

respectively. Now we find the conditions for the equivalence of these two representations. The representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu_0'\rho'}$ are called equivalent if there exists such an operator A which realizes an one-to-one mapping $D_\lambda$ onto $D_{\lambda'}$ that

$$T_g'A = AT_g \tag{3.1}$$

for any $g \in SL(2, C)$. From this definition we see immediately that if $\lambda = \lambda'$ (i. e. $\nu_0 = \nu_0'$, $\rho = \rho'$) then $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu_0'\rho'}$ are equivalent. This case is a trivial one and it is not of any interest, because here $D_\lambda$ and $D_{\lambda'}$ coincide, $A = 1$.

Let

$$f(\xi_1, \xi_2) \in D_\lambda, \qquad f'(\eta_1, \eta_2) \in D_{\lambda'}.$$

We represent the operator A in the form of an integral transformation with some kernel K:

$$f'(\eta_1, \eta_2) = Af(\xi_1, \xi_2) \equiv \frac{i}{2} \int K(\eta_1, \eta_2 ; \xi_1, \xi_2) f(\xi_1, \xi_2) d\omega_\xi \quad (3.2)$$

Then the condition of equivalence (3.1) can be rewritten explicitly in the form

$$T'_g f'(\eta_1, \eta_2) = \frac{i}{2} \int K(\eta_1, \eta_2 ; \xi_1, \xi_2) T_g f(\xi_1, \xi_2) d\omega_\xi \quad (3.3)$$

Using Eq. (2.1) we can rewrite last equation in the form:

$$f'(\eta'_1, \eta'_2) = \frac{i}{2} \int K(\eta_1, \eta_2 ; \xi_1, \xi_2) f(\xi'_1, \xi'_2) d\omega_{\xi'} \quad (3.4)$$

where $\eta'_a = \eta_b g_{ba}$, $\xi'_a = \xi_b g_{ba}$. Comparing Eqs (3.2) and (3.4) we get:

$$K(\eta'_1, \eta'_2 ; \xi'_1, \xi'_2) = K(\eta_1, \eta_2 ; \xi_1, \xi_2).$$

Thus, the kernel K must be an invariant function of $\xi_a$ and $\eta_a$. As it was well known in the theory of spinor representations of the group SL(2, C) from the variables $\xi_a$ and $\eta_a$ we can form the following invariant

$$\xi_1\eta_2 - \xi_2\eta_1 = \text{inv}.$$

The kernel K must be a function of this invariant combination $(\xi_1\eta_2 - \xi_2\eta_1)$. It must be also a homogeneous function since $f'(\eta_1, \eta_2)$ is a homogeneous one. Let K be a homogeneous function on two variables $(\xi_1, \xi_2)$ of degree $(\mu_1, \mu_2)$. Then putting $\xi_a = \sigma\xi'_a$ into (3.2) we get:

$$\int K(\eta_1, \eta_2 ; \xi_1, \xi_2) f(\xi_1, \xi_2) d\omega_\xi$$
$$= (\sigma)^{\mu_1+\lambda_1+2}(\bar{\sigma})^{\mu_2+\lambda_2+2} \int K(\eta_1, \eta_2 ; \xi'_1, \xi'_2) f(\xi'_1, \xi'_2) . d\omega_{\xi'}$$

for any complex $\sigma \neq 0$. Therefore we must have

$$\begin{aligned} \mu_1 &= -\lambda_1 - 2 = -\nu_0 - \frac{i\rho}{2} - 1 \\ \mu_2 &= -\lambda_2 - 2 = +\nu_0 - \frac{i\rho}{2} - 1 \end{aligned} \quad (3.5)$$

Thus, $K(\eta_1, \eta_2; \xi_1, \xi_2)$ must be an homogeneous function of $(\xi_1, \xi_2)$ of degree $\left(-\nu_0 - \frac{i\rho}{2} - 1,\ \nu_0 - \frac{i\rho}{2} - 1\right)$. On the other hand the kernel $K(\eta_1, \eta_2; \xi_1, \xi_2)$ is a function of the combination $(\xi_1\eta_2 - \xi_2\eta_1)$ and therefore it is also a homogeneous function of $(\eta_1, \eta_2)$ of the same degree $\left(-\nu_0 - \frac{i\rho}{2} - 1, \nu_0 - \frac{i\rho}{2} - 1\right)$. This means that if $\nu_0' = -\nu_0$, $\rho' = -\rho$ then the representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu_0'\rho'}$ are equivalent.

Consider now the splitting of irreducible unitary representations of the group SL(2, C) into direct sums of the irreducible representations of the maximal compact subgroup SU(2). In the theory of spinor representations of the group SL(2, C) we know that if $z_a$ is transformed as a spinor $\varphi_a$ then $\overline{z_a}$ is transformed as a spinor $\varphi^{\dot{a}}$ and the sum $z_a\overline{z_a}$ is transformed as the sum $\varphi_a\varphi^{\dot{a}}$ i. e. is invariant of the SU(2) subgroup. In the following for the convenience we denote $\overline{z_a}$ by $\overline{z}^{\dot{a}}$ or $\overline{z}^a$.

For the clearity we illustrate now our method on some simple examples. The general case will be considered in the following section. Consider firstly the representation $\mathfrak{C}_{0\rho}$. This representation is realized in the Hilbert space $D_{\left(\frac{i\rho}{2}-1, \frac{i\rho}{2}-1\right)}$ of homogeneous functions $f(z_1, z_2)$ of degree $\left(\frac{i\rho}{2} - 1, \frac{i\rho}{2} - 1\right)$. One of these functions is

$$f_{00}(z_1, z_2) \sim \left(z_1\overline{z}^1 + z_2\overline{z}^2\right)^{\frac{i\rho}{2}-1} \tag{3.6}$$

As it was noted, this function is invariant under the SU(2) subgroup and therefore characterizes the spin zero state. In order to get the functions corresponding to the states with non-zero spin we must construct them in such a manner that they contain some factors $z_a$ and $\overline{z}^b$ without summation. It is not difficult to see that for the spin 1 states we have the following functions:

$$f_{11}(z_1, z_2) \sim \left(z_1\overline{z}^1 + z_2\overline{z}^2\right)^{\frac{i\rho}{2}-2} z_1\overline{z}^2 \qquad \text{for} \quad j = 1, m = 1 \tag{3.9}$$

$$f_{10}(z_1, z_2) \sim \left(z_1\overline{z}^1 + z_2\overline{z}^2\right)^{\frac{i\rho}{2}-2}\left(z_2\overline{z}^2 - z_1\overline{z}^1\right) \qquad \text{for} \quad j = 1, m = 0 \tag{3.8}$$

$$f_{1,-1}(z_1, z_2) \sim \left(z_1\overline{z}^1 + z_2\overline{z}^2\right)^{\frac{i\rho}{2}-2} z_2\overline{z}^1 \qquad \text{for} \quad j = 1, m = -1 \tag{3.9}$$

($j$ and $m$ denote the spin and its projection on the $z$ — axis).

