ANNALES

DE

L'INSTITUT FOURIER

José Ignacio BURGOS GIL, Patrice PHILIPPON & Martín SOMBRA

Successive minima of toric height functions

Tome 65, nº 5 (2015), p. 2145-2197.

<http://aif.cedram.org/item?id=AIF_2015__65_5_2145_0>

Ann. Inst. Fourier, Grenoble
65, 5 (2015) 2145-2197

SUCCESSIVE MINIMA OF TORIC HEIGHT FUNCTIONS

**by José Ignacio BURGOS GIL,
Patrice PHILIPPON & Martín SOMBRA**

Abstract. — Given a toric metrized $\mathbb{R}$-divisor on a toric variety over a global field, we give a formula for the essential minimum of the associated height function. Under suitable positivity conditions, we also give formulae for all the successive minima. We apply these results to the study, in the toric setting, of the relation between the successive minima and other arithmetic invariants like the height and the arithmetic volume. We also apply our formulae to compute the successive minima for several families of examples, including weighted projective spaces, toric bundles and translates of subtori.

Résumé. — Étant donné un $\mathbb{R}$-diviseur torique métrisé d'une variété torique sur un corps global, nous démontrons une formule pour le minimum essentiel de la fonction hauteur associée. Sous des hypothèses de positivité convenables, nous donnons également des formules pour tous les minimums successifs. Nous appliquons ces résultats à l'étude, dans le cadre torique, des relations entre les minimums successifs et d'autres invariants arithmétiques comme la hauteur et le volume arithmétique. Nous appliquons aussi nos formules au calcul des minimums successifs de plusieurs familles d'exemples, incluant les espaces projectifs pondérés, les fibrés toriques et les translatés de sous-tores.

1. Introduction

The height is a tool that is ubiquitous in Diophantine geometry and approximation. It plays a central rôle in the proof of finiteness results on integral and rational points on curves and Abelian varieties like the theorems of Siegel, Mordell-Weil and Faltings, see for instance [19, 6]. It is also very useful in transcendence theory and in the context of Schmidt's subspace theorem.

Keywords: Height, essential minimum, successive minima, toric variety, toric metrized $\mathbb{R}$-divisor, concave function, Legendre-Fenchel duality.
Math. classification: 14G40, 14M25, 52A41.

Arakelov geometry provides a convenient framework to define and study heights. Let $\mathbb{K}$ be a global field, that is, a field which is either a number field or the function field of a projective curve, and let X be an algebraic variety over $\mathbb{K}$ of dimension n. To an (adelically) metrized $\mathbb{R}$-divisor $\overline{D}$ on X one can associate a real-valued height function

$$\mathrm{h}_{\overline{D}}\colon X(\overline{\mathbb{K}}) \longrightarrow \mathbb{R}$$

on the set of algebraic points of X, see § 3 for details. It is a generalization of the notion of height of algebraic points of projective varieties considered by Northcott, Weil and others.

Given $\eta \in \mathbb{R}$, we denote by $X(\overline{\mathbb{K}})_{\leqslant \eta}$ the set of algebraic points $p \in X(\overline{\mathbb{K}})$ with $\mathrm{h}_{\overline{D}}(p) \leqslant \eta$. For $i = 1, \dots, n+1$, the *i-th minimum* of X with respect to $\overline{D}$ is defined as

$$\upmu^{i}_{\overline{D}}(X) = \inf\{\eta \in \mathbb{R} \mid \dim\left(\overline{X(\overline{\mathbb{K}})_{\leqslant \eta}}\right) \geqslant n-i+1\}.$$

In particular, the first minimum is the infimum of the real numbers η such that the set $X(\overline{\mathbb{K}})_{\leqslant \eta}$ is dense. It is also called the *essential minimum* of X with respect to $\overline{D}$, and denoted $\upmu^{\mathrm{ess}}_{\overline{D}}(X)$.

These successive minima contain important information on the height function. The effective version of the generalized Bogomolov conjecture asks for an explicit lower bound for the essential minimum of certain varieties in terms of geometric and arithmetic data. Such lower bounds have been extensively studied and have several applications in Diophantine geometry and computer algebra, see for instance [2, 1].

Our aim in this text is to study the successive minima of height functions in the toric setting. Toric objects can be described in combinatorial terms, and their algebro-geometric properties can be expressed and studied in terms of this description. In particular, a proper toric variety X of dimension n over an arbitrary field is given by a fan Σ on a vector space $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Recall that a toric variety is called proper if it is proper as an algebraic variety. In combinatorial terms, this is equivalent to the fan being complete, that is, that the union of its cones covers the whole vector space. A toric $\mathbb{R}$-divisor D on a proper toric variety X defines a polytope Δ_D in the dual space $M_{\mathbb{R}} := N_{\mathbb{R}}^{\vee}$. There is a “toric dictionary” that translates algebro-geometric properties of the pair (X, D) into combinatorial properties of the fan and the polytope.

In [11, 10], we started a program to extend this toric dictionary to the arithmetic aspects of toric varieties. Suppose that X is a proper toric variety over the global field $\mathbb{K}$. Then, to a toric metrized $\mathbb{R}$-divisor $\overline{D}$ on X we associate a family of concave functions on the polytope $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$, indexed by the places $\mathfrak{M}_{\mathbb{K}}$ of $\mathbb{K}$. These functions are called the *local roof functions* of $\overline{D}$ and they are zero except for a finite set of places. The *global roof function* $\vartheta_{\overline{D}}$ is the concave function on Δ_D defined as a weighted sum over all places of these local roof functions. The main theme of this program is that the global roof function is the arithmetic analogue of the polytope and encodes a lot of information of the pair $(X, \overline{D})$. Among other results, we gave formulae for the height $\mathrm{h}_{\overline{D}}(X)$ and the arithmetic volumes $\widehat{\mathrm{vol}}(\overline{D})$ and $\widehat{\mathrm{vol}}_{\chi}(\overline{D})$ in terms of this function.

Our first main result in this text is that the essential minimum of a toric metrized $\mathbb{R}$-divisor is given by the maximum of the global roof function.

Theorem A (Corollary 3.9). — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then*

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x\in\Delta_D} \vartheta_{\overline{D}}(x).$$

Our second main result is that, under suitable positivity hypothesis on $\overline{D}$, not only the essential minimum, but all the succesive minima can be read from the global roof function.

Theorem B (Theorem 3.16). — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X with D ample. Then, for $i = 1, \dots, n+1$,*

$$\mu^{i}_{\overline{D}}(X) = \min_{F\in\mathcal{F}(\Delta_D)^{n-i+1}} \max_{x\in F} \vartheta_{\overline{D}}(x),$$

where $\mathcal{F}(\Delta_D)^{n-i+1}$ is the set of faces of the polytope Δ_D of dimension $n-i+1$.

Whereas there is a considerable amount of work on upper and lower bounds for the essential minimum, there are very few exact computations in the literature. By contrast, Theorems A and B are very concrete and well-suited for computations. For example, they allow to compute the successive minima of the canonical height on translates of subtori of a projective space as the maximum of a piecewise affine concave function on the polytope (Proposition 5.12). The following example illustrates this computation.

Example. — Let $C \subset \mathbb{P}^3_{\mathbb{Q}}$ be the cubic curve given as the image of the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad (t_0 : t_1) \longmapsto \Big(t_0^3 : 4\, t_0^2 t_1 : \frac{1}{3}\, t_0 t_1^2 : \frac{1}{2}\, t_1^3\Big).$$

Let $\overline{H}$ be the metrized divisor of $\mathbb{P}^3$ given by the hyperplane at infinity equipped with the canonical metric, and let $\overline{D}$ be the restriction of $\overline{H}$ to C.

Figure 1.1 shows the local roof functions associated to $\overline{D}$ for each place $v \in \mathfrak{M}_{\mathbb{Q}}$, and Figure 1.2 shows the global roof function. This global roof function is the sum of the local ones, and can be described as the minimal concave piecewise affine function on the interval $[0, 3]$ with lattice point values

$$\vartheta_{\overline{D}}(0) = 0, \qquad \vartheta_{\overline{D}}(1) = \frac{7}{3}\log(2) + \frac{1}{2}\log(3),$$
$$\vartheta_{\overline{D}}(2) = \frac{7}{6}\log(2) + \log(3), \qquad \vartheta_{\overline{D}}(3) = 0.$$

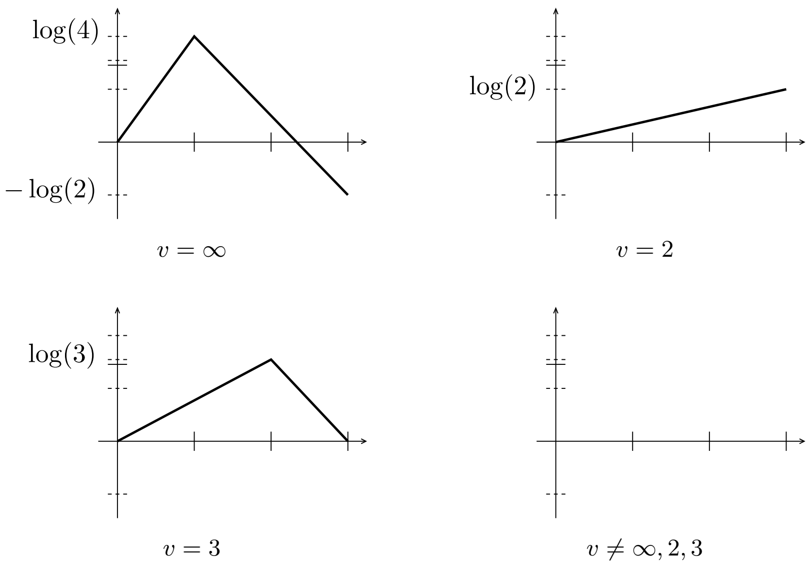

Figure 1.1. Local roof functions

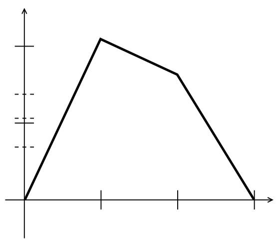

Figure 1.2. Global roof function

Theorem B then implies that $\mu^{\mathrm{ess}}_{\overline{D}}(C) = \frac{7}{3}\log(2) + \frac{1}{2}\log(3)$ and $\mu^{2}_{\overline{D}}(C) = 0$. We refer to § 5.5 for explanations on how to do this kind of computations.

Our results allow also to compute the successive minima of toric varieties with respect to weighted L^p-metrics and of translates of subtori of a projective space with the Fubini-Study metric, generalizing the computation of the successive minima of a subtori with the Fubini-Study metric in [25]. Another nice family of examples is given by toric bundles on a projective space, including Hirzebruch surfaces. We refer the reader to § 5 for the details and the explicit formulae.

A well-known theorem of Zhang shows that the successive minima of a height function can be estimated in terms of the height and the degree of the variety [29, 30], This result plays a key rôle in the proof of the Bogomolov conjecture for Abelian varieties and its ulterior developments, including the study of the distribution of Galois orbits of points of small height, see for instance [15, 28].

As a direct consequence of Theorems A and B and our previous results in [11, 10], we obtain a simple proof of Zhang's theorem in the toric case. This approach allows also to prove this result for an arbitrary global field and to relax the positivity hypothesis on the metrized $\mathbb{R}$-divisor.

Theorem C (Theorem 4.1). — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X with D big. Then*

$$\sum_{i=1}^{n+1} \mu^{i}_{\overline{D}}(X) \leqslant \frac{\mathrm{h}_{\overline{D}}(X)}{\deg_{D}(X)} \leqslant (n+1)\mu^{\mathrm{ess}}_{\overline{D}}(X).$$

Using our formulae, we can easily construct examples of semipositive toric metrics on the hyperplane divisor on $\mathbb{P}^n_{\mathbb{Q}}$ showing that almost every configuration of successive minima and height can actually happen.

Theorem D (Proposition 4.3). — *Let $n \geqslant 0$ and $\nu, \mu_1, \dots, \mu_{n+1} \in \mathbb{R}$ such that*

$$\mu_1 \geqslant \dots \geqslant \mu_{n+1} \quad \textit{and} \quad \sum_{i=1}^{n+1} \mu_i \leqslant \nu < (n+1)\mu_1. \tag{1.1}$$

Then there exists a semipositive toric metric on H, the hyperplane divisor on $\mathbb{P}^n_{\mathbb{Q}}$, such that

$$\mathrm{h}_{\overline{H}}(\mathbb{P}^n) = \nu \quad \textit{and} \quad \mu^{i}_{\overline{H}}(\mathbb{P}^n) = \mu_i \quad \textit{for } i = 1, \dots, n+1.$$

The case left aside in (1.1) deserves a separate explanation: we show that if, with the hypothesis of Theorem C, we have the equality

$$\frac{\mathrm{h}_{\overline{D}}(X)}{\deg_D(X)} = (n+1)\upmu^{\mathrm{ess}}_{\overline{D}}(X),$$

then the function $\vartheta_{\overline{D}}$ is constant and, necessarily, $\upmu^{i}_{\overline{D}}(X) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$ for all i, see Corollary E below. This observation is relevant when applying the known equidistribution results on Galois orbits of small points in the toric case.

By replacing the height of the variety by its χ-arithmetic volume, Zhang's lower bound for the essential minimum extends to the case when the metrics are not necessarily semipositive: if X is a proper variety of dimension n over a number field and $\overline{D}$ is a metrized divisor on X with D big, then

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(n+1)\,\mathrm{vol}(D)}, \tag{1.2}$$

see for instance [13, Lemme 5.1]. This lower bound is a key result in the study of the distribution of the Galois orbits of points of small height. Indeed, all known results in this direction are applicable only when the inequality (1.2) is an equality. This includes the equidistribution theorems of Szpiro, Ullmo and Zhang [26], Bilu [5], Favre and Rivera-Letelier [17], Chambert-Loir [12], Baker and Rumely [3], Yuan [28], Berman and Boucksom [4], and Chen [14].

In the toric case, we can also derive a simple proof of the inequality (1.2) for arbitrary global fields and toric metrics on a big toric $\mathbb{R}$-divisor. More importantly, we can characterize the cases when equality occurs. The following statement is a direct consequence of Propositions 4.4 and 4.6.

Corollary E. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X with D big. Then*

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(n+1)\,\mathrm{vol}(D)},$$

with equality if and only if $\vartheta_{\overline{D}}$ is constant. If this is the case, then $\upmu^{i}_{\overline{D}}(X) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$ for all i.

The condition that the roof function is constant is *very* strong, and it is equivalent to the fact that the v-adic metrics in $\overline{D}$ can be derived from the canonical metric by translation and scaling (Remark 4.8). In particular, these are the only toric metrics to which the equidistribution theorems mentioned above can be applied.

In collaboration with Juan Rivera-Letelier, we are currently studying the equidistribution properties of Galois orbits of points of small height in the toric setting. We plan to expose our results in a subsequent paper.

Acknowledgement. — We thank Dominique Bernardi, Pierre Dèbes, Luis Dieulefait and Juan Carlos Naranjo for useful discussions and pointers to the literature.

Part of this work was done while the authors met at the Universitat de Barcelona, the Instituto de Ciencias Matemáticas (Madrid) and the Institut de Mathématiques de Jussieu (Paris).

2. Preliminary results

In this section we gather some preliminary results on global fields and on convex analysis.

2.1. Global fields

A *global field* $\mathbb{K}$ is either a number field or the function field of a smooth projective curve over an arbitrary field, equipped with a set of places $\mathfrak{M}_{\mathbb{K}}$. Each place $v \in \mathfrak{M}_{\mathbb{K}}$ is a pair consisting of an absolute value $|\cdot|_v$ on $\mathbb{K}$ and a positive weight $n_v \in \mathbb{Q}_{>0}$, defined as follows.

The places of the field of rational numbers $\mathbb{Q}$ consist of the Archimedean and the p-adic absolutes values, normalized in the standard way, and with all weights equal to 1. For the function field $\mathrm{K}(C)$ of a smooth projective curve C over a field k, the set of places is indexed by the closed points of C. For each closed point $v_0 \in C$, we consider the absolute value and weight given, for $\alpha \in \mathrm{K}(C)^{\times}$, by

$$|\alpha|_{v_0} = c_k^{-\mathrm{ord}_v(\alpha)}, \quad n_{v_0} = [k(v_0) : k],$$

where $\mathrm{ord}_{v_0}(\alpha)$ denotes the order of α in the discrete valuation ring $\mathcal{O}_{C,v_0}$ and

$$c_k = \begin{cases} \mathrm{e} & \text{if } \#k = \infty, \\ \#k & \text{if } \#k < \infty. \end{cases}$$

Let $\mathbb{K}_0$ denote either $\mathbb{Q}$ or $\mathrm{K}(C)$, and let $\mathbb{K}$ be a finite extension of $\mathbb{K}_0$. The set of places of $\mathbb{K}$ is then formed by the pairs $v = (|\cdot|_v, n_v)$ where $|\cdot|_v$ is an absolute value on $\mathbb{K}$ extending an absolute value $|\cdot|_{v_0}$ on $\mathbb{K}_0$ and

$$n_v = \frac{[\mathbb{K}_v : \mathbb{K}_{0,v_0}]}{[\mathbb{K} : \mathbb{K}_0]} n_{v_0}, \tag{2.1}$$

where $\mathbb{K}_v$ denotes the completion of $\mathbb{K}$ with respect to $|\cdot|_v$, and similarly for $\mathbb{K}_{0,v_0}$. By [20, Proposition XII.6.1], the weight n_v can be also written as

$$n_v = \frac{e(v/v_0)f(v/v_0)}{[\mathbb{K}:\mathbb{K}_0]} n_{v_0}, \tag{2.2}$$

where $e(v/v_0)$ is the ramification degree and $f(v/v_0)$ is the residual degree of v over v_0. Therefore, the notion of global field in this paper is compatible with that in [11, Definition 1.5.12].

In the function field case, the extension $\mathbb{K}/\mathbb{K}_0$ corresponds to a dominant morphism $B \to C$ of smooth projective curves and $\mathbb{K} = \mathrm{K}(B)$. However, the set of places of $\mathbb{K}$ depends on the extension and not just on the field $\mathbb{K}$.

Given $v \in \mathfrak{M}_{\mathbb{K}}$ and $v_0 \in \mathfrak{M}_{\mathbb{K}_0}$, we write $v \mid v_0$ whenever $|\cdot|_v$ extends $|\cdot|_{v_0}$. The set of places of $\mathbb{K}$ satisfies, for all $v_0 \in \mathfrak{M}_{\mathbb{K}_0}$,

$$\sum_{v|v_0} n_v = n_{v_0} \tag{2.3}$$

and the *product formula*

$$\prod_{v\in\mathfrak{M}_{\mathbb{K}}} |\alpha|_v^{n_v} = 1 \quad \text{for all } \alpha \in \mathbb{K}^{\times}.$$

Both properties are well-known in the case of number fields. In the function field case, the equality (2.3) follows from the projection formula [21, Proposition 9.2.11], whereas the product formula for $\mathbb{K}$ follows from (2.3) and the product formula for $\mathbb{K}_0$.

We will first construct finite extensions of $\mathbb{K}$ of arbitrary degree that are totally split over a given set of places. To this end, we need the following consequence of Hilbert's irreducibility theorem.

Lemma 2.1. — *Let $f(x) \in \mathbb{K}[x]$ be a separable polynomial of positive degree, $S \subset \mathfrak{M}_{\mathbb{K}}$ a finite subset of places of $\mathbb{K}$ and $(\varepsilon_v)_{v\in S}$ a collection of positive real numbers. Then there exists an element $c \in \mathbb{K}$ such that the polynomial $f(x) + c$ is irreducible in $\mathbb{K}[x]$ and $|c|_v < \varepsilon_v$ for all $v \in S$.*

Proof. — We want to use Hilbert's irreducibility theorem for fields with a product formula in [16, Theorem 3.4]. To this end, we need to show that any global field satisfies its hypothesis. The first hypothesis is that the field is either of characteristic zero or imperfect of positive characteristic. Since, if $\mathrm{char}(\mathbb{K}) > 0$, then $\mathbb{K}$ is the function field of a curve over a field of positive characteristic and so it is not perfect. Thus, this hypothesis holds for global fields. The second one is a density hypothesis that, when $\mathbb{K}$ is a number field, follows from the strong approximation theorem and, when $\mathbb{K}$

is a function field, follows from the Riemann-Roch theorem for curves over an arbitrary field given in [21, Theorem 7.3.17].

