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THE DETERMINANT OF THE LAX–PHILLIPS SCATTERING OPERATOR

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ABSTRACT. — Let M denote a finite volume, non-compact hyperbolic surface without elliptic points, and let B denote the Lax–Phillips scattering operator. Using the superzeta function approach due to Voros, we define a Hurwitz-type zeta function $\zeta_B^{\pm}(s,z)$ constructed from the resonances associated to $zI - [(1/2)I \pm B]$. We prove the meromorphic continuation in s of $\zeta_B^{\pm}(s,z)$ and, using the special value at $s = 0$, define a determinant of the operators $zI - [(1/2)I \pm B]$. We obtain expressions for Selberg's zeta function and the determinant of the scattering matrix in terms of the operator determinants.

RÉSUMÉ. — Soit M une surface hyperbolique non compacte à volume fini sans points elliptiques, et soit B l'opérateur de diffusion de Lax–Phillips. En utilisant l'approche due à Voros sur la fonction superzeta, nous définissons une fonction zêta de type Hurwitz $\zeta_B^{\pm}(s,z)$ construite à partir des résonances associées à $zI - [(1/2)I \pm B]$. Nous prouvons le prolongement méromorphe en le paramètre s de $\zeta_B^{\pm}(s,z)$ et, en utilisant la valeur spéciale à $s = 0$, définissons un déterminant des opérateurs $zI - [(1/2)I \pm B]$. Nous obtenons des expressions pour la fonction zêta de Selberg et le déterminant de la matrice de diffusion en termes de déterminants d'opérateurs.

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1. Introduction

1.1. Determinant of the Laplacian and analytic torsion

To begin, let M denote a compact, connected Riemannian manifold of real dimension n with Laplace operator Δ_M. Following the seminal article [24], one defines the determinant of the Laplacian, which we denote by $\det^* \Delta_M$, as follows. Let $e^{-t\Delta_M}$ be the heat operator associated to Δ_M. Since M is compact, the operator $e^{-t\Delta_M}$ is a Hilbert–Schmidt type operator with the kernel $K_M(t; x, y)$, the heat kernel associated to Δ_M. Moreover, using the semi-group property of $e^{-t\Delta_M}$ one can show that $e^{-t\Delta_M}$ is actually of trace-class. Therefore, we can consider the trace of the operator $e^{-t\Delta_M}$, which we refer to as the *trace of the heat kernel*[1] which is defined by

$$(1.1) \qquad \mathrm{Tr}(e^{-t\Delta_M}) = \mathrm{Tr}\, K_M(t) := \int_M K_M(t; x, x)\mathrm{d}\mu_M(x),$$

where $\mathrm{d}\mu_M(x)$ is the volume form on M. As shown in [24], the parametrix construction of the heat kernel implies that its trace $\mathrm{Tr}\, K_M(t)$ admits a certain asymptotic behavior as t approaches zero, and separately one can prove that $\mathrm{Tr}\, K_M(t)$ is continuous in t and bounded as t tends to infinity. As a result, one can define and study various integral transforms of the heat kernel. In particular, for $s \in \mathbb{C}$ with real part $\mathrm{Re}(s)$ sufficiently large, the spectral zeta function $\zeta_M(s)$ is defined from the Mellin transform of the trace of the heat kernel. Specifically, one sets

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\mathrm{Tr}\, K_M(t) - 1\right) t^s \frac{\mathrm{d}t}{t}$$

where $\Gamma(s)$ is the classical Gamma function. The asymptotic expansion of $\mathrm{Tr}\, K_M(t)$ as t approaches zero allows one to prove the meromorphic continuation of $\zeta_M(s)$ to all $s \in \mathbb{C}$ which is holomorphic at $s = 0$. Subsequently, the determinant of the Laplacian is defined by

$$(1.2) \qquad\qquad \det^* \Delta_M := \exp\left(-\zeta_M'(0)\right).$$

There are several generalizations of the above considerations. For example, let E be a flat vector bundle on M, metrized so that one can define the action of a Laplacian $\Delta_{E,k}$ which acts on k-forms that take values in E. Analogous to the above discussion, one can use properties of an associated heat kernel and obtain a definition of the determinant of the Laplacian

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[1] When M is non-compact, the operator $e^{-t\Delta_M}$ is not even Hilbert–Schmidt type. See [11, Theorem 3.7] for more details.

$\det^* \Delta_{E,k}$. Going further, by following [29] and [30], one can consider linear combinations of determinants yielding, for example, the analytic torsion $\tau(M, E)$ of E on M which is given by

$$(1.3) \qquad \tau(M, E) := \frac{1}{2} \sum_{k=1}^{n} (-1)^k k \det^* \Delta_{E,k}.$$

At this time, one understands (1.3) to be a spectral invariant associated to the de Rham cohomology of E on M. If instead one considers compact, connected complex manifolds with metrized holomorphic vector bundles, one obtains a similar definition for analytic torsion stemming from Dolbeault cohomology.

There are many manifestations of the determinant of the Laplacian, and more generally analytic torsion, throughout the mathematical literature. For the sake of space, we will not survey some of the ways in which determinants of the Laplacian have been studied.

1.2. Non-compact hyperbolic Riemann surfaces

If M is a non-compact Riemannian manifold, then it is often the case that the integral (1.1) is divergent for any value of $t > 0$. Hence, the above approach to define a determinant of the Laplacian does not get started. This assertion is true in the case when M is a finite volume, connected, hyperbolic Riemann surfaces, which will be the setting considered in this article. The first attempt to define a determinant of the Laplacian for non-compact, finite volume, hyperbolic Riemann surfaces is due to I. Efrat in [4, 5]. Efrat's approach began with the Selberg trace formula, which in the form Efrat employed does not connect directly with a differential operator. In [17] the authors defined a regularized difference of traces of heat kernels, which did yield results analogous to theorems proved in the setting of compact hyperbolic Riemann surfaces. In [27], W. Müller generalized the idea of a regularized difference of heat traces to other settings. Following this approach, J. Friedman in [11] defined a regularized determinant of the Laplacian for any finite-volume three-dimensional hyperbolic orbifolds with finite-dimensional unitary representations, which he then related to special values of the Selberg zeta-function.

The concept of a regularized quotient of determinants of Laplacian has found important applications. For example, the dissertation of T. Hahn [13] studied Arakelov theory on non-compact finite volume Riemann surfaces

using the regularized difference of heat trace approach due to Jorgenson–Lundelius and Müller; see also [10] where the study is approached differently and somewhat more generally. In [6] the authors used the regularized difference of determinants together with the metric degeneration concept from [18] in their evaluation of the sum of Lyapunov exponents of the Kontsevich–Zorich cocycle with respect to $\mathrm{SL}(2,\mathbb{R})$ invariant measures.

1.3. Our results

It remains an open, and potentially very important, question to define determinants of Laplacians, or related spectral operators, on non-compact Riemannian manifolds.

In the present article we consider a general finite volume hyperbolic Riemann surface without fixed points. Scattering theory, stemming from work due to Lax and Phillips (see [22] and [23]), provides us with the definition of a scattering operator, which we denote by B. The scattering operator is defined using certain Hilbert space extensions of the so-called Ingoing and Outgoing spaces; see Section 3 below. Lax and Phillips have shown that B has a discrete spectrum; unfortunately, one cannot define a type of heat trace associated to the spectrum from which one can use a heat kernel type approach to defining the determinant of B. Instead, we follow the superzeta function technique of regularization due to A. Voros in order to define and study the zeta functions $\zeta_B^{\pm}(s,z)$ constructed from the resonances associated to $zI - [(1/2)I \pm B]$. We prove the meromorphic continuation in s of $\zeta_B^{\pm}(s,z)$ and, using the special value at $s = 0$, define a determinant of the operators $zI - [(1/2)I \pm B]$.

Our main results are as follows. First, we obtain expressions for the Selberg zeta function and the determinant of the scattering matrix in terms of the special values of $\zeta_B^{\pm}(s,z)$ at $s = 0$; see Theorem 6.2. Furthermore, we express the special value $Z'(1)$ of the Selberg zeta function in terms of the determinants of the operator $-B + (1/2)I$; see Theorem 6.7.

Regarding Theorem 6.7, it is important to note the structure of the constants which relate the regularized determinant of $-B + (1/2)I$ and $Z'(1)$. Specifically, we now understand the nature of the corresponding constant from [31] in terms of the R-class of Arakelov theory. As it turns out, the multiplicative constant which appears in Theorem 6.7 has a similar structure. Finally, one should note that Müller [26] was the first to study the regularized determinant, and associated zeta function of the Lax–Phillips

operator. He studied the function $\zeta_B^+(s, z)$ for $\mathrm{Re}(z) > 1$ and gave a mero-morphic extension to $s \in \mathbb{C}$. The method of proof in [26] utilizes the zeta regularization approach from [4, 5], whereas we develop an alternative approach to define a regularized product, see Section 4 below.

1.4. Outline of the paper

The article is organized as follows. In Section 2 we recall various back-ground material from the literature and establish the notation which will be used throughout the paper. This discussion continues in Section 3 where we recall results from Lax–Phillips scattering theory. In Section 4 we establish the meromorphic continuation of superzeta functions in very general con-text. From the general results from Section 4, we prove in Section 5 that the superzeta functions $\zeta_B^\pm(s, z)$ admit meromorphic continuations, with appropriate quantifications. Finally, in Section 6, we complete the proof of the main results of the paper, as cited above.

Acknowledgement

2. Background material

2.1. Basic notation

Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be torsion free Fuchsian group of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy \,, y > 0\}$. Let M be the quotient space $\Gamma \backslash \mathbb{H}$ and g the genus of M. Denote by $\mathbf{c}$ number of inequivalent cusps of M.