Consider now the representations $\mathfrak{C}_{\nu_0\rho}(\nu_0 \neq 0)$. Since the representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{-\nu_0, -\rho}$ are equivalent then we can assume that $\nu_0 > 0$. We choose the basis elements of the representations $\mathfrak{C}_{\nu_0\rho}$ in the form of the products of the quantity $(z_1\overline{z}^1 + z_2\overline{z}^2)^{\frac{i\rho}{2}-n}$ $(n \geqslant \nu_0 + 1)$ and some factors $z_a$

and $\bar{z}^b$ without summation. The products with the minimal number of free factors $z_a$ and $\bar{z}^b$ are of the form

$$f(z_1, z_2) \sim \left(z_1\bar{z}^1 + z_2\bar{z}^2\right)^{\frac{i\rho}{2}-1-\nu_0} z_{a_1}z_{a_2}\ldots z_{a_{2\nu_0}} \tag{3.10}$$

These functions describe the states with spin $j_0 = \nu_0$. The other functions correspond to the states with spins $j = \nu_0 + 1,\ \nu_0 + 2, \ldots$ Thus, the representation $\mathfrak{C}_{\nu_0\rho}$ splits into the direct sum of irreducible finite-dimensional representations of the SU(2) subgroup. Each of them is contained in given representation $\mathfrak{C}_{\nu_0\rho}$ once and describes state with definite spin $j = j_0 + n,\ n = 0, 1, 2, \ldots$

## § 4. MATRIX ELEMENTS OF FINITE TRANSFORMATIONS

As it was noted in the introduction, in studying the structure of the vertex parts and the scattering amplitudes we must use the matrix elements of the finite transformations of the group S and in particular of the group SL(2, C). Note that this problem was first consider in the paper by Dolghinov and Toptyghin [*13*] for the case with $\nu_0 = 0$. These authors choose the analytic continuations of the 4-dimensional spherical functions as the basis functions (see [*13*] [*14*]). Our method is based on the results obtained in § 2.

The matrix element $D^{\nu_0\rho}_{jm;j'm'}(g)$ corresponding to the representation $g \to U_g$ is defined in the following manner:

$$U_g \mid \nu_0\rho\,; jm\rangle = \sum_{j'm'} D^{\nu_0\rho}_{jm;j'm'}(g) \mid \nu_0\rho\,; j'm'\rangle, \tag{4.1}$$

where $\mid \nu_0\rho; jm\rangle$ is the canonical basis of representation $\mathfrak{C}_{\nu_0\rho}$; $j$ and $m$ are the spin and its projection on the $z$-axis. Generalizing the obtained results (see (3.6)-(3.10)) we determine firstly the canonical basis in the space of homogeneous functions $D_{\lambda\equiv\left(\nu_0+\frac{i\rho}{2}-1,\ -\nu_0+\frac{i\rho}{2}-1\right)}$ in the following manner:

$$\mid \nu_0\rho\,; jj\rangle \to f^{\nu_0\rho}_{jj}(z_1, z_2) = c_{jj}\left(z_1\bar{z}^1 + z_2\bar{z}^2\right)^{\frac{i\rho}{2}-1-j}(z_1)^{j+\nu_0}\left(\bar{z}^2\right)^{j-\nu_0} \tag{4.2}$$

where $c_{jj}$ are the normalization constants. Using (2.10) we get:

$$c_{jj} = \left\{\frac{i}{2}\int \left|f^{\nu_0\rho}_{jj}(z)\right|^2 dz d\bar{z}\right\}^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}\left\{\frac{(2j+1)\,!}{(j+\nu_0)\,!\,(j-\nu_0)\,!}\right\}^{1/2} \tag{4.3}$$

From $| \nu_0\rho; jj \rangle$ we can find $| \nu_0\rho; jm \rangle$ :

$$| \nu_0\rho ; jm \rangle \rightarrow f^{\nu_0\rho}_{jm}(z_1, z_2) = N_{jm}(J^-)^{j-m} f^{\nu_0\rho}_{jj}(z_1, z_2), \tag{4.4}$$

where

$$N_{j,m} = \left\{ \frac{(j+m)\,!}{(2j)\,!\,(j-m)\,!} \right\}^{1/2} \tag{4.5}$$

$$J^- = z_2 \frac{\partial}{\partial z_1} - \bar{z}^1 \frac{\partial}{\partial \bar{z}^2} \tag{4.6}$$

From (4.2)-(4.6) we get the final expression for $f^{\nu_0\rho}_{jm}$ $(z_1, z_2)$:

$$f^{\nu_0\rho}_{jm}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left\{(2j+1)(j+m)!(j-m)!(j+\nu_0)!(j-\nu_0)!\right\}^{1/2} \left(z_1\bar{z}^1 + z_2\bar{z}^2\right)^{\frac{i\rho}{2}-1-j}$$
$$\cdot \sum_d (-1)^d \frac{1}{d!(j-m-d)!(\nu_0+m+d)!(j-\nu_0-d)!} (z_1)^{\nu_0+m+d} (z_2)^{j-m-d} (\bar{z}^1)^d (\bar{z}^2)^{j-\nu_0-d} \tag{4.7}$$

Having the explicit expression for the canonical basis $| \nu_0\rho; jm \rangle$ we can find the matrix elements $D^{\nu_0\rho}_{jm;\, j'm'}(g)$. It is well known that every matrix $g$ can be represented in the form

$$g = u_1 \varepsilon u_2$$

where $u_1$ and $u_2$ are the unitary unimodular matrices which corresponding to the space rotation;

$$\varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \qquad (\varepsilon\text{—real number})$$

and corresponds to the pure Lorentz transformation in the plane $(x_3, x_4)$. Thus without losing the generality we can consider only the matrix element $D^{\nu_0\rho}_{jm;\, j'm'}(\varepsilon)$.

From (4.1) and from the orthonormalization relations

$$\frac{i}{2} \int f^{\nu_0\rho}_{jm}(z) \overline{f^{\nu_0\rho}_{j'm'}}(z) dz d\bar{z} = \delta_{jj'} . \delta_{mm'}$$

we have

$$D^{\nu_0\rho}_{jm;j'm'}(\varepsilon) = \frac{i}{2} \int T_\varepsilon f^{\nu_0\rho}_{jm}(z) . \overline{f^{\nu_0\rho}_{j'm'}}(z) dz d\bar{z}. \tag{4.8}$$