Consider the polynomial

$$F(x,y) = f(x) + y \in \mathbb{K}[x,y].$$

Being irreducible in $\mathbb{K}[x,y]$ and of positive degree in x, it is also irreducible in $\mathbb{K}(y)[x]$. Then [16, Theorem 3.4] implies that there exists $c \in \mathbb{K}$ such that $F(x,c) = f(x) + c$ is irreducible in $\mathbb{K}[x]$ and $|c|_v \leqslant \varepsilon_v$ for all $v \in S$ as stated. □

Lemma 2.2. — *Let $d \geqslant 1$ be an integer and $S \subset \mathfrak{M}_{\mathbb{K}}$ a finite subset of places of $\mathbb{K}$. There exists an extension $\mathbb{L}/\mathbb{K}$ of degree d such that, for all $v \in S$, there are d different extensions of the absolute value $|\cdot|_v$ to $\mathbb{L}$.*

Proof. — Let $\beta_1, \dots, \beta_d \in \mathbb{K}$ such that $\beta_i \neq \beta_j$ for $i \neq j$. Set $f(x) = \prod_{j=1}^{d}(x - \beta_i) \in \mathbb{K}[x]$, which is a separable polynomial of positive degree. For $v \in S$, put

$$\varepsilon_v = \Big(\frac{1}{4}\min_{i\neq j} |\beta_i - \beta_j|_v\Big)^d.$$

By Lemma 2.1, there is an element $c \in \mathbb{K}$ such that $f(x) + c$ is irreducible and $|c|_v < \varepsilon_v$ for $v \in S$. Set

$$\mathbb{L} = \mathbb{K}[x]/(f(x) + c).$$

Since $f(x)$ is monic of degree d, so is $f(x) + c$. Denote by $\alpha_1, \dots, \alpha_d$ the roots of $f(x) + c$ in an algebraic closure of $\mathbb{K}$. For $v \in S$ and $i \in \{1, \dots, d\}$ we have that $|f(\beta_i) + c|_v = |c|_v$, from which it follows that there exists an index $\sigma(v,i) \in \{1, \dots, d\}$ satisfying

$$|\alpha_{\sigma(v,i)} - \beta_i|_v \leqslant |c|_v^{1/d} < \varepsilon_v^{1/d}. \tag{2.4}$$

By the choice of ε_v, we deduce that $\sigma(v,i) \neq \sigma(v,j)$ for $i \neq j$, and so $\sigma(v,\cdot)$ is a bijection. Let $\tau(v,\cdot)$ denote the inverse bijection. Then, using (2.4) and the definition of ε_v, we obtain, for $i \in \{1, \dots, d\}$ and $j \neq i$,

$$|\alpha_i - \alpha_j|_v > |\beta_{\tau(v,i)} - \beta_{\tau(v,j)}|_v - 2\varepsilon_v^{1/d} \geqslant 2\varepsilon_v^{1/d} > 2|\alpha_i - \beta_{\tau(v,i)}|_v. \tag{2.5}$$

This implies that $f(x) + c$ is separable. Moreover, the inequality (2.5) also implies that, for each $i \in \{1, \dots, d\}$, we have $\mathbb{K}_v(\alpha_i) = \mathbb{K}_v(\beta_{\tau(v,i)}) = \mathbb{K}_v$. If v is non-Archimedean, this follows from Krasner's lemma [22, page 152]. If v is Archimedean, we only need to see that, if $\mathbb{K}_v = \mathbb{R}$, then $\mathbb{K}_v(\alpha_i) = \mathbb{R}$. Assume that, on the contrary, $\mathbb{K}_v(\alpha_i) = \mathbb{C}$. Since the coefficients of $f(x)+c$

are real, there is $j \neq i$ such that α_j is the complex conjugate of α_i. By hypothesis, $\beta_{\tau(v,i)} \in \mathbb{K}_v = \mathbb{R}$ and so

$$|\alpha_j - \alpha_i|_v \leqslant 2|\alpha_i - \beta_{\tau(v,i)}|_v < \min_{\substack{1\leqslant j\leqslant d\\ j\neq i}} |\alpha_i - \alpha_j|_v,$$

which is a contradiction.

Thus, for all $v \in S$, the polynomial $f(x) + c$ splits completely in $\mathbb{K}_v[x]$. Then, by [22, Proposition 8.2], this implies that there are d distinct places of the extension $\mathbb{L} = \mathbb{K}[x]/(f(x) + c)$ over v, completing the proof. □

Let $\mathbb{G}_m$ be the multiplicative group over $\mathbb{K}$ and $\mathbb{T} \simeq \mathbb{G}_m^n$ a split torus of dimension n over $\mathbb{K}$. Let $N = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the lattice of cocharacters of $\mathbb{T}$ and write $N_\mathbb{R} = N \otimes \mathbb{R}$. We fix a splitting $\mathbb{T} \simeq \mathbb{G}_m^n$, which induces isomorphisms $\mathbb{T}(\mathbb{K}) \simeq (\mathbb{K}^\times)^n$ and $N_\mathbb{R} \simeq \mathbb{R}^n$. Given elements $x \in \mathbb{T}(\mathbb{K})$ and $u \in N_\mathbb{R}$, we denote by x_i and u_i, $i = 1, \dots, n$, the components of the image of x and u under the previous isomorphisms. Consider the space $\bigoplus_{v\in\mathfrak{M}_\mathbb{K}} N_\mathbb{R}$ with the norm given by

$$\|(u_v)_v\| = \sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \sum_{i=1}^{n} |u_{v,i}|.$$

The induced topology is called the L^1-*topology*. It does not depend on the choice of the splitting of $\mathbb{T}$. We denote by $H_\mathbb{K} \subset \bigoplus_{v\in\mathfrak{M}_\mathbb{K}} N_\mathbb{R}$ the subspace defined by

$$H_\mathbb{K} = \Big\{(u_v)_v \in \bigoplus_{v\in\mathfrak{M}_\mathbb{K}} N_\mathbb{R} \,\Big|\, \sum_v n_v u_v = 0\Big\} \tag{2.6}$$

with the induced L^1-topology.

For each $v \in \mathfrak{M}_\mathbb{K}$, there is a map $\mathrm{val}_v \colon \mathbb{T}(\mathbb{K}) \to N_\mathbb{R}$, given, in the fixed splitting, by

$$\mathrm{val}_v(x_1, \dots, x_n) = (-\log|x_1|_v, \dots, -\log|x_n|_v). \tag{2.7}$$

This map does not depend on the choice of the splitting. By the product formula, we can define a map $\mathrm{val}\colon \mathbb{T}(\mathbb{K}) \to H_\mathbb{K}$ as

$$\mathrm{val}(x) = (\mathrm{val}_v(x))_{v\in\mathfrak{M}_\mathbb{K}}.$$

This is a group homomorphism, and so it can be extended to a map

$$\mathrm{val}\colon \mathbb{T}(\mathbb{K}) \otimes \mathbb{Q} \to H_\mathbb{K}.$$

Dirichlet's unit theorem does not hold for general global fields. Nevertheless, the following result, that in the case of number fields is a consequence of Dirichlet's unit theorem, is true in general.

Lemma 2.3. — *The set* $\mathrm{val}(\mathbb{T}(\mathbb{K}) \otimes \mathbb{Q})$ *is dense in* $H_{\mathbb{K}}$ *with respect to the* L^1*-topology.*

Proof. — Since the torus is split, working component-wise, it is enough to treat the case $n = 1$. Thus $\mathbb{T}(\mathbb{K}) = \mathbb{K}^{\times}$.

First suppose that $\mathbb{K}$ is a number field or the function field of a curve over a finite field. For each finite subset $S \subset \mathfrak{M}_{\mathbb{K}}$ we put

$$H_{\mathbb{K},S} = \{(u_v)_{v \in \mathfrak{M}_{\mathbb{K}}} \in H_{\mathbb{K}} \mid u_v = 0 \text{ for } v \notin S\},$$
$$\mathbb{K}_S = \{\alpha \in \mathbb{K} \mid |\alpha|_v = 1 \text{ for } v \notin S\} \subset \mathbb{K}^{\times}.$$

Dirichlet S-unit theorem [27, Chapter IV, § 4, Theorem 9] states that $\mathrm{val}(\mathbb{K}_S)$ is a lattice in $H_{\mathbb{K},S}$. Let $u \in H_{\mathbb{K}}$. Then there exists a finite subset S such that $u \in H_{\mathbb{K},S}$. Let $\varepsilon > 0$. By the density of rational numbers, we can find an element $u' \in \mathrm{val}(\mathbb{K}_S \otimes \mathbb{Q}) \subset H_{\mathbb{K},S}$ with $\|u - u'\| < \varepsilon$, proving the lemma in this case.

Now let $B \to C$ be a dominant morphism of smooth projective curves over an infinite field k and set $\mathbb{K} = \mathrm{K}(B)$ with the induced structure of global field. In this case, Dirichlet S-unit theorem may not hold and the lemma is a consequence of the Riemann-Roch theorem.

Let $(u_v)_v \in H_{\mathbb{K}}$ and $\varepsilon > 0$. We have to show that there is an element $x \in \mathbb{K}^{\times} \otimes \mathbb{Q}$ such that

$$\|(u_v)_v - \mathrm{val}(x)\| < \varepsilon.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may assume without loss of generality that $u_v \in \mathbb{Q}$ for all $v \in \mathfrak{M}_{\mathbb{K}}$. Since there is a finite subset S such that $u \in H_{\mathbb{K},S}$, we can choose an integer $q \geqslant 1$ such that $qu_v \in \mathbb{Z}$ for all $v \in \mathfrak{M}_{\mathbb{K}}$.

Recall that the set of places of $\mathbb{K}$ is indexed by the set of closed points of B. With notation as in (2.2), we consider the Weil divisor on B given by

$$D = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} e(v/v_0) q u_v [v].$$

By the definition of the weights n_v and the product formula,

$$\deg(D) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} e(v/v_0) q u_v [k(v) : k] = q[\mathbb{K} : \mathbb{K}_0] \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v u_v = 0.$$

Let E be an effective Weil divisor on B with $\deg(E) = r \geqslant g(B)$, where $g(B)$ is the genus of B, and choose an integer

$$l > \frac{2r}{\varepsilon q[\mathbb{K} : \mathbb{K}_0]}.$$

Since $\deg(lD + E) = l \deg(D) + \deg(E) = r \geqslant g(B)$, by the Riemann-Roch theorem [21, Theorem 7.3.17] we can find an element $\alpha \in \mathbb{K}^{\times}$ and an effective divisor E' on B with $\deg(E')=r$ such that

$$lD+E=E'+\mathrm{div}(\alpha). \tag{2.8}$$

Writing $E'-E=\sum_v a_v[v]$, the equation (2.8) reads

$$a_v=e(v/v_0)lqu_v-\mathrm{ord}_v(\alpha)=e(v/v_0)lq\Big(u_v-\frac{1}{lq}\mathrm{val}_v(\alpha)\Big)\quad \text{for all } v. \tag{2.9}$$

Put $x=\alpha^{\frac{1}{lq}}=\alpha\otimes\frac{1}{lq}\in\mathbb{K}^\times\otimes\mathbb{Q}$. Using that E and E' are effective divisors of degree r, we deduce that

$$\sum_{v\in\mathfrak{M}_\mathbb{K}}[k(v):k]|a_v|\leqslant\deg(E)+\deg(E')=2r$$

and, using (2.9),

$$\begin{aligned}\|(u_v)_v-\mathrm{val}(x)\|=\Big\|(u_v)_v-\frac{1}{lq}\mathrm{val}(\alpha)\Big\|=\frac{1}{lq}\Big\|\Big(\frac{a_v}{e(v/v_0)}\Big)_v\Big\|\\ =\frac{1}{lq[\mathbb{K}:\mathbb{K}_0]}\sum_{v\in\mathfrak{M}_\mathbb{K}}[k(v):k]|a_v|\leqslant\frac{2r}{lq[\mathbb{K}:\mathbb{K}_0]}<\varepsilon,\end{aligned}$$

obtaining the result. □

Remark 2.4. — The space $H_\mathbb{K}$ has another natural topology, the *direct sum topology*. A subset $U\subset H_\mathbb{K}$ is open for the direct sum topology if and only if its intersections with all the subsets $H_{\mathbb{K},S}$ are open. The direct sum topology is finer than the L^1-topology. In fact, a sequence $(u_j)_{j\geqslant 1}$ of elements of $H_\mathbb{K}$ converges to $u\in H_\mathbb{K}$ in the direct sum topology if and only it converges in the L^1-topology and there is a finite subset $S\subset\mathfrak{M}_\mathbb{K}$ such that $u_j\in H_{\mathbb{K},S}$ for all $j\geqslant 1$.

The proof of Lemma 2.3 for number fields and function fields over a finite field shows the stronger result that the set $\mathrm{val}(\mathbb{T}(\mathbb{K})\otimes\mathbb{Q})$ is dense in $H_\mathbb{K}$ for the direct sum topology. By contrast, the proof of Lemma 2.3 for general function fields only shows density for the L^1-topology because we have no control on the support of the divisor E' in the equation (2.8).

Although the L^1-topology is coarser than the direct sum topology, the next result shows that it will be enough for our purposes.

Lemma 2.5. — *Let $\Psi\colon N_\mathbb{R}\to\mathbb{R}$ be a continuous function with $\Psi(0)=0$ and Lipschitz at 0. Let $(\psi_v)_{v\in\mathfrak{M}_\mathbb{K}}$ be a collection of continuous functions on $N_\mathbb{R}$ such that there is a finite subset $S\subset\mathfrak{M}_\mathbb{K}$ with $\psi_v=\Psi$ for $v\notin S$. Then the map $\bigoplus_{v\in\mathfrak{M}_\mathbb{K}}N_\mathbb{R}\to\mathbb{R}$ given by*

$$(u_v)_v\longmapsto\sum_{v\in\mathfrak{M}_\mathbb{K}}n_v\psi_v(u_v)$$

is continuous with respect to the L^1-topology.

Proof. — First note that the function in the lemma is well-defined because the sum only involves a finite number of nonzero terms. Within this proof, we will indistinctly denote by $\|\cdot\|$ the L^1-norm on $N_{\mathbb{R}} \simeq \mathbb{R}^r$ or on $\bigoplus_v N_{\mathbb{R}}$.

Since Ψ is Lipschitz at 0, there are constants $B > 0$ and $\varepsilon_0 > 0$ such that, for $u' \in N_{\mathbb{R}}$ with $\|u'\| \leqslant \varepsilon_0$,

$$|\Psi(u')| \leqslant B\|u'\|,$$

Fix $(u_v)_v \in \bigoplus_v N_{\mathbb{R}}$ and $\varepsilon > 0$. Write

$$S' = S \cup \{v \in \mathfrak{M}_{\mathbb{K}} \mid u_v \neq 0\}.$$

Since ψ_v is continuous in u_v, we can choose $0 < \delta < \min(\varepsilon/2B, \varepsilon_0)$ such that, for all $v \in S'$,

$$n_v\|u_v - u'_v\| < \delta \Longrightarrow n_v|\psi_v(u_v) - \psi_v(u'_v)| < \frac{\varepsilon}{2\#S'}.$$

If $\|(u'_v)_v - (u_v)_v\| < \delta$ then $n_v\|u_v - u'_v\| < \delta$ for all $v \in \mathfrak{M}_{\mathbb{K}}$, and $\sum_{v\notin S'} n_v\|u'_v\| < \delta$. Therefore

$$\begin{aligned}\Bigg|\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v\psi_v(u'_v) - \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v\psi_v(u_v)\Bigg| \\ \leqslant \sum_{v\in S'} n_v|\psi_v(u'_v) - \psi_v(u_v)| + \sum_{v\notin S'} n_v|\Psi(u'_v)| \\ < \sum_{v\in S'} \frac{\varepsilon}{2\#S'} + B\sum_{v\notin S'} n_v\|u'_v\| < \frac{\varepsilon}{2} + B\delta < \varepsilon.\end{aligned}$$

This shows the continuity at the point $(u_v)_v$. Since this point is arbitrary we obtain the lemma. □

2.2. Concavification of functions

We next introduce the concavification of a function and study its basic properties.

Definition 2.6. — *Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function. The* concavification *of f, denoted* $\mathrm{conc}(f)$, *is the smallest concave function on $N_{\mathbb{R}}$ that is bounded below by f.*

The concavification of a function may not exists but if it exists, it is unique. Recall from [10, Definition A.1] that the *stability set* of the function f is the subset of $M_{\mathbb{R}}$ given by

$$\mathrm{stab}(f) = \{x \in M_{\mathbb{R}} \mid x - f \text{ is bounded below}\}. \tag{2.10}$$

Lemma 2.7. — *Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function. Then* $\mathrm{conc}(f)$ *exists if and only if* $\mathrm{stab}(f) \neq \emptyset$. *If this is the case, then for $u \in N_{\mathbb{R}}$,*

$$\mathrm{conc}(f)(u) = \sup \sum_{j=1}^{\ell} \nu_j f(u_j),$$

where the supremum is over all expressions of u as a convex combination of points of $N_{\mathbb{R}}$, that is, all expressions of the form $u = \sum_{j=1}^{\ell} \nu_j u_j$ with $\ell \in \mathbb{N}$, $\nu_j \geqslant 0$ for all j, $\sum_{j=1}^{\ell} \nu_j = 1$ and $u_j \in N_{\mathbb{R}}$.

Proof. — Clearly, $\mathrm{conc}(f)$ exists if and only if there exists a concave function $g\colon N_{\mathbb{R}} \to \mathbb{R}$ with $f \leqslant g$.

Assume that $\mathrm{stab}(f) \neq \emptyset$. Let $x \in \mathrm{stab}(f)$. Then, there exists $c \in \mathbb{R}$ such that $f(u) \leqslant \langle x, u\rangle + c$ for all $u \in N_{\mathbb{R}}$. Since the function $\langle x, u\rangle + c$ is concave, we deduce that $\mathrm{conc}(f)$ exists. Conversely, assume that $\mathrm{conc}(f)$ exists. Since $\mathrm{conc}(f)$ is concave, $\mathrm{stab}(\mathrm{conc}(f)) \neq \emptyset$. Therefore $\mathrm{stab}(f) \supset \mathrm{stab}(\mathrm{conc}(f))$ is not empty.

The expression for $\mathrm{conc}(f)(u)$ follows from [24, Theorem 5.3], see *loc. cit.* page 36. □

If the function f is locally bounded below, we can assume that the numbers ν_j of the previous lemma are rational numbers.

Lemma 2.8. — *Let $f\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function such that* $\mathrm{stab}(f) \neq \emptyset$ *and which is locally bounded below. Then, for $u \in N_{\mathbb{R}}$,*

$$\mathrm{conc}(f)(u) = \sup \frac{1}{d} \sum_{j=1}^{d} f(u_j),$$

where the supremum is over all expressions of the form $u = \frac{1}{d}\sum_{j=1}^{d} u_j$ with $u_j \in N_{\mathbb{R}}$.

Proof. — By Lemma 2.7, it is clear that

$$\mathrm{conc}(f)(u) \geqslant \sup \frac{1}{d} \sum_{j=1}^{d} f(u_j).$$

Thus, we only need to show the other inequality. Let $\varepsilon > 0$. By Lemma 2.7 we can find a convex combination $u = \sum_{j=1}^{k} \nu_j u_j$ with $\nu_j > 0$ for all j and

$\sum_{j=1}^{\ell} \nu_j = 1$ such that

$$\operatorname{conc}(f)(u) \leqslant \sum_{j=1}^{k} \nu_j f(u_j) + \varepsilon/2. \tag{2.11}$$

Fix an isomorphism $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and consider the associated L^1-norm, that we denote by $\|\cdot\|$. Since $\operatorname{stab}(f) \neq \emptyset$ and f is locally bounded below, the function $|f|$ is bounded on compact subsets. Therefore, there is a constant $B > 0$ such that $|f(v)| \leqslant B$ for all $v \in N_{\mathbb{R}}$ with $\|v\| \leqslant 2\sum_{j=1}^{k} \|u_j\|$. In particular, $|f(u_j)| \leqslant B$.

