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Section A :
Physique théorique.

The place of random processes and random fields in Quantum Theory

by

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SUMMARY. — The states of a system in Quantum Theory can be represented by wave functions which are elements of a Hilbert space. All the different representations are unitarily equivalent. However, there exists a particular representation where the wave functions can be treated as random variables. It is obtained by the use of a generalized random field (GRF) which associates with every $\psi \in L^2(R^N)$ the random variable $F(\psi, \omega) \in L^2(\Omega, \mathcal{B}, P)$. We prove that the GRF which has to be used is uniquely characterized by the basic principles of Quantum Theory and this is « white noise » defined as the derivative, in the sense of the theory of distributions, of Wiener process.

Once we are dealing with random variables, we can define random events whose probabilities are the same as the probabilities of occurrence of different eigenstates as they are postulated in ordinary Quantum Theory.

Hence a mathematical formalism is established which can be used to support those theories which provide a statistical interpretation of Quantum Theory based on thermodynamical fluctuations where Brownian motion has a role to play.

SOMMAIRE. — Les états d'un système en Mécanique Quantique peuvent être représentés par des fonctions d'onde qui sont des éléments d'un espace de Hilbert. Toutes ces représentations sont équivalentes à une transforma-

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tion unitaire près. Il existe cependant une représentation particulière telle que les fonctions d'onde peuvent être considérées comme des variables aléatoires. On l'obtient en utilisant un champ aléatoire généralisé (GRF) qui associe à tout élément $\psi \in L^2(R^N)$ la variable aléatoire

$$F(\psi, \omega) \in L^2(\Omega, \mathcal{B}, P).$$

Nous démontrons que le processus stochastique qui doit être utilisé est caractérisé d'une façon unique par les principes de base de la Mécanique Quantique. On trouve qu'il s'agit du « bruit blanc » défini comme étant la dérivée, au sens de la théorie des distributions, du processus brownien de Wiener.

A partir du moment où nous avons affaire à des variables aléatoires, il est possible de définir des événements aléatoires dont les probabilités sont les mêmes que celles qui sont postulées en Mécanique Quantique pour la mesure des états propres d'un système.

De cette façon on établit un formalisme mathématique qui peut servir de base aux théories qui cherchent une interprétation statistique de la Mécanique Ondulatoire s'appuyant sur des considérations de fluctuations thermo-dynamiques dans lesquelles le mouvement brownien joue un rôle.

INTRODUCTION

The Brownian motion functions $x(t, \alpha)$ have been defined rigorously by Norbert Wiener and Paul Levy. When the time t varies from $-\infty$ to $+\infty$, $x(t, \alpha)$ represents the physical Brownian motion, projected on an arbitrary axis, of a particle suspended in a fluid and subject to collisions with the surrounding molecules. The parameter α varies from 0 to 1 and except for a set of values of α of Lebesgue measure equal to zero, there is a one-to-one correspondence between the values of α and the Brownian trajectories of the particle. If we consider the interval [0, 1] as a sample space ([0, 1], $\mathcal{B}$, Lebesgue), $x(t, \alpha)$ is a random process called Wiener process.

The increment $x(dt, \alpha)/dt$ of $x(t, \alpha)$ over any interval $(t, t + dt)$ is another random process which is called « white noise ». Here we must notice that the quantity $x(dt, \alpha)$ has no meaning as an ordinary function. Indeed it is well-known that for almost all α, the functions $x(t, \alpha)$ are continuous and they have no derivative. However, $x(dt, \alpha)$ is well defined as a generalized function (distribution). Therefore the « white noise » exists as a generalized random process which is the derivative (in the sense of the theory of distributions) of the Wiener process.

This concept of increments of the Brownian motion functions can be extended by replacing the time variable t by a variable s in a Euclidean space R^N. R^N will be for instance a configuration space

$$\mathcal{C} = R^n \ni s = (q_1, \ldots, q^n)$$

or a Gibbs phase space

$$\Gamma = R^{2n} \ni s = (q_1, \ldots, q_n ; p_1, \ldots, p_n).$$

Thus we have a N dimensional « white noise » $x(ds, \alpha)$ and in this case we call it a Generalized Random Field (GRF) whereas the term Generalized Random Process (GRP) is reserved for the case where $s \in R^1$.