From (2.1) and (4.7) we get:

$$D^{\nu_0\rho}_{jm;j'm'}(\varepsilon) = \delta_{mm'} \cdot \frac{1}{\pi}\{(2j+1)(2j'+1).(j+m)!\,(j-m)!\,(j+\nu_0)!\,(j-\nu_0)!\,(j'+m')!\,(j'-m)!\,(j'+\nu_0)!\,(j'-\nu_0)!\}^{1/2}$$

$$\cdot \sum_{d,d'} (-1)^{d+d'}\{d!\,d'!\,(j-m-d)!\,(j'-m-d')!\,(\nu_0+m+d)!\,(\nu_0+m+d')!\,(j-\nu_0-d)!\,(j'-\nu_0-d')!\}^{-1}$$

$$.\varepsilon^{-2\left(2d+m+\nu_0+1-\frac{i\rho}{2}\right)} \cdot \frac{i}{2}\int dz d\bar{z}\,|z|^{2(d+d'+m+\nu_0)}(1+|z|^2)^{-\frac{i\rho}{2}-1-j'}(1+\varepsilon^{-4}|z|^2)^{\frac{i\rho}{2}-1-j}, \tag{4.9}$$

where $d$ and $d'$ can take any integer number which does not make each factor under the factorial to become a negative number. By putting $z = \sqrt{v}e^{i\varphi}(0 \leqslant v < \infty;\ 0 \leqslant \varphi \leqslant 2\pi)$ the integral in (4.9) can be rewritten in the form

$$J = \pi\int_0^\infty dv . v^{d+d'+m+\nu_0}(1+v)^{-\frac{i\rho}{2}-1-j'}(1+\varepsilon^{-4}v)^{\frac{i\rho}{2}-1-j}$$

$$= \pi . \varepsilon^{4(d+d'+m+\nu_0+1)}\frac{(d+d'+m+\nu_0)!\,(j+j'-d-d'-m-\nu_0)!}{(j+j'+1)!}$$

$$F\left(j'+1+\frac{i\rho}{2},\ d+d'+m+\nu_0+1\,;j+j'+2\,;\ 1-\varepsilon^4\right), \tag{4.10}$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. By setting (4.10) into (4.9) we get the final result:

$$D^{\nu_0\rho}_{jm;j'm'}(\varepsilon) = \frac{\delta_{mm'}}{(j+j'+1)!}$$

$$\{(2j+1)(2j'+1)(j+m)!(j-m)!(j+\nu_0)!(j-\nu_0)!(j'+m)!(j'-m)!(j'+\nu_0)!(j'-\nu_0)!\}^{1/2}$$

$$\cdot \sum_{d,d'}(-1)^{d+d'}\frac{(d+d'+m+\nu_0)!\,(j+j'-d-d'-m-\nu_0)!}{d!\,d'!\,(j-m-d)!\,(j'-m-d')!\,(\nu_0+m+d)!\,(\nu_0+m+d')!\,(j-\nu_0-d)!\,(j'-\nu_0-d')!}$$

$$.\varepsilon^{2\left(2d'+m+\nu_0+1+\frac{i\rho}{2}\right)}F\left(j'+1+\frac{i\rho}{2},\ d+d'+m+\nu_0+1\,;j+j'+2\,;1-\varepsilon^4\right). \tag{4.11}$$

This result exactly coincides with the result obtained earlier by the authors in another way [*15*].

Now we note some simple properties of $D^{\nu_0\rho}_{jm;j'm'}(\varepsilon)$.

1. Putting in (4.11) $\varepsilon = 1$ and taking into account $F(\alpha, \beta; \gamma; 0) = 1$ we have:

$$D^{\nu_0\rho}_{jm;j'm'}(1) = \delta_{jj'} \cdot \delta_{mm'} \tag{4.12}$$

This is a trivial relation. It means that we deal here with the identity transformation.

2. Making the permutation of $jm$ and $j'm'$ in (4.11) and using the properties of the homogeneous functions

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F\left(\gamma-\alpha, \beta; \gamma; \frac{z}{z-1}\right) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right), \tag{4.13}$$

we obtain

$$D^{\nu_0\rho}_{j'm';jm}(\varepsilon) = \overline{D^{\nu_0\rho}_{jm;j'm'}(\varepsilon^{-1})} \tag{4.14}$$

3. Making the substitution $m \to -m$, $m' \to -m'$ in (4.14), putting $j - \nu_0 - d \equiv d_1$, $j' - \nu_0 - d' \equiv d'_1$ and using (4.13) we can prove that

$$D^{\nu_0\rho}_{jm;j'm'}(\varepsilon) = (-1)^{j+j'-2\nu_0} D_{j,-m;j',-m'}(\varepsilon^{-1}). \tag{4.15}$$

4. From (4.12) (4.14) and from group property of $D_{jm;j'm'}$ we get:

$$\sum_{jm} D^{\nu_0\rho}_{jm;j'm'}(\varepsilon) \overline{D^{\nu_0\rho}_{jm;j''m''}(\varepsilon)} = \delta_{j'j''} \cdot \delta_{m'm''}$$

This last relation means the unitarity condition of the representation.

## § 5. GENERALIZED TENSORS

From the canonical basis which was given in (4.7) we go to the other basis called generalized tensors of the SU(2, C) group. They are constructed in the following manner:

$$f^{b_1 b_2 \ldots b_{j-\nu_0}}_{a_1 a_2 \ldots a_{j+\nu_0}}(z_1 z_2) = \left(z_c \overset{-c}{z}\right)^{\frac{i\rho}{2}-1-j} \Phi^{b_1 b_2 \ldots b_{j-\nu_0}}_{a_1 a_2 \ldots a_{j+\nu_0}}, \quad (a, b = 1, 2) \tag{5.1}$$

where

$$\Phi^{b_1 b_2 \ldots b_t}_{a_1 a_2 \ldots a_{t+k}} = \sum_{s=0}^{t} \alpha(t, s, k) \left(z_c \overset{-c}{z}\right)^s S \delta^{b_1}_{a_1} \delta^{b_2}_{a_2} \ldots \delta^{b_s}_{a_s} z_{a_{s+1}} \ldots z_{a_{t+k}} \overset{-b_{s+1}}{z} \ldots \overset{-b_t}{z}$$

$$\alpha(t, s, k) = (-1)^s \frac{t!\,(t+k)!\,(2t+k-s)!}{s!\,(t-s)!\,(t+k-s)!\,(2t+k)!} \tag{5.2}$$

S denotes the symmetrization in the upper and lower indices $a$ and $b$ separately:

$$\mathrm{S}\left(\mathrm{T}_{a_1a_2\ldots a_i}^{b_1b_2\ldots h_j}\right) \equiv \frac{1}{i\,!\,j\,!}\sum_{\mathrm{P}(a;b)} \mathrm{T}_{a_1a_2\ldots a_i}^{b_1b_2\ldots b_j}$$

$\left(\sum_{\mathrm{P}(a;b)}\right.$ stands for summation over all permutations of $a$ and all permutations of $b$).