Set $\eta = \min\{\frac{\varepsilon}{4kB}, \nu_1, \dots, \nu_k\} > 0$ and choose integers $d \geqslant 1$ and $a_j \geqslant 1$, $j = 1, \dots, k$, such that

$$\frac{\eta}{2} < \nu_j - \frac{a_j}{d} < \eta. \tag{2.12}$$

Put

$$a_{k+1} = d - \sum_{j=1}^{k} a_j, \quad u_{k+1} = \frac{d}{a_{k+1}}\left(u - \sum_{j=1}^{k} \frac{a_j}{d} u_j\right),$$

so that $\sum_{j=1}^{k+1} \frac{a_j}{d} = 1$ and $\sum_{j=1}^{k+1} \frac{a_j}{d} u_j = u$ holds. Moreover, the inequalities in (2.12) imply that $\frac{k\eta}{2} < \frac{a_{k+1}}{d} < k\eta$ and

$$\|u_{k+1}\| \leqslant \left\| \frac{d}{a_{k+1}}\left(u - \sum_{j=1}^{k} \frac{a_j}{d} u_j\right)\right\| \leqslant \frac{2}{k\eta}\eta \sum_{j=1}^{k} \|u_j\| \leqslant 2\sum_{j=1}^{k} \|u_j\|.$$

Thus $|f(u_{k+1})| \leqslant B$. Now we compute

$$\begin{aligned}\left|\sum_{j=1}^{k} \nu_j f(u_j) - \sum_{j=1}^{k+1} \frac{a_j}{d} f(u_j)\right| &\leqslant \left|\sum_{j=1}^{k}\left(\nu_j - \frac{a_j}{d}\right) f(u_j) - \frac{a_{k+1}}{d} f(u_{k+1})\right| \\ &\leqslant \eta k B + \eta k B \leqslant \varepsilon/2.\end{aligned}$$

Combining this with (2.11),

$$\operatorname{conc}(f)(u) \leqslant \frac{1}{d}\sum_{j=1}^{k+1} a_j f(u_j) + \varepsilon,$$

which proves the result. □

Remark 2.9. — In the previous lemma, the hypothesis that f is locally bounded below is necessary because there exist non-concave functions that satisfy the concavity condition for rational convex combinations. For instance, a discontinuous $\mathbb{Q}$-linear function from $\mathbb{R}$ to $\mathbb{R}$ is not concave because it is not continuous, but it satisfies the concavity condition for rational combinations because it is $\mathbb{Q}$-linear.

The condition of being locally bounded below is trivially satisfied if f is continuous.

The next result gives a criterion for the stability set of a conic function to be nonempty and, *a fortiori*, for the existence of its concavification.

Lemma 2.10. — *Let* $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ *be a conic function. Then* $\operatorname{stab}(\Psi) \neq \emptyset$ *if and only if, for all collections of points* $u_j \in N_{\mathbb{R}}$, $j = 1, \dots, \ell$, *such that* $\sum_{j=1}^{\ell} u_j = 0$, *we have*

$$\sum_{j=1}^{\ell} \Psi(u_j) \leqslant 0. \tag{2.13}$$

Proof. — Suppose that, for all zero-sum families of points $u_j \in N_{\mathbb{R}}$, $j = 1, \dots, \ell$, the inequality (2.13) holds. For $u \in N_{\mathbb{R}}$, set

$$\Phi(u) = \sup \sum_{j=1}^{\ell} \Psi(w_j) \in \mathbb{R} \cup \{\infty\}, \tag{2.14}$$

where the supremum is over all $w_j \in N_{\mathbb{R}}$, $j = 1, \dots, \ell$, such that $u = \sum_j w_j$. By (2.13) applied to the points $-u$ and w_j, $j = 1, \dots, \ell$,

$$\sum_{j=1}^{\ell} \Psi(w_j) \leqslant -\Psi(-u)$$

and so the supremum in (2.14) is finite. Hence, (2.14) defines a conic function $\Phi\colon N_{\mathbb{R}} \to \mathbb{R}$. By construction, Φ is concave and $\Phi \geqslant \Psi$. Hence, $\operatorname{stab}(\Psi) \supset \operatorname{stab}(\Phi) \neq \emptyset$.

Conversely, assume that $\operatorname{stab}(\Psi) \neq \emptyset$. Let $x \in \operatorname{stab}(\Psi)$. Since Ψ is conic, we have $\Psi(u) \leqslant \langle x, u\rangle$. Thus

$$\sum_{j=1}^{\ell} \Psi(u_j) \leqslant \sum_{j=1}^{\ell} \langle x, u_j\rangle = 0.$$

□

Lemma 2.11. — *Let* $f\colon N_{\mathbb{R}} \to \mathbb{R}$ *be a function with* $\operatorname{stab}(f) \neq \emptyset$ *and* $g\colon N_{\mathbb{R}} \to \mathbb{R}$ *another function with* $|f - g|$ *bounded. Then* $\operatorname{stab}(g) \neq \emptyset$ *and* $|\operatorname{conc}(f) - \operatorname{conc}(g)|$ *is bounded.*

Proof. — If $|f - g|$ is bounded, then $\operatorname{stab}(f) = \operatorname{stab}(g)$, which gives the first statement. Since $|f - g|$ is bounded we can choose $B > 0$ such that $|f(u) - g(u)| \leqslant B$ for all $u \in N_{\mathbb{R}}$. Fix a point $u \in N_{\mathbb{R}}$ and consider a convex combination of points of $N_{\mathbb{R}}$

$$u=\sum_{j=1}^{\ell}\nu_j u_j.$$

Then

$$\sum_{j=1}^{\ell}\nu_j f(u_j)-\operatorname{conc}(g)(u)\leqslant\sum_{j=1}^{\ell}\nu_j f(u_j)-\sum_{j=1}^{\ell}\nu_j g(u_j)\leqslant B.$$

Since this is true for any convex combination as above, we deduce

$$\operatorname{conc}(f)(u)-\operatorname{conc}(g)(u)\leqslant B.$$

By symmetry $\operatorname{conc}(g)(u)-\operatorname{conc}(f)(u)\leqslant B$ and the second statement follows. □

3. Successive minima of toric metrized $\mathbb{R}$-divisors

In this section, we give the formulae for the successive minima of the height function associated to a toric metrized $\mathbb{R}$-divisor on a proper toric variety over a global field. We will use the notations and results in [11, 10] although, for the convenience of the reader, we recall below some of them.

Let $\mathbb{K}$ be a global field as in the previous section and X a variety over $\mathbb{K}$, that is, a reduced and irreducible separated scheme of finite type over $\mathbb{K}$. The elements of $X(\overline{\mathbb{K}})$ will be called the *algebraic points* of X. For each place $v\in\mathfrak{M}_{\mathbb{K}}$, we denote by X_v^{an} the v-adic analytification of X. If v is Archimedean, this is a complex space (equipped with an anti-linear involution if $\mathbb{K}_v\simeq\mathbb{R}$) and, if v is non-Archimedean, it is a Berkovich space.

Given a (quasi-algebraic) metrized $\mathbb{R}$-divisor $\overline{D}$ on X as in [10, Definition 3.3], we consider the associated height function

$$\mathrm{h}_{\overline{D}}\colon X(\overline{\mathbb{K}})\to\mathbb{R}$$

defined as follows.

For each $p\in X(\overline{\mathbb{K}})$ choose a function $f\in\mathrm{K}(X)^{\times}_{\mathbb{R}}=\mathrm{K}(X)^{\times}\otimes\mathbb{R}$ such that $p\notin|D-\operatorname{div}(f)|$, the support of $D-\operatorname{div}(f)$. For instance, when D is a Cartier divisor, we can take f as a local equation of D at p.

Choose a finite extension $\mathbb{F}$ of $\mathbb{K}$ such that $p\in X(\mathbb{F})$. To f, we can associate a metrized $\mathbb{R}$-divisor $\widehat{\operatorname{div}}(f)$ and we consider the metrized $\mathbb{R}$-divisor $\overline{D}-\widehat{\operatorname{div}}(f)$ on X. For simplicity, we also denote by $\overline{D}-\widehat{\operatorname{div}}(f)$ the metrized $\mathbb{R}$-divisor on $X_{\mathbb{F}}$ obtained by base change. To each place $w \in \mathfrak{M}_{\mathbb{F}}$ is associated a w-adic Green function

$$g_{\overline{D}-\widehat{\operatorname{div}}(f),w} \colon (X_{\mathbb{F}}^{\mathrm{an}})_w \setminus |D - \operatorname{div}(f)| \to \mathbb{R},$$

see [10, Definitions 3.3 and 3.4]. For instance, if $\overline{D}$ is a metrized Cartier divisor on X and $p \notin |D|$, we have that $g_{\overline{D},w}(p) = -\log \|s_D(p)\|_w$ with s_D the canonical rational section of the line bundle $\mathcal{O}(D)$ and $\|\cdot\|_w$ the w-adic metric on $\mathcal{O}(D)_w^{\mathrm{an}}$ obtained from the extension of $\overline{D}$ on $X_{\mathbb{F}}$ by base change. We denote by $\iota_w \colon X(\mathbb{F}) \to (X_{\mathbb{F}})_w^{\mathrm{an}}$ the inclusion of the $\mathbb{F}$-rational points of X into the v-adic analytification.

Definition 3.1. — *With the previous notations, the* height *of p with respect to $\overline{D}$ is given by*

$$\mathrm{h}_{\overline{D}}(p) = \sum_{w \in \mathfrak{M}_{\mathbb{F}}} n_w g_{\overline{D}-\widehat{\operatorname{div}}(f),w}(\iota_w(p)).$$

The height is independent of the choice of the rational function f and of the extension $\mathbb{F}$.

Remark 3.2. — This definition is the natural extension to metrized $\mathbb{R}$-divisor of the height functions of points from Arakelov geometry as in [7, 30, 18, 12, 11]. Observe that, to define the height of cycles of arbitrary dimension in [11], we need the variety to be proper and the metrics to be DSP, but these conditions are not needed in the case of points. The reason is that Definition 3.1 is equivalent to first restricting the metrized divisor to the point and then computing the height of the point with respect to this restriction, together with the observation that a point is proper and that every metric on a point is semipositive.

Definition 3.3. — *Let X be a variety over $\mathbb{K}$ and $W \subset X$ a locally closed subset. For $\eta \in \mathbb{R}$, consider the subset of algebraic points of W given by*

$$W(\overline{\mathbb{K}})_{\leqslant \eta} = \{p \in W(\overline{\mathbb{K}}) \mid \mathrm{h}_{\overline{D}}(p) \leqslant \eta\}.$$

Let $d = \dim(W)$. For $i = 1, \dots, d+1$, the i-th successive minimum *of W with respect to $\overline{D}$ is defined as*

$$\upmu^i_{\overline{D}}(W) = \inf \left\{\eta \in \mathbb{R} \mid \dim \left(\overline{W(\overline{\mathbb{K}})_{\leqslant \eta}}\right) \geqslant d - i + 1\right\}.$$

We set $\upmu^{\mathrm{abs}}_{\overline{D}}(W) = \upmu^{d+1}_{\overline{D}}(W)$ and $\upmu^{\mathrm{ess}}_{\overline{D}}(W) = \upmu^1_{\overline{D}}(W)$ for the absolute minimum *and the* essential minimum *of W with respect to $\overline{D}$, respectively.*

Clearly,

$$\upmu^{\mathrm{ess}}_{\overline{D}}(W) = \upmu^1_{\overline{D}}(W) \geqslant \upmu^2_{\overline{D}}(X) \geqslant \cdots \geqslant \upmu^{d+1}_{\overline{D}}(W) = \upmu^{\mathrm{abs}}_{\overline{D}}(W). \tag{3.1}$$

The following result shows that the successive minima are stable with respect to finite maps.

Proposition 3.4. — *Let $f\colon X\to Y$ be a dominant morphism of varieties over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on Y.*

(1) If f is generically finite then $\mu^{\mathrm{ess}}_{f^*\overline{D}}(X)=\mu^{\mathrm{ess}}_{\overline{D}}(Y)$.

(2) If f is finite then $\mu^{i}_{f^*\overline{D}}(X)=\mu^{i}_{\overline{D}}(Y)$ for $i=1,\dots,\dim(Y)+1$.

Proof. — For $p\in X(\overline{\mathbb{K}})$, the equality $\mathrm{h}_{f^*\overline{D}}(p)=\mathrm{h}_{\overline{D}}(f(p))$ holds, see [11, Theorem 1.5.11(2)] and Remark 3.2. It follows that, for any real number η, we have $X(\overline{\mathbb{K}})_{\leqslant\eta}=f^{-1}Y(\overline{\mathbb{K}})_{\leqslant\eta}$.

We first prove (2). Being finite, the morphism f is proper and, since it is dominant, it is also surjective. Hence $Y(\overline{\mathbb{K}})_{\leqslant\eta}=f(X(\overline{\mathbb{K}})_{\leqslant\eta})$.

Now let $1\leqslant i\leqslant \dim(Y)+1$ and suppose that $\mu^{i}_{f^*\overline{D}}(X)>\eta$. Then there exists a closed subset $V\subset X$ of dimension bounded by $n-i+1$ and containing $X(\overline{\mathbb{K}})_{\leqslant\eta}$. The image $f(V)$ is a closed subset of dimension bounded by $n-i+1$ and containing $Y(\overline{\mathbb{K}})_{\leqslant\eta}$. Hence, $\mu^{i}_{\overline{D}}(Y)>\eta$ and, since this holds for all real numbers below the i-th minimum of X, it follows that $\mu^{i}_{f^*\overline{D}}(X)\leqslant\mu^{i}_{\overline{D}}(Y)$.

Conversely, suppose that $\mu^{i}_{\overline{D}}(Y)>\eta$ and let $W\subset Y$ be a closed subset of dimension bounded by $n-i+1$ which contains $Y(\overline{\mathbb{K}})_{\leqslant\eta}$. Since f is finite, the preimage $f^{-1}(W)$ is a closed subset of dimension bounded by $n-i+1$ which contains $X(\overline{\mathbb{K}})_{\leqslant\eta}$. Hence, $\mu^{i}_{f^*\overline{D}}(X)>\eta$ and we conclude that $\mu^{i}_{f^*\overline{D}}(X)=\mu^{i}_{\overline{D}}(Y)$.

The statement (1) follows from (2) by restricting f to open dense subsets of X and Y where it is finite. □

We now specialize to the toric case. Let $\mathbb{T}\simeq\mathbb{G}_m^n$ be a split torus of dimension n over $\mathbb{K}$. Let $N=\mathrm{Hom}(\mathbb{G}_m,\mathbb{T})$ be the lattice of cocharacters of $\mathbb{T}$, $M=\mathrm{Hom}(\mathbb{T},\mathbb{G}_m)=N^\vee$ the lattice of characters, and write $N_\mathbb{R}=N\otimes\mathbb{R}$ and $M_\mathbb{R}=M\otimes\mathbb{R}$.

Let X be a proper toric variety over $\mathbb{K}$ with torus $\mathbb{T}$, described by a complete fan Σ on $N_\mathbb{R}$. Recall that, to each cone $\sigma\in\Sigma$ correspond an open affine subset X_σ and an orbit $O(\sigma)$. In particular, for $\sigma=\{0\}$ we obtain the principal open subset X_0 that, in this case, agrees with the orbit $O(0)$. It is canonically isomorphic to the split torus $\mathbb{T}$ that acts on the toric variety X. The action of $\mathbb{T}$ on X will be denoted by $(t,p)\mapsto t\cdot p$.

A toric $\mathbb{R}$-divisor on X is an $\mathbb{R}$-divisor invariant under the action of $\mathbb{T}$. Such a divisor D defines a function $\Psi_D\colon N_\mathbb{R}\to\mathbb{R}$ whose restriction to each cone of the fan Σ is linear, and which is called a “virtual support function”.

The toric $\mathbb{R}$-divisor D is nef if and only if Ψ_D is concave. One can also associate to D the subset $\Delta_D \subset M_{\mathbb{R}}$ given as $\Delta_D = \operatorname{stab}(\Psi_D)$, the stability set of Ψ_D as in (2.10). If D is pseudo-effective, Δ_D is a polytope and, otherwise, it is the empty set.

For each place $v \in \mathfrak{M}_{\mathbb{K}}$, we associate to the torus $\mathbb{T}$ an analytic space $\mathbb{T}_v^{\mathrm{an}}$ and we denote by $\mathbb{S}_v^{\mathrm{an}}$ its compact subtorus. In the Archimedean case, it is isomorphic to $(S^1)^n$. In the non-Archimedean case, it is a compact analytic group, see [11, § 4.2] for a description. Then, a metrized $\mathbb{R}$-divisor $\overline{D}$ on X is *toric* if D is a toric $\mathbb{R}$-divisor and its v-adic Green function $g_{\overline{D},v}$ is invariant with respect to the action of $\mathbb{S}_v^{\mathrm{an}}$ or, equivalently, if its v-adic metric $\|\cdot\|_v$ is invariant with respect to the action of $\mathbb{S}_v^{\mathrm{an}}$, for all v.

A toric metrized $\mathbb{R}$-divisor $\overline{D}$ on X defines an adelic family of continuous functions $\psi_{\overline{D},v}\colon N_{\mathbb{R}} \to \mathbb{R}$ indexed by the places of $\mathbb{K}$. For $v \in \mathfrak{M}_{\mathbb{K}}$, this function is given, for $p \in \mathbb{T}_v^{\mathrm{an}}$, by

$$\psi_{\overline{D},v}(\operatorname{val}_v(p)) = \log \|s_D(p)\|_v, \tag{3.2}$$

where val_v is the valuation map in (2.7) and s_D is the canonical rational $\mathbb{R}$-section of D as in [10, § 3].

The family of functions associated to $\overline{D}$ satisfies that, for all $v \in \mathfrak{M}_{\mathbb{K}}$, the function $|\psi_{\overline{D},v} - \Psi_D|$ is bounded and, for all v except for a finite number, $\psi_{\overline{D},v} = \Psi_D$. In particular, the stability set of $\psi_{\overline{D},v}$ coincides with Δ_D. The toric metrized $\mathbb{R}$-divisor $\overline{D}$ is semipositive if and only if $\psi_{\overline{D},v}$ is concave for all v.

Example 3.5. — Let X be a proper toric variety over $\mathbb{K}$ and D a toric $\mathbb{R}$-divisor on X. The *canonical metric* on D is the metric defined, for each $v \in \mathfrak{M}_{\mathbb{K}}$ and $p \in \mathbb{T}_v^{\mathrm{an}}$, by

$$\log \|s_D(p)\|_{\mathrm{can},v} = \Psi_D(\operatorname{val}_v(p)),$$

see [11, Proposition-Definition 4.3.15]. We denote the resulting toric metrized $\mathbb{R}$-divisor by $\overline{D}^{\mathrm{can}}$. In this case, $\psi_{\overline{D}^{\mathrm{can}},v} = \Psi_D$ for all v. In particular, $\overline{D}^{\mathrm{can}}$ is semipositive if and only if D is nef.

For each $v \in \mathfrak{M}_{\mathbb{K}}$, we consider the *local roof function* $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$ that is given, for $x \in \Delta_D$, by

$$\vartheta_{\overline{D},v}(x) = \psi_{\overline{D},v}^{\vee}(x) = \inf_{u \in N_{\mathbb{R}}} (\langle x, u\rangle - \psi_{\overline{D},v}(u)).$$

When $\psi_{\overline{D},v}$ is concave, the function $\vartheta_{\overline{D},v}$ coincides with the Legendre-Fenchel dual of $\psi_{\overline{D},v}$. This gives an adelic family of continuous concave functions on Δ_D which are zero except for a finite number of places.

