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ISOSPECTRAL DEFORMATIONS OF THE LAGRANGIAN GRASSMANNIANS

by Jacques GASQUI & Hubert GOLDSCHMIDT

Abstract. — We study the special Lagrangian Grassmannian $SU(n)/SO(n)$, with $n \geqslant 3$, and its reduced space, the reduced Lagrangian Grassmannian X. The latter is an irreducible symmetric space of rank $n-1$ and is the quotient of the Grassmannian $SU(n)/SO(n)$ under the action of a cyclic group of isometries of order n. The main result of this paper asserts that the symmetric space X possesses non-trivial infinitesimal isospectral deformations. Thus we obtain the first example of an irreducible symmetric space of arbitrary rank $\geqslant 2$, which is both reduced and non-infinitesimally rigid. Our result may be viewed as a generalization of the construction which we had given previously for the reduced Grassmannian of 3-planes in $\mathbb{R}^6$; in fact, this space is isometric to the reduced space of $SU(4)/SO(4)$.

Résumé. — Nous étudions la grassmannienne lagrangienne spéciale $SU(n)/SO(n)$, avec $n \geqslant 3$, et son espace réduit X, qui est l'espace symétrique irréductible de rang $n-1$ quotient de $SU(n)/SO(n)$ par l'action d'un groupe cyclique d'isometries d'ordre n. Notre résultat principal est la construction de déformations infinitésimales isospectrales non triviales de X. Nous obtenons ainsi les premiers exemples en rang quelconque $\geqslant 2$ d'espaces symétriques irréductibles réduits et non infinitésimalement rigides. Notre résultat peut être vu comme une généralisation de la construction que nous avions donnée dans un précédent papier pour la grassmannienne réduite des 3-plans de $\mathbb{R}^6$, espace qui est en fait isométrique à l'espace réduit de $SU(4)/SO(4)$.

Introduction

In [2], we introduced the space $I(X)$ of infinitesimal isospectral deformations of a Riemannian symmetric space (X, g) of compact type. We were motivated by a criterion due to Guillemin [3] for the infinitesimal isospectral rigidity of such a space. Our definition of $I(X)$ involves the Radon

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transform for symmetric 2-forms, considered in [1] and defined in terms of integration over the maximal flat totally geodesic tori of X; we say that a symmetric 2-form on X satisfies the Guillemin condition if it belongs to the kernel $\mathcal{N}_2$ of this Radon transform. The space $I(X)$ is the orthogonal complement of the space of Lie derivatives of the metric g in $\mathcal{N}_2$. If $I(X)$ vanishes, or equivalently if every symmetric 2-form on X satisfying the Guillemin condition is a Lie derivative of the metric, we say that (X, g) is rigid in the sense of Guillemin; in this case, a isospectral deformation of the metric g is trivial to first-order.

We shall henceforth suppose that X is irreducible. The reduced space of X constructed in [2] is a symmetric space covered by X and which is not the cover of another symmetric space. We say that X is reduced if it is equal to its reduced space. If X is not reduced, in [2] we proved that the space $I(X)$ does not vanish. The fundamental problem concerning infinitesimal isospectral deformations for our class of symmetric spaces may be formulated as follows: determine the space of infinitesimal isospectral deformations of an irreducible reduced space. We refer the reader to the introduction of the paper [2] for more details concerning the definitions and results mentioned above.

The Grassmannian $G^{\mathbb{K}}_{m,n}$ of m-planes in $\mathbb{K}^{m+n}$, where $\mathbb{K}$ is a division algebra over $\mathbb{R}$, is a symmetric space of compact type which is both irreducible and reduced if and only if $m \neq n$ and $m, n \geqslant 1$; if this last condition holds, we know that this Grassmannian is rigid (see [1]).

In [2], we gave the first example of an irreducible reduced symmetric space which is not rigid in the sense of Guillemin; it is the reduced space $\bar{G}^{\mathbb{R}}_{3,3}$ of the Grassmannian $G^{\mathbb{R}}_{3,3}$ of 3-planes in $\mathbb{R}^6$.

Let $n \geqslant 3$ be a given integer and consider the group $G = SU(n)$ and its subgroup $K = SO(n)$. The special Lagrangian Grassmannian $X = G/K$ is an irreducible symmetric space. Its reduced space Y, which we call the reduced Lagrangian Grassmannian, is the quotient of X by the action of a cyclic group Σ of order n consisting of isometries which commute with the action of G on X; it is an irreducible symmetric space of rank $n-1$. We prove that this reduced space Y is not rigid in the sense of Guillemin. Thus we obtain the first example of an irreducible symmetric space of arbitrary rank $\geqslant 2$, which is both reduced and non-infinitesimally rigid. Since the universal cover of the Grassmannian $G^{\mathbb{R}}_{3,3}$ is isometric to the Lagrangian Grassmannian $SU(4)/SO(4)$, we recover the non-rigidity of $\bar{G}^{\mathbb{R}}_{3,3}$.

We now describe our construction of non-trivial infinitesimal isospectral deformations of the reduced Lagrangian Grassmannian Y. We consider an explicit subspace $\mathcal{F}_Y$ of the space of real-valued functions on Y which is of finite codimension and orthogonal to the subspace of constant functions, and then show that there exists an injective mapping

$$\mathcal{F}_Y \to I(Y).$$

For $p \geqslant 2$, the symmetric space X carries a natural symmetric p-form σ_p which is invariant under its group of isometries and is therefore parallel. Indeed, the tangent space of X at the coset of the identity element of G is isomorphic as a K-module to the subspace $\mathfrak{p}_0$ of the Lie algebra $\mathfrak{su}(n)$ consisting of all purely imaginary $n \times n$ matrices with trace zero. The form σ_p is induced by the G-invariant homogeneous polynomial q_p of degree p on the space $\mathfrak{p}_0$ defined by

$$q_p(A) = (-i)^p \operatorname{Tr} A^p,$$

for all $A \in \mathfrak{p}_0$. It is well-known that the algebra of all K-invariant polynomials on $\mathfrak{p}_0$ is generated by the $n-1$ algebraically independent homogeneous polynomials $\{q_p\}$, with $2 \leqslant p \leqslant n$. In fact, the form σ_2 is equal to the Riemannian metric g of X and the form σ_3 is up to a constant the only G-invariant symmetric 3-form on X (see [2, §2]). Also the form σ_p induces a G-invariant symmetric p-form $\sigma_{Y,p}$ on Y.

In [1], we introduced the Guillemin condition for symmetric forms of arbitrary degree on a symmetric space. Let σ be a symmetric p-form, with $p \geqslant 3$, on a symmetric space Z; the form σ determines a mapping $\tilde{\sigma}$ from the space of 1-forms on Z to the space of symmetric $(p-1)$-forms on Z. We now suppose that the pair (Z, σ) is equal either to (X, σ_p) or to $(Y, \sigma_{Y,p})$; then the mapping $\tilde{\sigma}$ is injective. We show that a 1-form θ on Z satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}(\theta)$ on Z satisfies the Guillemin condition. Thus if f is a real-valued function on Z, the symmetric $(p-1)$-form $\tilde{\sigma}(df)$ satisfies the Guillemin condition. Moreover if $p = 3$, the element $\tilde{\sigma}(df)$ gives rise to an element of $I(Z)$. If f is a non-zero element of $\mathcal{F}_Y$, we prove that the symmetric 2-form $\tilde{\sigma}_{Y,3}(df)$ on Y is not a Lie derivative of the metric of Y and thus this function f gives rise to a non-zero element of $I(Y)$.

The symmetric 3-form σ on the universal cover of $G^{\mathbb{R}}_{3,3}$, which we introduced in [2], can be viewed as a constant multiple of the symmetric 3-form on $SU(4)/SO(4)$ induced by σ_3, when we identify these two spaces (see [2, §11]). Our construction of infinitesimal isospectral deformations of the reduced space of $SU(4)/SO(4)$ is totally equivalent to the one given in [2] for the space $\bar{G}^{\mathbb{R}}_{3,3}$.

Harmonic analysis on the homogeneous space X of the group G plays an important role in the proof of our result concerning the isospectral deformations of Y. We consider the G-module $C^\infty(X)$ of complex-valued functions on X and its irreducible submodules. We introduce explicit functions on X which arise from K-invariant functions on the group G and show that the highest weight vectors of these irreducible submodules occur among these functions. We determine all the highest weight vectors of an isotypic component of the space of complex 1-forms on X corresponding to one of these irreducible submodules of $C^\infty(X)$, and express these vectors in terms of our family of functions. Our description allows us to say which of these highest weight vectors arise from objects defined on the quotient space Y of X. We prove that the space $\mathcal{F}'$ of functions $f \in C^\infty(X)$ for which the symmetric 2-form $\tilde{\sigma}_3(df)$ is a Lie derivative of the metric g is the sum of two irreducible G-submodules of $C^\infty(X)$. In order to demonstrate this fact, it suffices to consider the 1-forms on X corresponding to the highest weight vectors of the irreducible G-submodules of $C^\infty(X)$. The necessary verifications are carried out in §9; we require an elementary algebraic result presented in §8, a section which can be read independently of the rest of this paper. We thus obtain a subspace of $I(X)$ isomorphic to the infinite-dimensional space of real-valued functions on X orthogonal to $\mathcal{F}'$. In fact, all of the functions belonging to $\mathcal{F}'$ are induced by functions on Y; the latter form a finite-dimensional space $\mathcal{F}'_Y$ of functions on Y. Consequently, for the reduced Lagrangian Grassmannian Y, the space $\mathcal{F}_Y$ giving rise to elements of $I(Y)$ is the space of real-valued functions on Y orthogonal to $\mathcal{F}'_Y$.

Finally, we wish to point out that the only results of [2] which we require here are Lemma 1.1 and Propositions 1.2 and 10.1.

1. Riemannian manifolds

Let X be a differentiable manifold, whose tangent and cotangent bundles we denote by $T = T_X$ and $T^* = T^*_X$, respectively. We consider the space of complex-valued functions $C^\infty(X)$ (resp. real-valued functions $C^\infty_{\mathbb{R}}(X)$) on X. Let $\mathbb{R}(X)$ denote the subspace of $C^\infty_{\mathbb{R}}(X)$ consisting of the constant functions on X. Let E be a vector bundle over X; we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of E over X and by $C^\infty(E)$ the space of global sections of E over X. By $\bigotimes^k E$, $S^l E$, $\bigwedge^j E$, we shall mean the k-th tensor product, the l-th symmetric product and the j-th exterior product of the vector bundle E, respectively. We shall identify $S^k T^*$ and

$\bigwedge^k T^*$ with sub-bundles of $\bigotimes^k T^*$ as in §1, Chapter I of [1]. In particular, if $\alpha, \beta \in T^*$, the symmetric product $\alpha \cdot \beta$ is identified with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If u is a section of $S^p T^*$ over X, we consider the morphism of vector bundles

$$u^\flat : T \to S^{p-1}T^*,$$

defined by

$$(u^\flat \xi)(\eta_1, \dots, \eta_{p-1}) = u(\xi, \eta_1, \dots, \eta_{p-1}),$$

for $\xi, \eta_1, \dots, \eta_{p-1} \in T$.

Let g be a Riemannian metric on X. We denote by $g^\sharp : T^* \to T$ the inverse of the isomorphism $g^\flat : T \to T^*$. If u is a section of $S^p T^*$ over X, we consider the morphism of vector bundles

$$\tilde{u} = u^\flat \cdot g^\sharp : T^* \to S^{p-1}T^*.$$

We also consider the scalar products on the spaces $C^\infty(X)$, $C^\infty(T)$ and $C^\infty(S^2T^*)$, defined in terms of the Riemannian measure of X and the scalar products on the vector bundles T and S^2T^* induced by the metric g. We denote by $C^\infty_{\mathbb{R},0}(X)$ the orthogonal complement of the subspace $\mathbb{R}(X)$ of $C^\infty_{\mathbb{R}}(X)$.

Let ∇ be the Levi-Civita connection of (X, g). We consider the symmetrized covariant derivative

$$D^1 : \mathcal{T}^* \to S^2\mathcal{T}^*,$$

defined by

$$(D^1\theta)(\xi, \eta) = \tfrac{1}{2}((\nabla\theta)(\xi, \eta) + (\nabla\theta)(\eta, \xi)),$$

for $\theta \in \mathcal{T}^*$, $\xi, \eta \in T$. If f is a real-valued function on X, the Hessian Hess f of f is equal to $D^1 df = \nabla df$. The Killing operator

$$D_0 : \mathcal{T} \to S^2\mathcal{T}^*$$

of (X, g), which sends a vector field ξ into the Lie derivative $\mathcal{L}_\xi g$ of g along ξ of g along ξ, and the operator D^1 are related by the formula

$$\tfrac{1}{2} D_0\xi = D^1 g^\flat(\xi), \tag{1.1}$$

for $\xi \in \mathcal{T}$. We easily see that

$$D^1(f_1 df_2) = \tfrac{1}{2}\, df_1 \cdot df_2 + f_1 \,\mathrm{Hess}\, f_2, \tag{1.2}$$

for all $f_1, f_2 \in C^\infty(X)$. We also consider the divergence operator

$$\mathrm{div} : S^2\mathcal{T}^* \to \mathcal{T}^*,$$

as defined in §1, Chapter I of [1]; we recall that the formal adjoint of D_0 is equal to $2g^\sharp \cdot \mathrm{div} : S^2\mathcal{T}^* \to \mathcal{T}$. When X is compact, since the operator D_0 is elliptic, we therefore have the orthogonal decomposition

$$C^\infty(S^2T^*) = D_0C^\infty(T) \oplus \big\{h \in C^\infty(S^2T^*) \mid \mathrm{div}\; h = 0\big\} \tag{1.3}$$

given by the relation (1.11) of [1]; we denote by

$$P : C^\infty(S^2T^*) \to \big\{h \in C^\infty(S^2T^*) \mid \mathrm{div}\; h = 0\big\}$$

the projection determined by the decomposition (1.3).

