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Commutants of Certain Operator Algebras on Fock Space (*)

by

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H. Ekstein recently proposed to replace the usual assumptions on invariance under a group of internal symmetries by a simple postulate on the commutant of the S matrix [1]. For this he needs the following theorem which we propose to prove in this note.

THEOREM. — Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$ the respective Hilbert spaces of symmetric and antisymmetric tensors of order n over $\mathcal{H}$ (i. e., the symmetric and antisymmetric parts, respectively, of $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$, the tensor product in which $\mathcal{H}$ appears n times as a factor). Furthermore, let $\mathcal{H}^{\vee}$ and $\mathcal{H}^{\wedge}$, respectively, denote the symmetric and Grassmann algebras over $\mathcal{H}$ [7], i. e.,

$$(1) \quad \mathcal{H}^{\vee} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n},$$

$$(2) \quad \mathcal{H}^{\wedge} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n},$$

where $\mathcal{H}^{\vee}$ and $\mathcal{H}^{\wedge}$ are, respectively, the Fock spaces of bosons and fermions with wave functions in $\mathcal{H}$. Also, let $B_1 = L_1 \otimes I_{\mathcal{H}_2}$ and $B_2 = I_{\mathcal{H}_1} \otimes L_2$ be the von Neumann algebras over $\mathcal{H}$, where L_1 and L_2 are arbitrary bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; and let $B_1^{\vee n}$, $B_1^{\wedge n}$ ($B_2^{\vee n}$, $B_2^{\wedge n}$)

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and $B_2^{\wedge n}$) denote the von Neumann algebras induced in $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$, respectively, by $B_1(B_2)$. Analogously, let $B_1^\vee$ and $B_1^\wedge$ ($B_2^\vee$ and $B_2^\wedge$) denote the von Neumann algebras induced in $\mathcal{H}^\vee$ and $\mathcal{H}^\wedge$, respectively, by $B_1(B_2)$. Then $B_1^{\vee n}$ and $B_2^{\vee n}$ (and likewise $B_1^\vee$ and $B_2^\vee$) are the commutants of one another; and corresponding statements hold for $B_1^{\wedge n}$ and $B_2^{\wedge n}$ (and for $B_1^\wedge$ and $B_2^\wedge$). That is,

$$(3) \quad (B_1^{\vee n})' = B_2^{\vee n} \quad \text{in } \mathcal{H}^{\vee n},$$

$$(4) \quad (B_1^\vee)' = B_2^\vee \quad \text{in } \mathcal{H}^\vee,$$

$$(5) \quad (B_1^{\wedge n})' = B_2^{\wedge n} \quad \text{in } \mathcal{H}^{\wedge n},$$

$$(6) \quad (B_1^\wedge)' = B_2^\wedge \quad \text{in } \mathcal{H}^\wedge.$$

In fact, the result needed in [I] refers to the action on $\mathcal{H}_2$ of the group $SU(p)$ (where p , supposed finite, is the dimension of $\mathcal{H}_2$) but the problem of reduction in irreducible tensors is known to be the same for $SU(p)$ and the general linear group. If the group is $SO(p)$ instead of $SU(p)$, it is known that there is a further decomposition (as shown in [4] and in chap. 10 of [5]).

Proof of the Theorem. — Let $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_2^{\otimes n}$ be the tensor n th powers of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then it is obvious that

$$(7) \quad \mathcal{H}^{\otimes n} = \mathcal{H}_1^{\otimes n} \otimes \mathcal{H}_2^{\otimes n}.$$

On the other hand, if S and A denote the symmetrizing and antisymmetrizing operators, respectively (i. e., the Hermitean projectors whose action on $\mathcal{H}^{\otimes n}$ produces $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$, respectively), then

$$(8) \quad \mathcal{H}^{\vee n} = S\mathcal{H}^{\otimes n},$$

$$(9) \quad \mathcal{H}^{\wedge n} = A\mathcal{H}^{\otimes n}.$$

Now let $B_1^{\otimes n}$ and $B_2^{\otimes n}$, the tensor n th powers of B_1 and B_2 , respectively, be von Neumann algebras on $\mathcal{H}^{\otimes n}$ (chap. I, § 2, section 4 of [6]). By definition, one has

$$(10) \quad B_1^{\vee n} = (B_1^{\otimes n})_S, \quad B_2^{\vee n} = (B_2^{\otimes n})_S,$$

$$(11) \quad B_1^{\wedge n} = (B_1^{\otimes n})_A, \quad B_2^{\wedge n} = (B_2^{\otimes n})_A.$$