The *global roof function* $\vartheta_{\overline{D}}\colon \Delta_D \to \mathbb{R}$ is defined as the weighted sum

$$\vartheta_{\overline{D}} = \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \vartheta_{\overline{D},v}.$$

As it is customary in convex analysis, we can also consider $\vartheta_{\overline{D}}$ as a function from the whole of $M_{\mathbb{R}}$ to the extended real line $\mathbb{R}\cup\{-\infty\}$ by writing $\vartheta_{\overline{D}}(x) = -\infty$ for $x \notin \Delta$. With this convention, D is not pseudo-effective if and only if $\vartheta_{\overline{D}} \equiv -\infty$ on $M_{\mathbb{R}}$.

In the case of toric varieties, the height of an algebraic point can be expressed in terms of the family of functions $\{\psi_{\overline{D},v}\}_{v\in\mathfrak{M}_{\mathbb{K}}}$. Let p be an algebraic point in the principal open subset X_0. Since D is a toric $\mathbb{R}$-divisor, p is not in the support of D. Choose a finite extension $\mathbb{F}$ of $\mathbb{K}$ such that $p \in X_0(\mathbb{F})$. For simplicity, we will also denote by $\overline{D}$ the toric metrized $\mathbb{R}$-divisor on $X_{\mathbb{F}}$ obtained by base change. Then, by the definition of the height and the definition of these functions in (3.2),

$$\begin{aligned}(3.3)\qquad \mathrm{h}_{\overline{D}}(p) &= -\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \log\|s_D(p)\|_w \\ &= -\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \psi_{\overline{D},w}(\mathrm{val}_w(p)) = -\sum_{v\in\mathfrak{M}_{\mathbb{K}}}\sum_{w\mid v} n_w\psi_{\overline{D},v}(\mathrm{val}_w(p)),\end{aligned}$$

since $\psi_{\overline{D},w} = \psi_{\overline{D},v}$ for all $w \mid v$ [11, Proposition 4.3.8].

The following is the key technical result to study successive minima of toric varieties.

Theorem 3.6. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then*

$$\mu^{\mathrm{abs}}_{\overline{D}}(X_0) = \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x).$$

Proof. — We first show that

$$\mu^{\mathrm{abs}}_{\overline{D}}(X_0) \geqslant \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x). \tag{3.4}$$

For shorthand we write $\Psi = \Psi_D$, $\Delta = \Delta_D$, $\psi_v = \psi_{\overline{D},v}$ and $\vartheta_v = \vartheta_{\overline{D},v}$. Let p be an algebraic point of X_0 and choose a finite extension $\mathbb{F}$ of $\mathbb{K}$ such that $p \in X_0(\mathbb{F})$. We have $\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \mathrm{val}_w(p) = 0$ and recall that, for each $w \in \mathfrak{M}_{\mathbb{F}}$ and $x \in \Delta$,

$$\vartheta_w(x) = \inf_{u\in N_{\mathbb{R}}} \left(\langle x,u\rangle - \psi_w(u)\right).$$

Hence, by (3.3), for any $x \in \Delta$,

$$\begin{aligned}\mathrm{h}_{\overline{D}}(p) &= -\sum_{w} n_w \psi_w(\mathrm{val}_w(p)) \\ &= \sum_{w} n_w(\langle x, \mathrm{val}_w(p)\rangle - \psi_w(\mathrm{val}_w(p))) \\ &\geqslant \sum_{w} n_w \vartheta_w(x) \\ &= \vartheta_{\overline{D}}(x).\end{aligned}$$

We conclude that $\mu^{\mathrm{abs}}_{\overline{D}}(X_0) \geqslant \vartheta_{\overline{D}}(x)$ for all $x \in \Delta$. Since $\mu^{\mathrm{abs}}_{\overline{D}}(X_0) \geqslant -\infty = \vartheta_{\overline{D}}(x)$ for all $x \notin \Delta$, we obtain the inequality (3.4).

We now prove

$$\mu^{\mathrm{abs}}_{\overline{D}}(X_0) \leqslant \max_{x \in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x). \tag{3.5}$$

Let S be a nonempty subset of places $v \in \mathfrak{M}_{\mathbb{K}}$ that contains all Archimedean places and all places v such that $\psi_v \not\equiv \Psi$. In particular, S contains all the places where $\vartheta_v \not\equiv 0$.

Suppose first that D is pseudo-effective. By [10, Proposition 4.9(2)], this is equivalent to the fact that $\Delta \neq \emptyset$. By Lemma 2.7, this implies that $\mathrm{conc}(\psi_v)$ exists for all v.

Let $x_0 \in \Delta$ such that $\vartheta_{\overline{D}}(x_0) = \max_{x \in \Delta} \vartheta_{\overline{D}}(x)$. By [24, Theorem 23.8],

$$0 \in \sum_{v \in S} n_v \partial \vartheta_v(x_0).$$

Choose a collection $u_v \in N_{\mathbb{R}}$, $v \in S$, such that

$$u_v \in \partial\vartheta_v(x_0) \quad \text{and} \quad \sum_{v \in S} n_v u_v = 0.$$

For $v \notin S$ put $u_v = 0$. Since $\vartheta_v = \psi_v^{\vee} = \mathrm{conc}(\psi_v)^{\vee}$ and $\mathrm{conc}(\psi_v)$ is concave,

$$\vartheta_v(x_0) = \langle x_0, u_v\rangle - \mathrm{conc}(\psi_v)(u_v). \tag{3.6}$$

Let $\varepsilon > 0$. Using Lemma 2.8, we deduce that there exists $d \geqslant 1$ and, for all $v \in S$, there exists $u_{v,j} \in N_{\mathbb{R}}$, $j = 1, \dots, d$, such that

$$\frac{1}{d}\sum_{j=1}^{d} u_{v,j} = u_v \quad \text{and} \quad \mathrm{conc}(\psi_v)(u_v) \leqslant \frac{1}{d}\sum_{j=1}^{d} \psi_v(u_{v,j}) + \frac{\varepsilon}{2\sum_{v \in S} n_v}.$$

For $v \notin S$, put $u_{v,j} = 0$, $j = 1, \dots, d$. Since for $v \notin S$ we have $\psi_v = \Psi$ and hence $\psi_v(0) = \mathrm{conc}(\psi_v)(0) = 0$, we deduce

$$\sum_{v \in \mathfrak{M}_{\mathbb{K}}} \sum_{j=1}^{d} \frac{n_v}{d} \psi_v(u_{v,j}) \geqslant \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \, \mathrm{conc}(\psi_v)(u_v) - \frac{\varepsilon}{2}. \tag{3.7}$$

Let $\mathbb{F}/\mathbb{K}$ be an extension of degree d such that all places in S split completely, as given by Lemma 2.2. For each $v \in S$ and $w \in \mathfrak{M}_{\mathbb{F}}$ such that $w \mid v$, we have $n_w = n_v/d$. We number the places above a given place $v \in S$ and write them as $w(v,j)$, $j = 1, \dots, d$.

Let $p = \alpha^r = \alpha \otimes r \in \mathbb{T}(\mathbb{F}) \otimes \mathbb{Q}$ with $\alpha \in \mathbb{T}(\mathbb{F})$ and $r \in \mathbb{Q}$. Such an element may be viewed as a point of $\mathbb{T}$ defined over some radical extension of $\mathbb{F}$. Hence, $\mathrm{val}_w(p) = r\mathrm{val}_w(\alpha)$ is the common value at p of the valuation maps associated to the places of this extension over w.

Recall that $H_{\mathbb{F}} \subset \bigoplus_{w\in\mathfrak{M}_{\mathbb{F}}} N_{\mathbb{R}}$ is the hyperplane defined by the equation

$$\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w z_w = 0$$

as in (2.6).

The functions Ψ and ψ_v satisfy the hypothesis of Lemma 2.5. Hence, we deduce from Lemmas 2.3 and 2.5 that there exists $p = \alpha \otimes r \in \mathbb{T}(\mathbb{F}) \otimes \mathbb{Q}$ with $\alpha \in \mathbb{T}(\mathbb{F})$ and $r \in \mathbb{Q}$ such that $\mathrm{val}_w(p) = 0$ for w above $v \notin S$, with $\mathrm{val}_{w(v,j)}(p)$ sufficiently close to $u_{v,j}$ for all $v \in S$ and $j = 1, \dots, d$. Therefore

$$\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \psi_w(\mathrm{val}_w(p)) \geqslant \sum_{v\in\mathfrak{M}_{\mathbb{K}}} \sum_{j=1}^{d} \frac{n_v}{d} \psi_v(u_{v,j}) - \frac{\varepsilon}{2}. \tag{3.8}$$

From (3.7) and (3.8), we deduce that

$$\mathrm{h}_{\overline{D}}(p) = -\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \psi_w(\mathrm{val}_w(p)) \leqslant -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \,\mathrm{conc}(\psi_v)(u_v) + \varepsilon.$$

Using $\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v u_v = 0$ and (3.6), we obtain

$$\mathrm{h}_{\overline{D}}(p) \leqslant \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v(\langle x_0, u_v\rangle - \mathrm{conc}(\psi_v)(u_v)) + \varepsilon = \vartheta_{\overline{D}}(x_0) + \varepsilon.$$

From this, we deduce that $\mu^{\mathrm{abs}}_{\overline{D}}(X_0) \leqslant \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x) + \varepsilon$ for all $\varepsilon > 0$ proving the inequality (3.5) in the case when D is pseudo-effective.

If D is not pseudo-effective, then $\mathrm{stab}(\Psi) = \emptyset$ and $\max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x) = -\infty$. By Lemma 2.10, there exist $u_j \in N_{\mathbb{R}}$, $j = 1, \dots, \ell$, such that

$$\sum_{j=1}^{\ell} u_j = 0 \quad \text{and} \quad \sum_{j=1}^{\ell} \Psi(u_j) > 0.$$

Using Lemmas 2.5 and 2.3, there exists $p = \alpha \otimes r \in \mathbb{T}(\mathbb{K}) \otimes \mathbb{Q}$ such that

$$\eta := \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \Psi(\mathrm{val}_v(p)) > 0.$$

For $l \geqslant 1$ such that $lr \in \mathbb{N}$, we view $p_l := \alpha \otimes lr$ as a point of $\mathbb{T}(\mathbb{K})$. Then

$$\mathrm{h}_{\overline{D}^{\mathrm{can}}}(p_l) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} -n_v \Psi(\mathrm{val}_v(p_l)) = l \sum_{v \in \mathfrak{M}_{\mathbb{K}}} -n_v \Psi(\mathrm{val}_v(p)) = -l\eta,$$

where $\overline{D}^{\mathrm{can}}$ denotes the $\mathbb{R}$-divisor D equipped with the canonical metric as in Example 3.5. Since the difference between the functions $\mathrm{h}_{\overline{D}^{\mathrm{can}}}$ and $\mathrm{h}_{\overline{D}}$ is bounded, it follows that $\lim_{l \to \infty} \mathrm{h}_{\overline{D}}(p_l) = -\infty$. Hence $\upmu^{\mathrm{abs}}_{\overline{D}}(X_0) = -\infty$, which completes the proof. □

Corollary 3.7. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then $\upmu^{\mathrm{abs}}_{\overline{D}}(X_0) > -\infty$ if and only if D is pseudo-effective.*

Proof. — By [10, Proposition 4.9(2)], D is pseudo-effective if and only if Δ is not empty. The result follows then from Theorem 3.6. □

The next lemma shows that the successive minima of a toric variety with respect to a toric metrized $\mathbb{R}$-divisor can be computed in terms of the absolute minima of the orbits under the action of $\mathbb{T}$.

Lemma 3.8. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X.*

(1) Let $\sigma \in \Sigma$. Then $\upmu^{i}_{\overline{D}}(O(\sigma)) = \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma))$ for $i = 1, \dots, \dim(O(\sigma))+1$.

(2) For $i = 1, \dots, n+1$,

$$\upmu^{i}_{\overline{D}}(X) = \min_{\sigma \in \Sigma^{\leqslant i-1}} \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)),$$

where $\Sigma^{\leqslant i-1}$ denotes the set of cones of Σ of dimension $\leqslant i-1$.

Proof. — We first prove (1). Let $\mathbb{T}(\overline{\mathbb{K}})_{\mathrm{tors}}$ denote the subgroup of torsion points of the group of algebraic points of $\mathbb{T}$. Under the identification $\mathbb{T}(\overline{\mathbb{K}}) = \mathrm{Hom}(M, \overline{\mathbb{K}}^{\times})$, this subgroup corresponds to $\mathrm{Hom}(M, \upmu_{\infty})$, the homomorphisms from M to the group of roots of unity. This implies that, if $t \in \mathbb{T}(\overline{\mathbb{K}})_{\mathrm{tors}}$ and $p \in X(\overline{\mathbb{K}})$, then $\mathrm{h}_{\overline{D}}(t \cdot p) = \mathrm{h}_{\overline{D}}(p)$.

Now let $\varepsilon > 0$ and choose an algebraic point p of the orbit $O(\sigma)$ such that $\mathrm{h}_{\overline{D}}(p) \leqslant \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)) + \varepsilon$. Then, $\mathbb{T}(\overline{\mathbb{K}})_{\mathrm{tors}} \cdot p$ is a dense subset of algebraic points of $O(\sigma)$ of the same height as p. Hence,

$$\upmu^{\mathrm{ess}}_{\overline{D}}(O(\sigma)) \leqslant \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)) + \varepsilon.$$

We deduce that $\upmu^{\mathrm{ess}}_{\overline{D}}(O(\sigma)) \leqslant \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma))$. If follows that the chain of inequalities in (3.1), applied to the variety $O(\sigma)$, shrinks to the equalities in the statement.

Now we consider (2). Let $\sigma \in \Sigma^{\leqslant i-1}$. Then, $O(\sigma)$ is of dimension $\geqslant n-i+1$ and so

$$\upmu^{i}_{\overline{D}}(X) \leqslant \upmu^{\mathrm{ess}}_{\overline{D}}(O(\sigma)).$$

Using (1), we deduce that $\upmu^{i}_{\overline{D}}(X) \leqslant \min_{\sigma\in\Sigma^{\leqslant i-1}} \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma))$.

For the reverse inequality, observe that, for a cone $\sigma \in \Sigma$ of dimension $\geqslant i$, the orbit $O(\sigma)$ is of dimension $\leqslant n-i$. Using the decomposition of X into orbits, we deduce that

$$\upmu^{i}_{\overline{D}}(X) = \upmu^{i}_{\overline{D}}\Big(X \setminus \bigcup_{\sigma\in\Sigma^{\geqslant i}} O(\sigma)\Big) = \upmu^{i}_{\overline{D}}\Big(\bigcup_{\sigma\in\Sigma^{\leqslant i-1}} O(\sigma)\Big).$$

Hence,

$$\upmu^{i}_{\overline{D}}(X) \geqslant \upmu^{\mathrm{abs}}_{\overline{D}}\Big(\bigcup_{\sigma\in\Sigma^{\leqslant i-1}} O(\sigma)\Big) = \min_{\sigma\in\Sigma^{\leqslant i-1}} \upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)),$$

which proves the result. □

Theorem A in the introduction is a direct consequence of the previous results.

Corollary 3.9. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then*

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x).$$

Proof. — From Lemma 3.8(2) and Theorem 3.6 we obtain

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \upmu^{\mathrm{abs}}_{\overline{D}}(X_0) = \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x).$$

□

Using this result, we can deduce some relations between the essential minimum and the positivity properties of $\overline{D}$ in the toric setting.

Corollary 3.10. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. Then*

(1) $\overline{D}$ is pseudo-effective if and only if $\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geqslant 0$;
(2) $\overline{D}$ is big if and only if $\dim(\Delta_D) = n$ and $\upmu^{\mathrm{ess}}_{\overline{D}}(X) > 0$.

Proof. — This follows from Corollary 3.9 and [10, Theorem 2(3,4)]. □

It is possible to reformulate Corollary 3.9 to give a formula for the essential minimum in terms of the functions $\psi_{\overline{D},v}$ that is useful when computing the essential minimum in concrete situations as those considered in § 5.

For convenience we recall the definition of sup-convolution of two concave functions [11, § 2.3]. Let $\Delta \subset M_{\mathbb{R}}$ be a polytope and ψ_1, ψ_2 two concave functions on $N_{\mathbb{R}}$ whose stability set is Δ. Then

$$(\psi_1 \boxplus \psi_2)(u) = \sup_{u_1+u_2=u} \psi_1(u_1) + \psi_2(u_2).$$

This is a concave function on $N_{\mathbb{R}}$ with stability set Δ. The sup-convolution is an associative operation. In fact it corresponds to the pointwise addition by Legendre-Fenchel duality [11, Proposition 2.3.1]. That is,

$$\psi_1 \boxplus \psi_2 = (\psi_1^{\vee} + \psi_2^{\vee})^{\vee}.$$

Moreover, if $\psi_2 = \Psi$ is the support function of Δ, then $\psi_1 \boxplus \Psi = \psi_1$.

Recall also the right multiplication of a concave function by a scalar.

$$(\psi_1\lambda)(u) = \lambda\psi_1(u/\lambda).$$

This operation is dual of the usual left multiplication

$$\psi_1\lambda = (\lambda\psi_1^{\vee})^{\vee}.$$

Corollary 3.11. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X.*

(1) If D is pseudo-effective, then

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \mu^{\mathrm{abs}}_{\overline{D}}(X_0) = -\Big(\boxplus_{v\in\mathfrak{M}_{\mathbb{K}}} \operatorname{conc}(\psi_{\overline{D},v} n_v)\Big)(0).$$

(2) If $\overline{D}$ is semipositive and its v-adic metric agrees with the canonical metric for all places except one place v_0, then

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \mu^{\mathrm{abs}}_{\overline{D}}(X_0) = -n_{v_0}\psi_{\overline{D},v_0}(0).$$

Proof. — By Theorem 3.6, $\mu^{\mathrm{abs}}_{\overline{D}}(X_0) = \max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x)$. From the definition of Legendre-Fenchel duality, $\max_{x\in M_{\mathbb{R}}} \vartheta_{\overline{D}}(x) = -\vartheta^{\vee}_{\overline{D}}(0)$. By the duality between the sum and the sup-convolution, and that between the right and the left multiplication,

$$\vartheta^{\vee}_{\overline{D}} = \boxplus_{v\in\mathfrak{M}_{\mathbb{K}}} \operatorname{conc}(\psi_{\overline{D},v} n_v).$$

Hence we obtain the first statement.

If $\overline{D}$ is semipositive and its metric agrees with the canonical metric for all places except one place v_0, then $\psi_{\overline{D},v} = \Psi_D$ for all $v \neq v_0$ and $\psi_{\overline{D},v_0}$ is concave. Since the semipositivity of $\overline{D}$ implies that D is pseudo-effective, the first statement implies that

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = -(\psi_{\overline{D},v_0} n_{v_0})(0) = -n_{v_0}\psi_{\overline{D},v_0}(0/n_{v_0}) = -n_{v_0}\psi_{\overline{D},v_0}(0),$$

which proves the second statement. $\square$

Remark 3.12. — If $\overline{D}$ is semipositive and its v-adic metric agrees with the canonical metric for all places except one place v_0, then we can identify a dense set of points whose height agrees with $\mu^{\mathrm{ess}}_{\overline{D}}(X)$. Namely, each point

$$p \in \mathbb{T}(\overline{\mathbb{K}})_{\mathrm{tors}} \subset X_0(\overline{\mathbb{K}})$$

satisfies $\mathrm{val}_v(p) = 0$ for all $v \in \mathfrak{M}_{\mathbb{K}}$. Hence

$$\mathrm{h}_{\overline{D}}(p) = -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \psi_{\overline{D},v}(0) = -n_{v_0}\psi_{\overline{D},v_0}(0) = \mu^{\mathrm{ess}}_{\overline{D}}(X).$$

We now want to extend Corollary 3.9 to the other successive minima. To do this, we have to describe the restriction to an orbit of a toric metrized $\mathbb{R}$-divisor in terms of its roof function.