We now suppose that X is a symmetric space of compact type. We say that a symmetric p-form u on X satisfies the Guillemin condition if, for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z u(\zeta, \zeta, \dots, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z. We consider the subspace $\mathcal{N}_p$ of $C^\infty(S^pT^*)$ consisting of all symmetric p-forms satisfying the Guillemin condition and recall that $D_0C^\infty(T)$ is a subspace of $\mathcal{N}_2$ (see Lemma 2.10 of [1]). We then define the space of infinitesimal isospectral deformations of g by

$$I(X) = \{h \in \mathcal{N}_2 \mid \mathrm{div}\; h = 0\}.$$

>From the decomposition (1.3), we obtain the orthogonal decomposition

$$\mathcal{N}_2 = D_0C^\infty(T) \oplus I(X); \tag{1.4}$$

moreover, the orthogonal projection of $\mathcal{N}_2$ onto $I(X)$ is equal to the restriction of the projection P to $\mathcal{N}_2$. Thus the vanishing of the space $I(X)$ is equivalent to the fact that the space X is rigid in the sense of Guillemin. Moreover if there exists a symmetric 2-form on X belonging to $\mathcal{N}_2$ which is not equal to a Lie derivative of the metric g, the space $I(X)$ does not vanish.

The connected component G of the group of isometries of X is a compact semi-simple Lie group. Let σ be a G-invariant symmetric p-form on X, with $p \geqslant 2$; clearly, σ is parallel and so we have

$$\nabla\sigma = 0.$$

The morphisms

$$\sigma^\flat : T \to S^{p-1}T^*, \qquad \tilde{\sigma} : T^* \to S^{p-1}T^*$$

induced by σ are G-equivariant; if X is irreducible and σ is non-zero, they are monomorphisms of vector bundles.

We now suppose that $p = 3$ and assume that the following is true: if a 1-form φ on X satisfies the Guillemin condition, the symmetric 2-form $\tilde{\sigma}(\varphi)$ also satisfies the Guillemin condition. Then if f is an element of $C^\infty_{\mathbb{R}}(X)$, the symmetric 2-form $\tilde{\sigma}(df)$ satisfies the Guillemin condition. Thus if P is the orthogonal projection corresponding to the decomposition (1.3), the mapping

$$P_\sigma : P\tilde{\sigma}d : C^\infty_{\mathbb{R}}(X) \to I(X)$$

is well-defined. Clearly, if f is an element of $C^\infty_{\mathbb{R}}(X)$, then $\tilde{\sigma}df$ is a Lie derivative of the metric if and only if $P_\sigma f = 0$.

2. A decomposition of a space of tensors

If V is a real finite-dimensional vector space, we denote by $\bigotimes^k V$, $S^l V$, $\bigwedge^j V$ the k-th tensor product, the l-th symmetric product and the j-th exterior product of V, respectively; we shall identify $S^k V^*$ and $\bigwedge^k V^*$ with subspaces of $\bigotimes^k V^*$. Let $n \geqslant 3$ be a given integer and let U be a real vector space of dimension n endowed with a positive definite scalar product q. The scalar product q induces a scalar product on an arbitrary subspace of $\bigotimes^k U^*$. We consider the group $SL(U)$ consisting of all automorphisms of U whose determinants are equal to 1 and its subgroup $SO(U)$ consisting of those elements of $SL(U)$ which preserve the scalar product q.

Let $B(U)$ be the subspace of $\bigwedge^2 U^* \otimes \bigwedge^2 U^*$ consisting of those elements v of $\bigwedge^2 U^* \otimes \bigwedge^2 U^*$ which satisfy the first Bianchi identity

$$v(\xi_1, \xi_2, \xi_3, \xi_4) + v(\xi_2, \xi_3, \xi_1, \xi_4) + v(\xi_3, \xi_1, \xi_2, \xi_4) = 0,$$

for all $\xi_1, \xi_2, \xi_3, \xi_4 \in U$; it is well-known that $B(U)$ is an irreducible $SL(U)$-submodule of $S^2(\bigwedge^2 U^*)$ equal to the image of the morphism

$$\tau : S^2U^* \otimes S^2U^* \to S^2(\textstyle\bigwedge^2 U^*)$$

of $SL(U)$-modules defined by

$$\begin{aligned}(\tau u)(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{2}\big(&u(\xi_1, \xi_3, \xi_2, \xi_4) + u(\xi_2, \xi_4, \xi_1, \xi_3) \\ &- u(\xi_1, \xi_4, \xi_2, \xi_3) - u(\xi_2, \xi_3, \xi_1, \xi_4)\big),\end{aligned}$$

for all $u \in S^2U^* \otimes S^2U^*$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in U$. The morphism

$$\psi : \textstyle\bigwedge^2 U^* \otimes \bigwedge^2 U^* \to S^2(S^2U^*)$$

of $SL(U)$-modules is well-defined by

$$(\psi v)(\xi_1,\xi_2,\xi_3,\xi_4) = \frac{1}{2}\big(v(\xi_1,\xi_3,\xi_2,\xi_4) + v(\xi_2,\xi_4,\xi_1,\xi_3) + v(\xi_1,\xi_4,\xi_2,\xi_3) + v(\xi_2,\xi_3,\xi_1,\xi_4)\big),$$

for all $v \in \bigwedge^2 U^* \otimes \bigwedge^2 U^*$ and $\xi_1,\xi_2,\xi_3,\xi_4 \in U$, and its restriction to the subspace $B(U)$ of $\bigwedge^2 U^* \otimes \bigwedge^2 U^*$ is given by

$$(\psi v)(\xi_1,\xi_2,\xi_3,\xi_4) = v(\xi_1,\xi_4,\xi_2,\xi_3) + v(\xi_1,\xi_3,\xi_2,\xi_4),$$

for all $v \in B(U)$ and $\xi_1,\xi_2,\xi_3,\xi_4 \in U$. We then verify that the restriction of the morphism $\frac{1}{3}\tau \circ \psi$ to $B(U)$ is equal to the identity mapping of $B(U)$. Hence the morphism $\psi : B(U) \to S^2(S^2U^*)$ is injective.

For $k \geqslant 2$, the kernels $S^k_0 U^*$ and $B^0(U)$ of the trace mappings

$$\mathrm{Tr} : S^kU^* \to S^{k-2}U^*, \qquad \mathrm{Tr} : B(U) \to S^2U^*,$$

defined by

$$(\mathrm{Tr}\ u)(\xi_1,\dots,\xi_{k-2}) = \sum_{j=1}^{n} u(t_j,t_j,\xi_1,\dots,\xi_{k-2}),$$

$$(\mathrm{Tr}\ v)(\eta_1,\eta_2) = \sum_{j=1}^{n} v(t_j,\eta_1,t_j,\eta_2),$$

for $u \in S^kU^*$, $v \in B(U)$ and $\xi_1,\dots,\xi_{k-2},\eta_1,\eta_2 \in U$, where $\{t_1,\dots,t_n\}$ is an orthonormal basis of U, are $SO(U)$-modules of S^kU^* and $B(U)$, respectively. In fact, $S^k_0U^*$ is an irreducible $SO(U)$-submodule of S^kU^*. Thus the image of $S^4_0U^*$ under the natural monomorphism

$$S^4U^* \to S^2(S^2U^*)$$

of $SL(U)$-modules is an irreducible $SO(U)$-submodule of $S^2(S^2_0U^*)$; we shall identify $S^4_0U^*$ with this submodule.

It is easily verified that the morphism

$$\phi : S^2U^* \to S^2(S^2U^*)$$

of $SO(U)$-modules defined by

$$\begin{aligned}(\phi h)(\xi_1,\xi_2,\xi_3,\xi_4) = n\big(&q(\xi_1,\xi_3)h(\xi_2,\xi_4) + q(\xi_2,\xi_4)h(\xi_1,\xi_3)\\ &+ q(\xi_1,\xi_4)h(\xi_2,\xi_3) + q(\xi_2,\xi_3)h(\xi_1,\xi_4)\big)\\ &- 4\big(q(\xi_1,\xi_2)h(\xi_3,\xi_4) + q(\xi_3,\xi_4)h(\xi_1,\xi_2)\big),\end{aligned}$$

for all $h \in S^2U^*$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in U$, is injective. The image of $S^2_0U^*$ under the morphism ϕ is a submodule of $S^2(S^2_0U^*)$; thus $\phi(S^2_0U^*)$ is an irreducible $SO(U)$-submodule of $S^2(S^2_0U^*)$.

We know that $B^0(U)$ vanishes when $n = 3$ and is an irreducible $SO(U)$-module when $n \geqslant 5$; on the other hand, when $n = 4$, the space $B^0(U)$ admits a decomposition

$$B^0(U) = B^0_+(U) \oplus B^0_-(U), \tag{2.1}$$

where $B^0_+(U)$ and $B^0_-(U)$ are irreducible $SO(U)$-submodules. We easily verify that $\psi(B^0(U))$ is an $SO(U)$-submodule of $S^2(S^2_0U^*)$.

We identify the scalar product on $S^2_0U^*$ induced by q with an element Q of $S^2(S^2_0U^*)$; the one-dimensional subspace $\{Q\}$ of $S^2(S^2_0U^*)$ generated by Q is a trivial $SO(U)$-submodule. The sum of the dimensions of the $SO(U)$-modules $\{Q\}$, $S^2_0U^*$, $S^4_0U^*$ and $B^0(U)$ is equal to the dimension of $S^2(S^2_0U^*)$. The $SO(U)$-modules $\{Q\}$, $S^2_0U^*$, $S^4_0U^*$ are irreducible and pairwise non-isomorphic; when $n \geqslant 5$, the irreducible $SO(U)$-module $B^0(U)$ is not isomorphic to any one of these modules. Moreover when $n = 4$, the $SO(U)$-modules $\{Q\}$, $S^2_0U^*$, $S^4_0U^*$, $B^0_+(U)$ and $B^0_-(U)$ are irreducible and pairwise non-isomorphic. Thus we obtain the direct sum decomposition

$$S^2(S^2_0U^*) = \{Q\} \oplus S^4_0U^* \oplus \psi(B^0(U)) \oplus \phi(S^2_0U^*) \tag{2.2}$$

of $S^2(S^2_0U^*)$ into $SO(U)$-submodules.

3. The special unitary group

Let n be a given integer $\geqslant 3$. We consider the special unitary group $G = SU(n)$ and we suppose that $X = G$. If B denotes the Killing form of the Lie algebra $\mathfrak{g}_0 = \mathfrak{su}(n)$, we endow X with the bi-invariant Riemannian metric g_0 induced by $-B$. Endowed with this metric g_0, the manifold X is an irreducible symmetric space. As usual, we identify the G-module $\mathfrak{g}_0$ with the tangent space of X at the identity element $e_0 = I_n$ of G.

If $k \geqslant 1$ is a given integer, we consider the space M_k of all $k \times k$ complex matrices. For $1 \leqslant j,k \leqslant n$, let $E_{jk} = (c_{lr})$ be the element of M_n determined by $c_{jk} = 1$ and $c_{lr} = 0$ whenever $(l,r) \neq (j,k)$. If $1 \leqslant j,k \leqslant n$ and $1 \leqslant l \leqslant n-1$ are integers, with $j \neq k$, the matrices

$$A_{jk} = E_{jk} - E_{kj}, \qquad B_{jk} = i(E_{jk} + E_{kj}), \qquad C_l = i(E_{ll} - E_{l+1,l+1})$$

of M_n belong to $\mathfrak{g}_0$; in fact, the set of all these matrices $\{A_{jk}, B_{jk}, C_l\}$, with $1 \leqslant j < k \leqslant n$ and $1 \leqslant l \leqslant n-1$, form a basis of $\mathfrak{g}_0$. For $1 \leqslant j \leqslant n$, we consider the element

$$\widetilde{C}_j = \frac{1}{n}\Big(\sum_{k=j}^{n-1}(n-k)C_k - \sum_{k=1}^{j-1} kC_k\Big)$$

of $\mathfrak{g}_0$; then we verify that

$$C_j = \widetilde{C}_j - \widetilde{C}_{j+1}, \tag{3.1}$$

for $1 \leqslant j \leqslant n-1$.

For $p \geqslant 2$, the homogeneous polynomial Q_p on $\mathfrak{g}_0$ defined by

$$Q_p(\xi) = (-i)^p \operatorname{Tr} \xi^p,$$

for all $\xi \in \mathfrak{g}_0$, is G-invariant, non-zero and real-valued; therefore it gives rise to a non-zero bi-invariant symmetric p-form σ'_p on X. We know that the metric g_0 is equal to the symmetric 2-form $2n \cdot \sigma'_2$ (see [2, §2]).

We shall always consider the symmetric space $X = SU(n)$, with $n \geqslant 3$, endowed with the Riemannian metric $g' = \sigma'_2$. We easily verify that the product of matrices $C_j \cdot C_k$ is equal to 0, for all $1 \leqslant j,k \leqslant n-1$, with $j < k+1$, and hence that

$$g'(C_j, C_j) = 2, \qquad g'(C_l, C_{l+1}) = -1, \qquad g'(C_k, C_q) = 0, \tag{3.2}$$

for all $1 \leqslant j,k,q \leqslant n-1$ and $1 \leqslant l \leqslant n-2$, with $q \geqslant k+2$.

We identify an element of $\mathfrak{g}_0$ with the left-invariant vector field on G that it determines. Throughout the remainder of this section, by C_j and $\widetilde{C}_l$ we shall always mean the left-invariant vector fields on G determined by the corresponding elements of $\mathfrak{g}_0$.

