ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 4, nº 1 (1966), p. 1-20
<http://www.numdam.org/item?id=AIHPA_1966__4_1_1_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. IV, n° 1, 1966, p. 1-20. *Physique théorique.*

Conservation laws (*)

by

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I. — INTRODUCTION

In the present paper I propose to expound in general terms some of the main attempts which have been made to establish conservation equations in general relativity, emphasizing the difficulties which still remain in the way of a satisfactory and useful definition of the concepts of gravitational momentum and energy. From the start I want to apologise for the incompleteness of my exposition mainly due to the necessity of keeping close to a central thread. In the second part of my talk I shall present some personal considerations.

Since Einstein's introduction of his canonical pseudotensor much work has been done. Whilst on the one hand the structure of Einstein's complex has been more closely examined (Tolman [*99*], Freud [*48*]) and conditions have been established whereby the correct global quantities may be obtained from it in various physical conditions (Einstein [*44*], Klein [*57*], Fock [*46*], Trautman [*101*]), on the other critiques have been formulated with regard to its effective physical content (Schrödinger [*95*], Bauer [*4*]). Other complexes have been discovered by Landau-Lifshitz [*62*] and by Møller [*71*]; whole families of such complexes including all the preceding ones have been found by Goldberg [*53*] without however any of them presenting all the requirements which, *a priori,* one would have desired to be present in a « good » energy-momentum complex.

(*) Conférence générale faite à la 5^{e} Réunion Internationale sur la Gravitation et la Relativité Générale (Londres, 1-9 juillet 1965).

Much light has been thrown on the significance of such an abundant choice, by the work of Bergmann and his collaborators [*8*] [*9*] [*10*], who with the trend initiated by Noether in 1918 [*78*] directly associated it with the general covariance of Einstein's formalism. Bergmann's work, which fitted the infinite number of conservation laws of general relativity into an organic scheme, showed how every law of conservation is in fact subordinate to the introduction of a supplementary structure (vector field or otherwise) into the space-time manifold. The conservation equations subordinated to the introduction of a tetrad field (Møller [*75*] [*76*], Pellegrini-Plebanski [*83*]) or of a second metric (Rosen [*90*] [*91*] [*92*], Papapetrou [*79*], Graiff [*54*], Bonazzola [*16*] [*17*]) or of a type of absolute parallelism based on the Levi-Civita's transport along geodesics (Rayski [*87*]) confirm this opinion.

At the same time Bergmann's schemes marked the beginning of a research for single index conserved complexes. It was this trend that enabled Komar [*58*] to achieve local covariance with the minimum of auxiliary elements (vector field ξ^i). At the present time, bearing in mind the important aspects pointed out by Pirani [*86*], among the various known complexes Komar's current vector seems to possess the most satisfactory qualities. In the present paper we shall have occasion to observe that there exists a possible alternative to Komar's vector which may bring some possible improvements.

The arbitrariness of the auxiliary structures on which the different complexes depend, poses for each of them two types of problems which are not completely separate: (*a*) What limitations must be applied to the auxiliary structures themselves in order to conform to the different physical situations at infinity in such a way as to obtain correct global quantities (*b*). Possible research on preferred structures intended to single out the magnitudes of general relativity which may justly be interpreted as the energy and momentum equivalents of a Lorentz-covariant theory. Although a great deal of work has been done on these problems there does not yet seem to be a satisfactory answer to them, either through a lack of covariance or because of insufficient physical justification. Komar has put forward an interesting if questionable criterion intended to give a definite positive character to energy. Another possible criterior will ben indicated in the present paper. Both these criteria are linked to the important question of the sign to be attributed to the pure gravitational energy.

Among the attempts which have been made to find an answer to the problem of the gravitational energy it is worth mentioning separately the attempt culminating in the four index symmetrical tensor discovered independently by Bel [5] [6] and by Robinson [89]. In spite of the appreciable properties of this tensor which make it similar to an energy momentum tensor of a Lorentz covariant theory and of the superenergy which can be constructed with it, the latter lacks true properties of conservation so that there is still doubt as to its exact physical significance. According to a recent paper of Bonazzola, it is possible that the Bel-Robinson tensor is linked to the variations in gravitational energy, rather than to the energy itself.

Another thing which still remains uncertain is the question of the decomposition of the total energy of a space-time into an energy of pure matter and into a gravitational energy. These problems will be mentioned at the end of the present paper.

II. — DOUBLE INDEX COMPLEXES

The various double index complexes which have been successively discovered (Einstein [42], Landau-Lifshitz [62], Møller [71], Goldberg [53]) have a common structure which can be summarized by the following points.