Let X be a proper toric variety and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor with D nef. Let Σ be the fan of X and Δ_D the polytope associated to D. Recall that, as in [11, Example 2.5.13], to a cone $\sigma \in \Sigma$ we can associate a face $F_\sigma \subset \Delta_D$ given by

$$F_\sigma = \{x \in \Delta_D \mid \langle y - x, u\rangle \geqslant 0 \text{ for all } y \in \Delta_D, u \in \sigma\}.$$

Choose an element $m_\sigma \in M_{\mathbb{R}}$ in the affine space generated by the face F_σ. Then the $\mathbb{R}$-divisor $D - \mathrm{div}(\chi^{-m_\sigma})$ intersects the orbit closure $V(\sigma)$ properly, see [11, Proposition 3.3.14] for the case of Cartier divisors. We set

$$(3.9)\quad D_\sigma = (D - \mathrm{div}(\chi^{-m_\sigma}))|_{V(\sigma)} \quad \text{and} \quad \overline{D}_\sigma = (\overline{D} - \widehat{\mathrm{div}}(\chi^{-m_\sigma}))|_{V(\sigma)}$$

with $\widehat{\mathrm{div}}(\chi^{-m_\sigma})$ as in [10, Definition 3.4]. In this situation, $\Delta_{D_\sigma} = F_\sigma - m_\sigma$.

Proposition 3.13. — *Let X be a proper toric variety and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor with D nef. Let Σ be the fan of X and $\sigma \in \Sigma$. With notation as in* (3.9), *for all $x \in F_\sigma - m_\sigma$,*

$$\vartheta_{\overline{D}_\sigma}(x) \geqslant \vartheta_{\overline{D}}(x + m_\sigma).$$

Moreover, if $\overline{D}$ is semipositive, the equality holds.

Proof. — For short, write $\Psi = \Psi_D$ and $\psi_v = \psi_{\overline{D},v}$. Since D is nef, the function Ψ is concave. Since $m_\sigma \in M_{\mathbb{R}}$ belongs to the affine space generated by the face F_σ, this implies that $\Psi|_\sigma = m_\sigma$. Let N_σ be the partial compactification of $N_{\mathbb{R}}$ in the direction of σ [11, (4.1.5)]. Recall that there is an inclusion $N(\sigma)_{\mathbb{R}} \subset N_\sigma$. The function $\Psi - m_\sigma$ extends to a continuous function on N_σ and thus can be restricted to $N(\sigma)$. We denote by $\Psi(\sigma)$ this restriction. For each place $v \in \mathfrak{M}_{\mathbb{K}}$, the function $\psi_v - m_\sigma$ can also be extended to a continuous function on N_σ and restricted to $N(\sigma)$. We denote by $\psi_v(\sigma)$ this restriction.

By the analogue of [11, Proposition 3.3.14] for $\mathbb{R}$-divisors, the support function of D_σ is $\Psi(\sigma)$. By the commutative diagram in [11, Proposition 4.1.6] the metric induced on D_σ by the metric of D is given by the family of functions $\{\psi_v(\sigma)\}_{v\in\mathfrak{M}_{\mathbb{K}}}$.

Assume that $\overline{D}$ is semipositive. Hence for every $v \in \mathfrak{M}_{\mathbb{K}}$ the function ψ_v is concave. We identify $\operatorname{stab}(\Psi(\sigma))$ with F_σ by means of the translation by $-m_\sigma$. By the analogue for $\mathbb{R}$-divisors of [11, Proposition 4.8.9], we have that, for all $x \in F_\sigma - m_\sigma$,

$$\psi_v(\sigma)^\vee(x) = \vartheta_{\overline{D},v}(x + m_\sigma).$$

Summing up for $v \in \mathfrak{M}_{\mathbb{K}}$, we obtain the equality in this case.

We now drop the hypothesis of $\overline{D}$ being semipositive. In this case, the functions ψ_v are not necessarily concave. Since D is nef, the function Ψ is concave. By Lemma 2.11, for each $v \in \mathfrak{M}_{\mathbb{K}}$, we have that $|\operatorname{conc}(\psi_v) - \Psi|$ is bounded. By [10, Proposition 4.19(1)], the family of functions $\{\operatorname{conc}(\psi_v)\}_{v\in\mathfrak{M}_{\mathbb{K}}}$ determines a semipositive toric metric on D. We denote by $\overline{D}'$ the corresponding semipositive metrized $\mathbb{R}$-divisor. Since for each function f with nonempty stability set

$$\operatorname{conc}(f)^\vee = f^\vee,$$

we have that $\vartheta_{\overline{D}'} = \vartheta_{\overline{D}}$. Since $\psi_v \leqslant \operatorname{conc}(\psi_v)$ on $N_{\mathbb{R}}$, then $\psi_v(\sigma) \leqslant \operatorname{conc}(\psi_v)(\sigma)$ on $N(\sigma)_{\mathbb{R}}$. This implies that $\vartheta_{\overline{D}_\sigma,v} \geqslant \vartheta_{\overline{D}'_\sigma,v}$ on $F_\sigma - m_\sigma$ for all v. Since $\overline{D}'$ is semipositive, it follows that, for all $x \in F_\sigma - m_\sigma$,

$$\vartheta_{\overline{D}_\sigma}(x) \geqslant \vartheta_{\overline{D}'_\sigma}(x) = \vartheta_{\overline{D}'}(x + m_\sigma) = \vartheta_{\overline{D}}(x + m_\sigma),$$

which concludes the proof. $\square$

Theorem 3.14. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X with D nef.*

(1) Let Σ be the fan of X and $\sigma \in \Sigma$. Then

$$\upmu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)) \geqslant \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x).$$

If $\overline{D}$ is semipositive, then the equality holds.

(2) For $i = 1, \dots, n+1$,

$$\upmu^{i}_{\overline{D}}(X) \geqslant \min_{\sigma\in\Sigma^{\leqslant i-1}} \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x).$$

If $\overline{D}$ is semipositive, then the equality holds.

Proof. — The first statement follows directly from Theorem 3.6 and Proposition 3.13. The second statement follows from the first one together with Lemma 3.8(2). $\square$

Remark 3.15. — Since $\vartheta_{\overline{D}}$ is a concave function, its minimum is attained at the vertices of Δ_D. Therefore, if $\overline{D}$ is semipositive, we deduce

$$\mu^{\mathrm{abs}}_{\overline{D}}(X) = \min_{x\in\Delta_D} \vartheta_{\overline{D}}(x).$$

This result was already implicit in [10, Theorem 2(2)]. It would be interesting to know if this formula and Corollary 3.9 can be generalized to non-toric varieties.

When the divisor D is ample, we are able to give a version of Theorem 3.14(2) in terms of the polytope Δ_D only.

THEOREM 3.16. — *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X with D ample. Then, for $i = 1,\dots,n+1$,*

$$\mu^{i}_{\overline{D}}(X) = \min_{F\in\mathcal{F}(\Delta_D)^{n-i+1}} \max_{x\in F} \vartheta_{\overline{D}}(x),$$

where $\mathcal{F}(\Delta_D)^{n-i+1}$ is the set of faces of dimension $n-i+1$ of the polytope.

Proof. — If D is ample, the correspondence $\sigma \mapsto F_\sigma$ is a bijection that sends cones of dimension $i-1$ to faces of dimension $n-i+1$. Thus, by Theorem 3.14(2), we deduce

$$\mu^{i}_{\overline{D}}(X) = \min_{F\in\mathcal{F}(\Delta_D)^{\geqslant n-i+1}} \max_{x\in F} \vartheta_{\overline{D}}(x).$$

where $\mathcal{F}(\Delta_D)^{\geqslant n-i+1}$ is the set of faces of dimension greater of equal to $n-i+1$. The concavity of $\vartheta_{\overline{D}}$ implies that the minimum in the right hand side is attained in faces of dimension $n-i+1$, hence the result. ☐

Example 3.17. — The positivity conditions on D and $\overline{D}$ in Theorems 3.14 and 3.16 are necessary, as it can be seen in the following examples.

(1) Let X be a toric surface over $\mathbb{K}$ and let $\overline{D}$ be a toric metrized $\mathbb{R}$-divisor such that the underlying divisor D is big but not nef and that there is a one dimensional orbit $O(\sigma)$ such that $\deg_D(\overline{O(\sigma)}) < 0$. In this case

$$\mu^{2}_{\overline{D}}(X) \leqslant \mu^{\mathrm{abs}}_{\overline{D}}(O(\sigma)) = -\infty$$

but

$$\min_{\sigma\in\Sigma^{\leqslant 1}} \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x) \geqslant \min_{F\in\mathcal{F}(\Delta)} \max_{x\in F} \vartheta_{\overline{D}}(x) > -\infty.$$

Thus the hypothesis D nef in Theorem 3.14(2) is necessary.

(2) Consider $X = \mathbb{P}^1_{\mathbb{Q}}$ and the divisor $\infty = (0 : 1)$. We consider the toric metric given by the canonical metric for $v \neq \infty$ and

$$\psi_\infty(u) = \begin{cases} u & \text{if } u \leqslant 0, \\ 0 & \text{if } 0 \leqslant u \leqslant 99, \text{ or } 101 \leqslant x, \\ u-99 & \text{if } 99 \leqslant u \leqslant 100, \\ 101-u & \text{if } 100 \leqslant u \leqslant 101. \end{cases}$$

Denote $\overline{D}$ the obtained metrized $\mathbb{R}$-divisor. The associated polytope is $\Delta = [0,1]$, and the roof function is

$$\vartheta_{\overline{D}}(x) = \begin{cases} 100x - 1 & \text{if } 0 \leqslant x \leqslant 1/100, \\ 0 & \text{if } 1/100 \leqslant x \leqslant 1. \end{cases}$$

Since $h_{\overline{D}}(p) \geqslant \mathrm{val}_\infty(p) - \psi_\infty(\mathrm{val}_\infty(p))$ we deduce $h_{\overline{D}}(p) \geqslant 0$ for all $p \in X(\overline{\mathbb{Q}})$ and then

$$\upmu^1_{\overline{D}}(X) = \upmu^2_{\overline{D}}(X) = 0,$$

but

$$\min_{\sigma\in\Sigma^{\leqslant 1}} \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x) = \min_{F\in\mathcal{F}(\Delta)^0} \max_{x\in F} \vartheta_{\overline{D}}(x) = -1.$$

Thus, we see that the hypothesis $\overline{D}$ semipositive is necessary for the equality in Theorem 3.14(2) to hold.

(3) Let X be the blow-up of $\mathbb{P}^2_{\mathbb{Q}}$ at the point $(1 : 0 : 0)$ and let $\overline{D}$ be the preimage of the metrized divisor given by the hyperplane at infinity with the canonical metric at the non-Archimedean places and the Fubini-Study metric at the Archimedean place.

Let $\sigma_0 \in \Sigma$ be the one dimensional cone corresponding to the exceptional divisor. Then F_{σ_0} is the vertex $(0,0)$ and has dimension zero. Thus

$$\upmu^2_{\overline{D}}(X) = \min_{\sigma\in\Sigma^{\leqslant 1}} \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x) = 0,$$

while

$$\min_{F\in\mathcal{F}(\Delta)^1} \max_{x\in F} \vartheta_{\overline{D}}(x) = \frac{1}{2}\log(2).$$

Hence the hypothesis D ample is necessary in Theorem 3.16.

4. On Zhang's theorem on successive minima

Zhang's theorem on successive minima [29, Theorem 5.2], [30, Theorem 1.10], shows that the successive minima of a metrized divisor can be estimated in terms of the height and the degree of the ambient variety. This result plays an important rôle in Diophantine geometry in the direction of the Bogomolov conjecture and the Lehmer problem and its generalizations. It also plays a rôle in the study of the distribution of Galois orbits of points of small height.

We start by giving a proof of a variant of Zhang's theorem in the toric setting (Theorem C in the introduction).

Theorem 4.1. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X such that D is big. Then*

$$\sum_{i=1}^{n+1} \mu^{i}_{\overline{D}}(X) \leqslant \frac{\mathrm{h}_{\overline{D}}(X)}{\deg_D(X)} \leqslant (n+1)\mu^{\mathrm{ess}}_{\overline{D}}(X). \tag{4.1}$$

Proof. — For short write $\Delta = \Delta_D$. Since $\overline{D}$ is a semipositive toric metrized divisor, necessarily D is generated by global sections [11, Corollary 4.8.5], hence, being toric $\deg_D(X) = n!\,\mathrm{vol}_M(\Delta)$. Since D is big, we also have $\mathrm{vol}(\Delta) > 0$. We first prove the inequality in the right hand side of (4.1). By Corollary 3.9, $\mu^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \vartheta_{\overline{D}}(x)$ for all $x \in \Delta$. Therefore, by [11, Theorem 5.2.5]

$$\begin{aligned}\mathrm{h}_{\overline{D}}(X) = (n+1)! \int_\Delta \vartheta_{\overline{D}}\,\mathrm{d}\mathrm{vol}_M \leqslant (n+1)! \int_\Delta \mu^{\mathrm{ess}}_{\overline{D}}(X)\,\mathrm{d}\mathrm{vol}_M \\ = (n+1)\mu^{\mathrm{ess}}_{\overline{D}}(X) n!\,\mathrm{vol}_M(\Delta) = (n+1)\mu^{\mathrm{ess}}_{\overline{D}}(X)\deg_D(X).\end{aligned}$$

We now prove the left inequality. For each face F of Δ we choose a point $x_F \in F$ such that

$$\vartheta_{\overline{D}}(x_F) = \max_{x\in F} \vartheta_{\overline{D}}(x).$$

For each flag of faces

$$\Xi = \{F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \Delta\}$$

with $\dim F_i = i$, we denote

$$\Delta_\Xi = \mathrm{conv}(x_{F_0}, \dots, x_{F_n}).$$

Then Δ_Ξ is a (possibly degenerate) simplex. Moreover

$$\Delta = \bigcup_\Xi \Delta_\Xi \quad \text{and} \quad \mathrm{int}\,\Delta_\Xi \cap \mathrm{int}\,\Delta_{\Xi'} = \emptyset, \text{ if } \Xi \neq \Xi'.$$

Let $f\colon \Delta \to \mathbb{R}$ be the function determined by

(1) For each complete flag Ξ, the restriction $f\,|_{\Delta_\Xi}$ is affine.
(2) If F is a face of Δ of dimension i, then $f(x_F) = \mu^{n-i+1}_{\overline{D}}(X)$.

Given a face F of Δ of dimension i, there exists a cone $\sigma \in \Sigma^{n-i}$ such that $F = F_\sigma$. Therefore, by Theorem 3.14(2),

$$f(x_F) = \mu^{n-i+1}_{\overline{D}}(X) \leqslant \max_{x\in F_\sigma} \vartheta_{\overline{D}}(x) = \vartheta_{\overline{D}}(x_F).$$

Since $\vartheta_{\overline{D}}$ is concave and f is affine in each simplex Δ_Ξ, we deduce $f(x) \leqslant \vartheta_{\overline{D}}(x)$ for all $x \in \Delta$. Therefore

$$\int_\Delta \vartheta_{\overline{D}}\, \mathrm{dvol}_M \geqslant \int_\Delta f\, \mathrm{dvol}_M = \sum_\Xi \int_{\Delta_\Xi} f\, \mathrm{dvol}_M\,.$$

Since

$$\sum_\Xi \int_{\Delta_\Xi} f\, \mathrm{dvol}_M = \sum_\Xi \frac{\sum_{i=0}^{n} \mu^{n-i+1}_{\overline{D}}(X)}{n+1} \mathrm{vol}(\Delta_\Xi) = \frac{\sum_{i=1}^{n+1} \mu^{i}_{\overline{D}}(X)}{n+1} \mathrm{vol}(\Delta),$$

we deduce

$$\begin{aligned} \mathrm{h}_{\overline{D}}(X) &= (n+1)! \int_\Delta \vartheta_{\overline{D}}\, \mathrm{dvol}_M \\ &\geqslant n! \sum_{i=1}^{n+1} \mu^{i}_{\overline{D}}(X)\, \mathrm{vol}(\Delta) = \sum_{i=1}^{n+1} \mu^{i}_{\overline{D}}(X)\, \mathrm{deg}_D(X), \end{aligned}$$

proving the result. □

Corollary 4.2. — *Suppose that $\overline{D}$ is nef. Then*

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) \leqslant \frac{\mathrm{h}_{\overline{D}}(X)}{\mathrm{deg}_D(X)} \leqslant (n+1)\mu^{\mathrm{ess}}_{\overline{D}}(X).$$

Proof. — Since $\overline{D}$ is nef, all the successive minima are non-negative. Then the corollary follows directly from Theorem 4.1. □

The following result improves [23, Théorème 1.4]. We show that, already for the universal line bundle on $\mathbb{P}^n_{\mathbb{Q}}$, almost every configuration for the successive minima and the height satisfying the inequalities in (4.1), can be realized.

Proposition 4.3. — *Let $r \geqslant 1$ and $\nu, \mu_1, \dots, \mu_{r+1} \in \mathbb{R}$ such that*

$$\mu_1 \geqslant \cdots \geqslant \mu_{r+1} \quad \text{and} \quad \sum_{i=1}^{r+1} \mu_i \leqslant \nu < (r+1)\mu_1.$$

Then there exists a semipositive toric metric on H, the divisor given by the hyperplane at infinity of $\mathbb{P}^r_{\mathbb{Q}}$, such that

$$\mu^{i}_{\overline{H}}(\mathbb{P}^r) = \mu_i,\ i = 1, \dots, r+1, \quad \text{and} \quad \mathrm{h}_{\overline{H}}(\mathbb{P}^r) = \nu.$$

Proof. — Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{R}^r$ and $\Delta^r = \operatorname{conv}(0, e_1, \dots, e_r)$ the standard simplex of $\mathbb{R}^r$. For $0 \leqslant t < 1$ consider the function $\theta_t \colon \Delta^r \to \mathbb{R}$ defined as the smallest concave function on Δ^r such that

$$\theta_t(x) = \begin{cases} \mu_1 & \text{for } x \in t\Delta^r, \\ \mu_i & \text{for } x = e_{i-1} \text{ and } i = 2, \dots, r+1. \end{cases}$$

Then the integral $\int_{\Delta^r} \theta_t \,\mathrm{d}x$ varies continuously in the interval

$$\left[\frac{1}{(r+1)!}\sum_{i=1}^{r+1}\mu_i, \frac{1}{r!}\mu_1\right).$$

In particular, there exists t such that the corresponding integral gives $\frac{\nu}{(r+1)!}$. Consider the semipositive toric metric $(\|\cdot\|_v)_v$ on H given by, for $v = \infty$, the toric metric associated to θ_t and, for $v \neq \infty$, the canonical metric. A straightforward calculation shows that this metric satisfies the required conditions. □

For the right hand inequality in Theorem 4.1, we can relax the hypothesis of semipositivity of the metrized $\mathbb{R}$-divisor, by replacing the height by the arithmetic volume or the χ-arithmetic volume of the divisor. In our present toric setting, the obtained lower bound of the essential minimum in terms of the χ-arithmetic volume extends [13, Lemme 5.1] to arbitrary global fields and metrized $\mathbb{R}$-divisors.

Proposition 4.4. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X such that D is big. Then*

$$\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \frac{\widehat{\operatorname{vol}}_\chi(\overline{D})}{(n+1)\operatorname{vol}(D)}. \tag{4.2}$$

If $\overline{D}$ is pseudo-effective, then

$$\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \frac{\widehat{\operatorname{vol}}(\overline{D})}{(n+1)\operatorname{vol}(D)}. \tag{4.3}$$

Proof. — For short write $\Delta = \Delta_D$. We first prove (4.2). By Corollary 3.9, $\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \vartheta_{\overline{D}}(x)$ for all $x \in \Delta$. Therefore, using the formula for the χ-arithmetic volume of a toric metrized $\mathbb{R}$-divisor in [10, Theorem 1] and the classical formula for the volume of a toric variety with respect to a toric divisor, we have

$$\begin{aligned}\widehat{\operatorname{vol}}_\chi(\overline{D}) &= (n+1)!\int_\Delta \vartheta_{\overline{D}}\,\mathrm{d}\operatorname{vol}_M \leqslant (n+1)!\int_\Delta \mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X)\,\mathrm{d}\operatorname{vol}_M \\ &= (n+1)\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X) n! \operatorname{vol}(\Delta) = (n+1)\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X)\operatorname{vol}(D),\end{aligned} \tag{4.4}$$

Since D is big, we have that $\mathrm{vol}(D) > 0$, and the inequality follows.