For $p \geqslant 3$, we consider the monomorphism

$$\tilde{\sigma}'_p : T^* \to S^{p-1}T^*$$

induced by the symmetric p-form σ'_p. We shall write $\sigma' = \sigma'_3$ and $\tilde{\sigma}' = \tilde{\sigma}'_3$. Let φ be an element of $T^*_{e_0}$ and let $A = (a_{jk})$ be the matrix $-ig'^{\sharp}(\varphi)$ of $\mathfrak{sl}(n,\mathbb{C})$; for $1 \leqslant j \leqslant n$, we know that a_{jj} is real. Then we see that

$$\varphi(C_j) = a_{jj} - a_{j+1,j+1},$$

for $1 \leqslant j \leqslant n-1$; it follows that

$$\varphi(\widetilde{C}_j) = a_{jj}, \tag{3.3}$$

for $1 \leqslant j \leqslant n$. If $p \geqslant 3$ is a given integer, by (3.3) we easily verify that

$$\tilde{\sigma}'_p(\varphi)(C_{j_1}, C_{j_2}, \dots, C_{j_{p-1}}) = 0, \tag{3.4}$$

for $1 \leqslant j_1, j_2, \dots, j_{p-1} \leqslant n-1$, with $j_1 > j_2 + 1$, and that

$$\tilde{\sigma}'_p(\varphi)(C_j, \dots, C_j) = \varphi(\widetilde{C}_j) + (-1)^{p+1}\varphi(\widetilde{C}_{j+1}), \tag{3.5}$$

for all $1 \leqslant j \leqslant n-1$; moreover, for all $1 \leqslant j \leqslant n-2$ and $1 \leqslant k \leqslant p-1$, we have

$$\tilde{\sigma}'_p(\varphi)(C_j, \dots, C_j, C_{j+1}, \dots, C_{j+1}) = (-1)^k \varphi(\widetilde{C}_{j+1}), \tag{3.6}$$

if the tangent vector C_j appears k times in the left-hand side of this equation. If p is an odd integer, by (3.3) we see that

$$\tilde{\sigma}'_p(\varphi)(A_{jk}, \dots, A_{jk}) = \tilde{\sigma}'_p(\varphi)(B_{jk}, \dots, B_{jk}) = \varphi(\widetilde{C}_j + \widetilde{C}_k), \tag{3.7}$$

for $1 \leqslant j < k \leqslant n$. >From the G-invariance of the form σ'_p, we now infer that the relations (3.4)–(3.7) hold on G for all $\varphi \in C^\infty(T^*)$.

We now consider the mapping

$$\iota' : \mathbb{R}^{n-1} \to G,$$

which sends $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^{n-1}$ into the diagonal matrix

$$\iota'(\theta) = \operatorname{diag}\left(e^{ix_1}, \dots, e^{ix_n}\right)$$

of G, where

$$x_1 = \theta_1, \qquad x_j = \theta_j - \theta_{j-1}, \qquad x_n = -\theta_{n-1}, \tag{3.8}$$

for $2 \leqslant j \leqslant n-1$. If $\{e'_1, \dots, e'_{n-1}\}$ is the standard basis of $\mathbb{R}^{n-1}$ and Λ' is the lattice of $\mathbb{R}^{n-1}$ generated by the basis $\{2\pi e'_j\}_{1 \leqslant j \leqslant n-1}$ of $\mathbb{R}^{n-1}$, the mapping ι' induces by passage to the quotient an imbedding

$$\iota' : \mathbb{R}^{n-1}/\Lambda' \to G.$$

The image of the mappings ι' is the maximal torus H of the group G which consists of all diagonal matrices of G and is therefore a maximal flat totally geodesic torus of G viewed as a symmetric space. Clearly we have $\iota'(0) = e_0$.

We consider the standard coordinate system $(\theta_1, \dots, \theta_{n-1})$ on $\mathbb{R}^{n-1}$ and endow this space with the Riemannian metric

$$\tilde{g} = \sum_{j=1}^{n-1} d\theta_j \cdot d\theta_j - \sum_{j=1}^{n-2} d\theta_j \cdot d\theta_{j+1}.$$

For $1 \leqslant j \leqslant n-1$, we consider the vector field $\xi_j = \partial/\partial\theta_j$ on $\mathbb{R}^{n-1}$. The vector field ζ'_j on H, determined by

$$\iota'_*(\xi_j(\theta)) = \zeta'_j(\iota'(\theta)), \tag{3.9}$$

for $\theta \in \mathbb{R}^{n-1}$, is invariant under the action of the group H; when we identify T_{e_0} with $\mathfrak{g}_0$, we easily verify that $\zeta'_j(e_0) = C_j$. It follows that ζ'_j is equal to the restriction of the vector field C_j to H. Since $\{C_1, \dots, C_{n-1}\}$ is a frame for the submanifold H of G, by (3.2) and (3.9) we see that

$$\iota'^* g' = \tilde{g}. \tag{3.10}$$

Hence the mapping $\iota' : \mathbb{R}^{n-1} \to H$ is an isometric imbedding. If f is a function on X, we now easily see that

$$\int_Z f\, dZ = \sqrt{n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\iota'(\theta))\, d\theta_1 \dots d\theta_{n-1}, \tag{3.11}$$

where $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, from the above remarks we infer that $\{\zeta'_1, \dots, \zeta'_{n-1}\}$ is a basis for the space of parallel vector fields on H. Since all maximal flat totally geodesic tori of X are conjugate under the left action of G, from the relations (3.1) and (3.4)–(3.6) we now deduce the following result:

Lemma 3.1. — *Let $n, p \geqslant 3$ be given integers and let X be the symmetric space $SU(n)$. A 1-form φ on X satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}'_p(\varphi)$ on X satisfies the Guillemin condition.*

4. Functions on the special unitary group

Let $k \geqslant 1$ be a given integer. For $1 \leqslant j, l \leqslant k$, we denote by z_{jl} the function on the space of matrices M_k which sends a matrix of M_k into its (j, l)-th entry. We also consider the complex-valued function $\Delta^{(k)}_{jl}$ on M_k defined as follows. If $k = 1$, the function $\Delta^{(1)}_{11}$ is identically equal to 1; if $k \geqslant 2$, the value of the function $\Delta^{(k)}_{jl}$ at a matrix $A \in M_k$ is the cofactor of the entry $z_{jl}(A)$ in A, which is equal to $(-1)^{j+l}$ times the determinant of the $(k-1) \times (k-1)$ matrix obtained from A by deleting its j-th row and its l-th column. We note that, if $A \in M_k$ is a symmetric matrix, then we have $\Delta^{(k)}_{jl}(A) = \Delta^{(k)}_{lj}(A)$. If $\det_k$ denotes the function on M_k which sends a matrix of M_k into its determinant, we recall that

$$\sum_{r=1}^{k} z_{jr}\, \Delta^{(k)}_{lr} = \delta_{jl} \det{}_k, \tag{4.1}$$

for all $1 \leqslant j,\, l \leqslant k$. Thus we obtain the relation

$$\frac{\partial}{\partial z_{jl}} \det{}_k = \Delta^{(k)}_{jl}, \tag{4.2}$$

for all $1 \leqslant j,\, l \leqslant k$.

We consider the group $G = SU(n)$, with $n \geqslant 3$, as a real submanifold of the complex manifold M_n. The left action of the group G on the manifold M_n induces a morphism Φ from $\mathfrak{g}_0$ to the Lie algebra of vector fields on M_n, which are tangent to the submanifold G of M_n. The mapping Φ extends to a $\mathbb{C}$-linear morphism from the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ to the space of all complex vector fields on M_n.

The functions $\{z_{jk}\}$ on M_n defined above, with $1 \leqslant j, k \leqslant n$, form a holomorphic coordinate system for M_n. We consider the $\mathbb{C}^n$-valued function Z_j on M_n which sends a matrix of M_n into its j-th row; then we have $Z_j = (z_{j1}, \dots, z_{jn})$. For $1 \leqslant j, k \leqslant n$, the complex vector field

$$\xi_{jk} = \sum_{l=1}^{n} z_{jl} \frac{\partial}{\partial z_{kl}}$$

on M_n satisfies

$$\xi_{jk} Z_l = \delta_{kl} Z_j, \qquad \xi_{jk} \bar{Z}_l = 0, \tag{4.3}$$

for all $1 \leqslant l \leqslant n$. For $1 \leqslant j, k \leqslant n$, with $j \neq k$, and $1 \leqslant l \leqslant n-1$, we verify that

$$\Phi(A_{jk}) = \bar{\xi}_{jk} - \bar{\xi}_{kj} + \xi_{jk} - \xi_{kj}, \qquad \Phi(B_{jk}) = i(\bar{\xi}_{jk} + \bar{\xi}_{kj} - \xi_{jk} - \xi_{kj}),$$
$$\Phi(C_l) = i(\bar{\xi}_{ll} - \bar{\xi}_{l+1,l+1} + \xi_{l+1,l+1} - \xi_{ll}).$$

If $1 \leqslant j, k \leqslant n$, with $j \neq k$, since E_{jk} is equal to $\frac{1}{2}(A_{jk} - iB_{jk})$, the complex vector field $\eta_{jk} = \Phi(E_{jk})$ on M_n is given by

$$\eta_{jk} = \bar{\xi}_{jk} - \xi_{kj},$$

and so we have

$$\bar{\eta}_{jk} = -\eta_{kj}. \tag{4.4}$$

If $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are elements of $\mathbb{C}^n$, we write

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j w_j.$$

Let $1 \leqslant k \leqslant n-1$ be a given integer. We consider the M_k-valued function A_k on M_n defined by

$$A_k = \big(\langle Z_j, Z_l \rangle\big)_{1 \leqslant j, l \leqslant k},$$

and the complex-valued function

$$f_k = \det A_k$$

on M_n. If $\iota' : \mathbb{R}^{n-1} \to G$ is the mapping defined in §3, for all $\theta \in \mathbb{R}^{n-1}$ and $1 \leqslant j, l \leqslant n$, we have

$$\langle Z_j, Z_l \rangle(\iota'(\theta)) = e^{2ix_j}\delta_{jl},$$

where x_j is given by (3.8); hence we have the equality

$$f_k(\iota'(\theta)) = e^{2i\theta_k}. \tag{4.5}$$

Lemma 4.1. — *Let $1 \leqslant k \leqslant n-1$ and $1 \leqslant j, l \leqslant n$ be given integers. The equalities*

$$\xi_{jl} f_k = 0$$

hold on M_n whenever $l > k$, and the equalities

$$\xi_{jl} f_k = 2\delta_{jl} f_k$$

hold on M_n whenever $1 \leqslant j, l \leqslant k$.

Proof. — The first equalities are immediate. We now suppose that the integers j, l satisfy $1 \leqslant j, l \leqslant k$. Since the function A_k takes its values in the space of symmetric matrices, according to (4.2) and (4.3) we have

$$\xi_{jl} f_k = \sum_{p,q=1}^{k} \Delta_{pq}^{(k)}(A_k) \cdot \xi_{jl}\langle Z_p, Z_q\rangle = 2\sum_{p,q=1}^{k} \Delta_{pq}^{(k)}(A_k)\,\langle Z_j, Z_q\rangle\,\delta_{pl};$$

by (4.1) it follows that

$$\xi_{jl} f_k = 2\sum_{q=1}^{k} (\Delta_{lq}^{(k)} z_{jq})(B_k) = 2\delta_{jl} f_k.$$

□

Let $1 \leqslant k \leqslant n-1$ and $1 \leqslant j < l \leqslant n$ be given integers. The preceding lemma implies that

$$\eta_{lj} f_k = -\xi_{jl} f_k = 0, \tag{4.6}$$

$$\Phi(C_j) f_k = -2i\delta_{jk} f_k. \tag{4.7}$$

5. Highest weight vectors

We consider the maximal torus H of the simple group G introduced in §3 and its Lie algebra $\mathfrak{h}_0$, and also the complexification $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of the Lie algebra $\mathfrak{g}_0$ of G. The complexification $\mathfrak{h}$ of $\mathfrak{h}_0$ is equal to the Cartan subalgebra of the simple Lie algebra $\mathfrak{g}$ consisting of all diagonal matrices of $\mathfrak{g}$, and the matrices $\{C_1, \ldots, C_{n-1}\}$ form a basis of $\mathfrak{h}_0$. For $1 \leqslant j \leqslant n$, the linear form $\lambda_j : \mathfrak{h} \to \mathbb{C}$, sending the diagonal matrix with $a_1, \dots, a_n \in \mathbb{C}$ as its diagonal entries into a_j, is purely imaginary on $\mathfrak{h}_0$. We write $\alpha_j = \lambda_j - \lambda_{j+1}$, for $1 \leqslant j \leqslant n-1$. Then

$$\{\lambda_j - \lambda_k \mid 1 \leqslant j, k \leqslant n \text{ and } j \neq k\}$$

is the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We take $\{\alpha_1, \dots, \alpha_{n-1}\}$ as a system of simple roots of $\mathfrak{g}$; the corresponding system of positive roots is

$$\Delta^+ = \{\lambda_j - \lambda_k \mid 1 \leqslant j < k \leqslant n\}.$$

If α is the root $\lambda_j - \lambda_k$, with $1 \leqslant j, k \leqslant n$ and $j \neq k$, the root subspace $\mathfrak{g}_\alpha$ corresponding to α is generated by E_{jk} (over $\mathbb{C}$). We have the decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}.$$

The corresponding fundamental weights are

$$\varpi_j = \lambda_1 + \cdots + \lambda_j,$$

with $1 \leqslant j \leqslant n-1$; in fact, we have

$$\varpi_k(C_j) = i\delta_{jk}, \tag{5.1}$$

for $1 \leqslant j, k \leqslant n-1$. The unique element w_0 of the Weyl group of $\mathfrak{g}$ satisfying

$$w_0(\Delta^+) = -\Delta^+$$

is the involutive automorphism determined by

$$w_0(\alpha_j) = -\alpha_{n-j},$$

for $1 \leqslant j \leqslant n-1$; we verify that

$$w_0(\varpi_j) = -\varpi_{n-j}, \tag{5.2}$$

for $1 \leqslant j \leqslant n-1$. A dominant integral form λ for the simply-connected group G is a linear form on $\mathfrak{h}$ which can be written in the form

$$\lambda = b_1\lambda_1 + b_2\lambda_2 + \cdots + b_n\lambda_n,$$

where $b_1, b_2 \dots, b_n$ are integers satisfying

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n;$$

if

$$\lambda = c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n$$

is another expression for the linear form λ, where $c_1, c_2, \dots, c_n$ are integers satisfying $c_1 \geqslant c_2 \geqslant \dots \geqslant c_n$, then there exists an integer c such that

$$c_j = b_j + c,$$

for all $1 \leqslant j \leqslant n$. Therefore a dominant integral form λ for G may be written in a unique way

$$\lambda = \gamma_{r_1,\dots,r_{n-1}} = r_1\varpi_1 + \dots + r_{n-1}\varpi_{n-1}, \tag{5.3}$$

where $r_1, \dots, r_{n-1}$ are non-negative integers. Thus the highest weight of an irreducible (complex) G-module has a unique expression of this form and we may identify the dual Γ of G with the set of all linear forms on $\mathfrak{h}$ which can be written in the form (5.3).