The fundamental element is a three index « superpotential » $U_i^{[kl]}$ or $U^{i[kl]}$, made up from the g^{rs} and their first derivatives, which is antisymmetrical with respect to a pair of indices. From the superpotential a double index complex is obtained by the operations of derivation and contraction

$$(2.1) \qquad \Upsilon_i^k \equiv \partial_l U_i^{[kl]}, \qquad \Upsilon^{ik} \equiv \partial_l U^{i[kl]}.$$

Because of the antisymmetry of U, this complex automatically satisfies a local identity of conservation

$$(2.2) \qquad \partial_k \Upsilon_i^k \equiv 0, \qquad \partial_k \Upsilon^{ik} \equiv 0.$$

The explicit espressions of known superpotential are the following (Einstein-Freud, Møller, Goldberg, Landau-Lifshitz):

$$(2.3) \qquad U_i^{[kl]} \equiv \begin{cases} \dfrac{1}{2\varkappa}(-g)^{-\frac{1}{2}} g_{in}\partial_m[(-g)(g^{kn}g^{lm} - g^{ln}g^{km})] & \text{(E)} \\ \dfrac{1}{\varkappa}(-g)^{\frac{1}{2}} g^{km}g^{ln}(\partial_m g_{in} - \partial_n g_{im}) & \text{(M)} \\ \dfrac{1}{2\varkappa}(-g)^{\frac{n}{2}} g_{in}\partial_m[(-g)(g^{kn}g^{lm} - g^{ln}g^{km})] & \text{(G)} \end{cases}$$

$$\text{(2.4)}\qquad U^{i[kl]} \equiv \begin{cases} \dfrac{1}{2\varkappa}\partial_m[(-g)(g^{ik}g^{lm} - g^{il}g^{km})] & \text{(L-L)} \\ \dfrac{1}{2\varkappa}(-g)^{\frac{n}{2}}\partial_m[(-g)(g^{ik}g^{lm} - g^{il}g^{km})]. & \text{(G)} \end{cases}$$

Neither these expressions nor those which derive from them have a real tensor character but behave like an affine (1) tensor density (2). The position of the indices derives from this.

The expressions under (2.2) are strong conservation laws, that is they are valid whatever may be the functions $g^{rs}(x)$ which intervene. From these it is easy to obtain weak conservation laws, that is subordinate to the condition that the $g^{rs}(x)$ satisfy the field equations

$$\text{(2.5)}\qquad G_{ik} \equiv R_{ik} - \frac{1}{2}Rg_{ik} = -\varkappa T_{ik}.$$

Limiting ourselves to a consideration of the complexes (E) and (L — L), for example, it is sufficient to introduce the quantities t_i^k or τ^{ik} (generally functions of g^{rs}, $g^{rs}{}_{,l}$, $g^{rs}{}_{,lm}$) thus defined

$$\text{(2.6)}\qquad t_i^k \equiv \frac{1}{\varkappa}G_i^k - (-g)^{-\frac{1}{2}}\Upsilon_i^k, \qquad \tau^{ik} \equiv \frac{1}{\varkappa}G^{ik} - (-g)^{-1}\Upsilon^{ik}$$

to recognise—taking into account (2.2) and (2.5)—the validity of the following weak conservation equations

$$\text{(2.7)}\qquad \partial_k\{\sqrt{-g}(T_i^k + t_i^k)\} = 0, \qquad \partial_k\{(-g)(T^{ik} + \tau^{ik})\} = 0$$

which in the empty regions become

$$\text{(2.8)}\qquad \partial_k\{\sqrt{-g}\,t_i^k\} = 0, \qquad \partial_k\{(-g)\tau^{ik}\} = 0.$$

Similar equations can be obtained for the other complexes.

Naturally for there to be an effective physical content directly or indirectly linked to a complex or to the global magnitudes obtained from it, it must display some relevant characteristics. Einstein's and Landau-Lifshitz's complexes present the exceptional property of t_i^k and τ^{ik} not depending on the second derivatives of the g^{rs}. Møller's complex does not have the

(1) That is with regard to the linear transformations of coordinates alone.
(2) The complexes deriving from (2.3) are ordinary affine densities that is having a weight 1. The complex (L — L) is an affine density with a weight 2. The last are of a weight growing with n.

same peculiarity but enjoys the notable property of giving rise to a density of energy which is unchanging for space-transformations of coordinates. The Landau-Lifshitz complex, like all those deriving from superpotentials (2.4), is symmetrical in the two indices i and k; this permits the establishment of a conservation law of angular momentum as well. On the contrary the global quantities that are obtained from them do not have all the desirable transformation properties.

Although all the above mentioned complexes are definable in every system of coordinates they do not constitute a geometrical object and it is therefore difficult to attribute a local physical content to them and to their continuity equations.

The same is not so from a global point of view. The character of affine densities in the various complexes considered enables global quantities of undoubted physical significance to be obtained from them in isolated physical systems. If we consider Einstein's complex, for example, they are defined thus

$$(2.9) \qquad P_i(x^4) \equiv \frac{1}{c}\int\int\int \Upsilon_i^4 d^3x = \lim \frac{1}{c} \oiint U_i^{[4l]} dS_l,$$

the first integration being carried out on the hypersurface $x^4 =$ const. and the second on a close 2-surface S belonging to the same hypersurface and expanding to infinity.

Subordinately to the asymptotic conditions of Einstein and Klein

$$(2.10) \qquad g_{ij} - \eta_{ij} = 0\left(\frac{1}{r}\right) \qquad \partial_r g_{ij} = 0\left(\frac{1}{r^2}\right)$$

($r =$ radial space coordinate) which at the same time imply hypotheses of a physical nature (absence of radiation) and limitations in the choice of the coordinates, the magnitudes (2.9), which are certainly convergent, satisfy all the requirements which allow us to interpret them as an energy-momentum free vector of the whole system.