The real GRF « white noise » is mathematically expressible by the following functional relation :

$$(1) \qquad F(\psi, \alpha) = \int_{R^N} \psi(s) x(ds, \alpha) ; \qquad \alpha \in [0, 1]$$

where $\psi(s)$ is a real function belonging to $L^2(R^N)$. The sample space of the random variable $F(\psi, \alpha)$ is as above $\{ \alpha ; \alpha \in ([0, 1], \mathcal{B}, \text{Lebesgue}) \}$. However, since we are going to deal with Quantum Theory we need a definition of the complex GRF « white noise ». In that case $\psi(s) \in L^2(R^N)$ is a complex function and we use the complex distribution

$$X(ds ; \beta, \gamma) = \frac{x(ds, \beta) + ix(ds, \gamma)}{2^{\frac{1}{2}}}.$$

We have now two independent random parameters β and γ varying in $[0, 1]$. There is no difficulty to replace the two-dimensional random parameter (β, γ) by a single one $\alpha \in [0, 1]$; this can be done by mapping isomorphically (except for sets of points of zero Lebesgue measure) the square $[0, 1] \times [0, 1]$ onto the interval $[0, 1]$. Hence we can write $X(ds; \beta, \gamma) = X(ds, \alpha)$ and we arrive finally to the following definition of the complex GRF « white noise » which generalizes the definition (1):

$$(2) \qquad F(\psi, \alpha) = \int_{R^N} \psi(s) X(ds, \alpha) ; \qquad \psi(s) \in L^2(R^N), \quad \alpha \in [0, 1]$$

where $F(\psi, \alpha)$ is now a complex random variable.

The transformation (2) maps the entire $L^2(R^N)$ space onto a space $\mathcal{F}$ of random variables. An important property of $\mathcal{F}$ is that it is a sub-Hilbert space of $L^2([0, 1])$ such that the correspondence $L^2(R^N) \Leftrightarrow \mathcal{F}$ is unitary. If $\psi(s)$ is a wave function, representative of the state of a physical system in Quantum Theory, $F(\psi, \alpha)$ can be considered as a new representation of that state which is unitarily equivalent to the first one. In other words we have a representation where the wave function is at the same time a random variable.

In the next section we shall recall a result, due to Wiener ([*1*], p. 78), which is the motivation of our work. Using the random variables $F(\psi, \alpha)$, Wiener defines a class of random events whose probabilities are the same as the probabilities of finding a physical system in a particular eigenstate after a measurement has been performed on that system. Thus a way is provided to relate the theory of measurement and, in particular, the so-called « reduction of wave packets » which occur in quantum experiments, to a series of random events belonging to the framework of classical probability theory.

The purpose of this article is to show that the random variables $F(\psi, \alpha)$ have a certain number of properties compatible with the fundamental principles on which Quantum Theory is based. Vice versa if one takes into account these properties, we shall see that there is a unique way to represent the system of states by a Hilbert space of random variables, and that is by the complex GRF « white noise ».

Having established the existence and the uniqueness of a GRF which enters naturally in quantum physics and which is so closely related to Brownian motion, it is appropriate to quote from a recent book of Louis de Broglie ([*2*], p. 106): « Or qui dit mouvement brownien dit aussi fluctuations et thermodynamique. »

We shall give in the conclusion of this article an account of the new ideas that L. de Broglie expresses in his book where he establishes a remarkable connection between the principle of least action in Mechanics and the principle of maximum entropy in Thermodynamics.

We shall mention there also an approach suggested recently by N. Wiener and G. Della Riccia ([*3*]), to the problem of finding a classical model for the probabilities which appear in Quantum Theory.

The present work is intended to provide a mathematical basis to the physical theories which introduce thermodynamical concepts and use the assumption that there exists an underlying Brownian motion in order to understand the statistical behavior of quantum systems.

§ 1. — SOME PROPERTIES OF « WHITE NOISE »

In this section we shall enumerate some properties of the complex « white noise » which are significant in Quantum Theory.

We shall not give the mathematical proofs. They can be found in Wiener [*1*].

1) The unitary property of « white noise ».

The transformation (2) given in the introduction is obviously linear:

$$F(c_1\psi_1 + c_2\psi_2, \alpha) = c_1F(\psi_1, \alpha) + c_2F(\psi_2, \alpha)$$

for all complex numbers c_1, c_2 and $\psi_1, \psi_2 \in L^2(R^N)$. As mentioned earlier (2) is also unitary. We have:

$$L^2(R^N) \ni \psi(s) \rightarrow F(\psi, \alpha) \in \mathcal{F} \subset L^2([0, 1], \mathcal{B}, \text{Lebesgue})$$

and

$$\| F(\psi, \alpha) \| = \| \psi(s) \|$$

where $\| \quad \|$ indicates the L^2 norm in the corresponding space.