Let $\gamma = \gamma_{r_1,r_2,\dots,r_{n-1}}$ be an element of Γ, where $r_1, \dots, r_{n-1}$ are non-negative integers; by (5.2), the unique element $\bar{\gamma}$ of Γ determined by

$$w_0(\gamma) = -\bar{\gamma}$$

is equal to $\gamma_{r_{n-1},\dots,r_2,r_1}$. In particular, if γ is the element ϖ_k of Γ, we have $\bar{\gamma} = \varpi_{n-k}$.

If E is a finite-dimensional G-module and v is a highest weight vector of E, then we know that $\mathfrak{n}^+ v = \{0\}$; thus we have

$$E_{jk}v = 0, \tag{5.4}$$

for all $1 \leqslant j < k \leqslant n$.

The action of G on the vector space $V = \mathbb{C}^n$ endows V with the structure of a G-module. We denote by ρ the representations of G and $\mathfrak{g}$ on the k-th tensor product $\bigotimes^k V$ of V. We shall consider the k-th symmetric product of the vector space V as a G-submodule of $\bigotimes^k V$; the k-th symmetric power v^k of $v \in V$ may then be viewed as an element of $\bigotimes^k V$. Let $\{e_1, \dots, e_n\}$ be the standard basis of V.

The center of G is the cyclic subgroup

$$S = \left\{ e^{2ik\pi/n} I_n \mid 0 \leqslant k \leqslant n-1 \right\} \tag{5.5}$$

of order n. If E is a G-module, we denote by E^S the G-submodule of E consisting of all S-invariant elements of E.

Lemma 5.1. — *Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers. Let E be an irreducible G-module corresponding to the element $\gamma_{r_1,\dots,r_{n-1}}$ of Γ and let v be a highest weight vector of E.*

(i) *If $r_{n-1} = 0$, then we have*

$$E_{n,n-1}v = 0, \qquad A_{n-1,n}v = 0. \tag{5.6}$$

(ii) *The equality $E = E^S$ holds if and only if*

$$r_1 + 2r_2 + \cdots + (n-1)r_{n-1} \equiv 0 \bmod n. \tag{5.7}$$

Proof. — For $1 \leqslant k \leqslant n-1$, we write $c_k = \sum_{j=k}^{n-1} r_j$ and $q = \sum_{k=1}^{n-1} c_k$. The Young symmetrizer Φ corresponding to the partition $(c_1, c_2, \ldots, c_{n-1})$ is an endomorphism of the G-module $\bigotimes^q V$. We know that the image W of Φ is a G-submodule of $\bigotimes^q V$ isomorphic to E and that the image of the element

$$w = e_1^{c_1} \otimes e_2^{c_2} \otimes \cdots \otimes e_{n-1}^{c_{n-1}}$$

of $\bigotimes^q V$ under the morphism Φ is a highest weight vector of W. If $r_{n-1} = 0$, we have $c_{n-1} = 0$ and $\rho(E_{n,n-1})w = 0$; the relations (5.6) are a consequence of the latter equality and (5.4). If s is the element $e^{2i\pi/n} I_n$ of S, we easily see that

$$\rho(s)w = e^{2il\pi/n} w,$$

where $l = r_1 + 2r_2 + \cdots + (n-1)r_{n-1}$. By Schur's lemma, we see that the element s of S acts on E by multiplication by the scalar $e^{2il\pi/n}$. Assertion (ii) is a direct consequence of this observation. □

The space $C^\infty(G)$ inherits a G-module structure from the left action of G on X; the corresponding representation π of G or of the Lie algebra $\mathfrak{g}$ on $C^\infty(G)$ is the left regular representation. If γ is an element of Γ and E_γ is an irreducible G-module corresponding to γ, according to the Peter-Weyl theorem the isotypic component $C^\infty_\gamma(G)$ of $C^\infty(G)$ corresponding to γ is isomorphic to k copies of E_γ, where the integer k is equal to the dimension of E_γ (over $\mathbb{C}$). For all $\xi \in \mathfrak{g}_0$ and $f \in C^\infty(G)$, we have

$$\pi(\xi) \cdot f = \Phi(\xi) \cdot f.$$

If $\gamma = \gamma_{r_1,\ldots,r_{n-1}}$ is an element of Γ, where $r_1, \ldots, r_{n-1}$ are non-negative integers, from Lemma 5.1, (ii) we deduce that $C^\infty_\gamma(G)$ is a G-submodule of $C^\infty(G)^S$ if and only if the integers $r_1, \ldots, r_{n-1}$ satisfy the relation (5.7).

If $\gamma \in \Gamma$, a linear form λ on $\mathfrak{h}$ is a weight of the G-module $C^\infty_\gamma(G)$ if and only if $-\lambda$ is a weight of the complex conjugate $\overline{C^\infty_\gamma(G)}$ of the space $C^\infty_\gamma(G)$. Therefore we have the equality

$$C^\infty_{\bar\gamma}(G) = \overline{C^\infty_\gamma(G)} \tag{5.8}$$

of G-modules.

Let $1 \leqslant k \leqslant n-1$ be a given integer. Throughout the remainder of this paper, by f_k we shall always often mean the restriction of the function f_k

on M_n to the submanifold G. From the relations (4.6), (4.7) and (5.1), we infer that the equalities

$$\Phi(C_j)f_k = -2\varpi_k(C_j)f_k, \qquad \Phi(E_{rj})f_k = 0, \tag{5.9}$$

hold on M_n for all integers $1 \leqslant j < r \leqslant n$. From the relations (5.9) and (4.4), it follows that

$$\pi(\xi)\bar{f}_k = 2\varpi_k(\xi)\bar{f}_k, \qquad \pi(\eta)\bar{f}_k = 0, \tag{5.10}$$

for all $\xi \in \mathfrak{h}_0$ and $\eta \in \mathfrak{n}^+$. Thus the function $\bar{f}_k$ is a highest weight vector of the isotypic component $C^\infty_{2\varpi_k}(G)$; moreover according to (5.8), we know that f_k is an element of $C^\infty_{2\varpi_{n-k}}(G)$.

Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers and γ be the element

$$\gamma^1_{r_1,\dots,r_{n-1}} = 2\gamma_{r_1,\dots,r_{n-1}}$$

of Γ; we consider the complex-valued function $f_{r_1,\dots,r_{n-1}}$ on G defined by

$$f_{r_1,\dots,r_{n-1}} = \prod_{k=1}^{n-1} f_k^{r_k}.$$

Since $\bar{f}_k$ is a highest weight vector of the G-module $C^\infty_{2\varpi_k}(G)$, we know that $\bar{f}_{r_1,\dots,r_{n-1}}$ is a highest weight vector of the G-module $C^\infty_\gamma(G)$.

6. The special Lagrangian Grassmannians

Let n be a given integer $\geqslant 3$. Let G be the group $SU(n)$ and let K be the subgroup $SO(n)$, which is equal to the set of fixed points of the involution s of G sending a matrix into its complex conjugate. Then (G, K) is a Riemannian symmetric pair. In the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of the Lie algebra $\mathfrak{g}_0$ of G corresponding to this involution, we know that $\mathfrak{k}_0$ is the Lie algebra of K and that the K-submodule $\mathfrak{p}_0$ is the space of all symmetric purely imaginary $n \times n$ matrices of trace zero. If B denotes the Killing form of the Lie algebra $\mathfrak{g}_0$, the restriction of $-B$ to $\mathfrak{p}_0$ induces a G-invariant Riemannian metric g_0 on the homogeneous space $X = G/K$. Endowed with this metric g_0, the manifold X is an irreducible symmetric space called the special Lagrangian Grassmannian. We identify the K-module $\mathfrak{p}_0$ with the tangent space of X at the coset x_0 of the identity element I_n of G.

The matrices $\{B_{jk}, C_l\}$, with $1 \leqslant j < k \leqslant n$ and $1 \leqslant l \leqslant n-1$, form a basis of $\mathfrak{p}_0$, while the matrices $\{A_{jk}\}$, with $1 \leqslant j < k \leqslant n$, form a basis of $\mathfrak{k}_0$. The restriction q_p of the homogeneous polynomial Q_p on $\mathfrak{g}_0$ defined in §3 to $\mathfrak{p}_0$ is K-invariant and therefore gives rise to a G-invariant symmetric p-form σ_p on X. It is well-known that the algebra of all K-invariant polynomials on $\mathfrak{g}_0$ is generated by the polynomials q_p, with $2 \leqslant p \leqslant n$, and that these polynomials are algebraically independent. We know that the metric g_0 is equal to the symmetric 2-form $2n \cdot \sigma_2$ and that σ_3 is up to a constant the unique G-invariant symmetric 3-form on X; in fact, we have

$$\sigma_3(\xi_1, \xi_2, \xi_3) = i \operatorname{Tr}(\xi_1 \cdot \xi_2 \cdot \xi_3),$$

for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{p}_0$ (see [2, §2]).

Throughout the remainder of this paper, we consider the irreducible symmetric space $X = SU(n)/SO(n)$, with $n \geqslant 3$, endowed with the Riemannian metric $g = \sigma_2$. We consider the line bundle $\{g\}$ generated by the section g of S^2T^*. According to (3.2), we have

$$g(C_j, C_j) = 2, \qquad g(C_l, C_{l+1}) = -1, \qquad g(C_k, C_q) = 0, \tag{6.1}$$

for all $1 \leqslant j, k, q \leqslant n-1$ and $1 \leqslant l \leqslant n-2$, with $q \geqslant k+2$.

For $p \geqslant 3$, we consider the G-equivariant monomorphism

$$\tilde{\sigma}_p : T^* \to S^{p-1}T^*$$

induced by the symmetric p-form σ_p. We shall write $\sigma = \sigma_3$ and $\tilde{\sigma} = \tilde{\sigma}_3$.

Let U be the vector space $\mathbb{R}^n$ endowed with its standard Euclidean scalar product q and let $\{e_1, \dots, e_n\}$ be the standard basis of U. We consider the objects which we associated with U and q in §2. We consider the morphism of K-modules

$$\mathfrak{p}_0 \to S^2U,$$

which sends the symmetric $n \times n$ matrix (ia_{jk}) of $\mathfrak{p}_0$, where $a_{jk} \in \mathbb{R}$, into the element $\sum_{j,k=1}^n a_{jk} e_j \otimes e_k$ of S^2U. The restriction to $S^2_0U^*$ of the dual of this morphism is an isomorphism

$$\mu : S^2_0U^* \to \mathfrak{p}_0^*$$

of K-modules. We shall consider the isomorphism

$$\mu' : S^2(S^2_0U^*) \to S^2T^*_{x_0}$$

of K-modules induced by μ; then if Q is the element of $S^2(S^2_0U^*)$ corresponding to the scalar product on $S^2_0U^*$ and if u is an element of $S^2_0U^*$, we see that the equalities

$$\mu'(Q) = g(x_0), \qquad 4n\tilde{\sigma}(\mu u) = \mu'\phi(u) \tag{6.2}$$

hold among elements of $S^2T^*_{x_0}$. By means of the isomorphism μ', the decompositions (2.1) and (2.2) give us a decomposition of the fiber of the vector bundle S^2T^* at x_0 into irreducible K-submodules. Since a K-submodule E^0 of $S^2T^*_{x_0}$ gives rise to a unique G-invariant sub-bundle E of S^2T^* such that $E_{x_0} = E^0$, from the decomposition (2.2) and the relations (6.2) we obtain the G-invariant decomposition

$$S^2T^* = \{g\} \oplus E_1 \oplus E_2 \oplus \tilde{\sigma}(T^*) \tag{6.3}$$

of the bundle S^2T^*, where E_1 and E_2 are the G-invariant sub-bundles of S^2T^* satisfying $E_{1,x_0} = \mu'(S^4_0U^*)$ and $E_{2,x_0} = \mu'\psi(B^0(U))$. When $n = 3$, we know that $E_2 = \{0\}$. On the other hand, when $n = 4$, the K-module E_{2,x_0} decomposes into the sum of two non-trivial irreducible K-modules neither of which is isomorphic to $S^2_0U^*$. We denote by $S^k\mathfrak{p}$ the k-th symmetric product of the complexification $\mathfrak{p}$ of the K-module $\mathfrak{p}_0$; since the K-modules $\mathfrak{p}_0$ and $\mathfrak{p}_0^*$ are isomorphic, from the preceding observations and the remarks made in §2 concerning the irreducible $SO(U)$-modules appearing in the decompositions (2.1) and (2.2), we obtain the following result:

Lemma 6.1. — *We have*

$$\dim \mathrm{Hom}_K(\mathfrak{p}, S^2\mathfrak{p}) = \dim \mathrm{Hom}_K(\mathfrak{p}, S^2T^*_{\mathbb{C},x_0}) = 1.$$

The lattice Λ of $\mathbb{R}^{n-1}$ generated by the basis $\{\pi e'_j\}_{1\leqslant j\leqslant n-1}$ of $\mathbb{R}^{n-1}$ contains the lattice Λ' of $\mathbb{R}^{n-1}$ defined in §3. We consider the mapping $\iota' : \mathbb{R}^{n-1} \to G$ whose image is the maximal torus H of the group G. We denote by $\rho : G \to X$ the natural projection. Two elements a and b of G, which belong to the image of ι', have the same image in X under ρ if and only if there exists an element $\lambda \in \Lambda$ such that $a = b\iota'(\lambda)$. Thus there is an injective mapping

$$\iota : \mathbb{R}^{n-1}/\Lambda \to X$$

such that the diagram

$$\begin{array}{ccc} \mathbb{R}^{n-1}/\Lambda' & \xrightarrow{\iota'} & G \\ \downarrow & & \downarrow{\scriptstyle\rho} \\ \mathbb{R}^{n-1}/\Lambda & \xrightarrow{\iota} & X \end{array}$$

is commutative. We also denote by ι the mapping $\rho \circ \iota' : \mathbb{R}^{n-1} \to X$. The subgroup $H' = H \cap K$ of H is equal to $\iota'(\Lambda)$ and the image Z of the mappings ι is a maximal flat totally geodesic torus of X, which is equal to $\rho(H) = H/H'$. Clearly we have $\iota(0) = \rho(e_0) = x_0$. If $\tilde{g}$ is the Riemannian metric on $\mathbb{R}^{n-1}$ defined in §3, by (3.10) we have the equality

$$\iota^*g = \tilde{g}; \tag{6.4}$$

hence the mapping $\iota : \mathbb{R}^{n-1}/\Lambda \to Z$ is an isometric imbedding. If f is a function on X, we now easily see that

$$\int_Z f\,dZ = \sqrt{n}\int_0^\pi \cdots \int_0^\pi f(\iota(\theta))\,d\theta_1 \dots d\theta_{n-1}, \tag{6.5}$$

where $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^{n-1}$.