The tensors $\Phi_{a_1a_2\ldots}^{b_1b_2\ldots}$ are symmetrical in upper and in lower indices and are traceless with respect to the contraction of any upper index with any lower index. They are irreducible under the SU(2) subgroup. Putting in (5.1), e. g., $a_1 = a_2 = \ldots = a_{j+\nu_0} = 1$, $b_1 = b_2 = \ldots = b_{j-\nu_0} = 2$ and using (5.1) (5.3) and (4.2) we get:

$$f_{\underbrace{11\ldots1}_{j+\nu_0\text{ times}}}^{\overbrace{22\ldots2}^{j-\nu_0\text{ times}}} = \sqrt{\pi}\left\{\frac{(j+\nu_0)\,!\,(j-\nu_0)\,!}{(2j+1)\,!}\right\}^{1/2} f_{jj}^{\nu_0\rho}$$

The inverse expansion is

$$z_{a_1}z_{a_2}\ldots z_{a_{t+k}}\overset{-b_1}{z}\overset{-b_2}{z}\ldots\overset{-b_t}{z} = \sum_{s=0}^{t}\beta(t,s,k)\left(z_c\overset{-c}{z}\right)^{t-s}\mathrm{S}\Phi_{a_1a_2\ldots a_{s+k}}^{b_1b_2\ldots b_s}\delta_{a_{s+k+1}}^{b_{s+1}}\ldots\delta_{a_{t+k}}^{b_t} \tag{5.4}$$

where

$$\beta(t,s,k) = \frac{t\,!\,(t+k)\,!\,(2s+k+1)\,!}{s\,!\,(s+k)\,!\,(t-s)\,!\,(t+k+s+1)\,!} \tag{5.5}$$

Under the transformation $g$ the tensor $f_{a_1a_2\ldots a_{j+\nu_0}}^{b_1b_2\ldots b_{j-\nu_0}}$ is transformed as

$$f_{a_1a_2\ldots a_{j+\nu_0}}^{b_1b_2\ldots b_{j-\nu_0}}(z_1z_2) \to \mathrm{T}_g f_{a_1a_2\ldots a_{j+\nu_0}}^{b_1b_2\ldots b_{j-\nu_0}}(z_1z_2)$$
$$= \sum_{j'} \mathrm{D}_{a_1a_2\ldots a_{j+\nu_0};d_1d_2\ldots d_{j'-\nu_0}}^{b_1b_2\ldots b_{j-\nu_0};c_1c_2\ldots c_{j'+\nu_0}}(g) f_{c_1c_2\ldots c_{j'+\nu_0}}^{d_1d_2\ldots d_{j'-\nu_0}}(z_1, z_2), \tag{5.6}$$

where the matrix elements $\mathrm{D}_{a_1a_2\ldots;d_1d_2\ldots}^{b_1b_2\ldots;c_1c_2\ldots}(g)$ are the generalization of $\mathrm{D}_{jm;j'm'}(g)$. The explicit expressions of these matrix elements will be given in the next section.

Any homogeneous function $\varphi(z_1, z_2)$ from the space $\mathrm{D}_{\left(\nu_0+\frac{i\rho}{2}-1,\ -\nu_0+\frac{i\rho}{2}-1\right)}$ can be represented in the form:

$$\varphi(z_1, z_2) = \sum_{j'} \varphi_{d_1d_2\ldots d_{j'-\nu_0}}^{c_1c_2\ldots c_{j'+\nu_0}} f_{c_1c_2\ldots c_{j'+\nu_0}}^{d_1d_2\ldots d_{j'-\nu_0}}(z_1, z_2). \tag{5.7}$$

The components $\varphi_{d_1d_2\dots}^{c_1c_2\dots}$ are also symmetrical in upper and lower indices and tranceless. These quantities will be called the generalized tensors. Under $g$ function $\varphi(z_1, z_2)$ transforms into $T_g\varphi(z_1, z_2)$ which can be also represented in the form of (5.7):

$$T_g\varphi(z_1, z_2) = \sum_j T_g\varphi_{b_1b_2\dots b_{j-\nu_0}}^{a_1a_2\dots a_{j+\nu_0}} \cdot f_{a_1a_2\dots a_{j+\nu_0}}^{b_1b_2\dots b_{j-\nu_0}}(z_1, z_2). \tag{5.8}$$

On the other hand, from (5.6) and (5.7) it follows that

$$T_g\varphi(z_1, z_2) = \sum_{j'j} \varphi_{d_1d_2\dots d_{j'-\nu_0}}^{c_1c_2\dots c_{j'+\nu_0}} D_{c_1c_2\dots c_{j'+\nu_0};b_1b_2\dots b_{j-\nu_0}}^{d_1d_2\dots d_{j'-\nu_0};a_1a_2\dots a_{j+\nu_0}}(g) f_{a_1a_2\dots a_{j+\nu_0}}^{b_1b_2\dots b_{j-\nu_0}} \tag{5.9}$$

Comparing (5.8) and (5.9) we find immediately the transformation law for tensors $\varphi_{b_1b_2\dots}^{a_1a_2\dots}$:

$$\varphi_{b_1b_2\dots b_{j-\nu_0}}^{a_1a_2\dots a_{j+\nu_0}} \to T_g\varphi_{b_1b_2\dots b_{j-\nu_0}}^{a_1a_2\dots a_{j+\nu_0}} = \sum_{j'} D_{c_1c_2\dots c_{j'+\nu_0};b_1b_2\dots b_{j-\nu_0}}^{d_1d_2\dots d_{j'-\nu_0};a_1a_2\dots a_{j+\nu_0}}(g)\varphi_{d_1d_2\dots d_{j'-\nu_0}}^{c_1c_2\dots c_{j'+\nu_0}} \tag{5.10}$$

## § 6. MATRIX ELEMENTS FOR GENERALIZED TENSORS

Now we determine the matrix elements $D_{a_1a_2\dots;\, d_1d_2\dots}^{b_1b_2\dots;\, c_1c_2\dots}(g)$ defined in (5.6). At first we rewrite (5.1) and (5.2) in the form

$$f_{a_1a_2\dots a_{j+\nu_0}}^{b_1b_1\dots b_{j-\nu_0}}(z_1, z_2) = \sum_{s=0}^{j-\nu_0} \alpha(j-\nu_0, s, 2\nu_0)(z_c\bar{z}^c)^{\frac{i\varrho}{2}-1-j+s} S\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\dots\delta_{a_s}^{b_s} z_{a_{s+1}}\dots z_{a_{j+\nu_0}}\bar{z}^{b_{s+1}}\dots\bar{z}^{b_{j-\nu_0}} \tag{6.1}$$