We denote by $\mathrm{d}s_{\mathrm{hyp}}^2(z)$ the line element and by $\mu_{\mathrm{hyp}}(z)$ the volume form corresponding to the hyperbolic metric on M which is compatible with the complex structure of M and has constant curvature equal to -1. Locally on M, we have

$$\mathrm{d}s_{\mathrm{hyp}}^2(z) = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \quad \text{and} \quad \mu_{\mathrm{hyp}}(z) = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2} \,.$$

We recall that the hyperbolic volume $\text{vol}_{\text{hyp}}(M)$ of M is given by the formula

$$\text{vol}(M) = 2\pi(2g - 2 + \mathbf{c}).$$

Let $\mathcal{V}^\Gamma$ denote the space of $\Gamma-$invariant functions $\varphi : \mathbb{H} \longrightarrow \mathbb{C}$. For $\varphi \in \mathcal{V}^\Gamma$, we set

$$\|\varphi\|^2 := \int_{\mathcal{F}} |\varphi(z)|^2 \mu_{\text{hyp}}(z),$$

whenever it is defined, where $\mathcal{F}$ is a fundamental domain of M, which we may assume is constructed as described in detail within Chapter 9 of [3]. We then introduce the Hilbert space

$$\mathcal{H}(\Gamma) := \left\{ \varphi \in \mathcal{V}^\Gamma \,\middle|\, \|\varphi\| < \infty \right\}$$

equipped with the inner product

$$\langle \varphi_1, \varphi_2 \rangle := \int_{\mathcal{F}} \varphi_1(z)\overline{\varphi_2(z)}\mu_{\text{hyp}}(z) \qquad (\varphi_1, \varphi_2 \in \mathcal{H}(\Gamma)).$$

The Laplacian

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

acts on the smooth functions of $\mathcal{H}(\Gamma)$ and extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}(\Gamma)$.

For $f(s)$ a meromorphic function, we define the *null set*, $N(f) = \{s \in \mathbb{C} \,|\, f(s) = 0\}$ counted with multiplicity. Similarly, $P(f)$ denotes the *polar set*.

2.2. Gamma function

Let $\Gamma(s)$ denote the gamma function. Its poles are all simple and located at each point of $-\mathbb{N}$, where $-\mathbb{N} = \{0, -1, -2, \ldots\}$. For $|\arg s| \leqslant \pi - \delta$ and $\delta > 0$, the asymptotic expansion [2, p. 20] of $\log \Gamma(s)$ is given by

$$(2.1) \quad \log \Gamma(s) = \frac{1}{2}\log 2\pi + \left(s - \frac{1}{2}\right)\log s - s$$

$$+ \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j-1)2j} \frac{1}{s^{2j-1}} + g_m(s).$$

Here B_i are the Bernoulli numbers and $g_m(s)$ is a holomorphic function in the right half plane $\text{Re}(s) > 0$ such that $g_m^{(j)}(s) = O(s^{-2m+1-j})$ as $\text{Re}(s) \to \infty$ for all integers $j \geqslant 0$, and where the implied constant depends on j and m.

2.3. Barnes double gamma function

The Barnes double gamma function is an entire order two function defined by

$$G(s+1) = (2\pi)^{s/2} \exp\left[-\frac{1}{2}\left[(1+\gamma)s^2 + s\right]\right] \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right)^n \exp\left[-s+\frac{s^2}{2n}\right],$$

where γ is the Euler constant. Therefore, $G(s+1)$ has a zero of multiplicity n, at each point $-n \in \{-1,-2,\ldots\}$.

For $s \notin -\mathbb{N}$, we have that (see [8, p. 114])

$$(2.2) \qquad \frac{G'(s+1)}{G(s+1)} = \frac{1}{2}\log(2\pi) + \frac{1}{2} - s + s\psi(s),$$

where $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ denotes digamma function. For $\mathrm{Re}(s) > 0$ and as $s \to \infty$, the asymptotic expansion of $\log G(s+1)$ is given in [7] or[2] [1, Lemma 5.1] by

$$(2.3) \quad \log G(s+1) = \frac{s^2}{2}\left(\log s - \frac{3}{2}\right) - \frac{\log s}{12} - s\,\zeta'(0)$$

$$+ \zeta'(-1) - \sum_{k=1}^{n} \frac{B_{2k+2}}{4\,k\,(k+1)\,s^{2k}} + h_{n+1}(s).$$

Here, $\zeta(s)$ is the Riemann zeta-function and

$$h_{n+1}(s) = \frac{(-1)^{n+1}}{s^{2n+2}} \int_0^{\infty} \frac{t}{\exp(2\pi t)-1} \int_0^{t^2} \frac{y^{n+1}}{y+s^2}\,\mathrm{d}y\,\mathrm{d}t.$$

By a close inspection of the proof of [1, Lemma 5.1] it follows that $h_{n+1}(s)$ is holomorphic function in the right half plane $\mathrm{Re}(s) > 0$ which satisfies the asymptotic relation $h_{n+1}^{(j)}(s) = O(s^{-2n-2-j})$ as $\mathrm{Re}(s) \to \infty$ for all integers $j \geqslant 0$, and where the implied constant depends upon j and n.

Set

$$(2.4) \qquad\qquad G_1(s) = \left(\frac{(2\pi)^s(G(s+1))^2}{\Gamma(s)}\right)^{\frac{\mathrm{vol}(M)}{2\pi}}$$

It follows that $G_1(s)$ is an entire function of order two with zeros at points $-n \in -\mathbb{N}$ and corresponding multiplicities $\frac{\mathrm{vol}(M)}{2\pi}(2n+1)$.

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[2] Note that (2.1) is needed to reconcile these two references.

2.4. Hurwitz zeta function

The Hurwitz zeta-function $\zeta_H(s, z)$ is defined for $\mathrm{Re}(s) > 1$ and $z \in \mathbb{C} \setminus (-\mathbb{N})$ by the absolutely convergent series

$$\zeta_H(s, z) = \sum_{n=0}^{\infty} \frac{1}{(z + n)^s}.$$

For fixed z, $\zeta_H(s, z)$ possesses a meromorphic continuation to the whole s−plane with a single pole at $s = 1$ of order 1 and with residue 1.

For fixed z, one can show that

$$\zeta_H(-n, z) = -\frac{B_{n+1}(z)}{n + 1},$$

where $n \in \mathbb{N}$, and B_n denotes the n−th Bernoulli polynomial.

For integral values of s, the function $\zeta_H(s, z)$ is related to derivatives of the digamma function in the following way:

$$\zeta_H(n + 1, z) = \frac{(-1)^{n+1}}{n!} \psi^{(n)}(z), \quad n = 1, 2, \ldots$$

2.5. Automorphic scattering matrix

Let $\phi(s)$ denote the determinant of the hyperbolic scattering matrix $\Phi(s)$, see [32, Section 3.5]. The function $\phi(s)$ is meromorphic of order two [32, Theorem 4.4.3]. It is regular for $\mathrm{Re}(s) > \frac{1}{2}$ except for a finite number of poles $\sigma_1, \sigma_2, \ldots \sigma_{\mathfrak{m}} \in (1/2, 1]$; each pole has multiplicity no greater than $\mathbf{c}$, the number of cusps of M.

We let ρ denote an arbitrary pole of $\phi(s)$. Since $\phi(s)\phi(1-s) = 1$, the set of zeros and poles are related by $N(\phi) = 1 - P(\phi)$, hence $1 - \sigma_1, 1 - \sigma_2, \ldots 1 - \sigma_{\mathfrak{m}}$ are the zeros in $[0, 1/2)$.

Each pole $\sigma_i \in (1/2, 1]$ corresponds to a Δ−eigenspace, $A_1(\lambda_i)$, with eigenvalue $\lambda_i = \sigma_i(1 - \sigma_i)$, $i = 1, \ldots, \mathfrak{m}$, in the space spanned by the incomplete theta series. For all $i = 1, \ldots, \mathfrak{m}$ we have [14, Equation 3.33 on p. 299]

(2.5)   [The multiplicity of the pole of $\phi(s)$ at $s = \sigma_i$]

$$\leqslant \dim A_1(\sigma_i(1 - \sigma_i)) \leqslant \mathbf{c}.$$

For $\mathrm{Re}(s) > 1$, $\phi(s)$ can be written as an absolutely convergent generalized Dirichlet series and Gamma functions; namely, we have that

(2.6)
$$\phi(s) = \pi^{\frac{\mathbf{c}}{2}} \left(\frac{\Gamma \left(s - \frac{1}{2} \right)}{\Gamma (s)} \right)^{\mathbf{c}} \sum_{n=1}^{\infty} \frac{d(n)}{g_n^{2s}}$$

where $0 < g_1 < g_2 < \dots$ and $d(n) \in \mathbb{R}$ with $d(1) \neq 0$.

We will rewrite (2.6) in a slightly different form. Let $c_1 = -2 \log g_1 \neq 0$, $c_2 = \log d(1)$, and let $u_n = g_n/g_1 > 1$. Then for $\mathrm{Re}(s) > 1$ we can write $\phi(s) = L(s)H(s)$ where

$$(2.7) \qquad L(s) = \pi^{\frac{\mathbf{c}}{2}} \left(\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \right)^{\mathbf{c}} e^{c_1 s + c_2}$$

and

$$(2.8) \qquad H(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{u_n^{2s}},$$

where $a(n) \in \mathbb{R}$ and the series (2.8) converges absolutely for $\mathrm{Re}(s) > 1$. From the generalized Dirichlet series representation (2.8) of $H(s)$ it follows that

$$(2.9) \qquad \frac{\mathrm{d}^k}{\mathrm{d}s^k} \log H(s) = O(\beta_k^{-\mathrm{Re}(s)}) \quad \text{when} \quad \mathrm{Re}(s) \to +\infty,$$

for some $\beta_k > 1$ where the implied constant depends on $k \in \mathbb{N}$.

2.6. Selberg zeta-function

The Selberg zeta function associated to the quotient space $M = \Gamma \backslash \mathbb{H}$ is defined for $\mathrm{Re}(s) > 1$ by the absolutely convergent Euler product

$$Z(s) = \prod_{\{P_0\} \in P(\Gamma)} \prod_{n=0}^{\infty} \left(1 - N(P_0)^{-(s+n)} \right),$$

where $P(\Gamma)$ denotes the set of all primitive hyperbolic conjugacy classes in Γ, and $N(P_0)$ denotes the norm of $P_0 \in \Gamma$. From the product representation given above, we obtain for $\mathrm{Re}(s) > 1$

$$(2.10) \qquad \begin{aligned} \log Z(s) &= \sum_{\{P_0\} \in P(\Gamma)} \sum_{n=0}^{\infty} \left(-\sum_{l=1}^{\infty} \frac{N(P_0)^{-(s+n)l}}{l} \right) \\ &= -\sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^s \log N(P)}, \end{aligned}$$

where $H(\Gamma)$ denotes the set of all hyperbolic conjugacy classes in Γ, and $\Lambda(P) = \frac{\log N(P_0)}{1 - N(P)^{-1}}$, for the (unique) primitive element P_0 conjugate to P.