For $1 \leqslant j \leqslant n-1$, we consider the vector field $\xi_j = \partial/\partial\theta_j$ on $\mathbb{R}^{n-1}$; the vector field ζ_j on Z, which is determined by

$$\iota_*(\xi_j(\theta)) = \zeta_j(\iota(\theta)), \tag{6.6}$$

for $\theta \in \mathbb{R}^{n-1}$, is invariant under the action of the group H on Z. The vector field ζ'_j on H is ρ-projectable and we have $\rho_*\zeta'_j = \zeta_j$. Moreover, $\{\zeta_1, \dots, \zeta_{n-1}\}$ is a basis for the space of parallel vector fields on Z. When we identify T_{x_0} with $\mathfrak{p}_0$, by (3.9) we see that

$$\zeta_j(x_0) = C_j, \tag{6.7}$$

and so $\zeta'_j(x_0)$ is equal to the vector $\zeta_j(x_0)$ of $\mathfrak{p}_0$ viewed as an element of $\mathfrak{g}_0$.

For $1 \leqslant j \leqslant n$, by (6.7) the parallel vector field

$$\eta_j = \frac{1}{n}\Bigg(\sum_{k=j}^{n-1}(n-k)\zeta_k - \sum_{k=1}^{j-1} k\zeta_k\Bigg)$$

on Z satisfies $\eta_j(x_0) = \widetilde{C}_j$; then we verify that

$$\zeta_j = \eta_j - \eta_{j+1}, \tag{6.8}$$

for $1 \leqslant j \leqslant n-1$, and that

$$\sum_{j=1}^n \eta_j = 0. \tag{6.9}$$

Let φ be an element of $T^*_{x_0}$; then there is a unique element $\zeta \in \mathfrak{p}_0$ such that $\varphi = g^\flat(\zeta)$. Since we have $g'(\mathfrak{p}_0, \mathfrak{k}_0) = 0$, the element $\rho^*\varphi$ of T^*_{G,e_0} is equal to $g'^\flat(\zeta')$, where ζ' is equal to the vector ζ viewed as an element of $\mathfrak{g}_0$. If $p \geqslant 3$ be a given integer, we therefore have the equalities

$$\varphi(y_1) = (\rho^*\varphi)(y'_1), \qquad \tilde\sigma_p(\varphi)(y_1, \dots, y_{p-1}) = \tilde\sigma'_p(\rho^*\varphi)(y'_1, \dots, y'_{p-1}),$$

for all $y_1, \dots, y_{p-1} \in \mathfrak{p}_0$, where y'_j is equal to the vector y_j considered as an element of $\mathfrak{g}_0$. Then from the equalities (3.4)–(3.6), we obtain the relation

$$\tilde\sigma_p(\varphi)(\zeta_{j_1}, \zeta_{j_2}, \dots, \zeta_{j_{p-1}}) = 0, \tag{6.10}$$

for $1 \leqslant j_1, j_2, \dots, j_{p-1} \leqslant n-1$, with $j_1 > j_2 + 1$, and the relations

$$\tilde\sigma_p(\varphi)(\zeta_j, \dots, \zeta_j) = \varphi(\eta_j) + (-1)^{p+1}\varphi(\eta_{j+1}), \tag{6.11}$$

for all $1 \leqslant j \leqslant n-1$; moreover, for all $1 \leqslant j \leqslant n-2$ and $1 \leqslant k \leqslant p-1$, we have

$$\tilde{\sigma}_p(\varphi)(\zeta_j, \dots, \zeta_j, \zeta_{j+1}, \dots, \zeta_{j+1}) = (-1)^k \varphi(\eta_{j+1}), \tag{6.12}$$

if the vector field ζ_j appears k times in the left-hand side of this equation. Because the vector fields ζ_j are invariant under the action of the group H, the relations (6.10)–(6.12) hold for all $\varphi \in C^\infty(T^*)$. Since $\{\zeta_1, \dots, \zeta_{n-1}\}$ is a basis for the space of parallel vector fields on Z, and since all maximal flat totally geodesic tori of X are conjugate under the action of G on X, from the relations (6.8) and (6.10)–(6.12) we deduce the following result:

Lemma 6.2. — *Let $n, p \geqslant 3$ be given integers and let X be the symmetric space $SU(n)/SO(n)$. A 1-form φ on X satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}_p(\varphi)$ on X satisfies the Guillemin condition.*

The equalities (11.1) of [2] and the preceding lemma, with $n = 4$ and $p = 3$, give us precisely the assertion of Lemma 4.1 of [2], with $n = 3$.

According to (6.9), (6.11) and (6.12), we easily see that an arbitrary element h of the sub-bundle $\tilde{\sigma}(T^*)$ of S^2T^* satisfies the relation

$$\sum_{j=1}^{n-1} h(\zeta_j, \zeta_j) + \sum_{j=1}^{n-2} h(\zeta_j, \zeta_{j+1}) = 0. \tag{6.13}$$

If φ is a 1-form on X, since the mapping ι is totally geodesic, by (6.6) and the definition of the operator D^1 we have the equality

$$2\,\iota^*(D^1\varphi)(\zeta_j, \zeta_k) = \xi_j \cdot \langle \xi_k, \iota^*\varphi\rangle + \xi_k \cdot \langle \xi_j, \iota^*\varphi\rangle \tag{6.14}$$

of functions on $\mathbb{R}^{n-1}$, for $1 \leqslant j, k \leqslant n-1$.

7. Functions on the special Lagrangian Grassmannians

As in §6, we consider the groups $G = SU(n)$ and $K = SO(n)$, with $n \geqslant 3$. If γ is an element of Γ, we consider an irreducible G-module E_γ corresponding to γ. We shall denote by Γ_0 the subset of Γ consisting of all elements $\gamma^1_{r_1,\dots,r_{n-1}}$ of Γ, where $r_1, \dots, r_{n-1} \geqslant 0$ are integers. Since the group G is a real form of the group $SL(n, \mathbb{C})$ and the subgroup K is equal to $G \cap SO(n, \mathbb{C})$, from Proposition 10.1 of [2] we deduce the following result:

Proposition 7.1. — *Let G be the group $SU(n)$ and K be its subgroup $SO(n)$, with $n \geqslant 3$. Let $\gamma = \gamma_{r_1,\dots,r_{n-1}}$ be an element of Γ, where $r_1, \dots, r_{n-1}$ are integers $\geqslant 0$. The multiplicity of the trivial K-module in* *the decomposition of the G-module E_γ, viewed as a K-module, is equal to* 1 *if γ belongs to Γ_0 and to* 0 *otherwise. If γ belongs to Γ_0, the multiplicity of the K-module $\mathfrak{p}$ in the decomposition of the G-module E_γ, viewed as a K-module, is equal to the number of non-zero integers r_j.*

We consider the Lagrangian Grassmannian $X = G/K$ and the natural projection $\rho : G \to X$. Let F be a G-invariant sub-bundle of S^pT^*. The spaces $C^\infty(X)$, $C^\infty_{\mathbb{R}}(X)$, $C^\infty(F)$ and $C^\infty(F_{\mathbb{C}})$ inherit structures of G-modules from the action of G on X. If γ is an element of Γ, we denote by $C^\infty_\gamma(X)$ (resp. $C^\infty_\gamma(F_{\mathbb{C}})$) the isotypic component of the (complex) G-module $C^\infty(X)$ (resp. $C^\infty(F_{\mathbb{C}})$) corresponding to γ. For $\gamma \in \Gamma$, we recall that the multiplicity of the G-module $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ is equal to the dimension of the weight subspace of the G-module $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ corresponding to γ (see §2, Chapter II of [1]).

If $\gamma \in \Gamma$, a linear form λ on $\mathfrak{h}$ is a weight of the G-module $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ if and only if $-\lambda$ is a weight of the complex conjugate $\overline{C^\infty_\gamma(S^pT^*_{\mathbb{C}})}$ of the space $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$; therefore we have the equality

$$C^\infty_{\bar\gamma}(S^pT^*_{\mathbb{C}}) = \overline{C^\infty_\gamma(S^pT^*_{\mathbb{C}})} \tag{7.1}$$

of G-modules.

If E is a G-submodule of $C^\infty(G)$, we denote by E^K the G-submodule of E consisting of all functions of E which are invariant under the right action of K on G. The natural projection $\rho : G \to X$ induces an isomorphism

$$\rho^* : C^\infty(X) \to C^\infty(G)^K$$

of G-modules, which sends a function $f \in C^\infty(X)$ into the function $\rho^* f$ on G. If γ is an element of Γ, this mapping ρ^* induces a monomorphism

$$\rho^* : C^\infty_\gamma(X) \to C^\infty_\gamma(G)^K.$$

A function f on G which is invariant under the right action of K on G determines a function $\tilde f$ on X satisfying $\rho^*\tilde f = f$.

Let $\mathcal{H}$ denote the space of functions on M_n generated by the complex-valued functions $\langle Z_j, Z_l\rangle$, with $1 \leqslant j, l \leqslant n$. A function f belonging to $\mathcal{H}$ is invariant under the right action of K on M_n; hence its restriction to G induces by passage to the quotient a function $\tilde f$ on X. Therefore for $1 \leqslant k \leqslant n-1$, the complex-valued function f_k on M_n is invariant under the right action of K on M_n; its restriction to G induces by passage to the quotient a function $\tilde f_k$ on X. The complex conjugate $\hat f_k$ of the function $\tilde f_k$ is equal to the function on X induced by the function $\bar f_k$ on G.

If $r_1, \dots, r_{n-1} \geqslant 0$ are integers, the function

$$\tilde{f}_{r_1,\dots,r_{n-1}} = \prod_{k=1}^{n-1} \tilde{f}_k^{r_k}$$

is equal to the function on X induced by the function $f_{r_1,\dots,r_{n-1}}$ on G; its complex conjugate $\hat{f}_{r_1,\dots,r_{n-1}}$ is equal to the function on X induced by the function $\bar{f}_{r_1,\dots,r_{n-1}}$ on G. If $r_1, \dots, r_{n-1} \in \mathbb{Z}$, when at least one of the integers is < 0, we set

$$\tilde{f}_{r_1,\dots,r_{n-1}} = 0.$$

If γ is an element of Γ, by the Frobenius reciprocity theorem the first assertion of Proposition 7.1 tells us that the isotypic component $C^\infty_\gamma(X)$ of $C^\infty(X)$ corresponding to γ is irreducible if γ belongs to Γ_0 and vanishes whenever γ does not belong to Γ_0.

If $r_1, \dots, r_{n-1} \geqslant 0$ are given integers and γ is the element $\gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0, since $\bar{f}_{r_1,\dots,r_{n-1}}$ is a highest weight vector of the G-module $C^\infty_\gamma(G)$, the function $\hat{f}_{r_1,\dots,r_{n-1}}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$. In particular, for $1 \leqslant k \leqslant n-1$, we know that $\hat{f}_k$ is a highest weight vector of the irreducible G-module $C^\infty_{2\varpi_k}(X)$.

If $\gamma \in \Gamma_0$, according to (5.8) or (7.1) we have the equality

$$C^\infty_{\bar{\gamma}}(X) = \overline{C^\infty_\gamma(X)} \tag{7.2}$$

of G-modules. Hence $\tilde{f}_k$ is an element of $C^\infty_{2\varpi_{n-k}}(X)$, for $1 \leqslant k \leqslant n-1$. The element $\gamma = \gamma^1_{1,0,\dots,0,1} = 2\varpi_1 + 2\varpi_{n-1}$ of Γ_0 satisfies $\bar{\gamma} = \gamma$; thus according to (7.2), the G-module $\mathcal{B} = C^\infty_\gamma(X)$ is invariant under complex conjugation, and hence is equal to the complexification of the G-submodule

$$\mathcal{B}_{\mathbb{R}} = \{ f \in \mathcal{B} \mid f = \bar{f} \}$$

of $C^\infty_{\mathbb{R}}(X)$. Since the function $\hat{f}_1 \cdot \hat{f}_{n-1}$ is a highest weight vector of $\mathcal{B}$, its complex conjugate $\tilde{f}_1 \cdot \tilde{f}_{n-1}$ is also an element of $\mathcal{B}$.

Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers and γ be the element $\gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0; by the Frobenius reciprocity theorem and the second assertion of Proposition 7.1, we see that the multiplicity of the isotypic component $C^\infty_\gamma(T^*_{\mathbb{C}})$ of $C^\infty(T^*_{\mathbb{C}})$ is equal to the number of non-zero integers $r_1, \dots, r_{n-1}$. Since $\hat{f}_k$ is not a constant function, we know that $d\hat{f}_k$ is a highest weight vector of the G-module $C^\infty_{2\varpi_k}(T^*_{\mathbb{C}})$. Therefore the section $\hat{f}_{r_1,\dots,r_{n-1}} d\hat{f}_k$ is a highest weight vector of the G-module $C^\infty_{\gamma'}(T^*_{\mathbb{C}})$, where

$$\gamma' = \gamma^1_{r_1,\dots,r_{k-1},r_k+1,r_{k+1},\dots,r_{n-1}} = 2\varpi_k + \gamma^1_{r_1,\dots,r_{n-1}}.$$

We consider the sections $\varphi_1, \dots, \varphi_{n-1}$ of $T^*_{\mathbb{C}}$ defined by

$$\varphi_k = \tilde{f}_{r_1,\dots,r_{k-1},r_k-1,r_{k+1},\dots,r_{n-1}} d\tilde{f}_k,$$

for $1 \leqslant k \leqslant n-1$. Note that φ_k is non-zero if and only if the integer r_k is non-zero. If φ_k is non-zero, we have just seen that the complex conjugate $\bar{\varphi}_k$ of φ_k is a highest weight vector of the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})$. In the next section, we shall verify that the non-zero elements of the family $\{\varphi_1, \dots, \varphi_{n-1}\}$ are linearly independent. We know that the number of such elements of this family is equal to the number of non-zero integers r_k. On the other hand, we remarked above that the latter number is equal to the multiplicity of the isotypic component $C^\infty_\gamma(T^*_{\mathbb{C}})$. This multiplicity is also equal to the dimension of the weight subspace $\mathcal{W}_\gamma$ of $C^\infty_\gamma(T^*_{\mathbb{C}})$ corresponding to its highest weight γ. Therefore we have the following result:

Lemma 7.2. — *Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers and let γ be the element $\gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0. Then the non-zero members of the family $\{\bar{\varphi}_1, \dots, \bar{\varphi}_{n-1}\}$ associated with the integers $r_1, \dots, r_{n-1}$ form a basis for the space $\mathcal{W}_\gamma$.*

We consider the element $\gamma_1 = 2\varpi_1$ of Γ_0. By Lemma 7.2, we know that the G-module $C^\infty_{\gamma_1}(T^*_{\mathbb{C}})$ is irreducible and we have the equality

$$C^\infty_{\gamma_1}(T^*_{\mathbb{C}}) = dC^\infty_{\gamma_1}(X). \tag{7.3}$$

The complexification $U_{\mathbb{C}}$ of $U = \mathbb{R}^n$ is a G-module, and so the k-th symmetric products $S^kU_{\mathbb{C}}$ and $S^kU^*_{\mathbb{C}}$ inherit structures of G-modules. In fact, the highest weight of the irreducible G-module $S^2U_{\mathbb{C}}$ is equal to γ_1 and this module viewed as a K-module is isomorphic to $S^2U^*_{\mathbb{C}}$. We consider the standard Euclidean scalar product of U as an element q of S^2U^*. If $\{q\}$ is the complex subspace of $S^2U^*_{\mathbb{C}}$ generated by q, we have the decomposition

$$S^2U^*_{\mathbb{C}} = S^2_0U^*_{\mathbb{C}} \oplus \{q\}$$

of the G-module $S^2U^*_{\mathbb{C}}$ into irreducible K-modules; according to remarks made in §§2 and 6, we then see that

$$\mathrm{Hom}_K(E_{1,x_0,\mathbb{C}}, S^2U^*_{\mathbb{C}}) = \mathrm{Hom}_K(E_{2,x_0,\mathbb{C}}, S^2U^*_{\mathbb{C}}) = \{0\}.$$

Therefore from the Frobenius reciprocity theorem, we obtain the following result:

Lemma 7.3. — *We have*

$$C^\infty_{\gamma_1}(E_{1,\mathbb{C}}) = C^\infty_{\gamma_1}(E_{2,\mathbb{C}}) = \{0\}.$$

>From Lemma 7.3, the equality (7.3) and the decomposition (6.3), we obtain the decomposition

$$C^\infty_{\gamma_1}(S^2T^*_{\mathbb{C}}) = C^\infty_{\gamma_1}(X)\cdot g \oplus \tilde\sigma d C^\infty_{\gamma_1}(X). \tag{7.4}$$

Thus the multiplicity of the isotypic component $C^\infty_{\gamma_1}(S^2T^*_{\mathbb{C}})$ is equal to 2 and the weight subspace of this G-module corresponding to its highest weight γ_1 is generated by the sections $\hat f_1 g$ and $\tilde\sigma(d\hat f_1)$.

8. An algebraic result

Let $n \geqslant 3$ be a given integer and let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers which are not all zero. We set

$$d_k = \frac{1}{n}\Big(\sum_{j=k}^{n-1}(n-j)r_j - \sum_{j=1}^{k-1} jr_j\Big), \tag{8.1}$$

for $1 \leqslant k \leqslant n$. We note that

$$d_1 + d_2 = \frac{1}{n}\Big((n-2)r_1 + 2\sum_{k=2}^{n-1}(n-k)r_k\Big),$$

$$d_{n-1} + d_n = -\frac{1}{n}\Big((n-2)r_{n-1} + 2\sum_{k=1}^{n-2} kr_k\Big);$$

thus the first of these two numbers is always positive, while the second one is always negative.

In the next section, we shall require the following result:

Proposition 8.1. — *Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers which are not all zero. Let $a_1, \dots, a_{n-1}$ be given complex numbers satisfying*

$$a_k r_j + a_j r_k = 0, \tag{8.2}$$

for all $1 \leqslant j, k \leqslant n-1$ with $j+2 \leqslant k \leqslant n-1$. Let c be a complex number satisfying the relations

$$2ia_k r_k = c(d_k + d_{k+1}), \tag{8.3}$$

for $1 \leqslant k \leqslant n-1$, and

$$i(a_k r_{k+1} + a_{k+1} r_k) = -c d_{k+1}, \tag{8.4}$$

for $1 \leqslant k \leqslant n-2$. Then either the complex number c vanishes or we have

$$r_k = 0, \qquad r_1 = r_{n-1}, \qquad a_1 = -a_{n-1}, \qquad c = 2ia_1,$$

for all $2 \leqslant k \leqslant n-2$.

We now begin our proof of Proposition 8.1, which will be in several steps. Let $a_1, \dots, a_{n-1}$ be given complex numbers. If $r_k \geqslant 1$, with $1 \leqslant k \leqslant n-1$, we write $c_k = a_k/r_k$. Let $1 \leqslant j, k \leqslant n-1$ be integers satisfying $k \geqslant j+2$; if the integers r_j and r_k are non-zero, then the relation (8.2) implies that

$$c_j = -c_k. \tag{8.5}$$

LEMMA 8.2. — *Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers which are not all zero, and let $a_1, \dots, a_{n-1}, c$ be given complex numbers. Then the coefficient c vanishes whenever one of the following two conditions holds:*

(i) *The relations* (8.3) *hold and at least one of the integers r_1 and r_{n-1} vanishes.*

(ii) *The relations* (8.2) *and* (8.3) *hold, the integers r_1 and r_{n-1} are non-zero and there exists an integer $3 \leqslant k \leqslant n-3$ such that $r_k \geqslant 1$.*

Proof. — Under the hypotheses of (i), the inequalities $d_1 + d_2 > 0$ and $d_{n-1} + d_n < 0$ imply that $c = 0$. Under the assumptions of (ii), according to the relations (8.5) the coefficients c_1, c_k and c_{n-1} vanish. Since we have $a_1 = 0$ and $d_1 + d_2 > 0$, from the equality (8.3), with $k = 1$, we then deduce the vanishing of c. □

We now suppose that the hypotheses of Proposition 8.1 hold; according to Lemma 8.2, we know that the assertions of this proposition hold except possibly in the following cases:

(i) We have $n \geqslant 5$ and the integers r_1, r_2, r_{n-2} and r_{n-1} are the only non-zero integers of the family $\{r_1, \dots, r_{n-1}\}$.

(ii) We have $n \geqslant 5$ and the integers r_1, r_{n-2} and r_{n-1} are the only non-zero integers of the family $\{r_1, \dots, r_{n-1}\}$.

(iii) We have $n \geqslant 5$ and the integers r_1, r_2 and r_{n-1} are the only non-zero integers of the family $\{r_1, \dots, r_{n-1}\}$.

(iv) We have $n = 4$ and the three integers r_1, r_2 and r_3 are non-zero.

(v) We have $n \geqslant 3$, the integers r_1 and r_{n-1} are the only non-zero integers of the family $\{r_1, \dots, r_{n-1}\}$ and $r_1 \neq r_{n-1}$.

(vi) We have $n \geqslant 3$, the integers r_1 and r_{n-1} are the only non-zero integers of the family $\{r_1, \dots, r_{n-1}\}$ and $r_1 = r_{n-1}$.

We now proceed to show that the complex number vanishes c whenever one of the hypotheses (i)–(v) holds. In the following lemmas, we shall always assume that the hypotheses of Proposition 8.1 hold.

LEMMA 8.3. — *Suppose that $n \geqslant 5$ and that $r_k = 0$, for $3 \leqslant k \leqslant n-3$. Assume that*

$$r_1 + 2r_2 = 2r_{n-2} + r_{n-1}. \tag{8.6}$$

Then if either $r_2 \neq 0$ or $r_{n-2} \neq 0$, the coefficient c vanishes.

Proof. — According to (8.3), we have

$$2ia_1r_1 = c(r_1 + 2r_2), \qquad 2ia_{n-1}r_{n-1} = -c(2r_{n-2} + r_{n-1}),$$
$$2ia_2r_2 = cr_2, \qquad 2ia_{n-2}r_{n-2} = -cr_{n-2};$$

by (8.4), we see that

$$i(a_1r_2 + a_2r_1) = -cr_2, \qquad i(a_{n-2}r_{n-1} + a_{n-1}r_{n-2}) = -cr_{n-2}.$$

Since the determinant of the matrix

$$\begin{pmatrix} 2r_1 & 0 & -r_1 - 2r_2 \\ 0 & 2r_2 & -r_2 \\ r_2 & r_1 & r_2 \end{pmatrix}$$

is equal to

$$2r_2(r_1 + r_2)(r_1 + 2r_2),$$

when r_2 is non-zero, the equalities

$$2ia_1r_1 = c(r_1 + 2r_2), \qquad 2ia_2r_2 = cr_2, \qquad i(a_1r_2 + a_2r_1) = -cr_2$$

imply that the coefficients a_1, a_2 and c vanish. The determinant D of the matrix

$$\begin{pmatrix} 2r_{n-1} & 0 & r_{n-1} + 2r_{n-2} \\ 0 & 2r_{n-2} & r_{n-2} \\ r_{n-2} & r_{n-1} & r_{n-2} \end{pmatrix}$$

satisfies

$$D = -2r_{n-2}(r_{n-1}^2 + 2r_{n-2}^2 - r_{n-1}r_{n-2}) \leqslant -2r_{n-2}^3;$$

thus when r_{n-2} is non-zero, the equalities

$$2ia_{n-1}r_{n-1} = -c(2r_{n-2} + r_{n-1}),$$
$$2ia_{n-2}r_{n-2} = -cr_{n-2} = i(a_{n-2}r_{n-1} + a_{n-1}r_{n-2})$$

imply that the coefficients a_{n-2}, a_{n-1} and c vanish. □

LEMMA 8.4. — *Suppose that $n \geqslant 5$ and that $r_k = 0$, for $3 \leqslant k \leqslant n-3$. Suppose that one of the following conditions holds:*

(i) *We have $r_2 = 0$ or $r_{n-2} = 0$.*
(ii) *We have $n \geqslant 6$.*
(iii) *We have $n = 5$ and the integers r_1, r_2, r_3, r_4 are non-zero.*

Then either the coefficient c vanishes or the equality (8.6) holds.

Proof. — By (8.3), with $k=2$ and $r_2=0$, we see that

$$c(d_2+d_3)=\frac{2c}{n}\,(-r_1+2r_{n-2}+r_{n-1})=0;$$

on the other hand, by (8.3), with $k=n-2$ and $r_{n-2}=0$, we see that

$$c(d_{n-2}+d_{n-1})=\frac{2c}{n}\,(-r_1-2r_2+r_{n-1})=0.$$

If $n\geqslant 6$, our hypothesis tells us that $r_{n-3}=0$; then the relation (8.3), with $k=n-3$, implies that

$$c(d_{n-3}+d_{n-2})=\frac{2c}{n}\,(-r_1-2r_2+2r_{n-2}+r_{n-1})=0.$$

Finally, suppose that (iii) holds. Then by (8.5), we have

$$c_2=-c_4=c_1=-c_3.$$

Hence from (8.4), we obtain

$$0=i(a_2r_3+a_3r_2)=-cd_3=\frac{c}{n}\,(r_1+2r_2-2r_{n-2}-r_{n-1}).$$

$\square$

Lemma 8.5. — *Suppose that $n=4$ and that the integers r_1 and r_3 are non-zero. Then either the coefficient c vanishes, or we have the equalities*

$$r_2=0,\qquad r_1=r_3.$$

Proof. — By (8.3), we have

$$4ia_1r_1=c(r_1+2r_2+r_3)=-4ia_3r_3 \tag{8.7}$$

$$4ia_2r_2=c(r_3-r_1); \tag{8.8}$$

by (8.4), we also see that

$$4i(a_1r_2+a_2r_1)=c(r_1-2r_2-r_3). \tag{8.9}$$

>From (8.7), we deduce that

$$4ic_1=\frac{c}{r_1^2}\,(r_1+2r_2+r_3),\qquad 4ic_3=-\frac{c}{r_3^2}\,(r_1+2r_2+r_3).$$

According to (8.5), we also have the equality

$$c_3=-c_1.$$

The previous equations imply that

$$\frac{c}{r_1^2}=\frac{c}{r_3^2};$$

thus either the coefficient c vanishes or the equality $r_1 = r_3$ holds. We now suppose that $r_1 = r_3$. According to (8.7), we have

$$2ia_1 r_1 = c(r_1 + r_2).$$

When $r_2 \neq 0$, by (8.8) we see that $a_2 = 0$. Then the equation (8.9) says that

$$c + 2ia_1 = 0.$$

>From the two preceding equations, we infer that

$$c(2r_1 + r_2) = 0,$$

and so the coefficient c also vanishes in this case. □

>From Lemmas 8.3–8.5, we deduce that the coefficient c vanishes whenever one of the hypotheses (i)–(iv) holds.