By Corollary 3.10(1), if $\overline{D}$ is pseudo-effective then $\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geqslant \max(0, \vartheta_{\overline{D}}(x))$ for all $x \in \Delta$. The inequality (4.3) follows similarly because, by using [10, Theorem 1] and Corollary 3.10(1),

$$\begin{aligned}(4.5)\qquad \widehat{\mathrm{vol}}(\overline{D}) &= (n+1)! \int_{\Delta} \max(0, \vartheta_{\overline{D}}) \,\mathrm{d}\mathrm{vol}_M \\ &\leqslant (n+1)! \int_{\Delta} \upmu^{\mathrm{ess}}_{\overline{D}}(X) \,\mathrm{d}\mathrm{vol}_M = (n+1)\upmu^{\mathrm{ess}}_{\overline{D}}(X)\,\mathrm{vol}(D).\end{aligned}$$

□

Now we will characterize when equality occurs in the lower bounds in Proposition 4.4. First we need a technical lemma.

Lemma 4.5. — *Let $\Psi \colon N_{\mathbb{R}} \to \mathbb{R}$ be a conic function such that* $\mathrm{stab}(\Psi)$ *has nonempty interior, and $f \colon N_{\mathbb{R}} \to \mathbb{R}$ a continuous function such that $|f - \Psi|$ is bounded. Let $u_0 \in N_{\mathbb{R}}$ and $\gamma \in \mathbb{R}$. The following conditions are equivalent:*

(1) $f^{\vee}(x) = \langle x, u_0\rangle + \gamma$ *for all* $x \in \mathrm{stab}(\Psi)$*;*
(2) $\mathrm{conc}(f)(u) = \mathrm{conc}(\Psi)(u - u_0) - \gamma$ *for all* $u \in N_{\mathbb{R}}$*;*
(3) $f(u_0) = -\gamma$ *and* $f(u) \leqslant \mathrm{conc}(\Psi)(u - u_0) - \gamma$ *for all* $u \in N_{\mathbb{R}}$*.*

Proof. — Set $\Delta = \mathrm{stab}(\Psi)$, which is a convex subset of $M_{\mathbb{R}}$ and agrees with $\mathrm{stab}(f)$ by the hypothesis $|f - \Psi|$ bounded.

(1) ⇒ (2): the function f is asymptotically conic in the sense of [10, Definition A.3]. Hence $\mathrm{stab}(\mathrm{conc}(f)) = \mathrm{stab}(f) = \Delta$ and $\mathrm{conc}(f) = f^{\vee\vee}$. Thus

$$(4.6)\qquad \mathrm{conc}(f)(u) = (f^{\vee})^{\vee} = \inf_{x \in \Delta} \langle x, u - u_0\rangle - \gamma.$$

Analogously

$$\mathrm{conc}(\Psi)(u) = (\Psi^{\vee})^{\vee} = \inf_{x \in \Delta} \langle x, u\rangle.$$

Thus, the right hand side of (4.6) agrees with $\mathrm{conc}(\Psi)(u - u_0) - \gamma$.

(2) ⇒ (1): This follows from the fact that $f^{\vee} = \mathrm{conc}(f)^{\vee}$

(1) ⇒ (3): Since, for any function with non empty stability set,

$$(4.7)\qquad f(u) \leqslant \mathrm{conc}(f)(u)$$

the bound for $f(u)$ follows from the implication (1) ⇒ (2). Thus we only have to show that $f(u_0) = -\gamma$.

By equations (4.6) and (4.7), we have $f(u_0) \leqslant -\gamma$. Thus assume that $f(u_0) = -\gamma - \varepsilon$ for some $\varepsilon > 0$.

Let x_0 be a point in the interior of Δ and choose a norm $\|\cdot\|$ on $N_{\mathbb{R}}$. By (4.6) and (4.7), and using that x_0 belongs to the interior of Δ, we deduce that there exists $K>0$ such that for all $u\in N_{\mathbb{R}}$

$$f(u)-\langle x_0,u-u_0\rangle \leqslant \inf_{x\in\Delta}\langle x-x_0,u-u_0\rangle-\gamma\leqslant -K\|u-u_0\|-\gamma. \tag{4.8}$$

By the continuity of f there is $\eta>0$ such that, if $\|u-u_0\|\leqslant\eta$ then

$$f(u)-\langle x_0,u-u_0\rangle\leqslant -\gamma-\varepsilon/2.$$

By the inequality (4.8), if $\|u-u_0\|\geqslant\eta$, then $f(u)-\langle x_0,u-u_0\rangle\leqslant -\gamma-\eta K$. Put $s=\min(\varepsilon/2,\eta K)>0$. Hence

$$f(u)\leqslant\langle x_0,u-u_0\rangle-\gamma-s$$

for all $u\in N_{\mathbb{R}}$. Thus

$$\begin{aligned} f^{\vee}(x_0)&=\inf_{u\in N_{\mathbb{R}}}\langle x_0,u\rangle-f(u)\\ &\geqslant\inf_{u\in N_{\mathbb{R}}}\langle x_0,u\rangle-\langle x_0,u-u_0\rangle+\gamma+s=\langle x_0,u_0\rangle+\gamma+s,\end{aligned}$$

contradicting (1). Therefore $f(u_0)=-\gamma$ finishing the proof of (3).

(3) $\Rightarrow$ (1): let $x\in\Delta$. We have that

$$f^{\vee}(x)=\inf_{u\in N_{\mathbb{R}}}\langle x,u\rangle-f(u).$$

Hence, the inequality in (3) implies that $f^{\vee}(x)\geqslant\langle x,u_0\rangle+\gamma$ and, on the other hand, $f^{\vee}(x)\leqslant\langle x,u_0\rangle-f(u_0)=\langle x,u_0\rangle+\gamma$, which implies the statement. □

Recall that $H_{\mathbb{F}}\subset\bigoplus_{w\in\mathfrak{M}_{\mathbb{F}}}N_{\mathbb{R}}$ is the hyperplane defined in (2.6).

Proposition 4.6. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X such that D is big.*

(1) The equality

$$\mu^{\mathrm{ess}}_{\overline{D}}(X)=\frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(n+1)\,\mathrm{vol}(D)}$$

holds if and only if there exist real numbers $(\gamma_v)_v\in\bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}}\mathbb{R}$, and vectors $(u_v)_v\in H_{\mathbb{K}}\subset\bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}}N_{\mathbb{R}}$, indexed by the set of places of $\mathbb{K}$, such that

(a) $\psi_{\overline{D},v}(u_v)=-\gamma_v$, for all $v\in\mathfrak{M}_{\mathbb{K}}$ and

(b) $\psi_{\overline{D},v}(u)\leqslant\operatorname{conc}(\Psi_D)(u-u_v)-\gamma_v$ for all $v\in\mathfrak{M}_{\mathbb{K}}$.

(2) If $\overline{D}$ is big, then the equality

$$\mu^{\mathrm{ess}}_{\overline{D}}(X)=\frac{\widehat{\mathrm{vol}}(\overline{D})}{(n+1)\,\mathrm{vol}(D)}$$

holds if and only if there exist $(\gamma_v)_v \in \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} \mathbb{R}$ *and* $(u_v)_v \in H_{\mathbb{K}} \subset \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} N_{\mathbb{R}}$ *such that*

(a) $\sum_v n_v\gamma_v > 0$,

(b) $\psi_{\overline{D},v}(u_v) = -\gamma_v$, *for all* $v \in \mathfrak{M}_{\mathbb{K}}$ *and*

(c) $\psi_{\overline{D},v}(u) \leqslant \operatorname{conc}(\Psi_D)(u-u_v) - \gamma_v$ *for all* $v \in \mathfrak{M}_{\mathbb{K}}$.

Proof. — For short we write $\Delta = \Delta_D$. We first prove (1). By (4.4), the equality for the essential minimum holds if and only if, for all $x \in \Delta$,

$$\vartheta_{\overline{D}}(x) = \mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X). \tag{4.9}$$

Since $\vartheta_{\overline{D}} = \sum_v n_v\vartheta_{\overline{D},v}$, the functions $\vartheta_{\overline{D},v}$ are concave and the weights n_v are positive, it follows that all the functions $\vartheta_{\overline{D},v}$ are affine and their linear parts add to zero. Hence, (4.9) holds if and only if there exists a collection of real numbers $\{\gamma_v\}_v$, with $\gamma_v = 0$ for all but a finite number of v and $(u_v)_v \in H_{\mathbb{K}}$ such that, for all $x \in \Delta$,

$$\vartheta_{\overline{D},v}(x) = \langle u_v, x\rangle + \gamma_v. \tag{4.10}$$

By [10, Proposition 4.16(1)], the functions $|\psi_{\overline{D},v} - \Psi_D|$ are bounded. Therefore, Lemma 4.5 implies that (4.10) is equivalent to the conditions (1a) and (1b), since $\mu^{\mathrm{ess}}_{\overline{D}}(X) = \vartheta_{\overline{D}} = \sum_v n_v\gamma_v$.

The proof of (2) is similar, but using equation (4.5) and observing that Corollary 3.10(2) implies the extra condition (2a). □

Proposition 4.6 also gives a criterion for when the right inequality in Theorem 4.1 is an equality.

Corollary 4.7. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X such that D is big. Then the equality*

$$\frac{\mathrm{h}_{\overline{D}}(X)}{\deg_D(X)} = (n+1)\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X)$$

holds if and only if there exist $(\gamma_v)_v \in \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} \mathbb{R}$ *and* $(u_v)_v \in H_{\mathbb{K}} \subset \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} N_{\mathbb{R}}$ *such that, for* $v \in \mathfrak{M}_{\mathbb{K}}$,

$$\psi_{\overline{D},v}(u) = \Psi_D(u-u_v) - \gamma_v \quad \textit{for all } u \in N_{\mathbb{R}}.$$

Proof. — Since $\overline{D}$ is assumed to be semipositive, we have that $\mathrm{h}_{\overline{D}}(X) = \widehat{\mathrm{vol}}_{\chi}(\overline{D})$ and all the functions $\psi_{\overline{D},v}$ are concave. Thus the corollary follows from Proposition 4.6(1) and Lemma 4.5. □

Remark 4.8. — If a metrized $\mathbb{R}$-divisor $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}_{\mathbb{K}}})$ satisfies the equivalent conditions of Corollary 4.7, then its metric is very close to the canonical metric. For instance, if there is an element $t \in \mathbb{T}(\mathbb{K})$ such that $\mathrm{val}_v(t) = u_v$ then

$$\|\cdot\|_v = \mathrm{e}^{-\gamma_v} t^* \|\cdot\|_{\mathrm{can},v}.$$

5. Examples

The previous results allow us to compute the successive minima of several examples. The difficulty of the computations increases with the number of places where the metric differs from the canonical one. The following subsections are ordered increasingly according to this level of difficulty.

5.1. Canonical metric

As a first example, we show that the essential minimum of a toric variety with respect to a pseudo-effective toric $\mathbb{R}$-divisor equipped with the canonical metric at all the places as in Example 3.5, is zero.

Proposition 5.1. — *Let X be a proper toric variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor with the canonical metric. Then*

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \begin{cases} 0 & \text{if } D \text{ is pseudo-effective,} \\ -\infty & \text{otherwise.} \end{cases}$$

Moreover, if D is nef, then $\mu^{i}_{\overline{D}}(X) = 0$ for $i = 1, \dots, n+1$.

Proof. — Since the metric of $\overline{D}$ is the canonical one, we have that, for all $v \in \mathfrak{M}_{\mathbb{K}}$, the local roof function $\vartheta_{\overline{D},v}$ is zero on Δ_D, and so the global roof function $\vartheta_{\overline{D}}$ is also zero on Δ_D. The result then follows from Corollary 3.9, since $\Delta_D \neq \emptyset$ if and only if D is pseudo-effective.

If D is nef, the result about the successive minima follows similarly from Theorem 3.14(2). □

5.2. Weighted L^p-metrics on toric varieties.

In the next three subsections we consider the case when only one metric (the Archimedean one in $\mathbb{Q}$) differs from the canonical one. This will allow us to use Corollary 3.11(2).

We introduce a general family of Archimedean metrics on toric varieties. To this end, let X be a proper toric variety of dimension n over $\mathbb{Q}$, with fan Σ, D a nef toric divisor on X and $\Delta = \Delta_D$ the polytope associated to D. The support function associated to D is the support function of Δ. It is given, for $u \in N_{\mathbb{R}}$, by

$$\Psi(u) = \min_{m\in\Delta\cap M} \langle m, u\rangle = \min_{m\in\mathcal{F}(\Delta)^0} \langle m, u\rangle. \tag{5.1}$$

Let $\boldsymbol{\alpha} = (\alpha_m)_{m\in M\cap\Delta}$ be a collection of non-negative real numbers such that, if m is a vertex of Δ, then $\alpha_m > 0$. Let $\Lambda > 0$ be a real number. We consider the metric on $\mathcal{O}(D)$ over $X_0(\mathbb{C})$ given, for $p \in X_0(\mathbb{C})$, by

$$\|s_D(p)\|_{\Lambda,\boldsymbol{\alpha}} = \left(\sum_{m\in\Delta\cap M} \alpha_m |\chi^m(p)|^{\Lambda}\right)^{\frac{-1}{\Lambda}}.$$

The function associated to this metric, $\psi_{\Lambda,\boldsymbol{\alpha}}\colon N_{\mathbb{R}} \to \mathbb{R}$, is given by

$$\psi_{\Lambda,\boldsymbol{\alpha}}(u) = \frac{-1}{\Lambda} \log\left(\sum_{m\in\Delta\cap M} \alpha_m \mathrm{e}^{-\Lambda\langle m,u\rangle}\right).$$

Proposition 5.2. — *The function $\psi_{\Lambda,\boldsymbol{\alpha}}$ is concave and $|\psi_{\Lambda,\boldsymbol{\alpha}} - \Psi|$ is bounded. Therefore, the metric $\|\cdot\|_{\Lambda,\boldsymbol{\alpha}}$ extends to a continuous semipositive metric on $\mathcal{O}(D)$ over $X(\mathbb{C})$.*

Proof. — Each function $\alpha_m \mathrm{e}^{-\Lambda\langle m,u\rangle}$ is log-convex. Since sums of log-convex functions are log-convex [8, § 3.5.2], we deduce that $\psi_{\Lambda,\boldsymbol{\alpha}}$ is concave.

For the second statement we first observe that

$$\begin{aligned}\min_{m\in\mathcal{F}(\Delta)^0} \alpha_m \max_{m\in\mathcal{F}(\Delta)^0} \mathrm{e}^{-\Lambda\langle m,u\rangle} &\leqslant \sum_{m\in\Delta\cap M} \alpha_m \mathrm{e}^{-\Lambda\langle m,u\rangle}\\ &\leqslant \#(\Delta\cap M) \max_{m\in\Delta\cap M} \alpha_m \max_{m\in\Delta\cap M} \mathrm{e}^{-\Lambda\langle m,u\rangle}.\end{aligned}$$

Using the equality (5.1), we deduce that $|\psi_{\Lambda,\boldsymbol{\alpha}} - \Psi|$ is bounded.

The last statement follows then from [11, Theorem 4.8.1]. □

Let $\overline{D}$ be the metrized divisor given by D, the metric $\|\cdot\|_{\Lambda,\boldsymbol{\alpha}}$ at the Archimedean place and the canonical metric at the non-Archimedean places. Hence, the adelic family of functions associated to $\overline{D}$ is given by $\psi_{\Lambda,\boldsymbol{\alpha}}$ at the Archimedean place and by Ψ at the non-Archimedean places.

Example 5.3. — When Δ is the standard simplex, the toric variety is the projective space and the divisor is the hyperplane at infinity. When $\Lambda = 2$ and $\alpha_m = 1$ for all $m \in \Delta\cap M = \mathcal{F}(\Delta)^0$ we recover the Fubini-Study metric. When $\Lambda = 2$ and α_m are arbitrary positive numbers, we recover the case of the weighted Fubini-Study metric as in [10, Example 6.5]. For general Λ we obtain weighted versions of the L^p metric.

Thus the metrics we are considering in this section are the natural generalization to arbitrary proper toric varieties over $\mathbb{Q}$ of the weighted Fubini-Study metric and weighted L^p-metric. In fact, they are the inverse images of the weighted Fubini-Study and weighted L^p-metrics on the projective space by a suitable toric morphism.

We first compute the absolute minima of the orbits of X.

Proposition 5.4. — *Let $\sigma \in \Sigma$ and $F_\sigma \subset \Delta$ the corresponding face. Then*

$$\mu^{\mathrm{abs}}(O(\sigma)) = \frac{1}{\Lambda} \log\Big(\sum_{m \in F_\sigma \cap M} \alpha_m \Big).$$

Proof. — Let $N(\sigma) = N/(\mathbb{R}\sigma \cap N)$ and $M(\sigma) \subset M$ be the dual lattice. Choose $m_0 \in F_\sigma \cap M$. The divisor $D' = D + \operatorname{div}(\chi^{m_0})$ intersects properly the closure of the orbit $V(\sigma) = \overline{O(\sigma)}$. The polytope of $D'|_{V(\sigma)}$ is $F_\sigma - m_0 \subset M(\sigma)_\mathbb{R}$. The metric of D induces a metric on $D'|_{V(\sigma)}$. By [11, Corollary 4.3.18] at every non-Archimedean place the induced metric is the canonical metric. Let $\pi_\sigma \colon N_\mathbb{R} \to N(\sigma)_\mathbb{R}$ be the projection. By [11, Proposition 4.8.9], the function associated to the metric on the Archimedean place is given, for $v \in N(\sigma)_\mathbb{R}$, by

$$\psi(v) = \sup_{u \in \pi_\sigma^{-1}(v)} \frac{-1}{\Lambda} \log\Bigg(\sum_{m \in \Delta \cap M} \alpha_m \mathrm{e}^{-\Lambda \langle m - m_0, u\rangle} \Bigg).$$

Fix $v \in M(\sigma)_\mathbb{R}$ and $u \in \pi_\sigma^{-1}(v)$. If $m \in F_\sigma$, then $\langle m - m_0, u\rangle$ does not depend on the choice of u and agrees with $\langle m - m_0, v\rangle$, when we consider $m - m_0 \in M(\sigma)$. If $m \notin F_\sigma$, we choose u_0 in the relative interior of σ. Then

$$\lim_{\lambda \to \infty} \langle m - m_0, u + \lambda u_0\rangle = \infty.$$

Hence, we deduce

$$\psi(v) = \frac{-1}{\Lambda} \log\Bigg(\sum_{m \in F_\sigma \cap M} \alpha_m \mathrm{e}^{-\Lambda \langle m - m_0, v\rangle} \Bigg).$$

By Corollary 3.11,

$$\mu^{\mathrm{abs}}(O(\sigma)) = -\psi(0) = \frac{1}{\Lambda} \log\Bigg(\sum_{m \in F_\sigma \cap M} \alpha_m \Bigg).$$

□

We can compute now the successive minima of X with respect to $\overline{D}$.

Theorem 5.5. — *Let notation be as above. Then, for $i = 1, \dots, n+1$,*

$$\mu^i_{\overline{D}}(X) = \min_{\sigma \in \Sigma^{i-1}} \frac{1}{\Lambda} \log\Big(\sum_{m \in F_\sigma \cap M} \alpha_m \Big).$$

If furthermore D is ample, then

$$\mu^{i}_{\overline{D}}(X) = \min_{F\in\mathcal{F}(\Delta)^{n-i+1}} \frac{1}{\Lambda}\log\Big(\sum_{m\in F\cap M}\alpha_m\Big).$$

Proof. — The first part follows directly from Proposition 5.4, Lemma 3.8(2) and the observation that, if $\sigma \subset \tau$, then $F_\tau \subset F_\sigma$. The second statement follows from the first and the fact that, when D is ample, the correspondence between cones of Σ and faces of Δ gives a bijection between cones of dimension $i-1$ and faces of dimension $n-i-1$. □

The example below and those in § 5.5 and 5.6 share a common setting that we summarize here.