Let P_{00} be the primitive hyperbolic conjugacy class in all of $P(\Gamma)$ with the smallest norm. Setting $\alpha = N(P_{00})^{\frac{1}{2}}$, we see that for $\operatorname{Re}(s) > 2$ and $k \in \mathbb{N}$ the asymptotic

$$(2.11) \qquad \frac{\mathrm{d}^k}{\mathrm{d}s^k} \log Z(s) = O(\alpha^{-\operatorname{Re}(s)}) \quad \text{when} \quad \operatorname{Re}(s) \to +\infty.$$

Here the implied constant depends on $k \in \mathbb{N}$.

The Selberg zeta function admits a meromorphic continuation to all $s \in \mathbb{C}$, and its divisor can be determined explicitly, as follows (see [33, p. 49] and [14, p. 499]):

(1) Zeros at the points s_j on the line $\operatorname{Re}(s) = \frac{1}{2}$ symmetric relative to the real axis and in $(1/2, 1]$. Each zero s_j has multiplicity $m(s_j) = m(\lambda_j)$ where $s_j(1 - s_j) = \lambda_j$ is an eigenvalue in the discrete spectrum of Δ;

(2) Zeros at the points $s_j = 1 - \sigma_j \in [0, 1/2)$ (see Section 2.5). Here, by (2.5), the multiplicity $m(s_j)$ is

$$[\text{multiplicity of the eigenvalue} \lambda_j = \sigma_j(1 - \sigma_j)]$$
$$- [\text{order of the pole of } \phi(s) \text{ at } s = \sigma_j] \geqslant 0;$$

(3) If $\lambda = \frac{1}{4}$ is an eigenvalue of Δ of multiplicity $d_{1/4}$, then $s = \frac{1}{2}$ is a zero (or a pole, depending on the sign of the following) of $Z(s)$ of multiplicity

$$2d_{1/4} - \frac{1}{2}\left(\mathbf{c} - \operatorname{tr}\Phi\left(\frac{1}{2}\right)\right);$$

(4) Zeros at each $s = \rho$, where ρ is a pole of $\phi(s)$ with $\operatorname{Re}(\rho) < \frac{1}{2}$;

(5) *Trivial* Zeros at points $s = -n \in -\mathbb{N}$, with multiplicities $\frac{\operatorname{vol}(M)}{2\pi} \times (2n + 1)$;

(6) Poles at $s = -n - \frac{1}{2}$, where $n = 0, 1, 2, \ldots$, each with multiplicity $\mathbf{c}$.

2.7. Selberg zeta function of higher order

For $\operatorname{Re}(s) > 1$ and $r \in \mathbb{N}$, following [21, Section 4.2.], we define the *Selberg zeta function of order r*, or the *poly-Selberg zeta function of degree r*, by the relation

$$Z^{(r)}(s) = \exp\left(-\sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^s (\log N(P))^r}\right).$$

This definition is consistent with the case $r = 1$ (see (2.10)), namely $Z^{(1)}(s) = Z(s)$.

Following [21, Section 4.2], it is easy to show that

$$Z^{(r)}(s) = \prod_{\{P_0\} \in P(\Gamma)} \prod_{n=0}^{\infty} H_r \left(N(P_0)^{-(s+n)} \right)^{(\log N(P_0))^{-(r-1)}},$$

for $\mathrm{Re}(s) > 1$, where $H_r(z) = \exp(\mathrm{Li}_r(z))$, and

$$\mathrm{Li}_r(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^r} \qquad (|z| < 1)$$

is the polylogarithm of a degree r.

The meromorphic continuation of $Z^{(r)}(s)$ follows inductively for $r \in \mathbb{N}$ from the differential ladder relation

$$\frac{\mathrm{d}^{r-1}}{\mathrm{d}z^{r-1}} \log Z^{(r)}(s) = (-1)^{r-1} \log Z(s).$$

See [21, Proposition 4.9] for more details. Note that [21] deals with compact Riemann surfaces, so one must modify the region Ω_Γ defined in [21, Proposition 4.9] by excluding the vertical lines passing through poles ρ of the hyperbolic scattering determinant ϕ; the other details are identical.

2.8. Complete zeta functions

In this subsection we define two zeta functions $Z_+(s)$ and $Z_-(s)$ associated with $Z(s)$ which are both entire functions of order two.

Set

$$Z_+(s) = \frac{Z(s)}{G_1(s)(\Gamma(s-1/2))^{\mathbf{c}}},$$

where $G_1(s)$ is defined by (2.4). Note that we have canceled out the trivial zeros and poles of $Z(s)$, hence the set $N(Z_+)$ consists of the following:

- At $s = \frac{1}{2}$, the multiplicity of the zero is $\mathfrak{a}$, where

$$\mathfrak{a} = 2d_{1/4} + \mathbf{c} - \frac{1}{2}\left(\mathbf{c} - \mathrm{tr}\,\Phi(\tfrac{1}{2})\right) = 2d_{1/4} + \frac{1}{2}\left(\mathbf{c} + \mathrm{tr}\,\Phi(\tfrac{1}{2})\right) \geqslant 0;$$

- Zeros at the points s_j on the line $\mathrm{Re}(s) = \frac{1}{2}$ symmetric relative to the real axis and in $(1/2, 1]$. Each zero s_j has multiplicity $m(s_j) = m(\lambda_j)$ where $s_j(1-s_j) = \lambda_j$ is an eigenvalue in the discrete spectrum of Δ;

- Zeros at the points $s_j = 1 - \sigma_j \in [0, 1/2)$ (see Section 2.5). Here, by (2.5), the multiplicity $m(s_j)$ is

[multiplicity of the eigenvalue $\lambda_j = \sigma_j(1 - \sigma_j)$]
$$- \text{[order of the pole of } \phi(s) \text{ at } s = \sigma_j] \geqslant 0;$$

- Zeros at each $s = \rho$, where ρ is a pole of $\phi(s)$ with $\text{Re}(\rho) < \frac{1}{2}$.

Set
$$Z_-(s) = Z_+(s)\phi(s).$$

Then it follows that $N(Z_-) = 1 - N(Z_+)$. That is, s is a zero of Z_+ iff $1 - s$ is a zero (of the same multiplicity) of Z_-.

3. Lax–Phillips scattering operator on M

Following [22] and [28] we will introduce the scattering operator B on M and identify its spectrum. Let $u = u(z, t)$ be a smooth function on $\mathbb{H} \times \mathbb{R}$. Consider the hyperbolic wave equation for $-\Delta$,

$$u_{tt} = Lu = -\Delta u - \frac{u}{4},$$

with initial values $f = \{f_1, f_2\} \in \mathcal{H}(\Gamma) \times \mathcal{H}(\Gamma)$, where

$$u(z, 0) = f_1(z) \quad \text{and} \quad u_t(z, 0) = f_2(z).$$

Recall that $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}(\Gamma)$. The energy form (norm) for the wave equation is

$$E(u) = \langle u, Lu \rangle + \langle \partial_t u, \partial_t u \rangle.$$

The energy form is independent of t, so in terms of initial values, an integration by parts yields

$$E(f) = \int_{\mathcal{F}} \left(y^2 |\partial f_1|^2 - \frac{|f_1|^2}{4} + |f_2|^2 \right) \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

where $\mathcal{F}$ denotes the Ford fundamental domain of Γ.

In general, the quadratic form E is not positive definite. To overcome this difficulty we follow [28] and modify E in the following manner: Choose a partition of unity $\{\psi_j \mid j = 0, \ldots, \mathbf{c}\}$ with ψ_0 of compact support and $\psi_j = 1$ in the jth cusp (transformed to ∞) for $y > a$, where a is fixed and sufficiently large. Set

$$E_j(f) = \int_{\mathcal{F}} \psi_j \left(y^2 |\partial f_1|^2 - \frac{|f_1|^2}{4} + |f_2|^2 \right) \frac{\mathrm{d}x\mathrm{d}y}{y^2},$$

so that $E = \sum_j E_j$. There exists a constant k_1 and a compact subset $K \subset \mathcal{F}$ so that

$$G(f) := E(f) + k_1 \int_K |f_1|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

is positive definite[3].

Define the Hilbert space $\mathcal{H}(\Gamma)_G$ as the completion with respect to G of C^∞ data $f = \{f_1, f_2\} \in C_0^\infty(\mathcal{F}) \times C_0^\infty(\mathcal{F})$ with compact support.

The wave equation may be written in the form $f_t = Af$ where

$$A = \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix},$$

defined as the closure of A, restricted to $C_0^\infty(\mathcal{F}) \times C_0^\infty(\mathcal{F})$. The operator A is the infinitesimal generator a unitary group $U(t)$ with respect to the energy norm E.

The Incoming and Outgoing subspaces of $\mathcal{H}(\Gamma)_G$ are defined as follows.

- The Incoming subspace $\mathcal{D}_-$ is the closure in $\mathcal{H}(\Gamma)_G$ of the set of elements of the form $\{y^{1/2}\varphi(y), y^{3/2}\varphi'(y)\}$, where φ is a smooth function of y which vanishes for $y \leqslant a$, and $\varphi' = \frac{\mathrm{d}}{\mathrm{d}y}\varphi$.
- The Outgoing subspace $\mathcal{D}_+$ is defined analogously as the closure of $\{y^{1/2}\varphi(y), -y^{3/2}\varphi'(y)\}$.

The subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ are G orthogonal. Let $\mathcal{K}$ denote the orthogonal complement of $\mathcal{D}_- \oplus \mathcal{D}_+$ in $\mathcal{H}(\Gamma)_G$ and let P denote the G-orthogonal (and E-orthogonal[4]) projection of $\mathcal{H}(\Gamma)_G$ onto $\mathcal{K}$ and set

$$\mathbf{Z}(t) = PU(t)P, \quad \text{for } t \geqslant 0.$$

The operators $\mathbf{Z}(t)$ form a strongly continuous semigroup of operators on $\mathcal{K}$ with infinitesimal generator B. For every λ in the resolvent set of B, $(B - \lambda I)^{-1}$ is a compact operator [23, Section 3]. Hence, B has a pure point spectrum of finite multiplicity and $(B - \lambda I)^{-1}$ is meromorphic in the entire complex plane. See also [22, Theorem 2.7].