Lemma 8.6. — *Suppose that $n = 3$ and that the integers r_1 and r_2 are non-zero. Then either the coefficient c vanishes, or we have the equalities*

$$r_1 = r_2, \qquad a_1 + a_2 = 0, \qquad c = 2ia_1.$$

Proof. — From (8.3) and (8.4), we obtain the equalities

$$\begin{gathered} 6ia_1 r_1 = c(r_1 + 2r_2), \qquad 6ia_2 r_2 = -c(2r_1 + r_2), \\ 3i(a_1 r_2 + a_2 r_1) = c(r_1 - r_2). \end{gathered} \tag{8.10}$$

Since the determinant of the matrix

$$\begin{pmatrix} r_1 & 0 & -r_1 - 2r_2 \\ 0 & r_2 & 2r_1 + r_2 \\ r_2 & r_1 & 2(r_2 - r_1) \end{pmatrix}$$

is equal to

$$(r_2 - r_1)(2r_1^2 + 3r_1 r_2 + 2r_2^2),$$

when $r_2 \neq r_1$, the coefficients a_1, a_2 and c vanish. If $r_1 = r_2$, then the equations (8.10) tell us that $a_1 + a_2 = 0$ and $c = 2ia_1$. □

Lemma 8.7. — *Suppose that $n \geqslant 3$. Assume that the integers r_1 and r_{n-1} are non-zero, and that $r_k = 0$, for $2 \leqslant k \leqslant n-2$. Then either the coefficient c vanishes, or we have the equalities*

$$r_1 = r_{n-1}, \qquad a_1 + a_{n-1} = 0, \qquad c = 2ia_1. \tag{8.11}$$

Proof. — Lemma 8.6 gives us the desired result when $n = 3$. We now suppose that $n \geqslant 4$. Then from Lemmas 8.4 and 8.5, either the coefficient c vanishes or we have the equality $r_1 = r_{n-1}$. We now suppose that this last relation is true; then by (8.2), we have $a_1 + a_{n-1} = 0$. On the other hand, the relation (8.3), with $k = 1$, says that $2ia_1 r_1 = cr_1$. □

Finally, note that Lemma 8.7 tells us that the assertions of Proposition 8.1 hold under the hypotheses (v) or (vi). This concludes the proof of this proposition.

9. Isospectral deformations of the Lagrangian Grassmannians

We pursue our study of the Lagrangian Grassmannian $X = G/K$. This section is devoted to the proof of the following proposition:

Proposition 9.1. — *We have*

$$D_0 C^\infty(T) \cap \tilde{\sigma} d C^\infty_{\mathbb{R}}(X) = \tilde{\sigma} d \mathcal{B}_{\mathbb{R}}, \qquad D_0 C^\infty(T_{\mathbb{C}}) \cap \tilde{\sigma} d C^\infty(X) = \tilde{\sigma} d \mathcal{B}.$$

We consider the orthogonal complement $\mathcal{F}$ of the finite-dimensional subspace $\mathcal{F}' = \mathbb{R}(X) \oplus \mathcal{B}_{\mathbb{R}}$ of $C^\infty_{\mathbb{R}}(X)$. According to Lemma 6.2, we know that the mapping

$$P_\sigma = P\tilde{\sigma} d : C^\infty_{\mathbb{R}}(X) \to I(X)$$

is well-defined. Proposition 9.1 tells us that the kernel of P_σ is the finite-dimensional space $\mathcal{F}'$ and that the mapping $P_\sigma : \mathcal{F} \to I(X)$ is injective.

Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers which are not all zero. We consider the element $\gamma = \gamma^1_{r_1,\dots,r_{n-1}}$ of Γ and the subspace $\mathcal{V}_\gamma$ of $C^\infty(T^*_{\mathbb{C}})$ generated by the 1-forms $\{\varphi_1, \dots, \varphi_{n-1}\}$, which we associated in §7 with the integers $r_1, \dots, r_{n-1}$. According to Lemma 7.2, the complex conjugate of the space $\mathcal{V}_\gamma$ is equal to the highest weight subspace $\mathcal{W}_\gamma$ of $C^\infty_\gamma(T^*_{\mathbb{C}})$. We consider the section

$$\varphi = \sum_{k=1}^{n-1} a_k \varphi_k$$

of $T^*_{\mathbb{C}}$, where $a_1, \dots, a_{n-1}$ are given complex numbers, and the 1-form

$$\vartheta = d\tilde{f}_{r_1,\dots,r_{n-1}} = \sum_{k=1}^{n-1} r_k \varphi_k$$

on X, which is also an element of $\mathcal{V}_\gamma$. For our proof of Proposition 9.1, we shall require the following result:

Lemma 9.2. — *Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers which are not all zero, and let $a_1, \dots, a_{n-1}$ be given complex numbers. Suppose that there is an element $c \in \mathbb{C}$ such that the 1-form*

$$\varphi = \sum_{k=1}^{n-1} a_k \varphi_k$$

satisfies the relation

$$D^1\varphi = c\tilde{\sigma}(d\tilde{f}_{r_1,\dots,r_{n-1}}). \tag{9.1}$$

Then either the coefficient c *vanishes or we have*

$$r_k = 0, \qquad r_1 = r_{n-2} = 1, \qquad \varphi = a(\tilde{f}_{n-1}\, d\tilde{f}_1 - \tilde{f}_1\, d\tilde{f}_{n-1}), \qquad c = -2ia,$$

for $2 \leqslant k \leqslant n-2$, *where* $a = a_1 = -a_{n-1}$.

We assume, without any loss of generality, that $a_k = 0$ whenever $r_k = 0$, for $1 \leqslant k \leqslant n-1$. We consider the function ψ on $\mathbb{R}^{n-1}$ defined by

$$\psi(\theta) = 2ie^{2i(r_1\theta_1+\cdots+r_{n-1}\theta_{n-1})},$$

for $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^{n-1}$; then we have

$$\xi_k \cdot \psi = 2ir_k\psi, \tag{9.2}$$

for $1 \leqslant k \leqslant n-1$. By (4.5), we see that

$$\iota^*\tilde{f}_{r_1,\dots,r_{n-1}} = \frac{1}{2i}\,\psi, \qquad \langle \xi_j, \iota^* d\tilde{f}_k\rangle = 2i\delta_{jk}\iota^*\tilde{f}_k,$$

for $1 \leqslant j, k \leqslant n-1$. Thus we have

$$\langle \xi_k, \iota^*\varphi\rangle = a_k\psi, \tag{9.3}$$

for $1 \leqslant k \leqslant n-1$. By formulas (6.14), (9.2) and (9.3), we obtain

$$\iota^*(D^1\varphi)(\zeta_k, \zeta_l) = i(a_k r_l + a_l r_k)\psi, \tag{9.4}$$

for $1 \leqslant k, l \leqslant n-1$.

If the 1-form φ vanishes, then according to formula (9.4), with $k = l$, we see that $a_k r_k = 0$, for all $1 \leqslant k \leqslant n-1$; hence in this case all the coefficients a_k vanish. In other words, the non-zero members of the family $\{\varphi_1, \dots, \varphi_{n-1}\}$ are linearly independent. This fact entered into the proof of Lemma 7.2.

According to (6.6) and (9.3), we have

$$\iota^*\vartheta(\zeta_k) = \langle \xi_k, \iota^*\vartheta\rangle = r_k\psi, \tag{9.5}$$

for $1 \leqslant k \leqslant n-1$; thus we obtain

$$\iota^*\vartheta(\eta_j) = d_j\psi, \tag{9.6}$$

for $1 \leqslant j \leqslant n$, where the number d_j is given by (8.1). We also consider the symmetric 2-form $h = \tilde{\sigma}(\vartheta)$. By (6.10), we have

$$h(\zeta_j, \zeta_k) = 0, \tag{9.7}$$

for $1 \leqslant j, k \leqslant n-1$, whenever $k \geqslant j+2$. Also by (6.11) and (6.12), we see that

$$h(\zeta_j, \zeta_j) = \vartheta(\eta_j + \eta_{j+1}), \qquad h(\zeta_k, \zeta_{k+1}) = -\vartheta(\eta_{k+1}), \tag{9.8}$$

for all $1 \leqslant j \leqslant n-1$ and $1 \leqslant k \leqslant n-2$.

We now begin the proof of Lemma 9.2. Let us suppose that there exists an element $c \in \mathbb{C}$ such that the equality (9.1) holds. Since the function ψ is everywhere non-vanishing, according to the equalities (9.4) and (9.6)–(9.8), we infer that the coefficients c and $a_1, \dots, a_{n-1}$ satisfy the relations (8.2)–(8.4). Hence by Proposition 8.1, we infer that the coefficient c vanishes unless the integers r_j and the coefficients a_j vanish, for all $2 \leqslant j \leqslant n-2$, and the relations

$$r_1 = r_{n-1} \geqslant 1, \qquad a_1 = -a_{n-1}, \qquad c = 2ia_1$$

hold. Thus if we consider the element

$$\beta = \tilde{f}_{n-1}\, d\tilde{f}_1 - \tilde{f}_1\, d\tilde{f}_{n-1}$$

of $C^\infty(T^*_{\mathbb{C}})$, we know that the coefficient c vanishes unless φ is a multiple of the 1-form $(\tilde{f}_1 \cdot \tilde{f}_{n-1})^r \beta$ and $\vartheta = d(\tilde{f}_1 \cdot \tilde{f}_{n-1})^{r+1}$, where $r = r_1 - 1 \geqslant 0$.

By (9.4), (9.6) and (9.8), we see that the relations

$$\begin{gathered} \iota^*(\mathrm{Hess}\ \tilde{f}_1)(\zeta_1, \zeta_1) = -4, \qquad \iota^*(\mathrm{Hess}\ \tilde{f}_1)(\zeta_1, \zeta_2) = 0, \\ \iota^*\tilde{\sigma}(d\tilde{f}_1)(\zeta_1, \zeta_1) = \iota^*\langle \eta_1 + \eta_2, d\tilde{f}_1\rangle = \frac{2i(n-2)}{n}, \\ \iota^*\tilde{\sigma}(d\tilde{f}_1)(\zeta_1, \zeta_2) = -\iota^*\langle \eta_2, d\tilde{f}_1\rangle = \frac{2i}{n} \end{gathered} \tag{9.9}$$

hold at the point 0 of $\mathbb{R}^{n-1}$.

Lemma 9.3.

(i) *We have the relations*

$$\mathrm{Hess}\ \tilde{f}_1 = -\frac{4}{n}\, \tilde{f}_1 g + 2i\tilde{\sigma}(d\tilde{f}_1), \tag{9.10}$$

$$\mathrm{Hess}\ \tilde{f}_{n-1} = -\frac{4}{n}\, \tilde{f}_{n-1} g - 2i\tilde{\sigma}(d\tilde{f}_{n-1}), \tag{9.11}$$

$$D^1\beta = 2i\tilde{\sigma}(d(\tilde{f}_1 \cdot \tilde{f}_{n-1})). \tag{9.12}$$

(ii) *For $\gamma = \gamma^1_{1,0,\dots,0,1}$, we have the inclusion*

$$\tilde{\sigma} d C^\infty_\gamma(X) \subset D^1 C^\infty(T^*_{\mathbb{C}}). \tag{9.13}$$

Proof. — Since the differential operator Hess is homogeneous and the G-module $C^\infty_{\gamma_1}(X)$ is irreducible, from the decomposition (7.4) we obtain the existence of constants $a, b \in \mathbb{C}$ such that

$$\text{Hess } f = afg + b\tilde{\sigma}(df),$$

for all $f \in C^\infty_{\gamma_1}(X)$. Since the complex conjugate $\hat{f}_1$ of $\tilde{f}_1$ is an element of $C^\infty_{\gamma_1}(X)$, from the relations (6.1), (6.7) and (9.9) we deduce that $a = -2/n$ and $b = -2i$, and so we obtain formula (9.10). As $\tilde{f}_{n-1}$ is an element of $C^\infty_{\gamma_1}(X)$, we have also verified the identity (9.11). By (1.2), we have

$$D^1\beta = \tilde{f}_{n-1} \text{ Hess } \tilde{f}_1 - \tilde{f}_1 \text{ Hess } \tilde{f}_{n-1};$$

the relation (9.12) is now a direct consequence of (9.10) and (9.11). Next the equality (9.12) implies that the symmetric 2-form $\tilde{\sigma}(d(\hat{f}_1 \cdot \hat{f}_{n-1}))$ belongs to $D^1C^\infty(T^*_{\mathbb{C}})$. Since the function $\hat{f}_1 \cdot \hat{f}_{n-1}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, where $\gamma = 2\varpi_1 + 2\varpi_{n-1}$, we obtain the inclusion (9.13). □

We denote by I_{n-2} the identity matrix of order $n-2$. For $\alpha \in \mathbb{R}$, we consider the element

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

of $SO(2)$ and the element $\phi_\alpha = (R_\alpha, I_{n-2})$ belonging to the subgroup

$$SO(2) \times SO(n-2)$$

of $K = SO(n)$; thus we have $\phi_\alpha(x_0) = x_0$. We consider the functions $\{f_{jl,\alpha}\}_{1 \leqslant j,l \leqslant n}$ on $\mathbb{R}^{n-1}$ defined by

$$f_{11,\alpha} = \cos^2\alpha \cdot e^{2ix_1} + \sin^2\alpha \cdot e^{2ix_2}, \quad f_{22,\alpha} = \sin^2\alpha \cdot e^{2ix_1} + \cos^2\alpha \cdot e^{2ix_2},$$

$$f_{12,\alpha} = f_{21,\alpha} = \cos\alpha \cdot \sin\alpha \cdot (e^{2ix_1} - e^{2ix_2}), \qquad f_{jl,\alpha} = e^{2ix_j}\delta_{jl}$$

whenever $1 \leqslant j \leqslant n$ and $3 \leqslant l \leqslant n$, or whenever $3 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant n$, where $\theta = (\theta_1, \dots, \theta_{n-1})$ and x_k is given by (3.8). We also consider the function $\check{f}_\alpha$ on $\mathbb{R}^{n-1}$ defined by

$$\check{f}_\alpha(\theta) = (f_{11,\alpha}f_{22,\alpha} - f^2_{12,\alpha})(\theta) \cdot e^{2i(\theta_{n-1}-\theta_2)},$$

for $\theta = (\theta_1, \dots, \theta_{n-1})$. Clearly the equalities

$$df_{11,\alpha} = 2i(\cos^2\alpha \cdot d\theta_1 + \sin^2\alpha \cdot (d\theta_2 - d\theta_1)), \qquad d\check{f}_\alpha = 2id\theta_{n-1} \tag{9.14}$$

hold at the point $0 \in \mathbb{R}^{n-1}$. We easily verify that

$$\iota^*\phi^*_\alpha\langle Z_j, Z_l\rangle = f_{jl,\alpha},$$

for all $1 \leqslant j, l \leqslant n$; it follows that

$$\iota^*\phi_\alpha^*\tilde{f}_1 = f_{11,\alpha}, \qquad \iota^*\phi_\alpha^*\tilde{f}_{n-1} = \check{f}_\alpha, \tag{9.15}$$

and hence that

$$(\phi_\alpha^*\tilde{f}_1)(x_0) = (\phi_\alpha^*\tilde{f}_{n-1})(x_0) = 1. \tag{9.16}$$

Thus according to (6.6), (9.14) and (9.15), the symmetric 2-form

$$h_\alpha = \phi_\alpha^*(d\tilde{f}_1 \cdot d\tilde{f}_1 - d\tilde{f}_{n-1} \cdot d\tilde{f}_{n-1})$$

on X satisfies

$$\left(\sum_{j=1}^{n-1} h_\alpha(\zeta_j, \zeta_j) + \sum_{j=1}^{n-2} h_\alpha(\zeta_j, \zeta_{j+1})\right)(x_0) = 24\cos^2\alpha \cdot \sin^2\alpha. \tag{9.17}$$

Clearly the function of α appearing on right-hand side of equation (9.17) is non-zero.