Let $\mathfrak{A}$ be a von Neumann algebra on the Hilbert space H and let E be a Hermitean projector in H belonging either to $\mathfrak{A}$ or its commutant $\mathfrak{A}'$. Here Dixmier's symbol $\mathfrak{A}_E$ (defined in chap. I, § 2, section 1 of [6]) is used

to denote that the set of operators $E A E$, in which A runs through $\mathfrak{A}$, is to be restricted to $E H$. In (10) and (11), it is clear that both A and S belong to the commutants $(B_1^{\otimes n})'$ and $(B_2^{\otimes n})'$ since $B_1^{\otimes n}$ and $B_2^{\otimes n}$ leave invariant the spaces $S\mathcal{H}^{\otimes n} = \mathcal{H}^{\vee n}$ and $A\mathcal{H}^{\otimes n} = \mathcal{H}^{\wedge n}$ of symmetric and antisymmetric tensors over $\mathcal{H}$.

We notice on the other hand that if $\mathfrak{L}(H)$ denotes the set of all bounded operators on a Hilbert space H and $C(H)$ denotes the multiples of the unit operator on H , then by Proposition 14 on p. 102 of [6] it follows that

$$(12) \quad B_1^{\otimes n} = \mathfrak{L}(\mathcal{H}_1)^{\otimes n} \otimes C(\mathcal{H}_2^{\otimes n}),$$

$$(13) \quad B_2^{\otimes n} = C(\mathcal{H}_1^{\otimes n}) \otimes \mathfrak{L}(\mathcal{H}_2)^{\otimes n}.$$

We can reduce the algebras $\mathfrak{L}(\mathcal{H}_1)^{\otimes n}$ and $\mathfrak{L}(\mathcal{H}_2)^{\otimes n}$ acting in the tensor spaces $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_2^{\otimes n}$ by decomposing these spaces into spaces of tensors that are irreducible under permutations. Specifically, if $\mathcal{H}_{1\chi}^n, (\mathcal{H}_{2\chi}^n)$ represents the subspace of $\mathcal{H}_1^{\otimes n}, (\mathcal{H}_2^{\otimes n})$ consisting of the tensors of symmetry character χ , where χ is any character of the symmetric group of n elements, then we have

$$(14) \quad \mathcal{H}_1^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}_{1\chi}^n,$$

$$(15) \quad \mathcal{H}_2^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}_{2\chi}^n.$$

Since the subspaces $\mathcal{H}_{1\chi}^n, (\mathcal{H}_{2\chi}^n)$ are the irreducible stable subspaces for the algebras $\mathfrak{L}(\mathcal{H}_1)^{\otimes n}$ and $\mathfrak{L}(\mathcal{H}_2)^{\otimes n}$, we have

$$(16) \quad \mathfrak{L}(\mathcal{H}_1)^{\otimes n} = \prod_{\text{all } \chi} \mathfrak{L}(\mathcal{H}_{1\chi}^n),$$

$$(17) \quad \mathfrak{L}(\mathcal{H}_2)^{\otimes n} = \prod_{\text{all } \chi} \mathfrak{L}(\mathcal{H}_{2\chi}^n).$$

Here Π denotes a product of von Neumann algebras ([6], chap. I, § 2, section 2) ⁽¹⁾.

According to (7) (14) and (15), we have

$$(18) \quad \mathcal{H}^{\otimes n} = \bigoplus_{\substack{\text{all pairs} \\ \chi_1 \chi_2}} (\mathcal{H}_{1\chi_1}^n \otimes \mathcal{H}_{2\chi_2}^n)$$

⁽¹⁾ It is *not* a tensor product, and is often called the direct sum of algebras.

and out of $\mathcal{H}^{\otimes n}$ we have to select the subspace $\mathcal{H}^{\vee n} = S\mathcal{H}^{\otimes n}$ ($\mathcal{H}^{\wedge n} = A\mathcal{H}^{\otimes n}$) of symmetric (antisymmetric) tensors. Since, according to (7), the representation of the symmetric group in $\mathcal{H}^{\otimes n}$ is evidently the tensor product of its representation in $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\otimes n}$, we know (see Appendix) that the only terms in (18) that contain symmetric (antisymmetric) tensors are those for which $\chi_2 = \chi_1$ ($\chi_2 = \varepsilon\chi_1$, where ε is the alternating character). Let P and Q, respectively, denote the projections in $\mathcal{H}^{\otimes n}$ on these subspaces; i. e., let

$$(19) \quad P\mathcal{H}^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}_{1\chi}^n \otimes \mathcal{H}_{2\chi}^n,$$

$$(20) \quad Q\mathcal{H}^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}_{1\chi}^n \otimes \mathcal{H}_{2\varepsilon\chi}^n.$$