Following [28], we define *the singular set* $\sigma(\Gamma)$. First, we define the multiplicity function $m(r)$ as follows:

(1) If $\mathrm{Im}(r) \leqslant 0$ and $r \neq 0$, the multiplicity $m(r)$ is the dimension of the eigenspace for $\lambda = \frac{1}{4} + r^2$ for Δ on $M = \Gamma \setminus \mathbb{H}$. Hence for $\mathrm{Im}(r) \leqslant 0, m(r) = 0$ outside of $(-\infty, \infty) \cup -i(0, \frac{1}{2}]$.

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[3] [28, p. 4] and [23, p. 265] differ in the y^{-2} term.
[4] Since the functions $\phi(y)$ are zero outside of the cusp sectors, the E and G forms agree.

(2) If $\text{Im}(r) > 0$, $m(r)$ is the multiplicity of the eigenvalue $\frac{1}{4} + r^2$ plus the order of the pole (or negative the order of the zero) of $\phi(s)$ at $s = \frac{1}{2} + ir$.

(3) For $r = 0$, $m(r)$ is twice the multiplicity of the cusp forms (with eigenvalue $\lambda = 1/4$) plus $(\mathbf{c} + \text{Tr}(\Phi(1/2)))/2$.

Then, the singular set $\sigma(\Gamma)$ is defined to be the set of all $r \in \mathbb{C}$ with $m(r) > 0$, counted with multiplicity. The singular set $\sigma(\Gamma)$ is closely related to the spectrum $\text{Spec}(B)$ of the operator B by the equation $\text{Spec}(B) = i\sigma(\Gamma)$, see [28].

Therefore, by setting $s = \frac{1}{2} + ir$ and referring to Section 2.8, we have

$$(3.1) \qquad \text{Spec}\left(\frac{1}{2}I + B\right) = N(Z_+).$$

and

$$(3.2) \qquad \text{Spec}\left(\frac{1}{2}I - B\right) = N(Z_-).$$

4. Process of zeta regularization

In the mathematical literature, there exist mainly three different approaches to zeta regularization. In the abstract approach, as in [15, 16, 19, 20] the authors start with a general sequence of complex numbers (generalized eigenvalues) and define criteria for the zeta regularization process. For example, in [16], a theta series is introduced and, under suitable conditions at zero and infinity, a possibly regularized zeta function is defined as the Laplace–Mellin transform of the theta series.

The second approach is based on a generalization of the Poisson summation formula or explicit formula. Starting with the truncated heat kernel, one defines a regularized zeta function as the Mellin transform of the trace of the truncated heat kernel modulo the factor $\frac{1}{\Gamma(s)}$. Variants of the second approach can be found in [4, 5, 11, 25, 26, 27, 30, 31], and many others.

The third approach, formulated by A. Voros in [34, 35, 36, 37] is based on the construction of the so-called superzeta functions, meaning zeta functions constructed over a set of zeros of the primary zeta function. In this setting, one starts with a sequence of zeros, rather than the sequence of eigenvalues, of a certain meromorphic function and then induces zeta regularization through meromorphic continuation of an integral representation of this function, valid in a certain strip. In this section we give a brief description of this methodology.

Let $\mathbb{R}^- = (-\infty, 0]$ be the non-positive real numbers. Let $\{y_k\}_{k \in \mathbb{N}}$ be the sequence of zeros of an entire function f of order 2, repeated with their multiplicities. Let

$$X_f = \{z \in \mathbb{C} \mid (z - y_k) \notin \mathbb{R}^- \text{ for all } y_k\}.$$

For $z \in X_f$, and $s \in \mathbb{C}$ (where convergent) consider the series

$$(4.1) \qquad \mathcal{Z}_f(s, z) = \sum_{k=1}^{\infty} (z - y_k)^{-s},$$

where the complex exponent is defined using the principal branch of the logarithm with $\arg z \in (-\pi, \pi)$ in the cut plane $\mathbb{C} \setminus \mathbb{R}^-$.

Since f is of order two, $\mathcal{Z}_f(s, z)$ converges absolutely for $\operatorname{Re}(s) > 2$. The series $\mathcal{Z}_f(s, z)$ is called the zeta function associated to the zeros of f, or the simply the *superzeta* function of f.

If $\mathcal{Z}_f(s, z)$ has a meromorphic continuation which is regular at $s = 0$, we define the *zeta regularized product* associated to f as

$$D_f(z) = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{Z}_f(s, z)|_{s=0}\right).$$

Hadamard's product formula allows us to write

$$f(z) = \Delta_f(z) = e^{g(z)} z^r \prod_{k=1}^{\infty} \left(\left(1 - \frac{z}{y_k}\right) \exp\left[\frac{z}{y_k} + \frac{z^2}{2y_k{}^2}\right]\right),$$

where $g(z)$ is a polynomial of degree 2 or less, r is the order of a zero of f at $z = 0$, and the other zeros y_k are listed with multiplicity. A simple calculation shows that when $z \in X_f$,

$$(4.2) \qquad \mathcal{Z}_f(3, z) = \frac{1}{2} \left(\log \Delta_f(z)\right)'''.$$

The following proposition is due to Voros ([34, 36, 37]). For completeness, we give a different proof.

PROPOSITION 4.1. — *Let f be an entire function of order two, and for $k \in \mathbb{N}$, let y_k be the sequence of zeros of f. Let $\Delta_f(z)$ denote the Hadamard product representation of f. Assume that for $n > 2$ we have the following asymptotic expansion:*

$$(4.3) \quad \log \Delta_f(z) = \widetilde{a}_2 z^2 \left(\log z - \frac{3}{2}\right) + b_2 z^2 + \widetilde{a}_1 z (\log z - 1)$$

$$+ b_1 z + \widetilde{a}_0 \log z + b_0 + \sum_{k=1}^{n-1} a_k z^{\mu_k} + h_n(z),$$

where $1 > \mu_1 > \cdots > \mu_n \to -\infty$, and $h_n(z)$ is a sequence of holomorphic functions in the sector $|\arg z| < \theta < \pi$, $(\theta > 0)$ such that $h_n^{(j)}(z) = O(|z|^{\mu_n - j})$, as $|z| \to \infty$ in the above sector, for all integers $j \geqslant 0$.

Then, for all $z \in X_f$, the superzeta function $\mathcal{Z}_f(s, z)$ has a meromorphic continuation to the half-plane $\mathrm{Re}(s) < 2$ which is regular at $s = 0$.

Furthermore, the zeta regularized product $D_f(z)$ associated to $\mathcal{Z}_f(s, z)$ is related to $\Delta_f(z)$ through the formula

$$(4.4) \qquad D_f(z) = e^{-(b_2 z^2 + b_1 z + b_0)} \Delta_f(z).$$

Proof. — For any $z \in X_f$, the series

$$(4.5) \qquad \mathcal{Z}_f(3, z + y) = \sum_{k=1}^{\infty} (z + y - y_k)^{-3},$$

which is obtained by setting $s = 3$ in (4.1), converges uniformly for $y \in (0, \infty)$. Furthermore, an application of [12, Formula 3.194.3], with $\mu = 3 - s$, $\nu = 3$ and $\beta = (z - y_k)^{-1}$ yields, for all y_k,

$$\int_0^{\infty} \frac{y^{2-s}\, \mathrm{d}y}{(z + y - y_k)^3} = \frac{1}{2}(z - y_k)^{-s} \Gamma(3 - s)\Gamma(s).$$

Absolute convergence of the series (4.1) for $\mathrm{Re}(s) > 2$ implies that

$$\mathcal{Z}_f(s, z) = \frac{2}{\Gamma(3 - s)\Gamma(s)} \int_0^{\infty} \mathcal{Z}_f(3, z + y) y^{2-s} \mathrm{d}y,$$

for $2 < \mathrm{Re}(s) < 3$. From the relation

$$\frac{1}{\Gamma(s)\Gamma(3 - s)} = \frac{1}{\Gamma(s)\Gamma(1 - s)(1 - s)(2 - s)} = \frac{\sin \pi s}{\pi(1 - s)(2 - s)},$$

(which is obtained by the functional equation and the reflection formula for the gamma function) we obtain

$$(4.6) \qquad \mathcal{Z}_f(s, z) = \frac{2 \sin \pi s}{\pi(1 - s)(2 - s)} \int_0^{\infty} \mathcal{Z}_f(3, z + y) y^{2-s} \mathrm{d}y,$$

for $2 < \mathrm{Re}(s) < 3$.

Next, we use (4.6) together with (4.3) in order to get the meromorphic continuation of $\mathcal{Z}_f(s, z)$ to the half plane $\mathrm{Re}(s) < 3$. We start with (4.2) and differentiate equation (4.3) three times to get

$$\mathcal{Z}_f(3, z + y) = \frac{\widetilde{a}_2}{(z + y)} - \frac{\widetilde{a}_1}{2(z + y)^2} + \frac{\widetilde{a}_0}{(z + y)^3}$$

$$+ \sum_{k=1}^{n-1} \frac{a_k \mu_k (\mu_k - 1)(\mu_k - 2)}{2(z + y)^{3 - \mu_k}} + \frac{1}{2} h_n'''(z + y),$$

for any $n > 2$.

Since $\mu_k \searrow -\infty$, for an arbitrary $\mu < 0$ there exists k_0 such that $\mu_k \leqslant \mu$ for all $k \geqslant k_0$, hence we may write

$$\mathcal{Z}_f(3, z+y)y^3 = y^3 \left(\frac{\widetilde{a}_2}{(z+y)} - \frac{\widetilde{a}_1}{2(z+y)^2} + \frac{\widetilde{a}_0}{(z+y)^3} \right.$$

$$\left. + \sum_{k=1}^{k_0-1} \frac{a_k\mu_k(\mu_k-1)(\mu_k-2)}{2(z+y)^{3-\mu_k}} \right) + g_\mu(z+y),$$

where $g_\mu(z+y) = \frac{1}{2}y^3 h'''_{k_0}(z+y)$.

Note that

(4.7) $g_\mu(z+y) = O(y^\mu)$ as $y \to \infty$, and $g_\mu(z+y) = O(y^3)$ as $y \searrow 0$.