Since under the transformation $g$

$$z_a \to z'_a = z_b g_{ba} \equiv z_b g_a^b$$

$$\bar{z}_a \to \bar{z}'^a = \bar{z}^b \bar{g}^{ba} \equiv \bar{z}^b g_b^{+a}$$

$$f(z_1, z_2) \to T_g f(z_1, z_2) = f(z'_1, z'_2)$$

then the tensor $f^{b_1b_2\cdots}_{a_1a_2\cdots}$ is transformed in the following manner:

$$f^{b_1b_2\ldots b_{j-\nu_0}}_{a_1a_2\ldots a_{j+\nu_0}} \to T_g f^{b_1b_2\ldots b_{j-\nu_0}}_{a_1a_2\ldots a_{j+\nu_0}}$$

$$= \sum_{s=0}^{j-\nu_0} \alpha(j-\nu_0, s, 2\nu_0) \left\{ z_p \bar{z}^q (g\overset{+}{g})^p_q \right\}^{\frac{i\rho}{2}-1-j+s} z_{c_{s+1}} z_{c_{s+2}} \ldots z_{c_{j+\nu_0}} \bar{z}^{d_{s+1}} \bar{z}^{d_{s+2}} \ldots \bar{z}^{d_{j-\nu_0}}$$

$$\cdot \underset{(a;b)}{S}\, \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\ldots\delta^{b_s}_{a_s} g^{c_{s+1}}_{a_{s+1}} g^{c_{s+2}}_{a_{s+2}} \ldots g^{c_{j+\nu_0}}_{a_{j+\nu_0}} \overset{+}{g}{}^{b_{s+1}}_{d_{s+1}} \overset{+}{g}{}^{b_{s+2}}_{d_{s+2}} \ldots \overset{+}{g}{}^{b_{j-\nu_0}}_{d_{j-\nu_0}} \tag{6.2}$$

We represent the $2 \times 2$ matrix $g\overset{+}{g}$ in terms of the Pauli matrices:

$$g\overset{+}{g} = \alpha_0\sigma_0 + \hat{\alpha} = \alpha_0(1 + \hat{\beta}); \qquad \hat{\beta} \equiv \frac{1}{\alpha_0}\hat{\alpha} = \frac{1}{\alpha_0}\vec{\alpha}.\vec{\sigma} \tag{6.3}$$

Putting this expression for $g\overset{+}{g}$ into $\left\{ z_p \bar{z}^q (g\overset{+}{g})^p_q \right\}^{\frac{i\rho}{2}-1-j+s}$ and performing some elementary expansion

$$\left\{ z_p \bar{z}^q (g\overset{+}{g})^p_q \right\}^{\frac{i\rho}{2}-1-j+s} = \alpha_0^{\frac{i\rho}{2}-1-j+s} \left\{ z_c \bar{z}^c + z_p \hat{\beta}^p_q \bar{z}^q \right\}^{\frac{i\rho}{2}-1-j+s}$$

$$= \alpha_0^{\frac{i\rho}{2}-1-j+s} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k!\,\Gamma\left(\frac{i\rho}{2}-j+s-k\right)} (z_c\bar{z}^c)^{\frac{i\rho}{2}-1-j+s-k} \left(z_p \hat{\beta}^p_q \bar{z}^q\right)^k \tag{6.4}$$

we can rewrite (6.2) in the form

$$T_g f^{b_1b_2\ldots b_{j-\nu_0}}_{a_1a_2\ldots a_{j+\nu_0}} = \sum_{s=0}^{j-\nu_0}\sum_{k=0}^{\infty} \alpha(j-\nu_0, s, 2\nu_0) \frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k!\,\Gamma\left(\frac{i\rho}{2}-j+s-k\right)} \alpha_0^{\frac{i\rho}{2}-1-j+s} (z_c\bar{z}^c)^{\frac{i\rho}{2}-1-j+s-k}$$

$$\cdot \hat{\beta}^{p_1}_{q_1}\hat{\beta}^{p_2}_{q_2}\ldots\hat{\beta}^{p_k}_{q_k} z_{p_1} z_{p_2}\ldots z_{p_k} z_{p_{k+1}} z_{p_{k+2}}\ldots z_{p_{k+j+\nu_0-s}} \bar{z}^{q_1}\bar{z}^{q_2}\ldots\bar{z}^{q_k}\bar{z}^{q_{k+1}}\bar{z}^{q_{k+2}}\ldots\bar{z}^{q_{k+j-\nu_0-s}}$$

$$\cdot \underset{(a;b)}{S}\, \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\ldots\delta^{b_s}_{a_s} g^{p_{k+1}}_{a_{s+1}} g^{p_{k+2}}_{a_{s+2}} \ldots g^{p_{k+j+\nu_0-s}}_{a_{j+\nu_0}} \overset{+}{g}{}^{b_{s+1}}_{q_{k+1}} \overset{+}{g}{}^{b_{s+2}}_{q_{k+2}} \ldots \overset{+}{g}{}^{b_{j-\nu_0}}_{q_{k+j-\nu_0-s}} \tag{6.5}$$

Here for convenience we had made some changes in the notations of the summation indices:

$$\begin{array}{ll} c_{s+1} \to p_{k+1} & d_{s+1} \to q_{k+1} \\ c_{s+2} \to p_{k+2} & d_{s+2} \to q_{k+2} \\ \cdots\cdots & \cdots\cdots \\ c_{j+\nu_0} \to p_{k+j+\nu_0-s} & d_{j-\nu_0} \to q_{k+j-\nu_0-s} \end{array}$$

Now we express the product $z_{p_1}z_{p_2} \dots z_{p_{k+j+\nu_0-s}}\bar{z}^{q_1}\bar{z}^{q_2} \dots \bar{z}^{q_{k+j-\nu_0-s}}$ in terms of $\Phi^{q_1q_2\dots}_{p_1p_2\dots}$ by means of (5.4) and we put these expressions into (6.5). We get:

$$
\begin{aligned}
&\mathrm{T}_g f^{b_1b_2\dots b_{j-\nu_0}}_{a_1a_2\dots a_{j+\nu_0}} \\
&= \sum_{s=0}^{j-\nu_0}\sum_{k=0}^{\infty}\sum_{r=0}^{k+j-\nu_0-s} \alpha(j-\nu_0), s, 2\nu_0).\beta(k+j-\nu_0-s, r, 2\nu_0)\frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k!\,\Gamma\left(\frac{i\rho}{2}-j+s-k\right)}\alpha_0^{\frac{i\rho}{2}-1-j-s} \\
&\qquad .\left(z_c\bar{z}^c\right)^{\frac{i\rho}{2}-1-\nu_0-r}\hat{\beta}^{p_1}_{q_1}\hat{\beta}^{p_2}_{q_2}\dots\hat{\beta}^{p_k}_{q_k} \\
&.\ \underset{(a;b)}{\mathrm{S}}\ \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}\dots\delta^{b_s}_{a_s}g^{p_{k+1}}_{a_{s+1}}g^{p_{k+2}}_{a_{s+2}}\dots g^{p_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\ g^{+b_{s+1}}_{q_{k+2}}g^{+b_{s+2}}_{q_{k+2}}\dots g^{+b_{j-\nu_0}}_{q_{k+j-\nu_0-s}} \\
&.\ \underset{(p;q)}{\mathrm{S}}\ \Phi^{q_1q_2\dots q_r}_{p_1p_2\dots p_{r+2\nu_0}}\delta^{q_{r+1}}_{p_{r+2\nu_0+1}}\delta^{q_{r+2}}_{p_{r+2\nu_0+2}}\dots\delta^{q_{k+j-\nu_0-s}}_{p_{k+j+\nu_0-s}}
\end{aligned}
\tag{6.6}
$$