One can define the distribution $X(ds, \alpha)$ in such a way that the random variables $F(\psi, \alpha) \in \mathcal{F}$ have all the same mean value μ equal to zero:

$$\mu = E\{ F(\psi, \alpha) \} = \int_0^1 F(\psi, \alpha) d\alpha = 0\,; \qquad \psi \in L^2(R^N).$$

With this choice of μ, the variance σ^2 is simply related to the norms:

$$\sigma^2 \{ F(\psi, \alpha) \} = E\{ | F(\psi, \alpha) |^2 \} = \int_0^1 F^*(\psi, \alpha) F(\psi, \alpha) d\alpha$$

(* indicates the complex conjugate). Hence, $\psi(s)$ and $F(\psi, \alpha)$ can be considered as two equivalent representations of the state function of a quantum system. But it is to be noticed that $F(\psi, \alpha)$ can be interpreted, in addition, as a random variable with mean $\mu = 0$ and variance

$$\sigma^2 = \| F(\psi, \alpha) \|^2 = \| \psi(s) \|^2.$$

2) $\mathcal{F}$ is a Gaussian system of complex random variables.

Let us start first with the real case. If $\psi(s) \in L^2(R^N)$ is a real function, the associated random variable

$$F(\psi, \alpha) = \int_{R^N} \psi(s) x(ds, \alpha)$$

is real and has a Gaussian distribution completely characterized by

$$\mu = 0 \quad \text{and} \quad \sigma^2 = \int_{R^N} \psi^2(s) ds.$$

In the complex case let us separate $F(\psi, \alpha)$ as well as $\psi(s)$ into their real and their imaginary parts by writing:

$$F(\psi, \alpha) = \frac{\xi(\psi, \alpha) + i\eta(\psi, \alpha)}{2^{\frac{1}{2}}}$$

$$\psi(s) = \psi_1(s) + i\psi_2(s)$$

where ξ and η are two real random variables and ψ_1, ψ_2 are two real $L^2(R^N)$ functions.

Let us recall that to obtain $F(\psi, \alpha)$ we have used the complex Brownian increments:

$$X(ds, \alpha) = \frac{x(ds, \beta) + tix(ds, \gamma)}{2^{\frac{1}{2}}}$$

where $(\beta, \gamma) \to \alpha$ is a measure preserving mapping of the square $([0, 1] \times [0, 1])$ onto the interval $[0, 1]$.

With these definitions it is not difficult to see that:

$$\text{(3)} \qquad \begin{aligned} \xi(\psi, \alpha) &= \int_{R^N} \psi_1(s) x(ds, \beta) - \int_{R^N} \psi_2(s) x(ds, \gamma) \\ \eta(\psi, \alpha) &= \int_{R^N} \psi_1(s) x(ds, \gamma) + \int_{R^N} \psi_2(s) x(ds, \beta). \end{aligned}$$

In account of the fact that each integral is a Gaussian random variable with mean zero, that β and γ vary independently, we can conclude that ξ and η have both the same Gaussian distribution with $\mu = 0$ and $\sigma^2 = \| \psi \|^2$. We notice next that:

$$E\{\xi . \eta\} = \int_0^1 \int_0^1 \xi . \eta d\beta d\gamma = 0.$$

Being Gaussian and uncorrelated, ξ and η are also mutually independent.

Hence $F(\psi, \alpha)$ is an isotropic Gaussian complex random variable with $\mu = 0$ and $\sigma^2 = \| \psi \|^2$.

We can prove that given any number n of random variables

$$\{ F_k \} \in \mathcal{F}, \ 1 \leq k \leq n,$$

the n-dimensional complex random variable $[F_1, \ldots, F_n]$ has a Gaussian distribution. This is to say that the space $\mathcal{F}$ is a Gaussian system of complex random variables.

If $\{ \psi_k \}$ is an orthonormal basis of $L^2(R^N)$, the associated random variables $\{ F_k \}$ form also an orthonormal basis of $\mathcal{F}$ (because of the unitarity of the transformation (2)). Furthermore it can be shown easily, by using relations (3), that if

$$E \{ F_i^* F_j \} = (F_i, F_j) = (\psi_i, \psi_j) = 0$$

where (,) indicates the scalar product in the corresponding Hilbert space, then F_i and F_j are mutually independent (this means that the four Gaussian random variables ξ_i, η_i, ξ_j, η_j are mutually independent). In other words orthogonality in $L^2(R^N)$ corresponds to orthogonality *and* independence in $\mathcal{F}$.