One thus has

$$(21) \quad P \geq S, \quad Q \geq A$$

and therefore

$$(22) \quad B_i^{\vee n} = (B_i^{\otimes n})_S = [(B_i^{\otimes n})_P]_S,$$

$i = 1, 2$

$$(23) \quad B_i^{\wedge n} = (B_i^{\otimes n})_A = [(B_i^{\otimes n})_Q]_A.$$

Obviously $B_i^{\otimes n}$ commutes with P and Q. To prove this formally, note that from (16) and (12) it follows that

$$(24) \quad \begin{aligned} B_1^{\otimes n} &= \left\{ \prod_{\chi_1} \mathcal{L}(\mathcal{H}_{1\chi_1}^n) \right\} \otimes C(\mathcal{H}_2^{\otimes n}), \\ &= \left\{ \prod_{\chi_1, \chi_2} \mathcal{L}(\mathcal{H}_{1\chi_1}^n) \right\} \otimes C(\mathcal{H}_{2\chi_2}^n), \end{aligned}$$

and analogous expressions follow for $B_2^{\otimes n}$. Now, we have

$$(25) \quad (B_1^{\otimes n})_P = \prod_{\chi} \left\{ \mathcal{L}(\mathcal{H}_{1\chi}^n) \otimes C(\mathcal{H}_{2\chi}^n) \right\},$$

$$(26) \quad (B_2^{\otimes n})_P = \prod_{\chi} \left\{ C(\mathcal{H}_{1\chi}^n) \otimes \mathcal{L}(\mathcal{H}_{2\chi}^n) \right\},$$

and hence

$$(27) \quad [(B_1^{\otimes n})_P]' = (B_2^{\otimes n})_P.$$

Analogously,

$$(25') \quad (B_1^{\otimes n})_Q = \prod_x \{ \mathcal{L}(\mathcal{H}_{1x}^n) \otimes C(\mathcal{H}_{2x}^n) \},$$

$$(26') \quad (B_2^{\otimes n})_Q = \prod_x \{ C(\mathcal{H}_{1x}^n) \otimes \mathcal{L}(\mathcal{H}_{2x}^n) \},$$

and hence

$$(27') \quad [(B_1^{\otimes n})_Q]' = (B_2^{\otimes n})_Q.$$

Now to (22) and (23) we apply the well known result $(\mathfrak{A}_\varepsilon)' = (\mathfrak{A}')_\varepsilon$ (Prop. 1(i) on p. 18 of [6]). Then

$$\begin{aligned} (B_1^{\vee n})' &= \{ [(B_1^{\otimes n})_P]_s \}' = \{ [(B_1^{\otimes n})_P]' \}_s \\ &= [(B_2^{\otimes n})_P]_s = B_2^{\vee n} \end{aligned}$$

and analogously

$$(B_1^{\wedge n})' = B_2^{\wedge n}.$$

Equations (4) and (6) now immediately result upon repeated application of Prop. 1(i) on p. 18 of [6] and making use of the remark that the projector on $\mathcal{H}^{\vee n}$ in $\mathcal{H}^\vee$ (on $\mathcal{H}^{\wedge n}$ in $\mathcal{H}^\wedge$) evidently belongs to the center of $B^\vee$ ($B^\wedge$).

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APPENDIX

In the course of the proof we have used the following well-known

LEMMA. — Let U_χ be the irreducible representation of the symmetric group of n objects corresponding to its character χ . The number of times the identical (or alternate) representation is contained in $U_\chi \otimes U_{\chi'}$, is equal to $\delta_{\chi, \chi'}$ (or $\delta_{\chi, \varepsilon \chi'}$), where

$$\delta_{\chi_1, \chi_2} = \begin{cases} 1 & \text{for } \chi_1 = \chi_2 \\ 0 & \text{for } \chi_1 \neq \chi_2 \end{cases}$$

and ε is the alternate character (equal to the parity of permutations).

Proof. — Let D_1 and D_2 be two representations of a compact group G of order h , and let χ_1 and χ_2 be the corresponding characters. The number of times n_1 that D_1 is contained in D_2 is given ([5], p. 105) by

$$n_1 = \frac{1}{h} \sum_{s \in G} \overline{\chi_2(s)} \chi_1(s).$$

In our particular case we have $h = n!$ and χ_2 is the character of $U_\chi \otimes U_{\chi'}$ and is equal to the product $\chi \chi'$. On the other hand, $\chi_1 = 1$ for the identity and $\chi_1 = \varepsilon$ for the alternate representation. The result then follows from the orthogonality relations of characters.

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