Note that the product

$$
\left\{\hat{\beta}^{p_1}_{q_1}\dots\hat{\beta}^{p_k}_{q_k}g^{p_{k+1}}_{a_{s+1}}g^{p_{k+2}}_{a_{s+2}}\dots g^{p_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\ g^{+b_{s+1}}_{q_{k+1}}g^{+b_{s+2}}_{q_{k+2}}\dots g^{+b_{j-\nu_0}}_{q_{k+j-\nu_0-s}}\right\}
.\sum_{\mathrm{P}(p;q)}\left\{\Phi^{q_1q_2\dots q_r}_{p_1p_2\dots p_{r+2\nu_0}}\delta^{q_{r+1}}_{p_{r+2\nu_0+1}}\delta^{q_{r+2}}_{p_{r+2\nu_0+2}}\dots\delta^{q_{k+j-\nu_0-s}}_{p_{k+j+\nu_0-s}}\right\}
$$

can be represented also in the following manner

$$
\Phi^{q_1q_2\dots q_r}_{p_1p_2\dots p_{r+2\nu_0}}\delta^{q_{r+1}}_{p_{r+2\nu_0+1}}\delta^{q_{r+2}}_{p_{r+2\nu_0+2}}\dots\delta^{q_{k+j-\nu_0-s}}_{p_{k+j+\nu_0-s}}
.\sum_{\mathrm{P}(p;q)}\hat{\beta}^{p_1}_{q_1}\dots\hat{\beta}^{p_k}_{q_k}g^{p_{k+1}}_{a_{s+1}}g^{p_{k+2}}_{a_{s+2}}\dots g^{p_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\ g^{+b_{s+1}}_{q_{k+1}}g^{+b_{s+2}}_{q_{k+2}}\dots g^{+b_{j-\nu_0}}_{q_{k+j-\nu_0-s}}
$$

Therefore we can rewrite (6.6) in a more convenient form

$$
\begin{aligned}
\mathrm{T}_g f^{b_1b_2\dots b_{j-\nu_0}}_{a_1a_2\dots a_{j+\nu_0}} &= \sum_{s=0}^{j-\nu_0}\sum_{k=0}^{\infty}\sum_{j'=\nu_0}^{k+j-s}\left(z\bar{z}^c\right)^{\frac{i\rho}{2}-1-j'}\Phi^{q_1q_2\dots q_{j'-\nu_0}}_{p_1p_2\dots p_{j'+\nu_0}}.\alpha_0^{\frac{i\rho}{2}-1-j+s} \\
&.\alpha(j-\nu_0, s, 2\nu_0)\beta(k+j-\nu_0-s, j'-\nu_0, 2\nu_0)\frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k!\,\Gamma\left(\frac{i\rho}{2}-j+s-k\right)}\delta^{q_{j'-\nu_0+1}}_{p_{j'+\nu_0+1}}\delta^{q_{j'-\nu_0+2}}_{p_{j'+\nu_0+2}} \\
&\qquad\dots\delta^{q_{k+j-\nu_0-s}}_{p_{k+j+\nu_0-s}} \\
&.\frac{1}{(j-\nu_0)!(j+\nu_0)!(k+j-\nu_0-s)!(k+j+\nu_0-s)!}\sum_{\mathrm{P}(a;b;p;q)}\hat{\beta}^{p_1}_{q_1}\dots\hat{\beta}^{p_k}_{q_k}\delta^{b_1}_{a_1}\dots\delta^{b_s}_{a_s}g^{p_{k+1}}_{a_{s+1}} \\
&\qquad\dots g^{p_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\ g^{+b_{s+1}}_{q_{k+1}}\dots g^{+b_{j-\nu_0}}_{q_{k+j-\nu_0-s}}
\end{aligned}
\tag{6.7}
$$

where we put $r + \nu_0 \equiv j'$. From (6.7) and from definitions (5.1) and (5.6) we get the following expression for $D^{b_1b_2\ldots,\,c_1c_2\ldots}_{a_1a_2\ldots,\,d_1d_2\ldots}(g)$:

$$
\begin{aligned}
&D^{b_1b_2\ldots b_{j-\nu_0};c_1c_2\ldots c_{j'+\nu_0}}_{a_1a_2\ldots a_{j+\nu_0};d_1d_2\ldots d_{j'-\nu_0}}(g)\\
&= \sum_{k=\max(0,j'-j)}^{\infty} \; \sum_{s=0}^{\min(j-\nu_0,j+k-j')} \alpha_0^{\frac{i\rho}{2}-1-j+s}\, \alpha(j-\nu_0, s, 2\nu_0)\\
&\qquad \beta(k+j-\nu_0-s, j'-\nu_0, 2\nu_0)\,\frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k!\,\Gamma\left(\frac{i\rho}{2}-j+s-k\right)}\\
&\cdot \frac{1}{(j-\nu_0)!\,(j+\nu_0)!\,(k+j-\nu_0-s)!\,(k+j+\nu_0-s)!}\,\delta^{d_{j'-\nu_0+1}}_{c_{j'+\nu_0+1}}\delta^{d_{j'-\nu_0+2}}_{c_{j'+\nu_0+2}}\ldots\delta^{d_{k+j-\nu_0-s}}_{c_{k+j+\nu_0-s}}\\
&\cdot \sum_{P(a;b;c;d)} \hat\beta^{c_1}_{d_1}\ldots\hat\beta^{c_k}_{d_k}\,\delta^{b_1}_{a_1}\ldots\delta^{b_s}_{a_s}\, g^{c_{k+1}}_{a_{s+1}}\ldots g^{c_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\, \overset{+}{g}{}^{b_{s+1}}_{d_{k+1}}\ldots \overset{+}{g}{}^{b_{j-\nu_0}}_{d_{k+j-\nu_0-s}} \qquad (6.8)
\end{aligned}
$$

Putting here the explicit expressions (5.3) and (5.5) for $\alpha$ and $\beta$ we obtain the final result:

$$
\begin{aligned}
&D^{b_1b_2\ldots b_{j-\nu_0};c_1c_2\ldots c_{j'+\nu_0}}_{a_1a_2\ldots a_{j+\nu_0};d_1d_2\ldots d_{j'-\nu_0}}(g)\\
&= \sum_{k=\max(0,j'-j)}^{\infty} \; \sum_{s=0}^{\min(j-\nu_0,j+k-j')} (-1)^s \frac{(2j-s)!\,(2j'+1)!}{s!\,k!\,(2j)!\,(j'-\nu_0)!\,(j'+\nu_0)!\,(j-\nu_0-s)!\,(j+\nu_0-s)!\,(k+j-j'-s)!\,(k+j+j'-s+1)!}\\
&\cdot \frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{\Gamma\left(\frac{i\rho}{2}-j+s-k\right)}\,\alpha_0^{\frac{i\rho}{2}-1-j+s}\,\delta^{d_{j'-\nu_0+1}}_{c_{j'+\nu_0+1}}\delta^{d_{j'-\nu_0+2}}_{c_{j'+\nu_0+2}}\ldots\delta^{d_{k+j-\nu_0-s}}_{c_{k+j+\nu_0-s}}\\
&\cdot \sum_{P(a;b;c;d)} \hat\beta^{c_1}_{d_1}\hat\beta^{c_2}_{d_2}\ldots\hat\beta^{c_k}_{d_k}\,\delta^{b_1}_{a_1}\ldots\delta^{b_s}_{a_s}\, g^{c_{k+1}}_{a_{s+1}}\ldots g^{c_{k+j+\nu_0-s}}_{a_{j+\nu_0}}\, \overset{+}{g}{}^{b_{s+1}}_{d_{k+1}}\ldots \overset{+}{g}{}^{b_{j-\nu_0}}_{d_{k+j-\nu_0-s}} \qquad (6.9)
\end{aligned}
$$