More specifically the P_i's (*a*) remain constant (*b*) are invariable with respect to all the changes of coordinates which do not operate at infinity (*c*) they change like the components of a free vector of M_4 with respect to all the linear transformations of coordinates (*d*) for a Schwarzschild space-time with a gravitational mass M_0, referred to asymptotically galileian rest coordinates, they assume the specific values:

$$(2.11) \qquad P_\alpha = 0, \qquad -cP_4 = M_0 c^2.$$

These are certainly essential qualities which must be required from quantities which set out to express the energy and momentum of an isolated system.

However Einstein's complex does not allow the construction of conserved global quantities, which can be interpreted as angular momentum.

Landau and Lifshitz complex on the contrary gives rise to such magnitudes; conversely the P_i's defined by this complex only constitute a vector in respect of linear transformations that leave the value of g unchanged. Møller's complex is subject to more pronounced limitations but it has particular local merits.

For an insular system, for which we do'nt exclude the possibility of irradiating gravitational energy, the Einstein conditions (2.10) can be substituted by Fock-Trautman's asymptotic conditions [*101*] [*102*]

$$(2.12)\qquad \begin{cases} g_{ij} - \eta_{ij} = 0\left(\dfrac{1}{r}\right) \\ \partial_r g_{ij} = h_{ij} k_r + 0\left(\dfrac{1}{r^2}\right), \qquad h_{ij} = 0\left(\dfrac{1}{r}\right), \qquad k_r k^r = 0 \\ \left(h_{ij} - \dfrac{1}{2}\eta_{ij}\eta^{rs} h_{rs}\right) k^j = 0\left(\dfrac{1}{r^2}\right), \end{cases}$$

which compensate for the slower rate of vanishing of the $\partial_r g_{ij}$ with the additional condition of asymptotic harmonicity imposed on the coordinates. Within the limits of the coordinate systems complying with (2.12) the P_i's constructed by means of Einstein's complex continue to satisfy the convergence and invariance requirements with respect to changes of coordinates not operating at infinity. Naturally in the new physical conditions accepted, one should not expect—at least in general—the conservation of the P_i's. The possible dependence of some of them on x^4 will indeed serve to reveal the cases of effective radiation.

The lack of covariance common to all the double index complexes that we have reviewed seems to be remedied by a second complex $\widehat{\Upsilon}_i^k$ constructed by Møller in 1961 [*75*] [*76*] and based on the introduction of a field of tetrads $\underset{a}{h}^i(x)$ into V_4. The process is based on the possibility of decomposing the gravitational Lagrangean $\sqrt{-g}R$

$$(2.13)\qquad \sqrt{-g}R \equiv \partial_k S^k + \widehat{\mathfrak{L}}$$

into a divergence type term which does not influence the variation of the action, and a term $\widehat{\mathfrak{L}}$ constructed by means of the $\underset{a}{h}^i$ and their covariant derivatives $\underset{a}{h^i}{;r}$ so that it turns out to be a scalar density with respect to the changes of coordinates. Using a classical method we thus reach a canonical complex

$$\widehat{t}_i^k \equiv \frac{1}{2\varkappa}(-g)^{-\frac{1}{2}}\left(\frac{\partial\widehat{\mathfrak{L}}}{\partial \underset{a}{h^r}_{,k}} \underset{a}{h^r}_{,i} - \delta_i^k \widehat{\mathfrak{L}}\right) \tag{2.14}$$

which satisfies a weak conservation law of the type (2.7) (2.8). This is naturally associated with a strong conservation law satisfied by a complex $\widehat{\Upsilon}_i^k$ which contains also the second derivatives of $\underset{a}{h^i}$ and can be derived from a superpotential

$$\left\{\begin{aligned} \widehat{\Upsilon}_i^k &\equiv \partial_l \widehat{U}_i^{[kl]} \\ \widehat{U}_i^{[kl]} &\equiv \frac{\sqrt{-g}}{2\varkappa}\left[\underset{a}{h^k}\overset{a}{h^l}{}_{;i} - \underset{a}{h^l}\overset{a}{h^k}{}_{;i} + 2\left(\delta_i^k \overset{a}{h^l} - \delta_i^l \overset{a}{h^k}\right)\underset{a}{h^s}{}_{;s}\right]. \end{aligned}\right. \tag{2.15}$$

Obviously $\widehat{U}_i^{[kl]}$, $\widehat{\Upsilon}_i^k$, $\widehat{t}_i^k$ have a covariant character with respect to all the changes of coordinates. However they lack covariance with respect to the local rotations of the tetrads and also only with respect to their purely space rotations; this covariance has been lost in decomposition (2.13)· Therefore, in order that the progress achieved be not wholly illusory, the tetrad field introduced into V_4 must be more precisely determined; this Møller achieves by means of certain differential and asymptotic supplementary conditions imposed on the $\underset{a}{h^i}$; these conditions are meant to substitute, in covariant form, the known conditions of harmonicity imposed on coordinates (de Donder, Fock). Perhaps there is not a sufficient physical justification for the differential conditions.