Let $r \geqslant 1$ be a given integer. We consider the function $f = \tilde{f}_1 \cdot \tilde{f}_{n-1}$ and, for $\alpha \in \mathbb{R}$, the symmetric 2-form

$$h'_\alpha = \phi_\alpha^*(f^{r-1}\, df \cdot \beta) \tag{9.18}$$

on X. Since we have

$$df \cdot \beta = \tilde{f}_{n-1}^2\, d\tilde{f}_1 \cdot d\tilde{f}_1 - \tilde{f}_1^2\, d\tilde{f}_{n-1} \cdot d\tilde{f}_{n-1},$$

by (9.16) we see that the equality

$$h'_\alpha = h_\alpha \tag{9.19}$$

holds at x_0.

Lemma 9.4. — *Let $r \geqslant 1$ be a given integer. Then the symmetric 2-form $D^1((\tilde{f}_1 \cdot \tilde{f}_{n-1})^r\beta)$ is not a section of $\tilde{\sigma}(T^*_{\mathbb{C}})$.*

Proof. — Suppose that $D^1(f^r\beta)$ is a section of $\tilde{\sigma}(T^*_{\mathbb{C}})$. According to formulas (9.12) and (1.2), we see that $f^{r-1}df \cdot \beta$ is also a section of $\tilde{\sigma}(T^*_{\mathbb{C}})$. Thus for all $\alpha \in \mathbb{R}$, the symmetric 2-form h'_α defined by (9.18) satisfies the relation (6.13). The equalities (9.17) and (9.19) now lead us to a contradiction. $\square$

Finally, in order to finish the proof of Lemma 9.2, we suppose that φ is a multiple of the 1-form $(\tilde{f}_1 \cdot \tilde{f}_{n-1})^r\beta$ and that $\vartheta = d(\tilde{f}_1 \cdot \tilde{f}_{n-1})^{r+1}$, where $r = r_1 - 1 \geqslant 0$. When $r \geqslant 1$, Lemma 9.4 tells us that $\varphi = 0$; because the form $\tilde{\sigma}(\vartheta)$ is non-zero, the coefficient c must also vanish in this case, and so we have completed the proof of Lemma 9.2.

In order to prove Proposition 9.1, by formula (1.1) it suffices to show that

$$D^1C^\infty(T^*_{\mathbb{C}}) \cap \tilde{\sigma} dC^\infty(X) = \tilde{\sigma} d\mathcal{B}.$$

Since the differential operators D^1 and $\tilde{\sigma} d$ are homogeneous, according to Proposition 2.1 of [1] and Lemma 9.3,(ii) the preceding equality holds if and only if

$$D^1C^\infty_\gamma(T^*_{\mathbb{C}}) \cap \tilde{\sigma} dC^\infty_\gamma(X) = \{0\}, \tag{9.20}$$

for all $\gamma \in \Gamma$, with $\gamma \neq \gamma^1_{1,0,\dots,0,1}$. We now proceed to verify that (9.20) holds and, in the process, complete the proof of Proposition 9.1.

If $\gamma \in \Gamma$ is equal to 0 or does not belong to Γ_0, we know that

$$dC^\infty_\gamma(X) = \{0\},$$

and so the equality (9.20) holds. Now let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers, which are not all zero and satisfy

$$(r_1, \dots, r_{n-1}) \neq (1, 0, \dots, 0, 1),$$

and consider the element $\gamma = \gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0. Suppose that the equality (9.20) does not hold for this element γ. Since the function $f = \hat{f}_{r_1,\dots,r_{n-1}}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, the inclusion

$$\tilde{\sigma} dC^\infty_\gamma(X) \subset D^1C^\infty_\gamma(T^*_{\mathbb{C}})$$

holds, and so there exists an element ψ of $\mathcal{W}_\gamma$ such that

$$D^1\psi = \tilde{\sigma}(df).$$

Then the element $\varphi = \bar{\psi}$ of $\mathcal{V}_\gamma$ satisfies the relation (9.1), with $c = 1$. Lemma 9.2 now leads us to a contradiction. Therefore the equality (9.20) holds for all $\gamma \in \Gamma$, and so we have proved Proposition 9.1.

10. The reduced Lagrangian Grassmannians

The center S of $G = SU(n)$ is the cyclic subgroup of order n given by (5.5). If s is the involution of G considered in §6 sending a matrix into its complex conjugate, the subgroup

$$K_S = \big\{ A \in G \mid A^{-1}s(A) \in S \big\}$$

of G contains the subgroup $K = SO(n)$ and we easily verify that

$$K_S = \big\{ A \in G \mid A = e^{ik\pi/n}B, \text{ with } B \in O(n) \text{ and } k \in \mathbb{Z} \big\}.$$

According to Corollary 9.3, Chapter VII of [4] (see also [2, §1]), we know that $Y = G/K_S$ is a symmetric space of compact type, which is the reduced space of X and which we call the reduced Lagrangian Grassmannian.

The diagonal matrix $B_0 = \operatorname{diag}(1, \ldots, 1, -1)$ is an element of $O(n)$ and the diagonal matrix $A_0 = e^{i\pi/n} B_0$ belongs to G. We easily see that A_0 generates a cyclic subgroup S' of G of order $2n$ and that A_0^k belongs to K if and only if $k \equiv 0 \bmod n$. Thus we have

$$K_S = K \cdot S' = \bigcup_{0 \leqslant k \leqslant n-1} K \cdot A_0^k.$$

If $B \in K$, the matrix

$$A_0^{-1} B A_0 = B_0 B B_0$$

also belongs to K. We consider the automorphism ϕ_0 of $\mathfrak{p}_0$ defined by

$$\phi_0(\xi) = B_0 \cdot \xi \cdot B_0,$$

for $\xi \in \mathfrak{p}_0$; then for $\xi \in \mathfrak{p}_0$, the element $A_0^{-1} \xi A_0$ of $\mathfrak{g}_0$ is clearly equal to the element $\phi_0(\xi)$ of $\mathfrak{p}_0$. According to these observations, the right action of A_0 on G passes to the quotient X and gives rise to an isometry τ of X. In fact, if we denote by A_{0*} the action on the tangent bundle of X induced by the left action of A_0 on X and identify $\mathfrak{p}_0$ with T_{x_0}, we see that

$$\tau_* \xi = A_{0*} \phi_0(\xi), \tag{10.1}$$

for $\xi \in \mathfrak{p}_0$. Clearly the group Σ of isometries of X generated by τ is a cyclic group of order n which acts freely on X; moreover, its action commutes with the action of G on X. The quotient of X by Σ is the symmetric space $Y = G/K_S$ and the natural projection $\varpi : X \to Y$ is a n-fold covering; moreover, the action of the group G on X passes to the quotient Y.

We consider the G-submodule $C^\infty(X)^\Sigma$ of $C^\infty(X)$ consisting of all Σ-invariant (or equivalently τ-invariant) functions on X. The space $C^\infty(Y)$ inherits a G-module structure from the action of G on Y and the projection ϖ induces an isomorphism

$$\varpi^* : C^\infty(Y) \to C^\infty(X)^\Sigma$$

of G-modules. If γ is an element of Γ, we denote by $C^\infty_\gamma(Y)$ the isotypic component of the G-module $C^\infty(Y)$ corresponding to γ and we write

$$C^\infty_\gamma(X)^\Sigma = C^\infty(X)^\Sigma \cap C^\infty_\gamma(X).$$

Then the isomorphism ϖ induces an isomorphism of G-modules

$$\varpi^* : C^\infty_\gamma(Y) \to C^\infty_\gamma(X)^\Sigma.$$

For $p \geqslant 2$, we easily see that

$$\sigma_p(\phi_0(\xi_1), \dots, \phi_0(\xi_p)) = \sigma_p(\xi_1, \dots, \xi_p),$$

for all $\xi_1, \dots, \xi_p \in \mathfrak{p}_0$. Therefore from the relation (10.1), we infer that

$$\tau^* \sigma_p = \sigma_p. \tag{10.2}$$

Thus the symmetric p-form σ_p induces an G-invariant symmetric p-form $\sigma_{Y,p}$ on Y such that

$$\sigma_p = \varpi^* \sigma_{Y,p}.$$

We shall always consider the symmetric space $Y = G/K_S$ endowed with the G-invariant Riemannian metric $g_Y = \sigma_{Y,2}$. For $p \geqslant 3$, we consider the monomorphism of vector bundles

$$\tilde{\sigma}_{Y,p} : T^*_Y \to S^{p-1} T^*_Y$$

induced by the symmetric p-form $\sigma_{Y,p}$. We write $\sigma_Y = \sigma_{Y,3}$ and $\tilde{\sigma}_Y = \tilde{\sigma}_{Y,3}$. If φ is a 1-form on Y, we have

$$\varpi^* \tilde{\sigma}_{Y,p}(\varphi) = \tilde{\sigma}_p(\varpi^* \varphi). \tag{10.3}$$

According to Lemma 1.1 of [2] and Lemma 6.2, we see that a 1-form φ on Y satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}_{Y,p}(\varphi)$ on Y satisfies the Guillemin condition.

If $B = (b_{lr})$ is an arbitrary element of M_n, we have

$$Z_j(BA_0) = e^{i\pi/n}(b_{j1}, \dots, b_{j,n-1}, -b_{jn}),$$

for $1 \leqslant j \leqslant n$; therefore we obtain the equality

$$\langle Z_j, Z_k \rangle (BA_0) = e^{2i\pi/n} \langle Z_j, Z_k \rangle (B),$$

for $1 \leqslant j, k \leqslant n$. It follows that

$$\tau^* \tilde{f} = e^{2i\pi/n} \tilde{f}, \qquad \tau^* \tilde{f}_k = e^{2ik\pi/n} \tilde{f}_k, \tag{10.4}$$

for all $f \in \mathcal{H}$ and $1 \leqslant k \leqslant n-1$.

Let $r_1, \dots, r_{n-1} \geqslant 0$ be given integers and consider the element $\gamma = \gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0. According to (10.4), the function $\tilde{f}_{r_1,\dots,r_{n-1}}$ on X is invariant under the isometry τ if and only if the relation (5.7) holds. Since the complex conjugate $\hat{f}_{r_1,\dots,r_{n-1}}$ of the function $\tilde{f}_{r_1,\dots,r_{n-1}}$ belongs to the irreducible G-module $C^\infty_\gamma(X)$, we infer that $C^\infty_\gamma(X)$ is a G-submodule of $C^\infty(X)^\Sigma$ if and only the relation (5.7) holds. For $1 \leqslant j \leqslant n-1$, a section φ_j of $T^*_{\mathbb{C}}$ associated with the integers $r_1, \dots, r_{n-1}$ is Σ-invariant if and only if the integers $r_1, \dots, r_{n-1}$ satisfy the relation (5.7).

We denote by Γ_1 the subset of Γ_0 consisting of all elements $\gamma^1_{r_1,\dots,r_{n-1}}$ of Γ_0, where $r_1,\dots,r_{n-1} \geqslant 0$ are integers satisfying the relation (5.7). Then by Proposition 2.1 of [1], we have the following result:

LEMMA 10.1.

(i) *Let $r_1,\dots,r_{n-1} \geqslant 0$ be given integers. The function $\tilde{f}_{r_1,\dots,r_{n-1}}$ on the special Lagrangian Grassmannian $X = G/K$ is induced by a function on the reduced space $Y = G/K_S$ of X if and only if the relation* (5.7) *holds.*

(ii) *The G-module $\bigoplus_{\gamma\in\Gamma_1} C^\infty_\gamma(X)$ is a dense submodule of $C^\infty(X)^\Sigma$ and the G-module $\bigoplus_{\gamma\in\Gamma_1} C^\infty_\gamma(Y)$ is a dense submodule of $C^\infty(Y)$.*

We consider the element $\gamma = \gamma^1_{1,0,\dots,0,1}$ of Γ_1; we know that

$$C^\infty_\gamma(X)^\Sigma = \mathcal{B}.$$

Therefore $\mathcal{B}_Y = C^\infty_\gamma(Y)$ is an irreducible G-module isomorphic to $\mathcal{B}$ and is invariant under conjugation; thus $\mathcal{B}_Y$ is equal to the complexification of the subspace

$$\mathcal{B}_{Y,\mathbb{R}} = \left\{ f \in \mathcal{B}_Y \mid f = \bar{f} \right\}$$

of $C^\infty_{\mathbb{R}}(Y)$ and the mapping ϖ induces an isomorphism $\varpi^* : \mathcal{B}_{Y,\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$.

If P denotes the orthogonal projection corresponding to the decomposition (1.3) on the space Y, according to Lemma 1.1 of [2] and Lemma 6.2 the mapping

$$P_{\sigma_Y} = P\tilde{\sigma}_Y\, d : C^\infty_{\mathbb{R}}(Y) \to I(Y) \tag{10.5}$$

is well-defined. We denote by $\mathcal{F}_Y$ the orthogonal complement of the finite-dimensional space $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_Y$ in $C^\infty_{\mathbb{R}}(Y)$. From Proposition 1.2 of [2] and Proposition 9.1, we obtain:

THEOREM 10.2. — *The reduced Lagrangian Grassmannian Y is not rigid in the sense of Guillemin. If f is a non-zero element of $\mathcal{F}_Y$, then the symmetric 2-form $\tilde{\sigma}_Y(df)$ on Y satisfies the Guillemin condition and is not a Lie derivative of the metric. Moreover, the relation*

$$D_0 C^\infty(T_Y) \cap \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y) = \tilde{\sigma}_Y d\mathcal{B}_Y$$

holds and the kernel of the mapping (10.5) *is the finite-dimensional space*

$$\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_{Y,\mathbb{R}}.$$

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