Usually in Quantum Theory we choose as vectors for the orthonormal basis $\{ \psi_k \}$, eigenfunctions of operators which are observables. We have seen that « white noise » transforms this basis into an orthonormal basis of $\mathcal{F}$ where the vectors have all a Gaussian distribution of the standard form and are all mutually independent.

3) Random events and probabilities occurring in Quantum Theory.

Let us consider a number n, not necessarily finite, of random variables

$$\{ F_1(\alpha), \ldots, F_n(\alpha) \} \in \mathcal{F},$$

and an arbitrary condition $\mathcal{E}$ imposed on these elements of $\mathcal{F}$.

The set S of values of α for which the condition $\mathcal{E}$ is satisfied is well defined and if it is Lebesgue measurable its measure $m(S)$ is necessarily a number lying between 0 and 1 since $\alpha \in [0, 1]$. Thus $\mathcal{E}$ can be considered as a random event, on the sample space ($[0, 1]$, $\mathcal{B}$, Lebesgue), with probability Prob $\{ \mathcal{E} \} = m(S)$.

Let $\{ \psi_k \}$ be an orthonormal basis of $L^2(R^N)$ and $\{ F_k \}$ the associated orthonormal basis of $\mathcal{F}$; let $\{ c_k \}$ be a sequence of complex numbers such that

$$\sum_k | c_k |^2 < \infty.$$

$$F(\alpha) = \sum_k c_k F_k(\alpha)$$

converges in $L^2([0, 1])$ and it is in fact the element $F(\psi, \alpha)$ of $\mathcal{F}$ corresponding to

$$\psi(s) = \sum_k c_k \psi_k(s) \in L^2(R^N).$$

The random events $\mathcal{E}_k$ introduced by Wiener, which play an essential role in connection with Quantum Theory are defined by the following inequalities:

$$\text{(4)} \quad \mathcal{E}_k : | c_k F_k(\alpha) | > | c_0 F_0(\alpha) + \ldots + c_{k-1} F_{k-1}(\alpha) + c_{k+1} F_{k+1}(\alpha) + \ldots | ; \quad k = 0, 1, \ldots$$

The remarkable result is that

$$\text{Prob} \{ \mathcal{E}_k \} = \frac{| c_k |^2}{\sum_k | c_k |^2}.$$

It is the same probability we find in Quantum Theory for the occurrence of the eigenstate ψ_k when we consider a system whose state function is precisely

$$\psi = \sum_k c_k \psi_k(s),$$

before we perform a measurement of the observable whose eigenfunctions are ψ_k.

Without metaphysics, the random events $\mathcal{E}_k$, defined by (4), provide a working model of the so-called « reduction of the wave packet » by a quantum measurement. Thus it is interesting to prove the uniqueness of such a model by showing that « white noise », derived from Brownian motion, is the only GRF which allows us to represent states by random variables having properties derived from the basic principles of Quantum Theory.

§ 2. — DEFINITION OF A GENERALIZED RANDOM FIELD CONSISTENT WITH QUANTUM THEORY

Now we arrive at our problem. We assume that the different states of a physical system form a space $\mathcal{F}$ of complex random variables generated by a GRF. The problem is to find the conditions which must be satisfied by this GRF and prove that these conditions characterize it uniquely.

First we shall recall some mathematical definitions using the terminology of Gel'fand. A generalized function (*synonym*: distribution) is a continuous linear functional on the space $\mathfrak{D}$ of infinitely differentiable functions $\varphi(s)$ having bounded support.

It is called a generalized random function if with every element $\varphi \in \mathfrak{D}$ there is associated a random variable $F(\varphi)$ which is continuous and linear in φ. The continuity of the functional is to be understood with respect to the usual topology on $\mathfrak{D}$.

In the case where $\mathfrak{D}$ consists of functions of one variable, the random function is called a generalized random process (GRP); in the case where $\mathfrak{D}$ is a space of functions of several variables, $F(\varphi)$ is called a generalized random field (GRF).

Let us consider a GRF which is supposed to satisfy the following conditions.

Condition I.

$F(\varphi)$ is a strictly homogeneous GRF with independent values at every point.