Consider now some particular cases:

1. If $g$ is a pure Lorentz transformation in the $(x_3, x_4)$ plane, i. e.

$$
g = \varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix}
$$

then

$$
\overset{+}{g}g = \begin{pmatrix} \varepsilon^{-2} & 0 \\ 0 & \varepsilon^{2} \end{pmatrix} = \frac{1}{2}(\varepsilon^{-2} + \varepsilon^{2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(\varepsilon^{-2} - \varepsilon^{2})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

and therefore

$$\alpha_0 = \frac{1}{2}(\varepsilon^{-2} + \varepsilon^2); \qquad \hat{\beta} = \frac{\varepsilon^{-2} - \varepsilon^2}{\varepsilon^{-2} + \varepsilon^2}\sigma_3 \tag{6.10}$$

Putting (6.10) into (6.9) we get:

$$D^{b_1 b_2 \ldots b_{j-\nu_0}; c_1 c_2 \ldots c_{j'+\nu_0}}_{a_1 a_2 \ldots a_{j+\nu_0}; d_1 d_2 \ldots d_{j'-\nu_0}}(\varepsilon)$$
$$= \sum_{k=\max(0,j'-j)}^{\infty} \sum_{s=0}^{\min(j-\nu_0+k-j')} \sum_{m=0}^{\infty} (-1)^{s+m} \frac{(2j-s)!(2j'+1)!}{m!s!k!(2j)!(j'-\nu_0)!(j'+\nu_0)!(j-\nu_0-s)!} \atop (j+\nu_0-s)!(k+j-j'-s)!(k+j+j'-s+1)!$$
$$\cdot \frac{1}{2^{k+m}} \cdot \frac{\Gamma\left(\frac{i\rho}{2} - j + s\right)}{\Gamma\left(\frac{i\rho}{2} - j + s - k - m\right)} \varepsilon^{2\left(j-s+1-\frac{i\rho}{2}\right)} (1-\varepsilon^4)^{m+k} \delta^{d_{j'-\nu_0+1}}_{c_{j'+\nu_0+1}} \delta^{d_{j'-\nu_0+2}}_{c_{j'+\nu_0+2}} \cdots \delta^{d_{k+j-\nu_0-s}}_{c_{k+j+\nu_0-s}}$$
$$\cdot \sum_{P(a;b;c;d)} (\sigma_3)^{c_1}_{d_1} \cdots (\sigma_3)^{c_k}_{d_k} \delta^{b_1}_{a_1} \cdots \delta^{b_s}_{a_s} \varepsilon^{c_{k+1}}_{a_{s+1}} \cdots \varepsilon^{c_{k+j+\nu_0-s}}_{a_{j+\nu_0}} \varepsilon^{b_{s+1}}_{d_{k+1}} \cdots \varepsilon^{b_{j-\nu_0}}_{d_{k+j-\nu_0-s}} \tag{6.11}$$

If we put $j = j'$ in Eq. (6.11) and

$$a_1 = a_2 = \ldots = a_{j+\nu_0} = c_1 = c_2 = \ldots = c_{j+\nu_0} = 1$$
$$b_1 = b_2 = \ldots = b_{j-\nu_0} = d_1 = d_2 = \ldots = d_{j-\nu_0} = 2,$$

then we have

$$D^{\nu_0\rho}_{jj;jj}(\varepsilon) \equiv D^{\overbrace{22\ldots2}^{j-\nu_0 \text{ times}}\,;\,\overbrace{11\ldots1}^{j+\nu_0 \text{ times}}}_{\underbrace{11\ldots1}_{j+\nu_0 \text{ times}}\,;\,\underbrace{22\ldots2}_{j-\nu_0 \text{ times}}}(\varepsilon)$$
$$= \varepsilon^{2\left(j-\nu_0+1-\frac{i\rho}{2}\right)} F\left(j+1-\frac{i\rho}{2}, j-\nu_0+1; 2j+2; 1-\varepsilon^4\right)$$

This result can be obtained also from (4.11).

2. If $g$ is a Lorentz transformation from the rest frame of a particle with mass $m_0$ to the frame in which this particle has momentum $p_\mu = (\vec{p}, iE)$ then

$$g = \frac{E + m_0 - \vec{\sigma}.\vec{p}}{\sqrt{2m_0(E+m_0)}}; \qquad \overset{+}{g}g = \frac{E - \vec{\sigma}\vec{p}}{m_0}$$

and therefore

$$\alpha_0 = \frac{E}{m_0}$$
$$\hat{\beta} = -\frac{1}{E}\vec{\sigma}\vec{p}$$

3. In the case $j = 0$ ($\nu_0 = 0$) formula (6.9) becomes

$$D^{;c_1c_2\dots c_{j'}}_{;d_1d_2\dots d_{j'}}(g) = \alpha_0^{\frac{i\rho}{2}-1} \cdot \sum_{k=j'}^{\infty} \frac{(2j'+1)!}{(j'!)^2(k-j')!\,(k+j'+1)!} \cdot \frac{\Gamma\left(\frac{i\rho}{2}\right)}{\Gamma\left(\frac{i\rho}{2}-k\right)}$$

$$\cdot\, \delta^{d_{j'+1}}_{c_{j'+1}} \delta^{d_{j'+2}}_{c_{j'+2}} \dots \delta^{d_k}_{c_k} \sum_{P(c)} \hat{\beta}^{c_1}_{d_1} \hat{\beta}^{c_2}_{d_2} \dots \hat{\beta}^{c_k}_{d_k}$$

## § 7. MOST DEGENERATE PRINCIPAL SERIES OF SL(n, C)

For the most degenerate principal series of SL($n$, $c$) we can immediately use the method developped in § 5, 6, for the group SL(2, C) with some light changes. Thus, instead of Eq. (5.1) we have

$$f^{b_1b_2\dots b_{j-\nu_0}}_{a_1a_2\dots a_{j+\nu_0}}(z_1, \dots, z_n) = (z_c\bar{z}^c)^{\frac{i\rho}{2}-\frac{n}{2}-j} \Phi^{b_1b_2\dots b_{j-\nu_0}}_{a_1a_2\dots a_{j+\nu_0}}, \qquad | a, b = 1, 2, \dots, n |$$

where $j$, $\nu_0$ are also integer of half-integer, however, $j$ does not still denote the spin. The formulæ (5.2) and (5.4) remain if instead of the expressions (5.3) and (5.5) for $\alpha$ and $\beta$ we take

$$\alpha(t, s, k) = (-1)^s \frac{t!\,(t+k)!\,(2t+k+n-2-s)!}{s!\,(t-s)!\,(t+k-s)!\,(2t+k+n-2)!}$$

$$\beta(t, s, k) = \frac{t!\,(t+k)!\,(2s+k+n-1)!}{s!\,(s+k)!\,(t-s)!\,(t+k+n+s-1)!}$$