The tetrad method on the contrary appears more satisfactory with regard to the behaviour of the global quantities P_i's, which, in virtue of the asymptotic conditions which impose parallelism at infinity on the tetrads, possess the correct vectorial behaviour with respect to the changes of the common orientation of the tetrads at infinity (Lorentz's transformations). The appearance of difficulties of the type made evident by Bauer [*4*] is now excluded, not on account of the covariance obtained with respect to the changes of coordinates, but because of the asymptotic conditions of parallelism imposed on the tetrads.

The principal merit of the tetrad method consists in my opinion not so much in its covariance, as in the extention it brings to Einstein's method.

The latter associates certain quantities Υ_i^k with every system of coordinates (the choice of which depends on 4 arbitrary functions of the x). The tetrad method associates the quantities $\hat{\Upsilon}_i^k$ with every orthonormal (generally non-holonomic) tetrad field the choice of which depends on 6 independent arbitrary functions. The tetrad procedure can probably be extended to fields of tetrads which are neither orthogonal nor normal; in such case the choice of tetrad fields and therefore of the quantities $\hat{\Upsilon}_i^k$ associated with them, would then indeed depend on 16 arbitrary functions. This greater scope would enable the tetrad method to contain all the determinations of Einstein's complex Υ_i^k, which would coincide with the determinations of the complex $\hat{\Upsilon}_i^k$ with regard to the holonomic tetrad fields alone (that is to those with which a system of coordinates can be associated).

The advantage of considering such a wide class of frames consists mainly in the opportunity offered by a wider field of research for the determination of a possible preferred frame. Indeed it is not evident that a possible preferred frame should necessarily be included among the holonomic frames.

The double metric method also, introduced by Rosen [*91*] [*92*], Papapetrou [*79*], Gupta [*55*], aims to remedy the lack of covariance in Einstein's complex. It proceeds formally like Einstein's method, but by virtue of a second minkowskian metric associated with the riemannian metric it is able to substitute the partial differentiation ∂_k by the covariant differentiation ∇_k with respect to the minkowskian metric. The arbitrary nature of the choice of the minkowskian metric, depending on the way in which the V_4 is mapped on M_4, exactly corresponds to the arbitrary nature of the choice of a system of coordinates in Einstein's method. Basically it seems to me that the double metric method is equivalent to interpreting the curvilinear coordinates, which are used in V_4, as cartesian coordinates of a flat space (cf. Synge [*96*]).

Rosen's method also can be justified when physically consistent criteria are provided for the unambiguous determination of the minkowskian metric.

III. — SINGLE INDEX COMPLEXES

Bergmann and some of his collaborators ([*8*]-[*12*]) have carried out research into the reasons for the abundance of conservation equations in general relativity showing how they are closely related to the general covariance of the theory. In accordance with Noether famous theorem which associates the classical conservation laws of the Lorentz-covariant theories with the properties of invariance of physical laws, Bergmann and his collaborators have demonstrated that in general covariant theories a quantity which is conserved, formally corresponds to each infinitesimal transformation of coordinates that is to every vector field $\xi^i(x)$. In spite of the fact that the general process presents some specific difficulty in the case of general relativity, Bergmann, associating the vector field ξ^i with Freud's superpotential succeeds in directly constructing a double index superpotential

$$\text{(3.1)} \qquad U^{[il]} \equiv \xi^k U_k^{[il]}$$

which in turn permits to define a « density of current »

$$\text{(3.2)} \qquad D^i[\xi] \equiv \partial_l U^{[il]}$$

which is conserved identically:

$$\text{(3.3)} \qquad \partial_i(\sqrt{-g}D^i) \equiv 0.$$

Other quantities C^i which are conserved weakly can then be associated with the quantities D^i. The former can be unambiguously determined by the property of not containing second derivatives of g^{rs}.

Since the formal covariance of general relativity does not imply, in itself, an intrinsic physical fact, Bergmann's work implicitly shows that not all the fields ξ^i can give rise to conserved quantities having a physical significance. Only those infinitesimal transformations of coordinates which show some intrinsic property of symmetry of space-time (Trautman [*102*] [*103*]) possibly only verified at infinity, can give rise to significant magnitudes.

Bergmann's work is important for other motives as well. The possibility which exists in the Lorentz-covariant theories of formulating the conservation of energy and momentum in a single local law using a single double-index tensor is closely linked to the linearity of Minkowski's space-time; this quality is lacking in the V_4 of general relativity. The work carried out by Bergmann and his school, with its *a priori* renunciation of the energy momentum synthesis, has opened the way for research into single-index conserved complexes which have a greater possibility of fitting to the specific situation of general relativity. The successive studies of Komar and Pirani support this approach.

It should also be recognised that Bergmann's work has given particular emphasis to the fact that also in general relativity the conservation laws are linked to the introduction of a supplementary structure into V_4; in Bergmann's formulation this structure is represented by a vector field, but in other formulations it can be a tetrad field (Møller) or a second metric (Rosen) or simply a system of coordinates (Einstein).

This established necessity reveals in my opinion, a profound aspect of the problem. Although the four dimensional formulation of Einstein's theory is perfect from a geometrical point of view, it cannot disclose all the physical aspects without the introduction of a physical frame of reference (preferred or not) which gives rise to the local separation of space from time. This frame which according to one's approach can be represented by a tetrad field or even by a simple unit vector field, is, in my opinion, an essential physical element and we should not be surprised that it is shown to be in some way necessary in the problem of conservation laws.