Strictly homogeneity (stationarity in the case of a GRP) means that if we translate the function $\varphi(s)$ by any vector $h \in R^N$, namely $\tau_h\varphi(s) = \varphi(s + h)$, then $F(\varphi)$ and $F(\tau_h\varphi)$ are two random variables having the same probability distribution function.

Independent values at every point means that the random variables $F(\varphi_1)$ and $F(\varphi_2)$ are mutually independent whenever $\varphi_1(s).\varphi_2(s) = 0$ for all $s \in R^N$.

Physically speaking the first assumption expresses the fact that the results of a measurement do not depend on the origin of the system of reference. The second assumption says that the results of measuring the random quantity F in disjoint regions of R^N are mutually independent. ($\varphi_1(s).\varphi_2(s) = 0$, for all $s \in R^N$, means in fact that φ_1 and φ_2 have disjoint supports).

Condition II.

A basic principle in Quantum Theory is the superposition principle. It says that the system of states form a linear manifold.

We know already from the linearity of $F(\varphi)$ that to a state which is of the from $\varphi = c_1\varphi_1 + c_2\varphi_2$ there corresponds the random variable

$$F(\varphi) = c_1F(\varphi_1) + c_2F(\varphi_2).$$

Since we are dealing with random variables we must also say how the probability distribution functions behave when we combine these random variables linearily. We assume that whenever $F(\varphi_1)$ and $F(\varphi_2)$ are two random variables mutually independent and with the *same* distribution law, there exists a constant B such that $F(\varphi_1) + F(\varphi_2) = BF$ defines a random variable F which has the same distribution as $F(\varphi_1)$ and $F(\varphi_2)$. Let us recall that $F(\varphi_1)$ and $F(\varphi_2)$ are mutually independent whenever φ_1 and φ_2 have disjoint supports and therefore it is reasonable to assume that if in addition $F(\varphi_1)$ and $F(\varphi_2)$ have the same distribution law, there exists B such that

$$F = \frac{F(\varphi_1) + F(\varphi_2)}{B} = \frac{F(\varphi_1 + \varphi_2)}{B}$$

has also that distribution law. This stability of the probability laws is in some sense a natural consequence of the superposition principle when we represent states by random variables. Finally it is admitted in Quantum Theory that φ and $e^{i\theta}\varphi$, where θ is arbitrary, represent the same state. Again we shall express that statement by a stability condition on the probability distributions, namely we shall assume that $F(\varphi)$ and

$$F(e^{i\theta}\varphi) = e^{i\theta}F(\varphi)$$

have the same probability distribution function. In other words we assume that the probability distribution of $F(\varphi)$ for all $\varphi \in \mathfrak{D}$ is isotropic. It follows immediately that $\mu = E\{F(\varphi)\} = 0$ for all $\varphi \in \mathfrak{D}$ if μ exists.

Condition III.

As mentioned earlier a GRF is mathematically defined as a functional on the space of functions $\mathfrak{D}$. Since we want our results to be valid in Quantum Theory where one deals essentially with Hilbert spaces of functions, we must extend the definition of $F(\varphi)$ over the entire space $L^2(R^N)$. In order to do that we assume that the variance of the random variable $F(\varphi)$ exists:

$$\sigma^2 \{ F(\varphi) \} = \int_\Omega F^*(\varphi, \omega) F(\varphi, \omega) P(d\omega) < \infty, \qquad \text{for all} \quad \varphi \in \mathfrak{D}$$

which means that $F(\varphi, \omega) \in L^2(\Omega, \mathfrak{B}, P)$ where we have indicated explicitely the random parameter ω and the probability measure space $(\Omega, \mathfrak{B}, P)$. Then we assume that, given any sequence of functions $\{ \varphi_n \} \in \mathfrak{D}$ which converges in the mean (L^2-convergence) to a function ψ, the associated sequence of random variables $F(\varphi_n)$, $\varphi_n \in \{ \varphi_n \}$, also converges in the mean. It is not difficult to show that the limit in the mean of $F(\varphi_n)$ is independent of the choice of the sequence $\{ \varphi_n \}$ which converges to ψ and therefore we can write:

$$\lim_{n \to \infty} \varphi_n = \psi \;\rightarrow\; \lim_{n \to \infty} F(\varphi_n) = F(\psi).$$

In other words condition III extends the definition of $F(\varphi)$ on the L^2 closure of $\mathfrak{D}$ which is the entire $L^2(R^N)$ (since $\mathfrak{D}$ is dense in $L^2(R^N)$). It states also that $\{ F(\psi, \omega); \psi \in L^2(R^N) \}$ is a Hilbert space $\mathcal{F} \subset L^2(\Omega, \mathfrak{B}, P)$ of random variables.