In order the get the formula for matrix elements

$$D^{b_1b_2\dots b_{j-\nu_0};c_1c_2\dots c_{j'+\nu_0}}_{a_1a_2\dots a_{j+\nu_0};d_1d_2\dots d_{j'-\nu_0}}(g)$$

corresponding to the transformation $z_a \to z'_a = z_b g_{ba}$ we make the analogous procedure as in § 6. The only difference is that we must now expand $g\overset{+}{g}$ in terms of the matrices generators $\lambda_i$ of the subgroup SU($n$). The results is:

$$D^{b_1b_2\dots b_{j-\nu_0};c_1c_2\dots c_{j'+\nu_0}}_{a_1a_2\dots a_{j+\nu_0};d_1d_2\dots d_{j'-\nu_0}}(g)$$

$$= \sum_{k=\max(0,j'-j)}^{\infty} \sum_{s=0}^{\min(j-\nu_0, j+k-j')} (-1)^s \frac{(2j+n-2-s)!(2j'+n-1)!}{s!k!(j-\nu_0-s)!(j+\nu_0-s)!(2j+n-2)!(j'-\nu_0)!}$$
$$(j'+\nu_0)!(k+j-j'-s)!(k+j+j'-s+n-1)!$$

$$\cdot \frac{\Gamma\left(\frac{i\rho}{2} - \frac{n}{2} - j + s + 1\right)}{\Gamma\left(\frac{i\rho}{2} - \frac{n}{2} - j + s + 1 - k\right)} \alpha_0^{\frac{i\rho}{2} - \frac{n}{2} - j + s} \cdot \delta_{c_{j'}+\nu_0+1}^{d_{j'}-\nu_0+1} \delta_{c_{j'}+\nu_0+2}^{d_{j'}-\nu_0+2} \cdots \delta_{c_{k+j+\nu_0-s}}^{d_{k+j-\nu_0-s}}$$

$$\cdot \sum_{P(a;b;c;d)} \hat{\beta}_{d_1}^{c_1} \hat{\beta}_{d_2}^{c_2} \cdots \hat{\beta}_{d_k}^{c_k} \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_s}^{b_s} g_{a_{s+1}}^{c_{k+1}} \cdots g_{a_{j+\nu_0}}^{c_{k+j+\nu_0-s}} \overset{+}{g}{}_{d_{k+1}}^{b_{s+1}} \cdots \overset{+}{g}{}_{d_{k+j-\nu_0-s}}^{b_{j-\nu_0}}$$

where

$$\overset{+}{g}g = \alpha_0 I + \sum_{i=1}^{n^2-1} \alpha_i \lambda_i = \alpha_0(1 + \hat{\beta}) \, ; \qquad \hat{\beta} = \frac{1}{\alpha_0} \sum_{i=1}^{n^2-1} \alpha_i \lambda_i$$

## § 8. SPACE REFLECTION FOR THE GROUP SL(2, C)

Now we identify the group SL(2, C) with the homogeneous proper Lorentz group and consider the space reflection P. There exist the following relations between P and the generators of the SL(2, C) group:

$$\begin{aligned} PM_j^i P^{-1} &= M_j^i \\ PN_j^i P^{-1} &= -N_j^i \\ P^2 &= 1, \end{aligned} \tag{8.1}$$

where $M_j^i$ and $N_j^i$ are compact and non-compact generators, respectively.

As it was known, the commutation relations for $M_j^i$ and $N_j^i$ are of the form:

$$\begin{aligned} [M_j^i, M_l^k] &= \delta_l^i M_j^k - \delta_j^k M_l^i \\ [N_j^i, N_l^k] &= -\delta_l^i M_j^k + \delta_j^k M_l^i \\ [M_j^i, N_l^k] &= \delta_l^i N_j^k - \delta_i^k N_l^i \end{aligned} \tag{8.2}$$

It is easy to see that in the space of homogeneous functions with canonical basis (4.7) these generators are [1]:

$$\begin{aligned} M_j^i &= z \frac{\partial}{\partial z_i} - \bar{z}^i \frac{\partial}{\partial \bar{z}^j} - \frac{1}{2} \delta_j^i \left( z_a \frac{\partial}{\partial z_a} - \bar{z}^a \frac{\partial}{\partial \bar{z}^a} \right) \\ i N_j^i &= z_j \frac{\partial}{\partial z_i} + \bar{z}^i \frac{\partial}{\partial \bar{z}^j} - \frac{1}{2} \delta_j^i \left( z_a \frac{\partial}{\partial z_a} + \bar{z}^a \frac{\partial}{\partial \bar{z}^a} \right) \end{aligned} \tag{8.3}$$

---

(1) The correspondence between our $M_j^i$, $N_j^i$ and H, F in reference [*12*] is following:

$$M_2^1 = H_- \ , \quad M_1^2 = H_+ \ , \quad M_1^1 = -M_2^2 = H_3$$
$$N_2^1 = F_- \ , \quad N_1^2 = F_+ \ , \quad N_1^1 = -N_2^2 = F_3$$

From eqs. (8.1) and (8.3) it follows that the operator P acts in such a way that

$$z_a \to \varepsilon_{ab} \bar{z}^b$$

$$\bar{z}^a \to z_b \varepsilon^{ba}$$

$$\varepsilon_{ab} = \varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and therefore

$$\mathrm{P} f(z_1, z_2\,;\, \bar{z}^1, \bar{z}^2) = f(\bar{z}^2, -\bar{z}^1\,;\, -z_2, z_1) \tag{8.4}$$

From eqs. (8.4) and (8.5) after some simple calculations we get:

$$\mathrm{P} f_{jm}^{\nu_0\rho}(z_1, z_2) = (-1)^{\frac{i\rho}{2}-1-j} f_{jm}^{-\nu_0\rho}(z_1, z_2) \tag{8.5}$$

Thus, under P the basis elements of representation $\mathfrak{C}_{\nu_0\rho}$ transform into the basis elements of $\mathfrak{C}_{-\nu_0\rho}$ which in its turn is equivalent to $\mathfrak{C}_{\nu_0, -\rho}$. This means that under P only the space $D_{\left(\frac{i\rho}{2}-1, \frac{i\rho}{2}-1\right)}$ ($\nu_0 = 0$) or the space $D_{(\nu_0-1, -\nu_0-1)}(\rho=0)$ transforms into itself. Moreover, it is seen from (8.5) that the parity of the basis vectors $f_{jm}$ differs one from another by a factor $(-1)^j$.

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