Application of [12, Formula 3.194.3] yields

(4.8) $\displaystyle\int_0^\infty \mathcal{Z}_f(3, z+y)y^{2-s}\mathrm{d}y$

$$= \widetilde{a}_2 z^{2-s}\Gamma(3-s)\Gamma(s-2) - \frac{\widetilde{a}_1}{2}z^{1-s}\Gamma(3-s)\Gamma(s-1) + \frac{\widetilde{a}_0}{2}\Gamma(3-s)\Gamma(s)$$

$$+ \sum_{k=1}^{k_0-1} a_k\mu_k(\mu_k-1)(\mu_k-2)\frac{\Gamma(3-s)\Gamma(s-\mu_k)}{2\Gamma(3-\mu_k)}z^{\mu_k-s}$$

$$+ \int_0^\infty g_\mu(z+y)y^{-s-1}\mathrm{d}y.$$

The integral on the right hand side of (4.8) is the Mellin transform of the function g_μ. By (4.7) this integral represents a holomorphic function in s for all s in the half strip $\mu < \mathrm{Re}(s) < 3$. The other terms on the right hand side of (4.8) are meromorphic in s, hence the right-hand side of (4.8) provides meromorphic continuation of integral $\int_0^\infty \mathcal{Z}_f(3, z+y)y^{2-s}\mathrm{d}y$ from the strip $2 < \mathrm{Re}(s) < 3$ to the strip $\mu < \mathrm{Re}(s) < 3$. Since $\mu < 0$ was chosen arbitrarily, we can let $\mu \to -\infty$ and obtain the meromorphic continuation of this integral to the half plane $\mathrm{Re}(s) < 3$.

Formula (4.8), together with (4.6), after multiplication with $\frac{2}{\Gamma(s)\Gamma(3-s)}$, now yields the following representation of $\mathcal{Z}_f(s, z)$, for an arbitrary, fixed $z \in X_f$, valid in the half plane $\mu < \mathrm{Re}(s) < 3$:

$$(4.9) \quad \mathcal{Z}_f(s,z) = \frac{2\widetilde{a}_2}{(s-1)(s-2)} z^{2-s} - \frac{\widetilde{a}_1}{(s-1)} z^{1-s}$$

$$+ \widetilde{a}_0 z^{-s} - \sum_{k=1}^{k_0-1} a_k \frac{\Gamma(s-\mu_k)}{\Gamma(s)\Gamma(-\mu_k)} z^{\mu_k-s}$$

$$+ \frac{1}{\Gamma(s)\Gamma(3-s)} \int_0^\infty h_{k_0}'''(z+y)y^{2-s} \mathrm{d}y.$$

From the decay properties of $h_{k_0}'''(z+y)$, it follows that $\mathcal{Z}_f(s,z)$ is holomorphic at $s=0$. Furthermore since $\frac{1}{\Gamma(s)}$ has a zero at $s=0$, the derivative of the last term in (4.9) is equal to

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{\Gamma(s)} \bigg|_{s=0} \right) \frac{1}{\Gamma(3)} \int_0^\infty h_{k_0}'''(z+y)y^2 \mathrm{d}y$$

$$= -\frac{1}{2} \int_0^\infty h_{k_0}'''(z+y)y^2 \mathrm{d}y = h_{k_0}(z),$$

where the last equality is obtained from integration by parts two times, and using the decay of $h_{k_0}(z+y)$ and its derivatives as $y \to +\infty$, for $\mu_{k_0} < 0$. Moreover, since

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{\Gamma(s-\mu_k)}{\Gamma(s)} \bigg|_{s=0} = \lim_{s\to 0} \frac{\Gamma(s-\mu_k)}{\Gamma(s)} \cdot \frac{\Gamma'}{\Gamma}(s) = -\Gamma(-\mu_k),$$

elementary computations yield

$$-\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{Z}_f(s,z) \bigg|_{s=0} = \widetilde{a}_2 z^2 \left(\log z - \frac{3}{2} \right) + \widetilde{a}_1 z (\log z - 1)$$

$$+ \widetilde{a}_0 \log z + \sum_{k=1}^{k_0-1} a_k z^{\mu_k} + h_{k_0}(z),$$

for z in the sector $|\arg z| < \theta < \pi$, $(\theta > 0)$. Finally, (4.4) follows from the uniqueness of analytic continuation. $\qquad\square$

5. Polar structure of superzeta functions associated to Z_+ and Z_-

Recall the definitions of Z_+, Z_-, G_1, and the null sets $N(Z_\pm)$.

Set $X_\pm = X_{Z_\pm}$, and for $z \in X_\pm$, set $\zeta_B^\pm(s,z) := \mathcal{Z}_{Z_\pm}(s,z)$, the superzeta functions of $Z_\pm$.

In this section we prove that $\zeta_B^\pm(s,z)$ has a meromorphic continuation to all $s \in \mathbb{C}$, with simple poles at $s = 2$ and $s = 1$, and we determine the corresponding residues.

Let $\mathcal{G}_1(s,z)$ be the superzeta function associated to the $G_1(s)$, defined for $z \in X_{G_1} = \mathbb{C} \setminus \mathbb{R}^-$, and $\mathrm{Re}(s) > 2$ by

$$(5.1) \quad \mathcal{G}_1(s,z) = \frac{\mathrm{vol}(M)}{2\pi} \sum_{n=0}^\infty \frac{(2n+1)}{(z+n)^s}$$

$$= \frac{\mathrm{vol}(M)}{\pi} [\zeta_H(s-1,z) - (z-1/2)\zeta_H(s,z)].$$

Equation (5.1) and the meromorphic continuation of $\zeta_H(s,z)$ immediately yield

PROPOSITION 5.1. — *For for $z \in \mathbb{C} \setminus \mathbb{R}^-$, function $\mathcal{G}_1(s,z)$ admits a meromorphic continuation (in the s variable) to $\mathbb{C}$ with simple poles at $s = 2$ and $s = 1$, with corresponding residues $\frac{\mathrm{vol}(M)}{\pi}$ and $-\frac{\mathrm{vol}(M)}{2\pi}(2z-1)$, respectively.*

Recall the divisor of the Selberg zeta-function $Z(s)$ in Section 2.6 and note that $\{z \in \mathbb{C} \mid (z - w_k) \notin \mathbb{R}^-$ for all $w_k\} = X_+$, where w_k is a zero or a pole of $Z(s)$. Analogously, the set $\{z \in \mathbb{C} \mid (z - w_k) \notin \mathbb{R}^-$ for all $y_k\}$, where y_k is a zero or a pole of $ZH(s)$ is equal to X_-. The polar structure of the superzeta function $\zeta_B^+(s,z)$ is given as follows:

THEOREM 5.2. — *Fix $z \in X_+$. The superzeta function $\zeta_B^+(s,z)$ has meromorphic continuation to all $s \in \mathbb{C}$, and satisfies*

$$(5.2) \quad \zeta_B^+(s,z) = -\mathcal{G}_1(s,z) + \mathbf{c}\zeta_H\left(s, z - \frac{1}{2}\right)$$

$$+ \frac{\sin \pi s}{\pi} \int_0^\infty \left(\frac{Z'}{Z}(z+y)\right) y^{-s} \mathrm{d}y.$$

Furthermore, the function $\zeta_B^+(s,z)$ has two simple poles at $s = 1$ and $s = 2$ with corresponding residues $\frac{\mathrm{vol}(M)}{\pi}(z-1/2) + \mathbf{c}$ and $-\frac{\mathrm{vol}(M)}{\pi}$ respectively.

Proof. — For $z \in X_+$ and $2 < \operatorname{Re}(s) < 3$, we apply (4.2) and (4.6) to get

$$(5.3) \quad \zeta_B^+(s,z) + \mathcal{G}_1(s,z)$$

$$= \frac{2\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty \left[\zeta_B^+(3, z+y) + \mathcal{G}_1(3, z+y) \right] y^{2-s} \mathrm{d}y$$

$$= \frac{\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty \left(\log F\left(z+y\right) \right)''' y^{2-s} \mathrm{d}y$$

$$= \frac{\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty y^{2-s} \mathrm{d}\left(\left(\log F\left(z+y\right) \right)'' \right),$$

where we put $F(x) = Z_+(x)G_1(x)$, hence, according to Section 2.8

$$\log(F(x)) = \log Z(x) - \mathbf{c}\log\left(\Gamma\left(x - \frac{1}{2} \right) \right),$$

and

$$\left(\log F\left(z+y\right) \right)'' = -\mathbf{c}\psi'(z+y-1/2) + \left(\frac{Z'(z+y)}{Z(z+y)} \right)'.$$

For fixed $z \in X_+$, it follows from (2.11) and (2.1) that

$$\left(\log F\left(z+y\right) \right)'' = O\left(\frac{1}{y} \right), \quad \text{as } y \to \infty$$

and

$$\left(\log F\left(z+y\right) \right)'' = O(1), \quad \text{as } y \searrow 0.$$

Therefore, for $1 < \operatorname{Re}(s) < 2$ we may integrate by parts and obtain

$$(5.4) \quad \frac{\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty y^{2-s} \mathrm{d}\left(\left(\log F\left(z+y\right) \right)'' \right)$$

$$= -\frac{\sin\pi s}{\pi(1-s)} \int_0^\infty \left(\frac{Z'(z+y)}{Z(z+y)} \right)' y^{1-s} \mathrm{d}y$$

$$+ \mathbf{c}\frac{\sin\pi s}{\pi(1-s)} \int_0^\infty \psi'(z+y-1/2)y^{1-s}\mathrm{d}y$$

$$= I_1(s,z) + I_2(s,z).$$

First, we deal with $I_1(s,z)$. By (2.11), $\frac{Z'(z+y)}{Z(z+y)} = O(y^{-n})$, for any positive integer n, as $y \to \infty$. Also, $\frac{Z'(z+y)}{Z(z+y)} = O(1)$, for fixed $z \in X_+$, as $y \to 0$. Hence we may apply integration by parts to the integral $I_1(s,z)$ and obtain, for $0 < \operatorname{Re}(s) < 1$ and $z \in X_+$,

$$I_1(s,z) = -\frac{\sin\pi s}{\pi(1-s)} \int_0^\infty y^{1-s} \mathrm{d}\left(\frac{Z'(z+y)}{Z(z+y)} \right)$$

$$= \frac{\sin\pi s}{\pi} \int_0^\infty \frac{Z'(z+y)}{Z(z+y)} y^{-s} \mathrm{d}y.$$

The integral $I_1(s, z)$, for $z \in X_+$ is actually a holomorphic function in the half plane $\text{Re}(s) < 1$. To see this, let $\mu \leqslant 0$ be arbitrary. Since $(\log Z(z+y))' = O(N(P_0)^{-\text{Re}(z+y)/2})$, as $y \to +\infty$, we have that $(\log Z(z+y))' = O(y^{-2+\mu})$, as $y \to +\infty$, where the implied constant may depend upon z and μ. Hence, $(\log Z(z + y))' y^{-s} = O(y^{-2})$, as $y \to +\infty$, for all s such that $\mu < \text{Re}(s) \leqslant 0$. Moreover, the bound $\frac{Z'(z+y)}{Z(z+y)} = O(1)$, for fixed $z \in X_+$ implies that $(\log Z(z + y))' y^{-s} = O(1)$, as $y \to 0$, for all s in the half plane $\text{Re}(s) \leqslant 0$. This shows that for $z \in X_+$ the integral $I_1(s, z)$ is absolutely convergent in the strip $\mu < \text{Re}(s) \leqslant 0$, hence represents a holomorphic function for all s in that strip. Since $\mu \leqslant 0$ was arbitrarily chosen, we have proved that $I_1(s, z)$, for $z \in X_+$, is holomorphic function in the half plane $\text{Re}(s) \leqslant 0$.