It was pursuing the way opened by Bergmann, that Komar [*58*], who was also inspired by an already quoted paper of Møller [*71*], succeeded in achieving a rigorously covariant single index complex. His procedure can be reduced to the following fundamental points:

(*a*) Introduction of an arbitrary vector field $\xi^i(x)$.

(*b*) Construction of a skew-symmetric tensorial superpotential

$$E^{[il]} \equiv \frac{2}{\varkappa}(\nabla^i \xi^l - \nabla^l \xi^i). \tag{3.4}$$

(*c*) Definition of a 4-density energy current (generalised)

$$E^i[\xi] \equiv \nabla_l E^{[il]} \tag{3.5}$$

satisfying the strong law

$$\nabla_i E^i \equiv 0. \tag{3.6}$$

One can then associate a vector $H^i \equiv 2\xi^m G^i_m + E^i$ to the vector E^i, free from second order time derivatives, which is weakly conserved.

From (3.6) it follows that, under suitable asymptotic conditions, the global scalar quantity

$$E[\xi] = \frac{1}{2}\iiint E^m d\sigma_m \tag{3.7}$$

(σ, space like hypersurface) is conserved.

In each particular case, particular and appropriate choices of the $\vec{\xi}$ field should give rise to the classic conserved quantities (energy, momentum, angular momentum). In a Schwarzschild universe, or asymptotically such, for $\xi^i = \delta^i_4$ (in almost galileian rest-coordinates) we obtain $E = M_0c^2$.

Specifying the field ξ^i in a field of time-like unit vector γ^i, Komar's vector E^i gives rise to a particularly significant vector, specifically linked to the energy of the field, that Pirani [*86*] obtained with different considerations. The introduction of a time-like unit vector field has a particular interest because it is equivalent to the introduction of a physical frame of reference in Møller's sense, *without any additional elements.* The vector $E^i[\gamma[$ then assumes the meaning of energy flux-density relative to the frame $\gamma^i(x)$. Its explicit expression and that of the corresponding energy density $e[\gamma]$ by means of the characteristic elements of the frame of reference ([1]), are respectively

$$(3.8)\qquad \begin{aligned} E^i[\gamma] &\equiv \frac{2}{\varkappa}\nabla_l(\nabla^i\gamma^l - \nabla^l\gamma^i) \\ &\equiv \frac{2}{\varkappa}\nabla_l(\tilde{\Omega}^{li} - \gamma^l C^i + \gamma^i C^l) \end{aligned}$$

$$(3.9)\qquad e[\gamma] \equiv E^i\gamma_i \equiv \frac{2}{\varkappa}\left(\tilde{\nabla}_l C^l + \frac{1}{2}\tilde{\Omega}_{ik}\tilde{\Omega}^{ik}\right).$$

Taking into account the field equations, the density of energy, may, in a vacuum, assume the following form:

$$(3.10)\qquad e[\gamma] = \frac{2}{\varkappa}\left\{\frac{1}{4}\tilde{\Omega}_{ik}\tilde{\Omega}^{ik} - \frac{1}{4}\tilde{K}_{ik}\tilde{K}^{ik} - C_iC^i + \frac{1}{2}\gamma^{ik}\mathfrak{L}\mathfrak{L}\gamma_{ik}\right\},$$

$\mathfrak{L}$ indicating the Lie-derivative with respect to the field γ^i.

In the case of a Schwarzschild space-time the global energy

$$(3.11)\qquad E[\gamma] = \iiint e[\gamma]d^3\sigma$$

calculated with a rest frame $\vec{\gamma}$, acquires the value M_0c^2.

Allow me to add here that the formulae (3.4) (3.5) are not the only ones which provide a conserved vector starting from an arbitrary field ξ^i.

([3]) The characteristic elements of the first order of the frame of reference $\gamma^i(x)$ are the following space tensors: $\tilde{K}_{ij}$, rate of deformation tensor; $\tilde{\Omega}_{ij}$, space-vortex tensor; C_i, curvature vector. $\tilde{\nabla}_l$ is the transverse derivative with respect to $\vec{\gamma}$ [*24*] [*28*].

Another possibility for example is the one which starts with the following superpotential (4):

$$\text{(3.12)} \qquad W^{[il]} \equiv \frac{2}{\varkappa}(\gamma^l \nabla_m \gamma^i - \gamma^i \nabla_m \gamma^l)\gamma^m$$

from which we deduce the vector

$$\text{(3.13)} \qquad W^i[\gamma] \equiv \nabla_l W^{[il]}$$

which, like Komar's vector, is linear in the second order covariant derivatives of the γ's, and is conserved ($\nabla_i W^i \equiv 0$) (5). Corresponding with it there is an energy density

$$\text{(3.14)} \qquad w[\gamma] \equiv W^i \gamma_i = \frac{2}{\varkappa} \tilde{\nabla}_l C^l$$

in which in a vacuum, by virtue of the field equations, one can also recognise the form:

$$\text{(3.15)} \quad w[\gamma] = -\frac{2}{\varkappa}\left\{\frac{1}{4}\tilde{K}_{ik}\tilde{K}^{ik} + \frac{1}{4}\tilde{\Omega}_{ik}\tilde{\Omega}^{ik} + C_l C^l - \frac{1}{2}\gamma^{ik}\mathfrak{L}\mathfrak{L}\gamma_{ik}\right\} \text{(6)}.$$

Comparing (3.9) (3.10) and (3.14) (3.15) we see that $w[\gamma]$ becomes identical with Pirani's energy density when the frame $\vec{\gamma}$ is irrotational.