§ 3. — CHARACTERIZATION OF « WHITE NOISE » BY CONDITIONS I, II AND III

We have said already that all the random variables $F(\varphi)$ have zero mean. On the basis of condition III we can easily show that this property

$$\mu = E \{ F(\psi) \} = 0$$

holds for all $\psi \in L^2(R^N)$ (The existence of μ is a consequence of the existence of σ).

We proceed now by showing that all the elements $F(\psi) \in \mathcal{F}$ are complex isotropic Gaussian random variables. Let us start with two random variables $F(\varphi_1)$ and $F(\varphi_2)$, where φ_1 is an arbitrary element of $\mathfrak{D}$ and $\varphi_2 \in \mathfrak{D}$ is simply $\varphi_2 = \tau_h \varphi_1 = \varphi_1(s + h)$ where the translation vector h is such that φ_1 and φ_2 have disjoint supports. From condition I it follows that the two random variables $F(\varphi_1)$ and $F(\varphi_2)$ are mutually independent and have the same probability distribution. We can apply to them condition II.

$$F(\varphi_1) + F(\varphi_2) = BF \tag{5}$$

where without any loss of generality we can assume that B is a real positive constant (since we assumed that we can change arbitrarily the phase of $BF = F(\varphi_1 + \varphi_2)$ without modifying the probability distribution).

The stability condition must hold separately for the real and the imaginary parts of (5); we have:

$$\xi_1 + \xi_2 = B\xi$$
$$\eta_1 + \eta_2 = B\eta$$

where ξ has the same distribution as ξ_1 and ξ_2, and η has the same distribution as η_1 and η_2. Now we use the well-known Paul Levy's theorem on stable laws ([4], p. 94) which gives the general form of the characteristic function C of a stable distribution.

Applying this theorem to ξ we have:

(6) $$C_\xi(z) = E\{e^{iz\xi}\} = \exp\left[\left(-c + id\frac{z}{|z|}\right)|z|^\alpha\right]$$

where c and d are two real constants, $c > 0$ and $0 < \alpha \leq 2$.

It is also well-known that the only stable distributions with finite variance are those corresponding to $\alpha = 2$; this is our case according to condition III.

The symmetry property of the distribution of every $F(\varphi)$ around the origin implies that:

(7) $$C_\xi(z) = C_\xi(-z)$$
$$C_\xi(z) = C_\eta(z)$$

and

(8) $$C_{\xi,\eta}(z_1, z_2) = E\{e^{i(z_1\xi + z_2\eta)}\} = C_{\xi,\eta}\left(\sqrt{z_1^2 + z_2^2}\right).$$

From (7) we deduce that in (6) we must take $d = 0$; therefore (with $\alpha = 2$) we obtain:

(8′) $$C_\xi(z) = C_\eta(z) = \exp(-cz^2).$$

Finally, because of (8) and (8′), we have

(9) $$C_{\xi,\eta}(z_1, z_2) = \exp\left[-c(z_1^2 + z_2^2)\right],$$

which shows that ξ and η are mutually independent random variables. Hence we have established that $F(\varphi)$, for all $\varphi \in \mathfrak{D}$, is a complex isotropic Gaussian random variable.

Furthermore any linear combination

$$\sum a_i F(\varphi_i)$$

of n elements $F(\varphi_1), \ldots, F(\varphi_n)$ in $\{ F(\varphi); \varphi \in \mathfrak{D} \}$ is also a similar Gaussian random variable since

$$\sum_i a_i F(\varphi_i) = F\Big(\sum_i a_i \varphi_i\Big) \quad \text{and} \quad \sum_{i=1}^{n} a_i \varphi_i \in \mathfrak{D}.$$

Because of that fact we can assert that $\{ F(\varphi); \varphi \in \mathfrak{D} \}$ forms a Gaussian system of random variables in the sense that for every n elements

$$F_1, \ldots, F_n \in \{ F(\varphi); \varphi \in \mathfrak{D} \},$$

the n-dimensional random variable $[F_1, \ldots, F_n]$ is Gaussian. If we take into account condition III, it is not difficult to show that when we extend $F(\varphi)$ on the L^2 closure of $\mathfrak{D}$, we still have a Gaussian system

$$\{ F(\psi); \psi \in L^2(R^N) \} = \mathcal{F} \subset L^2(\Omega, B, P)$$

where

$$F(\psi) = \frac{\xi(\psi) + i\eta(\psi)}{2^{\frac{1}{2}}},$$

is isotropic with variance $\sigma_2 \{ F(\psi) \} = E \{ | F(\psi) |^2 \} = 2c(\psi)$.