Next, we claim that $I_1(s, z)$, for $z \in X_+$, can be continued to the half-plane $\text{Re}(s) > 0$ as an entire function. For $z \in X_+$ and $0 < \text{Re}(s) < 1$ we put

$$\mathcal{I}_1(s, z) = \int_0^\infty \left(\frac{Z'}{Z}(z + y) \right) y^{-s} dy$$

and show that for $z \in X_+$ the integral $\mathcal{I}_1(s, z)$ can be meromorphically continued to the half-plane $\text{Re}(s) > 0$ with simple poles at the points $s = 1, 2, \ldots$ and corresponding residues

(5.5) $$\text{Res}_{s=n} \mathcal{I}_1(s, z) = -\frac{1}{(n-1)!} (\log Z(z))^{(n)}.$$

Since the function $\sin(\pi s)$ has simple zeros at points $s = 1, 2, \ldots$ this would prove that $I_1(s, z)$, for $z \in X_+$ is actually an entire function of s.

Let $\mu > 0$ be arbitrary, put $n = \lfloor \mu \rfloor$ to be the integer part of μ and let $\delta > 0$ (depending upon $z \in X_+$ and μ) be such that for $y \in (0, \delta)$ we have the Taylor series expansion

$$(\log Z(z + y))' = \sum_{j=1}^n \frac{(\log Z(z))^{(j)}}{(j-1)!} y^{j-1} + R_1(z, y),$$

where $R_1(z, y) = O(y^n)$, as $y \to 0$. Then, for $0 < \text{Re}(s) < 1$ we may write

$$\mathcal{I}_1(s, z) = \sum_{j=1}^n \frac{(\log Z(z))^{(j)}}{(j-1)!} \frac{\delta^{j-s}}{j-s}$$

$$+ \int_0^\delta R_1(z, y) y^{-s} dy + \int_\delta^\infty \left(\frac{Z'}{Z}(z + y) \right) y^{-s} dy.$$

The bound on $R_1(z, y)$ and the bound (2.11) imply that the last two integrals are holomorphic functions of s for $\text{Re}(s) \in (0, \mu)$. The first sum is meromorphic in s, for $\text{Re}(s) \in (0, \mu)$, with simple poles at $s = j$,

$j \in \{1, \ldots, n\}$ and residues equal to $-(\log Z(z))^{(j)}/(j-1)!$. Since $\mu > 0$ is arbitrary, this proves the claim. Therefore, we have proved that $I_1(s, z)$ is holomorphic function in the whole complex $s-$plane.

In order to evaluate integral $I_2(s, z)$ we use the fact that $\psi'(w) = \zeta_H(2, w)$ and that, for $1 < \operatorname{Re}(s) < 2$

$$\sum_{k=0}^{\infty} \int_0^{\infty} \left| \frac{y^{1-s}\mathrm{d}y}{(z+k-1/2+y)^2} \right| \ll \sum_{k=0}^{\infty} \frac{1}{|z+k-1/2|^{\operatorname{Re}(s)}} < \infty.$$

For $z - p \notin \mathbb{R}^-$, and $0 < \operatorname{Re}(s) < 2$, applying [12, Formula 3.194.3.] we get

$$\int_0^{\infty} \frac{y^{1-s}\mathrm{d}y}{(z+y-p)^2} = \int_0^{\infty} \frac{y^{s-1}\mathrm{d}y}{(1+y\,[z-p])^2} = \frac{1}{(z-p)^s} \cdot \frac{\Gamma(s)\Gamma(2-s)}{\Gamma(2)}$$
$$= \frac{(1-s)\Gamma(s)\Gamma(1-s)}{(z-p)^s} = \frac{\pi(1-s)}{\sin \pi s} \cdot \frac{1}{(z-p)^s},$$

hence the dominated convergence theorem yields

$$I_2(s, z) = \mathbf{c}\frac{\sin \pi s}{\pi(1-s)} \sum_{k=0}^{\infty} \int_0^{\infty} \frac{y^{1-s}\mathrm{d}y}{(z+k-1/2+y)^2} = \mathbf{c}\sum_{k=0}^{\infty} \frac{1}{(z+k-1/2)^s}$$
$$= \mathbf{c}\zeta_H(s, z-1/2).$$

This, together with the representation (5.1) of $\mathcal{G}_1(s, z)$ and formula (5.3) proves (5.2) for $z \in X_+$. Moreover, for $z \in X_+$, the function $I_2(s, z)$ is meromorphic in the whole $s-$plane, with a single simple pole at $s = 1$, with residue $\mathbf{c}$, hence the function $I_1(s, z) + I_2(s, z)$ is also meromorphic in the whole $s-$plane, with a single simple pole at $s = 1$, with residue $\mathbf{c}$.

Combining this with Proposition 5.1 completes the proof.   □

The polar structure of the superzeta function $\zeta_B^-(s, z)$, in the $s-$plane, for $z \in X_-$ is determined in the following theorem.

THEOREM 5.3. — For $z \in X_-$ the superzeta function $\zeta_B^-(s, z)$ can be represented as

$$(5.6) \quad \zeta_B^-(s, z) = -\mathcal{G}_1(s, z) + \mathbf{c}\zeta_H(s, z) + \frac{\sin \pi s}{\pi} \int_0^{\infty} \left(\frac{(ZH)'}{ZH}(z+y) \right) y^{-s}\mathrm{d}y.$$

Moreover, the superzeta function $\zeta_B^-(s, z)$, for $z \in X_-$, is a meromorphic function in variable s, with two simple poles at $s = 1$ and $s = 2$ with corresponding residues $\frac{\operatorname{vol}(M)}{\pi}(z-1/2) + \mathbf{c}$ and $-\frac{\operatorname{vol}(M)}{\pi}$.

Proof. — The proof is very similar to the proof of Theorem 5.2. We start with

$$(5.7) \qquad Z_-(s)G_1(s) = \pi^{\mathbf{c}/2} \exp(c_1 s + c_2)\Gamma(s)^{-\mathbf{c}}(ZH)(s),$$

where the left-hand side of the equation is entire function of order two. Proceeding analogously as above, for $2 < \mathrm{Re}(s) < 3$ we get

$$\zeta_B^-(s,z) + \mathcal{G}_1(s,z)$$

$$= \frac{2\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty \left[\zeta_B^-(3,z+y) + \mathcal{G}_1(3,z+y)\right] y^{2-s}\mathrm{d}y$$

$$= \frac{\sin\pi s}{\pi(1-s)(2-s)} \int_0^\infty y^{2-s}\mathrm{d}\left((\log T\,(z+y))''\right),$$

where

$$(\log T\,(z+y))'' = -\mathbf{c}\psi'(z+y) + \left(\frac{(ZH)'(z+y)}{ZH(z+y)}\right)'.$$

Bounds (2.9) and (2.11) imply that, for an arbitrary $\mu > 0$, positive integer k and $z \in X_-$ we have

$$\frac{\mathrm{d}^k}{\mathrm{d}y^k}(\log(ZH)(z+y)) = O(y^{-\mu}), \text{ as } y \to +\infty,$$

where the implied constant depends upon z and k. Moreover, from the series representation of $Z(s)$ and $H(s)$ it is evident that $(\log(ZH)(z+y))' = O(1)$, as $y \to 0$.

Therefore, repeating the steps of the proof presented above we deduce that (5.6) holds true and that the superzeta function $\zeta_B^-(s,z)$, for $z \in X_-$, possesses meromorphic continuation to the whole complex s−plane with simple poles at $s = 1$ and $s = 2$ with residues $\frac{\mathrm{vol}(M)}{\pi}(z - 1/2) + \mathbf{c}$ and $-\frac{\mathrm{vol}(M)}{\pi}$, respectively.   □

6. Regularized determinant of the Lax–Phillips operator B

After identifying the polar structure of the zeta functions $\zeta_B^\pm$, we are in position to state and prove our main results.

First, we express the complete zeta function $Z_+(z)$ as a regularized determinant of the operator $zI - (\frac{1}{2}I + B)$, modulo the factor of the form $\exp(\alpha_1 z + \beta_1)$, where $\alpha_1 = \mathrm{vol}(M)\log(2\pi)/\pi$ and $\beta_1 = \frac{\mathrm{vol}(M)}{4\pi}(4\zeta'(-1) - \log(2\pi)) + \frac{\mathbf{c}}{2}\log(2\pi)$ and obtain an analogous expression for the complete zeta function $Z_-(z)$, see Theorem 6.2. below.

Moreover, we prove that the scattering determinant $\phi(z)$ is equal to the product of $\exp(c_1 z + c_2 + \frac{\mathbf{c}}{2}\log\pi)$ and the quotient of regularized determinants of operators $zI - (\frac{1}{2}I - B)$ and $zI - (\frac{1}{2}I + B)$.

Then, we define the higher depth regularized determinant, i.e. the regularized determinant of depth $r \in \{1,2,\ldots\}$ and show that the determinant

of depth r of the operator $zI - (\frac{1}{2}I + B)$ can be expressed as a product of the Selberg zeta function of order r and the Milnor gamma functions of depth r, see Theorem 6.4. below.

Finally, we express $Z'(1)$ in terms of the (suitably defined) regularized determinant of $\frac{1}{2}I - B$.

6.1. Regularized product associated to G_1, Z_+, and Z_-

A simple application of Proposition 4.1 yields expressions for regularized products associated to G_1, Z_+, and Z_-. We start with $\mathcal{G}_1(s, z)$, which is regular at $s = 0$, hence we have the following proposition.