Naturally in the case of a Schwarzschild space-time the density $w(\gamma)$, as well like Pirani's density, gives rise to a global rest energy which has the value $M_0 c^2$.

IV. — PREFERRED DESCRIPTORS

There is the same problem for single index complexes as there is for those with two indices: namely the problem of research for possible prefe-

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(4) We have limited this definition to a unit vector.

(5) It should be noted that any vector field, which is a divergence of a double antisymmetrical tensor ($W_r \equiv \nabla_s W^{[rs]}$) is conserved. In fact :

$$\nabla_r(\nabla_s W^{[rs]}) = \frac{1}{\sqrt{-g}} \partial_r \left\{ \sqrt{-g} \left[\frac{1}{\sqrt{-g}} \partial_s (\sqrt{-g}\, W^{[rs]}) \right] \right\} = 0.$$

Other possible superpotentials which produce conserved vectors are the following :

$$\tilde{\Omega}^{[rs]} \equiv \gamma^{ru}\gamma^{sv}(\nabla_u \gamma_v - \nabla_v \gamma_u) \qquad (\gamma^{ru} = g^{ru} + \gamma^r\gamma^u)$$
$$U^{[rs]} \equiv \nabla^l \xi^r \nabla^s \xi_l - \nabla^l \xi^s \nabla^r \xi_l$$
$$V^{[rs]} \equiv \xi^l(\xi^r \nabla_m \xi^s - \xi^s \nabla_m \xi^r) \nabla_l \xi^m.$$

(6) The formulæ (3.10) and (3.15) are rapidly established starting from (3.9) and (3.14) by means of the formulæ of projection of Ricci's tensor established by I. Cattaneo-Gasparini [*30*] [*31*].

rential ξ^i fields, possibly unitary, which may give a sure physical significance to the global magnitude E[ξ]. As we are not longer dealing with a system of coordinates but with a vector field, there is no doubt that the possible preferential conditions may also have a covariant formulation. Komar [*59*] [*60*], starting from the fact that if V_4 admits a Killing field ξ_i the vector $\xi^m T^i_m$ is conserved [*102*], and that also in the Lorentz-covariant theories the existence of 10 constants of motion is associated with 10 independent Killing vectors, inclines to give to these vectors, when they exist, a preferential character. His conviction is further confirmed by the fact that a possible hypersurface orthogonal Killing field also includes the preferential properties of the minimum surfaces (Dirac [*41*], Arnowitt-Deser-Misner [*2*], Rayski [*87*] [*88*]) and of the harmonic coordinates (Fock [*46*]).

When V_4 does not admit Killing fields, which is the most likely case, Komar maintains that the descriptor of a physically significant global magnitude must possess—at least asymptotically—the qualities of a Killing hypersurface-orthogonal vector. He succeeds in formulating covariant asymptotic conditions which correspond to Fock-Trautman's asymptotic conditions, in such a way as also to include any possible radiative systems.

There is also proposed in Komar's work [*59*] a local type condition imposed on the descriptor ξ^i; this attributes a preferential feature to the so called semi-Killing vector fields. There remains some doubt however as to whether the condition imposes limitations not only on the field ξ^i but also on the manifold V_4.

We are also indebted to Komar [*60*] for a local preferential criterion for time like descriptors which still corresponds to the idea of approaching as near as possible to a Killing field, but with the precise condition too that the resultant energy, globally, and locally, is positive definite. This criterion agrees with a widely held opinion (Brill [*19*]; Arnowitt-Deser-Misner [*2*]; Peres, see Komar [*60*]) that a possible free gravitational energy, that is to say without a source, must have the same sign as matter energy.

With the aim of examining the question from a different point of view we shall employ as descriptor a unit vector field γ^i, which corresponds, as we know, to introducing into V_4 a physical frame in Møller's sense. With this choice of frame we shall assume as a definition of the local density of energy the formula (3.14) paying attention to the expression (3.15) that the same density $w[\gamma]$ takes on in a vacuum. From this it follows immediately that if one imposes the scalar second order condition on

$$\gamma^{ik}\mathfrak{L}\mathfrak{L}\gamma_{ik} = 0, \tag{4.1}$$

$w[\gamma]$ acquires the following properties: (*a*) it does not contain second order derivatives of γ^i's (*b*) it is negative definite:

$$(4.2) \qquad w[\gamma] = -\frac{2}{\varkappa}\left\{\frac{1}{4}\tilde{K}_{ik}\tilde{K}^{ik} + \frac{1}{4}\tilde{\Omega}_{ik}\tilde{\Omega}^{ik} + C_i C^i\right\}.$$

Moreover it has the noteworthy property that an eventual vanishing of this density in a frame $\vec{\gamma}$ satisfying the condition (4.1) implies the local vanishing of the Riemann's tensor (7).