Setting 5.6. — Let $\mathbb{K}$ be a global field, $\mathbb{P}^r$ the projective space of dimension r over $\mathbb{K}$ and H the divisor corresponding to the hyperplane at infinity. Then $\mathbb{P}^r$ is a toric variety and H is an ample toric divisor.

Let $e_1,\dots,e_r$ be the standard basis of $(\mathbb{R}^r)^\vee = \mathbb{R}^r$ and set also $e_0 = 0$. The polytope associated to H is the standard simplex of $\mathbb{R}^r$:

$$\Delta^r = \mathrm{conv}(e_0,\dots,e_r).$$

The toric divisor H corresponds to the support function of this polytope $\Psi_{\Delta^r}\colon \mathbb{R}^r \to \mathbb{R}$, that is

$$\Psi_{\Delta^r}(u_1,\dots,u_r) = \min(0,u_1,\dots,u_r).$$

Let N be a lattice of rank n and M the dual lattice. Let $\iota\colon N \hookrightarrow \mathbb{Z}^r$ be an injective linear map. We set $m_j = \iota^\vee e_j \in M$ for the j-th coordinate of ι, $j = 1,\dots,r$, and also $m_0 = \iota^\vee e_0 = 0$. Let $p = (p_0 : \cdots : p_r) \in \mathbb{P}^r_0(\mathbb{K}) \simeq (\mathbb{K}^\times)^r$ be a rational point and consider the monomial map $\varphi_{p,\iota}\colon \mathbb{T} \to \mathbb{P}^r$ given, for $t \in \mathbb{T}$, by

$$\varphi_{p,\iota}(t) = (p_0\chi^{m_0}(t) : \cdots : p_r\chi^{m_r}(t)).$$

The image $\mathrm{im}(\varphi_{p,\iota})$ is the translate of a subtorus of the open orbit $\mathbb{P}^r_0 \simeq \mathbb{G}^r_m$ by the point p. We set Y for its closure in $\mathbb{P}^r$.

Let Σ be the complete fan on $N_\mathbb{R}$ induced by ι and Σ_{Δ^r}, and set $\Psi = \iota^*\Psi_{\Delta^r}$. We denote by X and D the proper toric variety over $\mathbb{K}$ and the toric Cartier divisor on X associated to this data. Set $\Delta = \mathrm{conv}(m_0,\dots,m_r) \subset M_\mathbb{R}$. We can verify that Σ coincides with the normal fan of Δ and that Ψ is the support function of this polytope. In particular, Ψ is strictly concave on Σ, the divisor D is ample, and $\Delta_D = \Delta$.

Therefore, the monomial map $\varphi_{p,\iota}$ extends to a toric morphism $X \to \mathbb{P}^r$ that we denote also by $\varphi_{p,\iota}$ as in [11, (3.2.3)]. Let Y denote the image of $\varphi_{p,\iota}$. If $\iota(N)$ is a saturated sublattice of $\mathbb{Z}^r$, then X is the normalization of Y. In general, the map $X \to Y$ is finite and its degree is given by the index of the $\mathbb{Z}$-module $\iota(N)$ in its saturation. The definition of Ψ implies that $D = \varphi_{p,\iota}^* H$.

Example 5.7. — We place ourselves in the Setting 5.6 with $\mathbb{K} = \mathbb{Q}$ and $p = (1 : \cdots : 1)$ the distinguished point of the principal orbit of $\mathbb{P}^r$. Thus we consider the projective space $\mathbb{P}^r$ as a toric variety. We equip the divisor at infinity H with the Fubini-Study metric at the Archimedean place and the canonical metric at the non-Archimedean places. We denote $\overline{H}^{\mathrm{FS}}$ the obtained metrized divisor. As in Example 5.3 this corresponds to the standard simplex, $\Lambda = 2$ and $\alpha_m = 1$. Thus Theorem 5.5 implies

$$\mu^i_{\overline{H}^{\mathrm{FS}}}(\mathbb{P}^r) = \frac{1}{2}\log(r+2-i).$$

We consider now the metrized divisor on X given by

$$\overline{D}^{\mathrm{FS}} = \varphi_{p,\iota}^* \overline{H}^{\mathrm{FS}}.$$

Then, since D is ample and the map $X \to Y$ is finite, by Theorem 5.5 and Proposition 3.4(2),

$$\mu^i_{\overline{D}^{\mathrm{FS}}}(X) = \mu^i_{\overline{H}^{\mathrm{FS}}}(Y) = \min_{F \in \mathcal{F}(\Delta)^{n-i+1}} \frac{1}{2}\log(\#\{j \mid m_j \in F\}).$$

Hence we recover the computation of the successive minima of subtori with respect to the Fubini-Study metric in [25].

As an illustration, we consider the quadric $Q \subset \mathbb{P}^3$ defined as the image of the monomial map

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^3, \quad (t_0, t_1, t_2) \longmapsto (t_0 : t_0t_1 : t_0t_2 : t_1t_2).$$

The polytope Δ is the unit square $[0,1]^2$. Considering the lattice points in its different faces, we deduce that

$$\mu^1_{\overline{H}^{\mathrm{FS}}}(Q) = \log(2), \quad \mu^2_{\overline{H}^{\mathrm{FS}}}(Q) = \frac{1}{2}\log(2), \quad \mu^3_{\overline{H}^{\mathrm{FS}}}(Q) = 0.$$

5.3. Weighted projective spaces

Let $\Delta \subset M_{\mathbb{R}}$ be a lattice simplex of dimension n and (X, D) the associated polarized toric variety over $\mathbb{Q}$. Let $u_0, \dots, u_n$ be a set of vectors of $N_{\mathbb{R}}$, orthogonal to the faces of Δ and pointing inwards. The variety X is a weighted projective space if and only if the primitive vectors colinear to $u_0, \dots, u_n$ generate the lattice N while, for a general lattice simplex Δ, the toric variety X is a fake weighted projective space, see [9].

In this section we are going to consider a family of Archimedean metrics on this kind of polarized toric varieties. To this end choose a system of affine functions on $M_{\mathbb{R}}$, $\ell_i(x) = \langle u_i, x\rangle - \lambda_i$, $i = 0, \dots, n$, such that

$$\Delta = \{x \in M_{\mathbb{R}} \mid \ell_i(x) \geqslant 0, i = 0, \dots, n\}.$$

Let c_i, $i = 0, \dots, n$ be a collection of positive real numbers such that $\sum_{i=0}^n c_i u_i = 0$. We consider the function on Δ given by

$$\vartheta(x) := -\sum_{i=0}^n c_i \ell_i(x) \log(\ell_i(x)).$$

This function is concave [10, Lemma 6.2.1(1)]. We endow D with the canonical metric at all the finite places of $\mathbb{Q}$ and with the metric associated to ϑ under the correspondence of [11, Theorem 4.8.1(2)] at the Archimedean one. This is a particular case of the metrics associated to polytopes described in [11, § 6.2].

Let s_D be the toric section of $\mathcal{O}(D)$, $m_0, \dots, m_n$ the vertices of Δ, ordered in such a way that $\ell_i(m_i) > 0$, and $\Lambda := -\big(\sum_{i=0}^n c_i \lambda_i\big)^{-1}$. Note that $\Lambda > 0$ because $-\sum_{i=0}^n c_i\lambda_i = \sum_{i=0}^n c_i \ell_i(x) > 0$.

The next proposition shows that the metrics considered in this section are a particular case of the metrics considered in the previous section.

Proposition 5.8. — *The Legendre-Fenchel dual of ϑ is the concave function*

$$\psi(u) = \frac{-1}{\Lambda} \log\left(\sum_{i=0}^n \Lambda c_i \mathrm{e}^{-\Lambda\langle m_i, u\rangle}\right).$$

Therefore the metric at the Archimedan place is given, for $p \in X_0(\mathbb{C})$, by

$$\|s_D(p)\|_\infty = \left(\sum_{i=0}^n \Lambda c_i |\chi^{m_i}(p)|^{\Lambda}\right)^{\frac{-1}{\Lambda}}.$$

Proof. — We consider first the case of the simplex standard Δ^n and the concave function

$$\vartheta_0(x) = \frac{-1}{\Lambda}\sum_{i=0}^n x_i \log(x_i/\Lambda c_i),$$

where we write $x_0 = 1 - \sum_{i=1}^n x_i$. Arguing as in [11, Example 2.4.3], one checks that

$$\psi_0(u) := \vartheta_0^\vee(u) = \frac{-1}{\Lambda}\log\left(\sum_{i=0}^n \Lambda c_i \mathrm{e}^{-\Lambda u_i}\right),$$

where $u = (u_1, \dots, u_n)$ and $u_0 = 0$.

We now consider the function $\varphi\colon M_{\mathbb{R}} \to \mathbb{R}^n$ given by

$$\varphi(x) = \Big(\frac{\ell_1(x)}{\ell_1(m_1)}, \dots, \frac{\ell_n(x)}{\ell_n(m_n)}\Big).$$

This affine function sends Δ to the standard simplex. Note that, by the definition of c_i and Λ, we have

$$\ell_i(m_i) = \frac{1}{\Lambda c_i} \quad \text{and} \quad \sum_{i=0}^{n} \frac{\ell_i(x)}{\ell_i(m_i)} = 1.$$

Using these relations one can verify that $\vartheta = \varphi^*\vartheta_0$. We write $\varphi(x) = H(x) + a$, where H is a linear isomorphism and $a \in \mathbb{R}^n$. Then, by [11, Proposition 2.3.8(2)],

$$\psi(u) = (H^\vee)_*(\psi_0 - a)(u) = \psi_0((H^\vee)^{-1}u) - \langle H^{-1}a, u\rangle. \tag{5.2}$$

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and put $e_0 = 0$. Since φ^{-1} sends e_i to m_i, we deduce that

$$(H^\vee)^{-1}u = (\langle m_1 - m_0, u\rangle, \dots, \langle m_n - m_0, u\rangle)$$

and that $H^{-1}a = m_0$. Substituting this in equation (5.2) we obtain the first statement of the proposition. The second statement follows directly from the first. □

The faces of Δ are in one-to-one correspondence with the nonempty subsets $I \subset \{0, \dots, n\}$ by the formula

$$F_I = \{x \in \Delta \mid \ell_j(x) = 0, j \notin I\}.$$

Therefore, Proposition 5.8 and Theorem 5.5 imply that the successive minima of X are given by

$$\mathrm{\mu}^i_{\overline{D}}(X) = \min_{\substack{I \subset \{0,\dots,n\} \\ \#I = n-i+2}} \left(\frac{1}{\Lambda} \log\Big(\sum_{j\in I} \Lambda c_j\Big)\right), \quad i = 1, \dots, n+1.$$

In contrast with the previous example, for the metrics of this section we can also compute explicitly the height of X with respect to D [11, (6.2.4)]:

$$\frac{\mathrm{h}_{\overline{D}}(X)}{\deg_D(X)} = \frac{n+1}{\Lambda} \sum_{j=2}^{n+1} \frac{1}{j} + \frac{1}{\Lambda} \sum_{i=0}^{n} \log(\Lambda c_i).$$

5.4. Toric bundles

In this section, we compute the successive minima of the toric bundles that we considered in [11, § 7.2]. Let $n \geqslant 0$ and write $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Q}}$ for short. Let $a_r \geqslant \cdots \geqslant a_0 \geqslant 1$ be integers, consider the bundle $\mathbb{P}(E) \to \mathbb{P}^n$ of hyperplanes of the vector bundle

$$E = \mathcal{O}(a_0) \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) \longrightarrow \mathbb{P}^n,$$

where $\mathcal{O}(a_j)$ denotes the a_j-th power of the universal line bundle of $\mathbb{P}^n$. This bundle is a smooth toric variety over $\mathbb{Q}$ of dimension $n + r$.

We consider its universal line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, that is the line bundle corresponding to the Cartier divisor $D := a_0 D_0 + D_1$, where D_0 denotes the inverse image in $\mathbb{P}(E)$ of the hyperplane at infinity of $\mathbb{P}^n$ and $D_1 = \mathbb{P}(0 \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r))$. It is an ample Cartier divisor.

As explained in [11, § 7.2], there is a standard splitting $N_{\mathbb{R}} = \mathbb{R}^{n+r}$. This splitting gives us coordinates $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_r)$ on $M_{\mathbb{R}} = \mathbb{R}^{n+r}$. We set $y_0 = 1-\sum_{i=1}^{r} y_i$, $L(y) = \sum_{j=0}^{r} a_j y_j$ and $x_0 = L(y)-\sum_{i=1}^{n} x_i$. With this notation, the polytope associated to D is

$$\Delta_D := \left\{(x, y) \in \mathbb{R}^{n+r} \mid x_0, \dots, x_n, y_0, \dots, y_r \geqslant 0\right\}.$$

At the Archimedean place, we equip D with the smooth metric induced by the Fubini-Study metric in each summand of E. If we denote by s_D the toric section associated to D, this metric is given, for $(z, w) \in (\mathbb{C}^\times)^{n+r} \simeq \mathbb{P}(E)_0(\mathbb{C})$, by

$$\|s_D(z, w)\|_\infty = \left(\sum_{j=0}^{r} |w_j|^2 \left(\sum_{i=0}^{n} |z_i|^2\right)^{a_j}\right)^{-\frac{1}{2}} \tag{5.3}$$

with $w_0 = z_0 = 1$. We also equip D with the canonical metric at the non-Archimedean places and we denote by $\overline{D}$ the obtained semipositive metrized divisor. Clearly,

$$\|s_D(z, w)\|_\infty = \left(\sum_{m \in \Delta_D \cap M} \alpha_m |\chi^m(z, w)|^2\right)^{-\frac{1}{2}} \tag{5.4}$$

for certain weights $\alpha_m \in \mathbb{R}_{\geqslant 0}$. Hence, this is again a particular case of the metrics considered in § 5.2.

Proposition 5.9. — *With the previous notation*

$$\mu^{\mathrm{ess}}_{\overline{D}}(\mathbb{P}(E)) = \frac{1}{2} \log\left((n+1)^{a_0} + \cdots + (n+1)^{a_r}\right).$$

Proof. — By Theorem 5.5, we deduce from (5.4) that

$$\mu_{D}^{\mathrm{ess}}(\mathbb{P}(E)) = \frac{1}{2}\log\Big(\sum_{m\in\Delta_D}\alpha_m\Big).$$

To compute the sum inside the logarithm, it is enough to evaluate the expression for $\|s_D(z,w)\|_{\infty}^{-2}$ given by (5.3) at $w_j = z_i = 1$ for all j, i, which gives the stated formula. □

PROPOSITION 5.10. — *Let $1 \leqslant i \leqslant n+r+1$. Then*

$$\mu_{D}^{i}(\mathbb{P}(E)) = \min_{\max(0,i-r-1)\leqslant \ell\leqslant \min(i-1,n)}\left(\frac{1}{2}\log\Big(\sum_{j=0}^{r+1-i+\ell}(n+1-\ell)^{a_j}\Big)\right).$$

Proof. — By Theorem 5.5

$$\mu_{D}^{i}(\mathbb{P}(E)) = \min_{F\in\mathcal{F}(\Delta_D)^{n+r-i+1}}\frac{1}{2}\log\Big(\sum_{m\in F\cap M}\alpha_m\Big), \tag{5.5}$$

where the weights α_m in (5.4) are defined by the equation

$$\sum_{j=0}^{r}|w_j|^2\left(\sum_{i=0}^{n}|z_i|^2\right)^{a_j} = \sum_{m\in\Delta_D\cap M}\alpha_m|\chi^m(z,w)|^2 \tag{5.6}$$

with $z_0 = w_0 = 1$. In order to compute $\sum_{m\in F\cap M}\alpha_m$ easily without developing equation (5.6) we use the following trick. Let

$$m = (x_1,\dots,x_n,y_1,\dots,y_r)\in\Delta_D\cap M,$$

as before we put $y_0 = 1-\sum_{j=1}^{r}y_j$ and $x_0 = L(y)-\sum_{i=1}^{n}x_i$ and write

$$\chi_0^m(z_0,\dots,z_n,w_0,\dots,w_r) = \prod_{i=0}^{n}z_i^{x_i}\prod_{j=0}^{r}w_j^{y_j}.$$

We claim that

$$\sum_{j=0}^{r}|w_j|^2\left(\sum_{i=0}^{n}|z_i|^2\right)^{a_j} = \sum_{m\in\Delta_D\cap M}\alpha_m|\chi_0^m(z,w)|^2 \tag{5.7}$$

for all $(z_0,\dots,z_n,w_0,\dots,w_r)\in\mathbb{C}^{n+r+2}$. We consider the bigrading that gives z_i bidegree $(1,0)$ and w_j bidegree $(-a_j,1)$. Then both sides of equation (5.7) are bihomogeneous of bidegree $(0,1)$ and they agree whenever $z_0 = w_0 = 1$. Therefore they agree on $\mathbb{C}^{n+r+2}$.

The faces of Δ_D of dimension $n+r-h$ are the slices obtained cutting Δ_D by hyperplanes $x_i = 0$, $i\in I$ and $y_j = 0$, $j\in J$, with $I\subsetneq\{0,\dots,n\}$, $J\subsetneq\{0,\dots,r\}$ and $\#I+\#J = h$. We denote $F_{I,J}$ such a face. Consider the point $p_{I,J} \in \mathbb{C}^{n+r+2}$ given by $z_i = 0$ if $i \in I$, $z_i = 1$ if $i \notin I$, $w_j = 0$ if $j \in J$, $w_j = 1$ if $j \notin J$. This point satisfies

$$\chi_0^m(p_{I,J}) = \begin{cases} 1 & \text{if } m \in F_{I,J} \cap M, \\ 0 & \text{if } m \in (\Delta_D \setminus F_{I,J}) \cap M. \end{cases}$$

Evaluating (5.7) at the point $p_{I,J}$ we obtain

$$\sum_{j \notin J} (n+1-\#I)^{a_j} = \sum_{m \in F_{I,J} \cap M} \alpha_m.$$

Thus, by (5.5), $\mu^i_{\overline{D}}(\mathbb{P}(E))$ is the minimum of $\frac{1}{2}\log\left(\sum_{j\notin J}(n+1-\#I)^{a_j}\right)$ over all I, J satisfying $\#I + \#J = i-1$. We obtain the result by writing $\ell = \#I$ and taking into account that we ordered the a_j so that $a_r \geqslant \cdots \geqslant a_0 \geqslant 1$. $\square$

In particular, when $i = 1$ then ℓ necessarily takes the value 0 and we recover Proposition 5.9. Whereas for $n+1 \leqslant i \leqslant n+r+1$ it can be shown that the minimum is attained with $\ell = n$. For this value the sum inside the logarithm equals $n+r+2-i$ and we get

$$\mu^i_{\overline{D}}(\mathbb{P}(E)) = \frac{1}{2}\log(n+r+2-i) \text{ for } i = n+1, \dots, n+r+1.$$

Remarkably, as in § 5.3, in this example the roof function and the height of $\mathbb{P}(E)$ with respect to $\overline{D}$ are also computed, see [11, § 7.2].