PROPOSITION 6.1. — *For all $z \in \mathbb{C} \setminus (-\infty, 0]$, the zeta regularized product of $\mathcal{G}_1(s, z)$ is given by*

$$D_{G_1}(z) = \exp\left(-\frac{\mathrm{vol}(M)}{2\pi}\left[2z\log(2\pi) + (2\zeta'(-1) - \log(\sqrt{2\pi}))\right]\right) G_1(z).$$

Proof. — From (2.4) we get

$$\log G_1(z) = \frac{\mathrm{vol}(M)}{2\pi}\left(z\log(2\pi) + 2\log G(z+1) - \log\Gamma(z)\right),$$

upon applying (2.3) (2.1), (and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$), and after a straight-forward computation we obtain

$$\log G_1(z) = \frac{\mathrm{vol}(M)}{2\pi}\left[z^2\left(\log z - \frac{3}{2}\right) - z(\log z - 1) + (2\log(2\pi))z\right.$$
$$\left. + \frac{1}{3}\log z - \left(\frac{1}{2}\log(2\pi) - 2\zeta'(-1)\right)\right] + \sum_{j=1}^{m-1}\frac{c_j}{z^j} + h_m(z),$$

where c_j and $h_m(z)$ can be explicitly determined from (2.3) and (2.1) as $\mathrm{Re}(z) \to \infty$ in the sector $|\arg z| < \frac{\pi}{2} - \delta$, where $\delta > 0$. Applying Proposition 4.1 with

$$\widetilde{a_2} = \frac{\mathrm{vol}(M)}{2\pi}, b_2 = 0, \widetilde{a_1} = -\frac{\mathrm{vol}(M)}{2\pi}, b_1 = \frac{\mathrm{vol}(M)}{\pi}\log(2\pi),$$
$$\widetilde{a_0} = \frac{\mathrm{vol}(M)}{6\pi}, b_0 = \frac{\mathrm{vol}(M)}{2\pi}(2\zeta'(-1) - \log(\sqrt{2\pi}))$$

we obtain

$$(6.1) \quad \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}_1(s, z)\Big|_{s=0}\right)$$
$$= \exp\left(-\frac{\mathrm{vol}(M)}{2\pi}\left[2z\log(2\pi) + (2\zeta'(-1) - \log(\sqrt{2\pi}))\right]\right) G_1(z) \quad \square$$

Recall that $\text{Spec}\left(\frac{1}{2}I + B\right) = N(Z_+)$ and $\text{Spec}\left(\frac{1}{2}I - B\right) = N(Z_-)$, hence $\text{Spec}\left(zI - (\frac{1}{2}I + B)\right) = \{z - y_k \mid y_k \in N(Z_+)\}$ and $\text{Spec}\left(zI - (\frac{1}{2}I - B)\right) = \{z - y_k \mid y_k \in N(Z_-)\}$.

Therefore, for $z \in X_\pm$ we define

$$\det\left(zI - (\frac{1}{2}I + B)\right) = D_{Z_+}(z) = \exp\left(-\frac{d}{ds}\zeta_B^+(s, z)\Big|_{s=0}\right),$$

respectively

$$\det\left(zI - (\frac{1}{2}I - B)\right) = D_{Z_-}(z) = \exp\left(-\frac{d}{ds}\zeta_B^-(s, z)\Big|_{s=0}\right).$$

Our main result is

THEOREM 6.2. — For $z \in X_\pm$, the regularized product of $Z_\pm(z)$ is given by

$$(6.2) \quad \det\left(zI - (\frac{1}{2}I \pm B)\right) = \exp\left(-\frac{d}{ds}\zeta_B^\pm(s, z)\Big|_{s=0}\right) = \Upsilon_\pm(z)Z_\pm(z),$$

where

$$\Upsilon_+(z) = \exp\left[\frac{\text{vol}(M)}{2\pi}\left(2z\log(2\pi) + 2\zeta'(-1) - \frac{1}{2}\log(2\pi) + \frac{\mathbf{c}\pi}{\text{vol}(M)}\log(2\pi)\right)\right]$$

and

$$\Upsilon_-(z) = \exp\left[\left(\frac{\text{vol}(M)}{\pi}\log(2\pi) - c_1\right)z\right.$$

$$\left. + \frac{\text{vol}(M)}{2\pi}(2\zeta'(-1) - \log(\sqrt{2\pi})) - c_2 + \frac{\mathbf{c}}{2}\log 2\right].$$

Moreover, for $z \in X_+ \cap X_-$

$$(6.3) \qquad\qquad \phi(z) = (\pi)^{\frac{\mathbf{c}}{2}}e^{c_1 z + c_2}\frac{\det\left(zI - (\frac{1}{2}I - B)\right)}{\det\left(zI - (\frac{1}{2}I + B)\right)}.$$

Proof. — As $z \to \infty$, in $\text{Re}(z) > 0$, upon applying (2.3) and (2.1) we get

$$(6.4) \quad \log Z_+(z) = \log Z(z) - \log G_1(z) - \mathbf{c}\log\Gamma\left(z - \frac{1}{2}\right)$$

$$= \log Z(z) - \frac{\text{vol}(M)}{2\pi}\left[z^2\left(\log z - \frac{3}{2}\right) - z(\log z - 1) + 2z\log(2\pi)\right.$$

$$+ \frac{1}{3}\log z - \left(\frac{1}{2}\log(2\pi) - 2\zeta'(-1)\right)\Big]$$

$$- \mathbf{c}\left(\frac{1}{2}\log(2\pi) + (z - 1)\log z - z\right) + \sum_{j=1}^{m-1}\frac{c_j}{z^j} + h_m(z),$$

where the c_j and $h_m(z)$ can be calculated explicitly (with the help of Legendre's duplication formula). By (2.11), $\log Z(z)$ and its derivatives are of rapid decay, so it can be grouped with the last terms on the right.

Applying Proposition 4.1 with

$$\widetilde{a_2} = \frac{-\operatorname{vol}(M)}{2\pi}, \quad b_2 = 0, \quad \widetilde{a_1} = \frac{\operatorname{vol}(M)}{2\pi} - \mathbf{c}, \quad b_1 = \frac{-\operatorname{vol}(M)}{\pi}\log(2\pi),$$

$$\widetilde{a_0} = \frac{-\operatorname{vol}(M)}{6\pi} + \mathbf{c}, \quad b_0 = \frac{\operatorname{vol}(M)}{2\pi}(\log(\sqrt{2\pi}) - 2\zeta'(-1)) - \frac{\mathbf{c}}{2}\log(2\pi),$$

gives us the first part of (6.2).

Next, to study ζ_B^-, recall that $Z_- = \phi Z_+$. By (2.7), (2.9) and expansion

$$\log L(z) = \frac{\mathbf{c}}{2}\log\pi + c_1 z + c_2 + \mathbf{c}\log\Gamma(z - \frac{1}{2}) - \mathbf{c}\log\Gamma(z),$$

we have, as $z \to \infty$, in $\operatorname{Re}(z) > 0$,

(6.5)

$$\log Z_-(z) = \log Z(z) - \log G_1(z) - \mathbf{c}\log\Gamma(z) + \frac{\mathbf{c}}{2}\log\pi + c_1 z + c_2 + \log H(z)$$

$$= -\frac{\operatorname{vol}(M)}{2\pi}\left[z^2(\log z - \frac{3}{2}) - z(\log z - 1) + 2z\log(2\pi)\right.$$

$$+\frac{1}{3}\log z - (\frac{1}{2}\log(2\pi) - 2\zeta'(-1))\bigg] - \mathbf{c}\left(\frac{1}{2}\log(2\pi) + (z - \frac{1}{2})\log z - z\right)$$

$$+ \frac{\mathbf{c}}{2}\log\pi + c_1 z + c_2 + \log Z(z) + \log H(z) + \sum_{j=1}^{m-1}\frac{c_j}{z^j} + h_m(z)$$

Note that we can group $\log Z(z) + \log H(z)$ with the rapidly decaying remainder terms in (6.5). Applying Proposition 4.1 with

$$\widetilde{a_2} = -\frac{\operatorname{vol}(M)}{2\pi}, \qquad b_2 = 0,$$

$$\widetilde{a_1} = \frac{\operatorname{vol}(M)}{2\pi} - \mathbf{c}, \qquad b_1 = -\frac{\operatorname{vol}(M)}{2\pi}\cdot 2\log(2\pi) + c_1,$$

$$\widetilde{a_0} = -\frac{\operatorname{vol}(M)}{6\pi} + \frac{\mathbf{c}}{2}, \quad b_0 = -\frac{\operatorname{vol}(M)}{2\pi}(2\zeta'(-1) - \log(\sqrt{2\pi})) + c_2 - \frac{\mathbf{c}}{2}\log 2.$$

gives us the second part of (6.2).

It is left to prove (6.3). It follows after a straightforward computation from the relation $\phi(z) = Z_-(z)/Z_+(z)$ combined with (6.2).   □

Remark 6.3. — Equation (6.3) shows that the function

$$\phi(z)\cdot\exp\left(-c_1 z - c_2 - \frac{\mathbf{c}}{2}\log\pi\right)$$

is equal to a regularized determinant of the operator

$$\left(B + \left(z - \frac{1}{2}\right)I\right)\left(R_{(z-\frac{1}{2})}(B)\right),$$

for $z \in X_+ \cap X_-$, where $R_\lambda(B)$ denotes the resolvent of the operator B. This result is reminiscent of [9, Theorem 1], once we recall that B is the infinitesimal generator of the one-parameter family $\mathbf{Z}(t)$. Namely, the right hand side represents the quotient of regularized determinants of operators with infinitely many eigenvalues, while the left hand side is a determinant of a matrix operator.

Remark 6.4. — Since the Selberg zeta function $Z(s)$ possesses a non-trivial, simple zero at $s = 1$, it is obvious that $z = 1/2 \notin X_+$. However, inserting formally $z = 1/2$ into (6.3) and recalling the fact that $\exp(c_1/2 + c_2) = d(1)/g_1$, we conclude that (6.3) suggests that $\phi(1/2) = \text{sgn}(d(1))$, where $\text{sgn}(a)$ denotes the sign of a real, nonzero number a.