The possible adoption of the formula (4.2) as the density of free gravitational energy would put the latter in direct relationship with the typical magnitudes of the gravitational field (8) and the vanishing of the energy with the vanishing of the field.

The property of definiteness of sign satisfied by the density (4.2) is certainly a good property for an energy density. Naturally it remains to be seen if the negative sign is physically acceptable. An argument supporting the plausibility of this assumption will be given shortly.

With regard to the preferential condition (4.1) not to be confused with the condition of minimal area

$$\left(\gamma^{rs}\mathfrak{L}\gamma_{rs} = 0, \qquad \gamma_r = \partial_r\varphi/\sqrt{g^{rs}\partial_r\varphi\partial_s\varphi}\right),$$

it does not seem very restrictive.

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(7) Still with regard to the projection formulæ quoted in the previous footnote (6) one can recognise that if in an empty region D of V_4 C_i, $\tilde{\Omega}_{ij}$, $\tilde{K}_{ij}$ are simultaneously zero in value, Ricci's tensor of the three dimensional varieties orthogonal to the field $\vec{\gamma}$ is also zero; but for a three dimensional manifold, that implies the annihilation of the Riemann's tensor. Resorting finally to the projection formulæ for Riemann's tensor of V_4 one recognises that the preceding conditions imply in D the annihilation of the curvature tensor of V_4.

(8) According to the relative formulation of Einstein's theory of gravitation [*24*] [*29*] the gravitational field is locally represented by the tensor field $\nabla_i\gamma_j$ which is in turn represented by the space-tensors

$$\tilde{K}_{ij},\ \tilde{\Omega}_{ij},\ C_i\ :\ \nabla_i\gamma_j = \frac{1}{2}(\tilde{K}_{ij} + \tilde{\Omega}_{ij}) - \gamma_i C_j.$$

The tensor $\nabla_i\gamma_j$ has at the same time relative and absolute features. It varies with changes in the field $\vec{\gamma}$ but its possible annihilation in one frame implies its annihilation in every other as well as (in a vacuum) the annihilation of the curvature tensor of V_4.

V. — ON THE ENERGY OF PURE GRAVITATION

Whilst there can be no doubt, at least for stationary systems, on what is intended by total field energy, what is less certain is the possibility of decomposing this energy into two or more parts which have separately a physical significance, as for example a division into a purely matter energy and a purely gravitational energy, with the possible addition of the energy of interaction between the field of pure matter and the field of pure gravitation.

There are also doubts as to what sign is to be attributed to the energy of pure gravitation, provided, of course, that we can talk of the latter.

We can attempt to find our bearings on these important questions examining from close at hand the case of a Schwarzschild space-time; although it should be pointed out that this is a very peculiar case. Naturally I intend by this a complete space-time generated by a central spherical liquid mass having a uniform proper density μ_0 (9). In the zone occupied by the central nucleus the ds^2 has the form

$$(5.1)\quad \begin{cases} ds^2_{\text{int}} = \dfrac{dr^2}{1-\dfrac{r^2}{R^2}} + r^2(d\Theta^2 + \sin^2\Theta d\varphi^2) \\ \qquad + \left\{\dfrac{3}{2}\sqrt{1-\dfrac{r_1^2}{R^2}} - \dfrac{1}{2}\sqrt{1-\dfrac{r^2}{R^2}}\right\}^2 c^2dt^2 \end{cases}$$

where $R^2 = 3/\varkappa\mu_0 c^2$, and r_1 is the value of r on the surface of the nucleus. In the liquid there is a pressure

$$(5.2)\qquad p_0(r) = \frac{1}{\varkappa R^2}\,\frac{3\sqrt{1-\dfrac{r^2}{R^2}} - 3\sqrt{1-\dfrac{r_1^2}{R^2}}}{3\sqrt{1-\dfrac{r_1^2}{R^2}} - \sqrt{1-\dfrac{r^2}{R^2}}}.$$

In the external zone the ds^2, joining with the former, has the form

$$(5.3)\qquad ds^2_{\text{ext}} = \frac{dr^2}{1-\dfrac{\alpha}{r}} + r^2(d\Theta^2 + \sin^2\Theta d\varphi^2) - \left(1-\frac{\alpha}{r}\right)c^2dt^2$$

(9) Cf. for example Møller [70].

with

$$\alpha = \varkappa\mu_0 c^2 r_1^3/3 = r_1^3/R^2.$$

In the frame $\vec{\gamma}$ of rest the total energy has, concordantly in all the definitions, the value

$$E_{tot} = M_0 c^2 = \frac{4}{3}\pi r_1^3 \mu_0 c^2. \tag{5.4}$$

I shall now examine the possibility of defining an energy of pure gravitation.