Our last problem is to find the dependence of $c(\psi)$ with respect to ψ. In order to obtain the result we consider as a particular element $\psi \in L^2(R^N)$ the indicator function χ_I of an interval $I \in R^N$ and the associated random variable $F(\chi_I)$. Let $h \in R^N$ be a vector such that I and $I + h$ are two disjoint intervals and let us consider the two corresponding random variables $F(\chi_I)$ and $F(\chi_{I+h})$ which, as we know, are mutually independent with the same distribution function. Hence, we can write:

$$\sigma^2 \{ F(\chi_I + \chi_{I+h}) \} = \sigma^2 \{ F(\chi_I) \} + \sigma^2 \{ F(\chi_{I+h}) \} = 2\sigma^2 \{ F(\chi_I) \}.$$

On the other hand, it is obvious that the Lebesgue measure in R^N for disjoint intervals satisfies a similar relation:

$$m[I + (I + h)] = m(I) + m(I + h) = 2m(I).$$

As a consequence we can say that the variance of $F(\chi_I)$ where χ_I is the indicator function of a interval I is proportional to the Lebesgue measure $m(I)$ of that interval:

$$\sigma^2 \{ F(\chi_I) \} = \lambda m(I).$$

Without any loss of generality we shall choose the constant of proportionality λ to be equal to 1.

Actually what we have done is to associate to each finite interval $I \in R^N$; a random variable

$$F(\chi)_I = X(I, \omega) = \frac{x(I, \omega_1) + ix(I, \omega_2)}{2^{\frac{1}{2}}}\ ; \ \omega_1, \omega_2 \in (\Omega, \mathcal{B}, P)$$

where the properties of the random variables $x(I, \omega)$ are precisely those which are used in the mathematical definition of a Wiener process $x(t, \alpha)$. The only difference is that we have here a probability space $(\Omega, \mathcal{B}, P)$ more general than $([0, 1], \mathcal{B}, \text{Lebesgue})$. However, one of the authors of this article has proved [5] that in the case of the Wiener process $(\Omega, \mathcal{B}, P)$ has to be a Lebesgue space without atoms which means that $(\Omega, \mathcal{B}, P)$ is isomorphic, modulo sets of zero Lebesgue measure, to $([0, 1], \mathcal{B}, \text{Lebesgue})$.

After having shown that $c(\chi_I) = \frac{1}{2} m(I)$ it remains to find $c(\psi)$ for a general ψ. Again we shall proceed in several steps. First we consider the subset $\mathcal{A}$ of elements of $L^2(R^N)$ which are of the form

$$A = \sum_{i=1}^{n} a_i \chi_i,$$

where n is arbitrary but finite, and where $\{ \chi_i; 1 \leq i \leq n \}$ are indicator functions of intervals $\{ I_i \in R^N, 1 \leq i \leqslant n \}$ which are mutually disjoint two by two. For that set of functions we can easily see that:

$$F(A) = \sum_{i=1}^{n} a_i X(I_i, \omega)$$

$$\sigma^2 \{ F(A) \} = \sum_{i=1}^{n} | a_i |^2 m(I_i) = \| A \|^2 \quad \text{for all} \quad A \in \mathcal{A} \subset L^2(R^N).$$

But $\mathcal{A}$ is known to be dense in $L^2(R^N)$. Therefore any $\psi \in L^2(R^N)$ can be obtained as limit of a convergent sequence $\{ A_n \} \in \mathcal{A}$ in the L^2 sense. The corresponding sequence of random variables $\{ F(A_n) \}$ converges in the L^2 sense and we can write for the limit:

$$F(\psi, \alpha) = \int_{R^N} \psi(s) X(ds, \alpha)$$

and

$$\sigma^2 \{ F(\psi) \} = 2c(\psi) = \| \psi \|^2, \quad \text{for all} \quad \psi \in L^2(R^N)$$

where we now return to the previous notations used in (2): *ds* replacing I_t and α replacing ω.