Example 5.11. — The particular case $n = r = 1$ corresponds to the Hirzebruch surfaces: for $b \geqslant 0$, we have $\mathbb{F}_b = \mathbb{P}(\mathcal{O}(0) \oplus \mathcal{O}(b)) \simeq \mathbb{P}(\mathcal{O}(a_0) \oplus \mathcal{O}(a_0+b))$ for any $a_0 \geqslant 1$. Although the surface does not depend on the choice of a_0, the divisor does. We set $a_1 = a_0 + b$. Then we obtain

$$\mu^{\mathrm{ess}}_{\overline{D}}(\mathbb{F}_b) = \frac{1}{2}\log(2^{a_0} + 2^{a_1}), \quad \mu^2_{\overline{D}}(\mathbb{F}_b) = \frac{1}{2}\log(2), \quad \mu^{\mathrm{abs}}_{\overline{D}}(\mathbb{F}_b) = 0.$$

5.5. Translates of subtori with the canonical metric

In this section and the next one we study examples where more that one v-adic metric may be different from the canonical one. We place ourselves in Setting 5.6. We equip H with the canonical metric at all the places and denote $\overline{H}^{\mathrm{can}}$ the obtained toric metrized divisor. Write $\overline{D} = \varphi_{p,\iota}{}^*\overline{H}^{\mathrm{can}}$. Note that the metric induced on D is not necessarily the canonical one.

PROPOSITION 5.12. — *With the previous notation, for each $v \in \mathfrak{M}_{\mathbb{K}}$ let $\vartheta_v \colon \Delta \to \mathbb{R}$ be the function parametrizing the upper envelope of the polytope*

$$\widetilde{\Delta}_v := \operatorname{conv}((m_0, \log |p_0|_v), \dots, (m_r, \log |p_r|_v)) \subset M_{\mathbb{R}} \times \mathbb{R}$$

and set $\vartheta = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \vartheta_v$. Then, ϑ is the roof function of $\overline{D}$. In particular, for $i = 1, \dots, n+1$,

$$\mu^i_{\overline{D}}(X) = \mu^i_{\overline{D}}(Y) = \min_{F \in \mathcal{F}(\Delta)^{n-i+1}} \max_{x \in F} \vartheta(x).$$

Proof. — By [11, Example 5.1.16], the function ϑ coincides with the roof function of $\overline{D}$. Since D is ample and $\overline{D}$ is semipositive, Theorem 3.16 then gives the formula for the successive minima of X. The fact that the successive minima of X and Y coincide follows from Proposition 3.4(2). □

By Proposition 5.12, the computation of the successive minima of a translate of a subtori and of its normalization amounts to the computation of the maximum of a piecewice affine function over a polytope. This is a problem of linear programming. To do this in a concrete case, consider the polytopes $\widetilde{\Delta}_v$ and the functions ϑ_v in Proposition 5.12 and apply the following steps:

(a) for each v such that $\vartheta_v \not\equiv 0$, compute the regular subdivision Π_v of Δ given by the projection of the faces of the polytope $\widetilde{\Delta}_v$;
(b) compute a subdivision Π refining Π_v for all v. This subdivision can be constructed by intersecting all the polyhedra in the different Π_v as in [11, Definition 2.1.8];
(c) the function ϑ is affine on each polytope of Π. Hence, for each face F of Δ, the maximum $\max_{x \in F} \vartheta(x)$ is realized at a vertex of Π and to compute it we only need the values of ϑ at the finite set $F \cap \Pi^0$. Thus we obtain

$$\mu^i_{\overline{D}}(X) = \mu^i_{\overline{D}}(Y) = \min_{F \in \mathcal{F}(\Delta)^{n-i+1}} \max_{x \in F \cap \Pi^0} \vartheta(x).$$

Observe that, for each place v, the vertices of the subdivision Π_v in (a) are lattice points. If the dimension of Y is one, this implies that we can choose Π in (b) such that all its vertices are lattice points. This is the case in the example in the introduction. By contrast, in higher dimension, we may need Π to have non-lattice vertices as shown in the next example.

Example 5.13. — Consider the quadric $S \subset \mathbb{P}^3$ defined as the closure of the monomial map

$$\mathbb{T}^2 \longrightarrow \mathbb{P}^3, \quad (t_1, t_2) \longmapsto (1 : 2t_1 : 4t_2 : t_1 t_2).$$

As before let $\overline{D}$ be the restriction of the metrized divisor $\overline{H}$ to S. The corresponding v-adic roof functions are described by the diagram in Figure 5.1. These functions are the minimal concave piecewise affine functions on the square with the prescribed values at the vertices. The subdivisions Π_v are also given in the diagram.

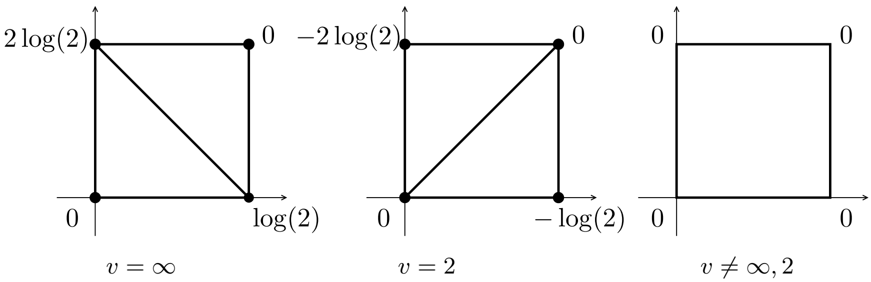

Figure 5.1. Local roof functions

The global roof function and the subdivision Π are given in Figure 5.2. From this picture, it follows that $\mu^{\mathrm{ess}}_{\overline{D}}(S) = \frac{3}{2}\log(2)$ and $\mu^{2}_{\overline{D}}(S) = \mu^{3}_{\overline{D}}(S) = 0$.

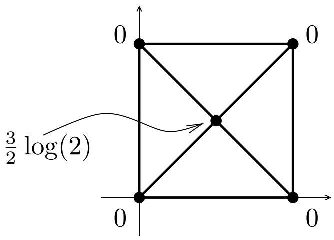

Figure 5.2. Global roof function

Remark 5.14. — The method used in this example can be applied to compute the successive minima of any toric variety over $\mathbb{K}$ with a semipositive toric metrized $\mathbb{R}$-divisor $\overline{D}$ such that D is ample and the associated functions $\psi_{\overline{D},v}$ are piecewise affine. In this case, for each v, the local roof function ϑ_v is not given by Proposition 5.12, but it is computed as the Legendre dual of $\psi_{\overline{D},v}$. Moreover, if one is only interested in the essential minimum, we can drop the ampleness and semipositiveness assumptions.

5.6. Translates of subtori with the Fubini-Study metric

We consider now the case when X is a toric variety over $\mathbb{Q}$ and $\overline{D}$ is a semipositive toric metrized $\mathbb{R}$-divisor, with D ample and such that, for every non-Archimedean place $v \in \mathfrak{M}_{\mathbb{K}}$, the function $\psi_{\overline{D},v}$ is piecewise affine and for $v = \infty$, the function $\psi_{\overline{D},v}$ is smooth. This is the case when the non-Archimedean metrics are defined by means of a model and the Archimedean metric is smooth, which is the situation classically considered in Arakelov geometry.

Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be the finite subset containing all non-Archimedean places with $\psi_{\overline{D},v} \neq \Psi_D$.

Lemma 5.15. — *With the previous notation, the essential minimum of X with respect to $\overline{D}$ is computed by applying the following steps.*

(a) *For each place $v \in S$ we compute the function $\vartheta_{\overline{D},v}$ as the Legendre-Fenchel dual of $\psi_{\overline{D},v}$.*
(b) *Set $\vartheta_S = \sum_{v\in S} \vartheta_{\overline{D},v}$ and compute its Legendre-Fenchel dual ψ_S.*
(c) *Find a value $u_0 \in N_{\mathbb{R}}$ such that*
$$\partial\psi_{\overline{D},\infty}(-u_0) \in \partial\psi_S(u_0).$$
In this condition, the left hand side is the differential of a smooth function and hence a vector, while the right hand side is the sup-differential of a concave piecewise affine function and hence is a set of vectors.
(d) *The essential minimum of X with respect to $\overline{D}$ is given by*
$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = -\psi_S(u_0) - \psi_{\overline{D},\infty}(-u_0).$$

Proof. — By Corollary 3.11, we know that
$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = -\Big(\big(\boxplus_{v\in S}\, \psi_{\overline{D},v}\big) \boxplus \psi_{\overline{D},\infty}\Big)(0).$$
By [11, Proposition 2.3.1], the sup-convolution is dual to the sum and so $\boxplus_{v\in S}\psi_{\overline{D},v} = \psi_S$. Hence
$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = -\Big(\psi_S \boxplus \psi_{\overline{D},\infty}\Big)(0) = -\sup_{u\in N_{\mathbb{R}}} \big(\psi_S(u) + \psi_{\overline{D},\infty}(-u)\big).$$
Since the stability sets of ψ_S and $\psi_{\overline{D},\infty}$ agree, by [24, Theorem 16.4], the supremum is attained at some point. By the concavity of the functions, the supremum is attained at any point u_0 satisfying the condition
$$0 \in \partial(\psi_S(u) + \psi_{\overline{D},\infty}(-u))(u_0),$$
which is equivalent to the condition given in step (c). □

We place ourselves again in Setting 5.6 with $\mathbb{K} = \mathbb{Q}$ and we equip H with the Fubini-Study metric at the Archimedean place and the canonical metric at the non-Archimedean places. We denote $\overline{H}^{\mathrm{FS}}$ the obtained toric metrized divisor and we set $\overline{D} = \varphi_{p,\iota}^{*}\overline{H}^{\mathrm{FS}}$.

In this case the pair $(X, \overline{D})$ satisfies the hypothesis of Lemma 5.15. Moreover, for each Archimedean place v, the function $\vartheta_{\overline{D},v}$ is given by the function ϑ_v in Proposition 5.12, hence step (a) is already done. For the Archimedean place the function $\psi_{\overline{D},\infty}$ is given by

$$\psi_{\overline{D},\infty}(u) = -\frac{1}{2}\log\Big(\sum_{j=0}^{r}|p_j|^2 \mathrm{e}^{-2\langle m_j, u\rangle}\Big).$$

We illustrate the recipe in Lemma 5.15 in the following examples, where $\overline{D}$ denotes the metrized divisor defined as before.

Example 5.16. — Let $C \subset \mathbb{P}^2_{\mathbb{Q}}$ be the quadric curve over $\mathbb{Q}$ given as the image of the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2, \quad (t_0 : t_1) \longmapsto \Big(t_0^2 : \frac{1}{4}\, t_0 t_1 : \frac{1}{2}\, t_1^2\Big).$$

Then, for $v \neq 2, \infty$, we have $\psi_{\overline{D},v} = \Psi_D$ and the corresponding metric agrees with the canonical metric. Moreover

$$\begin{aligned}\psi_{\overline{D},2}(u) &= \min(0, u - 2\log(2), 2u - \log(2)),\\ \psi_{\overline{D},\infty}(u) &= -\frac{1}{2}\log\Big(1 + \frac{1}{16}\mathrm{e}^{-2u} + \frac{1}{4}\mathrm{e}^{-4u}\Big).\end{aligned}$$

In this case $\psi_S = \psi_{\overline{D},2}$ and $\partial\psi_S$ is given by

$$\partial\psi_S(u) = \begin{cases} 2 & \text{if } u < -\log(2),\\ [1,2] & \text{if } u = -\log(2),\\ 1 & \text{if } -\log(2) < u < 2\log(2),\\ [0,1] & \text{if } u = 2\log(2),\\ 0 & \text{if } 2\log(2) < u.\end{cases}$$

Then, analyzing the function $\partial\psi_{\overline{D},\infty}(-u)$, we deduce that the point u_0 that satisfies the condition in step (c) belongs to the interval $-\log(2) < u < 2\log(2)$. Thus we have to solve the equation $\partial\psi_{\overline{D},\infty}(-u) = 1$, whose only solution is $u_0 = \frac{1}{2}\log(2)$. Thus

$$\mu^{\mathrm{ess}}_{\overline{D}}(C) = -\psi_{\overline{D},2}\Big(\frac{1}{2}\log(2)\Big) - \psi_{\overline{D},\infty}\Big(-\frac{1}{2}\log(2)\Big) = \frac{1}{2}\log(17).$$

Example 5.17. — Consider the quadric surface of Example 5.13, but recall that now $\overline{D}$ has the restriction of the Fubini-Study metric at the Archimedean place instead of the restriction of the canonical one.

For $v \neq 2, \infty$, the function $\psi_{\overline{D},v} = \Psi_D$. Hence $\psi_S = \psi_{\overline{D},2}$. The function ψ_S and its sup-differential is illustrated in Figure 5.3. In this figure, we see a polyhedral decomposition of the plane. The two vertices of this polyhedral decomposition are the points $(-\log(2), \log(2))$ and $(2\log(2), -2\log(2))$.

The function ψ_S is affine in each of the four maximal polyhedra and its value on each polyhedra is given in the figure. In the interior of each of these polyhedra, the sup-differential contains a single vector also given in the figure. The sup-differential at a point belonging to a non-maximal polyhedra is the convex envelope of the sup-differentials of the neighbouring maximal polyhedra. For instance,

$$\begin{aligned} \partial\psi_S(u_1,u_2) &= \operatorname{conv}((0,0),(1,0),(1,1)) &&\text{if } -\log(2) = u_1 = -u_2,\\ \partial\psi_S(u_1,u_2) &= \operatorname{conv}((0,0),(1,1)) &&\text{if } -\log(2) < u_1 = -u_2 < 2\log(2). \end{aligned}$$

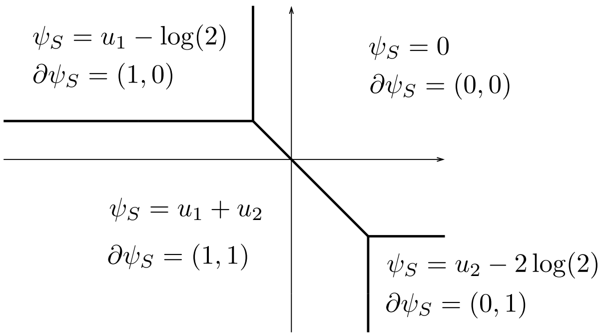

Figure 5.3. Function ψ_S and its gradient

The function $\psi_{\overline{D},\infty}$ is given by

$$\psi_{\overline{D},\infty}(u_1,u_2) = -\frac{1}{2}\log(1 + 4\mathrm{e}^{-2u_1} + 16\mathrm{e}^{-2u_2} + \mathrm{e}^{-2(u_1+u_2)}).$$

One checks that

$$0 < \frac{\partial\psi_{\overline{D},\infty}}{\partial u_1}(u_1,u_2), \frac{\partial\psi_{\overline{D},\infty}}{\partial u_2}(u_1,u_2) < 1.$$

This implies that a point u_0 satisfying the condition of step (c) belongs to the interval $-\log(2) < u_1 = -u_2 < 2\log(2)$. Thus we have to solve the equation

$$\frac{\partial\psi_{\overline{D},\infty}}{\partial u_1}(-u_1,-u_2)=\frac{\partial\psi_{\overline{D},\infty}}{\partial u_2}(-u_1,-u_2),\qquad \text{with } u_1=-u_2.$$

This equation has a single solution at the point $u_0=\frac{1}{2}(\log(2),-\log(2))$. Thus

$$\mu^{\mathrm{ess}}_{\overline{D}}(S)=-\psi_S(u_0)-\psi_{\overline{D},\infty}(-u_0)=\log(3\sqrt{2}).$$

BIBLIOGRAPHY

[1] F. Amoroso & E. Viada, "Small points on subvarieties of a torus", *Duke Math. J.* **150** (2009), no. 3, p. 407-442.

[2] M. Avendaño, T. Krick & M. Sombra, "Factoring bivariate sparse (lacunary) polynomials", *J. Complexity* **23** (2007), no. 2, p. 193-216.

[3] M. H. Baker & R. Rumely, "Equidistribution of small points, rational dynamics, and potential theory", *Ann. Inst. Fourier (Grenoble)* **56** (2006), no. 3, p. 625-688.

[4] R. Berman & S. Boucksom, "Growth of balls of holomorphic sections and energy at equilibrium", *Invent. Math.* **181** (2010), no. 2, p. 337-394.

[5] Y. Bilu, "Limit distribution of small points on algebraic tori", *Duke Math. J.* **89** (1997), no. 3, p. 465-476.

[6] E. Bombieri & W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006, xvi+652 pages.

[7] J.-B. Bost, H. Gillet & C. Soulé, "Heights of projective varieties and positive Green forms", *J. Amer. Math. Soc.* **7** (1994), no. 4, p. 903-1027.

[8] S. Boyd & L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, 2004, xiv+716 pages.

[9] W. Buczynska, *Fake weighted projective spaces*, Memoir, Warsaw Univ. (Poland), 2002, `http://arxiv.org/abs/0805.1211v1`.

[10] J. I. Burgos Gil, A. Moriwaki, P. Philippon & M. Sombra, "Arithmetic positivity on toric varieties", to appear in J. Alg. Geom., `http://arxiv.org/abs/1210.7692`.

[11] J. I. Burgos Gil, P. Philippon & M. Sombra, "Arithmetic geometry of toric varieties. Metrics, measures and heights", *Astérisque* (2014), no. 360, p. vi+222.

[12] A. Chambert-Loir, "Mesures et équidistribution sur les espaces de Berkovich", *J. Reine Angew. Math.* **595** (2006), p. 215-235.

[13] A. Chambert-Loir & A. Thuillier, "Mesures de Mahler et équidistribution logarithmique", *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 3, p. 977-1014.

[14] H. Chen, "Differentiability of the arithmetic volume function", *J. Lond. Math. Soc. (2)* **84** (2011), no. 2, p. 365-384.

[15] S. David & P. Philippon, "Minorations des hauteurs normalisées des sous-variétés des tores", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **28** (1999), no. 3, p. 489-543.

[16] P. Dèbes, "Density results for Hilbert subsets", *Indian J. Pure Appl. Math.* **30** (1999), no. 1, p. 109-127.

[17] C. Favre & J. Rivera-Letelier, "Équidistribution quantitative des points de petite hauteur sur la droite projective", *Math. Ann.* **335** (2006), no. 2, p. 311-361.

[18] W. Gubler, "Local and canonical heights of subvarieties", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **2** (2003), no. 4, p. 711-760.

[19] M. Hindry & J. H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000, An introduction, xiv+558 pages.

[20] S. Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002, xvi+914 pages.

[21] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications, xvi+576 pages.

[22] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder, xviii+571 pages.

[23] P. Philippon & M. Sombra, "Quelques aspects diophantiens des variétés toriques projectives", in *Diophantine approximation*, Dev. Math., vol. 16, SpringerWienNewYork, Vienna, 2008, p. 295-338.

[24] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970, xviii+451 pages.

[25] M. Sombra, "Minimums successifs des variétés toriques projectives", *J. Reine Angew. Math.* **586** (2005), p. 207-233.

[26] L. Szpiro, E. Ullmo & S. Zhang, "Équirépartition des petits points", *Invent. Math.* **127** (1997), no. 2, p. 337-347.

[27] A. Weil, *Basic number theory*, third ed., Springer-Verlag, New York-Berlin, 1974, Die Grundlehren der Mathematischen Wissenschaften, Band 144, xviii+325 pages.

[28] X. Yuan, "Big line bundles over arithmetic varieties", *Invent. Math.* **173** (2008), no. 3, p. 603-649.

[29] S. Zhang, "Positive line bundles on arithmetic varieties", *J. Amer. Math. Soc.* **8** (1995), no. 1, p. 187-221.

[30] ———, "Small points and adelic metrics", *J. Algebraic Geom.* **4** (1995), no. 2, p. 281-300.

Manuscrit reçu le 17 mars 2014,
accepté le 30 septembre 2014.

José Ignacio BURGOS GIL
Instituto de Ciencias Matemáticas
(CSIC-UAM-UCM-UCM3)
Calle Nicolás Cabrera 15
Campus UAB, Cantoblanco
28049 Madrid (Spain)
burgos@icmat.es

Patrice PHILIPPON
Institut de Mathématiques de Jussieu
U.M.R. 7586 du CNRS
Équipe de Théorie des Nombres.
Case 247, 4 place Jussieu
75252 Paris cedex 05 (France)
pph@math.jussieu.fr

Martín SOMBRA
ICREA & Universitat de Barcelona
Departament d'Àlgebra i Geometria.
Gran Via 585
08007 Barcelona (Spain)
sombra@ub.edu