Besides the total energy (of rest) we can introduce a purely matter energy of rest $\overline{M}_0 c^2$ calculated as a product of the constant density $\mu_0 c^2$ for the volume V_0 of the spherical nucleus ([10])

$$\begin{aligned} E_{mat} &\equiv \overline{M}_0 c^2 = \iiint T_{lm}\gamma^l\gamma^m d^3V = V_0\mu_0 c^2 \\ &= \frac{4}{3}\pi R^3\mu_0 c^2\left[\arcsin\frac{r_1}{R} - \frac{r_1}{R}\sqrt{1-\frac{r_1^2}{R^2}}\right]. \end{aligned} \tag{5.5}$$

It follows naturally to assume the difference $(M_0 - \overline{M}_0)c^2$ as energy directly linked to the gravitational field

$$E_g = (M_0 - \overline{M}_0)c^2 = \frac{4}{3}\pi\mu_0 c^2\left\{r_1^3 - \frac{3R^2}{2}\left[\arcsin\frac{r_1}{R} - \frac{r_1}{R}\sqrt{1-\frac{r_1^2}{R^2}}\right]\right\}. \tag{5.6}$$

It is negative definite as may be easily recognised by developping the quantity in brackets in power series of r_1/R:

$$\begin{aligned} E_g &= -\frac{4}{3}\pi r_1^3\mu_0 c^2\left\{\frac{3}{10}\frac{r_1^2}{R^2} + \frac{9}{56}\frac{r_1^4}{R^4} + \frac{5}{72}\frac{r_1^6}{R^6} + \dots\right\} \\ &= -\frac{16}{15}k\pi^2\mu_0^2 r_1^5 - \dots \end{aligned} \tag{5.7}$$

The value of first approximation, $-\frac{16}{15}k\pi^2\mu_0^2 r_1^5$, coincides with the value of the Newtonian gravitational energy for a sphere having a radius r_1 and a density μ_0. The following terms, which correspond to the increasing powers of k, constitute successive Einstein corrections; they come down

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([10]) Cf. Møller [*70*].

to zero for $c \to \infty$ and are negligible for $\frac{k\mu_0 r_1}{c^2} \lll 1$. It is fairly significant that E_g is negative ([11]).

Another criterion for introducing a global quantity linked to the gravitational field is to integrate the energy density (4.2) we obtained for empty zones for on the whole physical space, including of course the zones occupied by matter:

$$(5.9) \qquad E_{fg}[\vec{\gamma}] = -\iiint\left(\frac{1}{4}\tilde{K}_{rs}\tilde{K}^{rs} + \frac{1}{4}\tilde{\Omega}_{rs}\tilde{\Omega}^{rs} + C_r C^r\right)d^3\sigma$$

employing a frame $\vec{\gamma}$ adapted to the staticness of V_4. With easy calculations that I shall omit here one obtains:

$$(5.10) \qquad \begin{aligned} E_{fg} &= M_0 c^2 - \overline{M}_0 c^2 - \iiint 3p_0 d^3\sigma \\ &= -\frac{32}{15}k\pi^2\mu_0^2 r_1^5 - \dots \end{aligned}$$

The formula (5.10) which provides E_{fg} a value of first approximation double that of E_g suggests the possibility of decomposing the total energy M_0c^2 of Schwarzschild's space-time into an energy of pure matter $\overline{M}_0c^2$, a free gravitational energy E_{fg}, and an energy of interaction between the field of pure matter and the field of pure gravitation, $E_{int} = \iiint 3p_0 d^3\sigma$:

$$(5.11) \qquad E_{tot} = E_{mat} + E_{fg} + E_{int}.$$

([11]) The result now obtained is substantially confirmed by a second process which consists in carrying to infinity all the masses of the nucleus for successive spherical strata, thus destroying the gravitational field without destroying the matter; and in calculating the work carried out in this operation. The calculations, here omitted, give rise to a gravitational energy

$$(5.8) \qquad E'_g = 4\pi\mu_0 c^2 \int_0^{r_1} \frac{\rho^2 \log\sqrt{1-\frac{\rho^2}{R^2}}}{\sqrt{1-\frac{\rho^2}{R^2}}}\,d\rho = -\frac{16}{15}k\pi^2\mu_0^2 r_1^5 - \dots$$

which does not coincide exactly with the preceding one but has in common with it the term of first approximation. This coincidence comes down in favour of the substantial physical consistency of the two definitions. With regard to the exact physical significance to be attributed to E_g (or E'_g) I shall limit myself now to observing that it is linked to the existence of the gravitational field and that it would be lacking if k were equal to 0.

One gives rise to the same decomposition also in a more general case ($\tilde{\Omega}_{ij} = 0$ (4.1) satisfied) using the following definitions

$$\text{(5.12)} \qquad E_{tot} = \frac{2}{\varkappa} \iiint \nabla_i C^i d^3\sigma$$

$$\text{(5.13)} \qquad E_{mat} = \iiint T_{ik}\gamma^i\gamma^k d^3\sigma$$

$$\text{(5.14)} \qquad E_{fg} = -\frac{2}{\varkappa} \iiint \left\{ \frac{1}{4} \tilde{K}_{ik}\tilde{K}^{ik} + C_i C^i \right\} d^3\sigma$$

$$\text{(5.15)} \qquad E_{int} = \iiint T_{ik}\gamma^{ik} d^3\sigma.$$

The preceding is merely an attempt, to decompose total energy into its principal physical components.

ACKNOWLEDGEMENTS

I should like to thank my collaborators S. Bonazzola and G. Ferrarese for the useful discussions we have had on various aspects of this paper.

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Manuscrit reçu le 28 juillet 1965.