It is interesting to notice that if we did not extend the definition of $F(\varphi)$, $\varphi \in \mathfrak{D}$, over the entire $L^2(R^N)$ space, we could not characterize completely the GRF but only conclude that it is a generalized Gaussian field. In fact, from the form of the characteristic function we have obtained in (9), we can say that the characteristic functional $L(\varphi)$ of $F(\varphi)$ is:

$$L(\varphi) = E\left\{ e^{\frac{i[\xi(\varphi)+\eta(\varphi)]}{2^{\frac{1}{2}}}} \right\} = \exp\left[-c(\varphi)\right]$$

where

$$c(\varphi) = \frac{1}{2}\sigma^2 \{ F(\varphi) \}.$$

All generalized Gaussian fields have a characteristic functional which is of this form; these include not only « white noise » but also derivatives (in the sense of the theory of distributions) of « white noise »: $F'(\varphi)$, $F''(\varphi)$, ... respectively defined as $-F(\varphi')$, $F(\varphi'')$, ...

We have seen that we were able to select « white noise » out of all possible generalized Gaussian fields once the relation $\sigma^2 \{ F(\varphi) \} = \| \varphi \|^2$ was established. But in order to arrive at this relation we had to use $F(\chi_I)$ where χ_I is in $L^2(R^N)$ and not in $\mathfrak{D}$.

We may insist on the fact that this extension of $F(\varphi)$ over the entire $L^2(R^N)$ space, which is necessary for obtaining uniqueness, is itself required by the basic structure of Quantum Theory.

Incidentally let us notice that « white noise » $F(\psi)$ extended over $L^2(R^N)$, is not differentiable since ψ is not necessarily a differentiable function and therefore $F'(\psi)$ does not exist for all $\psi \in L^2(R^N)$.

CONCLUSION

Since almost the very beginning of Quantum Theory, during the years 1926-1927, L. de Broglie has suggested a description of the physical facts which differs from the usual one accepted by the great majority of scientists. Among the various points where the two interpretations disagree, that one which imports in our context is the problem of the localization of the particle in space. According to L. de Broglie the particle is permanently localized in space and it is incorporated in the wave in such a way that its motion is guided by the propagation of the wave. To be precise, the particle follows a line of current of a fluid whose flow is determined by the wave equation. Since 1951, when he started working again on his original idea, de Broglie [6] completed his theory by introducing in it a random factor which was needed in order to explain the statistical behaviour of the particle and to understand why $\rho = |\psi|^2$ where ψ is a solution of the wave equation should represent the probability density of presence of that particle. He used an assumption made by D. Bohm and J. P. Vigier [7] according to which there exists, at a subquantum level, a « hidden thermostat » which interacts with all physical systems we observe. As a result of this interaction, a particle is randomly scattered from one trajectory to another and one can derive the right probability density. No doubt that if Brownian motion has any role to play in the theory, it will appear in the description of the fluctuations imposed by the thermostat to the observable quantities. Going further in this direction of research, de Broglie [2] has built recently what he calls a « thermodynamics of the isolated particle » which shows how the classical dynamics of a particle can be associated with thermodynamical concepts. He gives a set of relations which establish a remarkable connection between the Hamilton principle of least action in Mechanics and the principle of maximum entropy in Thermodynamics.

The celebrated de Broglie relations formulated in 1923 had already emphasized the identity between the principle of least action and the principle of Fermat in the case where the approximation of Geometrical Optics can be used in Wave Mechanics. Now the three variational principles on which Physics is founded are all related.

A further understanding of the meaning of this unity will reveal perhaps new fundamental laws.

In a similar way, Wiener and Della Riccia [3] assume the permanent localization of the particle. The presence of the thermostat is made more explicit than in de Broglie's work by the direct use of the methods of Gibbs statistical mechanics.

The trajectories they consider are trajectories of states in phase space corresponding to solutions of Hamilton-Jacobi equations of motion. The probability waves are also in phase space and they represent the time evolution of probability distributions over the ensemble of all possible initial conditions (coordinates and momenta) of the particle. They show that if one perturbs slightly the thermal equilibrium distribution, the phase space wave can be expanded in a orthonormal set of modes of vibration such that when they are averaged over the momenta they become the usual eigenfunctions of Schroedinger equation.

It is then assumed that because of the randomness introduced by the thermostat, each mode of vibration has to be associated with a random variable defined by « white noise ». Then the random events we have mentioned in (4) lead to the right probabilities of occurrence of eigenstates postulated by ordinary Quantum Theory.

The theories we have briefly described all make use of thermodynamical fluctuations which rely on Brownian motion, or the derived « white noise ». It was our purpose in this work to show mathematically why it is reasonable to ignore fluctuations which are not of this nature.

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