Remark 6.5. — It is possible to relate the determinant $\det(zI - (\frac{1}{2}I + B))$ and the determinant $\det(\Delta + z(z-1)I)$ associated to the Laplacian. To do so, one considers the square of the expression (6.2) with plus signs. Observe that, in the notation of [4, 5, formula 1.6], one has the formula

$$Z_+(z) = Z(z)Z_\infty(z)\exp\left(-\frac{\text{vol}(M)}{\pi}z\log(2\pi)\right).$$

By comparing the main theorem in [4, 5] with (6.2) we immediately deduce that

$$\det{}^2\left(zI - (\frac{1}{2}I + B)\right) = \det{}^2(\Delta + z(z-1)I)\left(z - \frac{1}{2}\right)^{\mathbf{c}+\text{tr}\,\Phi(\frac{1}{2})}$$
$$\cdot \exp\left(\frac{\text{vol}(M)}{4\pi}(2z-1) + \mathbf{c}\log 2(2z+1)\right).$$

6.2. Higher-depth Determinants

By Theorems 5.2 and 5.3 functions $\zeta_B^+(s, z)$ and $\zeta_B^-(s, z)$ are holomorphic at $s = 0, -1, -2, \ldots$, hence it is possible to define higher-depth determinants of the operators $\left(zI - (\frac{1}{2}I + B)\right)$ and $\left(zI - (\frac{1}{2}I - B)\right)$, as in [21].

The determinant of the depth r (where $r = 1, 2, \ldots$) is defined for $z \in X_\pm$ as

(6.6) $\det_r \left(zI - \left(\frac{1}{2}I + B \right) \right) = \exp \left(-(\zeta_B^+)'(s, z) \big|_{s=1-r} \right)$

and

$$\det_r \left(zI - \left(\frac{1}{2}I - B \right) \right) = \exp \left(-(\zeta_B^-)'(s, z) \big|_{s=1-r} \right),$$

respectively. When $r = 1$, we obtain the classical (zeta) regularized determinant.

The higher depth determinants of the operator $\left(zI - (\frac{1}{2}I + B) \right)$ can be expressed in terms of the Selberg zeta function $Z^{(r)}(s)$ of order $r \geqslant 1$ and the Milnor gamma function of depth r, which is defined as

$$\Gamma_r(z) := \exp \left(\frac{\partial}{\partial w} \zeta_H(w, z) \Big|_{w=1-r} \right).$$

We have the following theorem

THEOREM 6.6. — *For $z \in X_+$, and $r \in \mathbb{N}$ one has*

$$\det_r \left(zI - (\frac{1}{2}I + B) \right) = \left(\Gamma_r \left(z - \frac{1}{2} \right) \right)^{-\mathbf{c}} \left(\frac{\Gamma_{r+1}(z)}{\Gamma_r(z)^{(z-\frac{1}{2})}} \right)^{\frac{\mathrm{vol}(M)}{\pi}}$$
$$\cdot \left[Z^{(r)}(z) \right]^{\left[(-1)^{r-1}(r-1)! \right]}.$$

Proof. — From Theorem 5.2, for $z \in X_+$ one has

$$\zeta_B^+(s, z) = -\frac{\mathrm{vol}(M)}{\pi} \left[\zeta_H(s-1, z) - \left(z - \frac{1}{2} \right) \zeta_H(s, z) \right]$$
$$+ \mathbf{c}\zeta_H(s, z - 1/2) + \frac{\sin \pi s}{\pi} \int_0^\infty \left(\frac{Z'}{Z}(z+y) \right) y^{-s} \mathrm{d}y$$

The right hand side is holomorphic, at $s = 0, -1, -2, \dots$.

Differentiating the above equation with respect to the variable s, inserting the value $s = 1 - r$, where $r \in \mathbb{N}$ and having in mind the definition of the Milnor gamma function of depth r we get

(6.7) $\dfrac{\partial}{\partial s} \zeta_B^+(s, z) \Big|_{s=1-r}$

$$= -\frac{\mathrm{vol}(M)}{\pi} \left[\log \Gamma_{r+1}(z) - \left(z - \frac{1}{2} \right) \log \Gamma_r(z) \right] + \mathbf{c} \log \Gamma_r \left(z - \frac{1}{2} \right)$$
$$+ (-1)^{r-1} \int_0^\infty \left(\frac{Z'}{Z}(z+y) \right) y^{r-1} \mathrm{d}y$$

for $z \in X_+$.

Assume for the moment that $\mathrm{Re}(z) > 1$. Then, for $y \geqslant 0$

$$\frac{Z'}{Z}(z+y) = \sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^{z+y}}.$$

The absolute and uniform convergence of the above sum for $\mathrm{Re}(z) > 1$ and $y \geqslant 0$ imply that, for $r \in \mathbb{N}$

$$\int_0^\infty \left(\frac{Z'}{Z}(z+y)\right) y^{r-1} \mathrm{d}y = \sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^z} \int_0^\infty y^{r-1} \exp(-y \log N(P)) \mathrm{d}y$$

$$= (r-1)! \sum_{P \in H(\Gamma)} \frac{\Lambda(P)}{N(P)^z \left(\log N(P)\right)^r}.$$

Equation (6.7), together with the above relation yield the formula

$$-\frac{\partial}{\partial s} \zeta_B^+(s,z)\Big|_{s=1-r} = \frac{\mathrm{vol}(M)}{\pi} \log\left(\frac{\Gamma_{r+1}(z)}{\Gamma_r(z)^{(z-\frac{1}{2})}}\right) - \mathbf{c}\log\Gamma_r\left(z - \frac{1}{2}\right)$$

$$+ (-1)^{r-1}(r-1)! \cdot \log Z^{(r)}(z),$$

for $\mathrm{Re}(z) > 1$. The statement of theorem follows by (6.6) and uniqueness of meromorphic continuation. □

6.3. An expression for $Z'(1)$ as a regularized determinant

Recall that for $z \in X_+$ we have

$$\det\left(zI - \left(\frac{1}{2}I + B\right)\right) = D_{Z_+}(z) = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\zeta_B^+(s,z)\Big|_{s=0}\right).$$

The above regularized product is not well defined at $z = 1$, since $z = 1$ corresponds to the constant eigenfunction ($\lambda = 0$) of Δ of multiplicity one, hence it does not belong to X_+. Therefore, for $\mathrm{Re}(s) > 2$ we define

$$(\zeta_B^+)^*(s,z) = \zeta_B^+(s,z) - (z-1)^{-s} = \sum_{\eta \in N(Z_+)\setminus\{1\}} \frac{1}{(z-\eta)^s}.$$

Meromorphic continuation of $(\zeta_B^+)^*(s,z)$ for $z \in X_+ \cup \{1\}$ to the whole s−plane is immediate consequence of Theorem 5.2 which implies that $(\zeta_B^+)^*(s,z)$ is holomorphic at $s = 0$. Moreover,

$$-\frac{\mathrm{d}}{\mathrm{d}s}(\zeta_B^+)^*(s,z) = -\frac{\mathrm{d}}{\mathrm{d}s}\zeta_B^+(s,z) - \frac{\log(z-1)}{(z-1)^s},$$

hence

$$(6.8) \quad \det{}^*\left(I - \left(\frac{1}{2}I + B\right)\right) = \lim_{z \to 1} \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}(\zeta_B^+)^*(s, z)\Big|_{s=0}\right)$$

$$= \lim_{z \to 1} \frac{1}{z-1} D_{Z_+}(z),$$

We give a direct proof of the following:

THEOREM 6.7. — *With the notation as above*

$$\det{}^*\left(-B + \frac{1}{2}I\right) = 2^{\frac{\mathbf{c}}{2}} \exp\left[\frac{\mathrm{vol}(M)}{2\pi}\left(2\zeta'(-1) + \frac{3}{2}\log(2\pi)\right)\right] Z'(1).$$

Proof. — From (6.8) and Theorem 6.2 it follows that

$$\det{}^*\left(\frac{1}{2}I - B\right)$$

$$= \exp\left[\frac{\mathrm{vol}(M)}{2\pi}\left(2\log(2\pi) + 2\zeta'(-1) - \frac{1}{2}\log(2\pi)\right.\right.$$

$$\left.\left.+ \frac{\mathbf{c}\pi}{\mathrm{vol}(M)}\log(2\pi)\right)\right] \cdot \lim_{z \to 1}\frac{Z_+(z)}{z-1}$$

$$= \exp\left[\frac{\mathrm{vol}(M)}{2\pi}\left(\frac{3}{2}\log(2\pi) + 2\zeta'(-1) + \frac{\mathbf{c}\pi}{\mathrm{vol}(M)}\log(2\pi)\right)\right]\frac{Z'(1)}{G_1(1)(\Gamma(1/2))^{\mathbf{c}}}$$

$$= 2^{\frac{\mathbf{c}}{2}} \exp\left[\frac{\mathrm{vol}(M)}{2\pi}\left(2\zeta'(-1) + \frac{3}{2}\log(2\pi)\right)\right] Z'(1). \qquad \square$$

Remark 6.8. — The above corollary may be viewed as a generalization of the result of [31, Corollary 1], for the determinant D_0 to the case of the non-compact, finite volume Riemann surface with cusps. Here, the role of the Laplacian is played by the operator $-B + \frac{1}{2}I$. In the case when $\mathbf{c} = 0$, the spectrum of $-B + \frac{1}{2}I$ consists of points $s = \frac{1}{2} + ir_n$, $s = \frac{1}{2} - ir_n$, $r_n \neq 0$, with multiplicities $m(\lambda_n)$ and the point $s = \frac{1}{2}$ with multiplicity $2d_{1/4}$. Therefore, formally speaking

$$\prod_{\lambda_n \neq 0}\left(\frac{1}{2} + ir_n\right)\left(\frac{1}{2} - ir_n\right) = \prod_{\lambda_n \neq 0}\lambda_n = \det{}'(\Delta_0),$$

in the notation of Sarnak. For $\mathbf{c} = 0$, Corollary 3 may be viewed as the statement

$$\overset{'}{\det}\det{}'(\Delta_0) = \exp\left[\frac{\mathrm{vol}(M)}{2\pi}\left(2\zeta'(-1) + \frac{3}{2}\log(2\pi)\right)\right] Z'(1).$$

This agrees with [31], the only difference being a constant term $\exp(-\frac{\mathrm{vol}(M)}{8\pi})$. It appears due to a different scaling parameter we use. Namely, in [31, Theorem 1], parameter is a (natural for the trace formula

setting) parameter $s(s-1)$, while we use s instead. This yields to a slightly different asymptotic expansion at infinity and produces a slightly different renormalization constant.

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