ANNALES

DE

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Tome 57, nº 5 (2007), p. 1509-1520.

<http://aif.cedram.org/item?id=AIF_2007__57_5_1509_0>

Ann. Inst. Fourier, Grenoble
57, 5 (2007) 1509-1520

A.E. CONVERGENCE OF SPECTRAL SUMS ON LIE GROUPS

**by Christopher MEANEY,
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Abstract. — Let $\mathcal{L}$ be a right-invariant sub-Laplacian on a connected Lie group G, and let $S_R f := \int_0^R dE_\lambda f,\ R \geqslant 0$, denote the associated "spherical partial sums," where $\mathcal{L} = \int_0^\infty \lambda \, dE_\lambda$ is the spectral resolution of $\mathcal{L}$. We prove that $S_R f(x)$ converges a.e. to $f(x)$ as $R \to \infty$ under the assumption $\log(2 + \mathcal{L})f \in L^2(G)$.

Résumé. — Soit $\mathcal{L}$ un sous-Laplacien invariant à droite sur un groupe de Lie G, et soit $S_R f := \int_0^R dE_\lambda f,\ R \geqslant 0$, l'opérateur "sommes sphériques partielles" associé, où $\mathcal{L} = \int_0^\infty \lambda \, dE_\lambda$ dénote la résolution spectrale de $\mathcal{L}$. Nous prouvons que $S_R f(x)$ converge vers $f(x)$ p.p. quand $R \to \infty$, si $\log(2 + \mathcal{L})f \in L^2(G)$.

1. Introduction

In [2] Carbery and Soria proved a.e. convergence of the spherical partial sums (or better integrals) of functions belonging to the logarithmic Sobolev space $\{f : \log(2 + \Delta)f \in L^2(\mathbb{R}^n)\}$, where $-\Delta$ denotes the Laplacian.

Recently the same result has been obtained [4] by a very short and simple proof which works not only over dilates of the sphere, but more generally over dilates of any fixed closed bounded region which is star shaped with respect to the origin and has the origin in its interior. The proof in [4] is based on the Rademacher-Menshov theorem. Since it makes use of very basic principles, it lends itself for generalisations. In this paper we extend it to arbitrary connected Lie groups, the "spherical partial sums" being defined in terms of the spectral resolution of a given sub-Laplacian $\mathcal{L}$. A

Keywords: Rademacher-Menshov theorem, sub-Laplacian, spectral theory.
Math. classification: 22E30, 43A50.

main tool is a Plancherel theorem for functions of $\mathcal{L}$, which had been proved by Hulanicki and Jenkins [6] in the case of stratified nilpotent Lie groups, with a modified proof by Christ in [3], which extends easily to the case of arbitrary connected Lie groups.

1.1. Preliminaries

Let G be a connected Lie group, with left-invariant Haar measure dx and right invariant Haar measure $\Delta(x)^{-1}dx$, where $\Delta(x)$ denotes the modular function. For $1 \leqslant p \leqslant \infty$, $L^p(G)$ will denote the Lebesgue space with respect to the left-invariant Haar measure dx. The following relations involving the modular function Δ will be used without further mentioning:

$$\begin{aligned} \int_G f(xg)dx &= \Delta(g)^{-1} \int_G f(x)dx \,, \\ \int_G f(x)dx &= \int_G f(x^{-1})\Delta(x)^{-1}dx \,. \end{aligned}$$

Assume $X_1, \ldots, X_k$ to be a family of right-invariant vector fields on G satisfying Hörmander's condition, which in this context means that $X_1, \ldots, X_k$ generate the Lie algebra $\mathfrak{g}$ of G. Let

$$\mathcal{L} := -\sum_{j=1}^{k} X_j^2$$

be the associated sub-Laplacian. $\mathcal{L}$ is right-invariant and essentially self-adjoint on $\mathcal{D}(G) \subset L^2(G)$. With a slight abuse of notation, its closure will also be denoted by $\mathcal{L}$. Moreover it is known (see, e.g., [7]) that there exist $h_t \in L^1(G),\ t > 0$, the heat kernels, such that for all $f \in L^2(G)$

$$e^{-t\mathcal{L}} f(x) = h_t * f(x) = \int_G h_t(y) f(y^{-1}x)\, dy. \tag{1.1}$$

The $\{h_t\}_{t>0}$ form a one-parameter semigroup of smooth probability measures with respect to convolution.

Proposition 1.1. — *Let G be as above, and denote by e the identity element of G. Then there exists a constant $c > 0$ such that*

$$0 < h_t(e) \leqslant ct^{-\alpha} \,, \quad 0 < t < 1 \,, \tag{1.2}$$

where $\alpha = \alpha(G) > 0$ is the "local homogeneous dimension" of G associated to the vector fields X_j.

Proof. — (1.2) follows from Theorem V.4.3 and Theorem IX.1.3 in [8]. □

Notice also that the Gaussian estimates for h_t in [8] imply in particular that $h_t \in L^2(G)$ for $t > 0$.

PROPOSITION 1.2. — *The following relations hold true:*

$$h_{2t}(e) = h_t * h_t(e) = \int_G h_t(y) h_t(y^{-1})\, dy; \tag{1.3}$$

$$h_t(y) = h_t(y^{-1})\Delta(y)^{-1}; \tag{1.4}$$

$$\int_G h_t(y)^2 \Delta(y)\, dy = \int_G h_t(y)^2\, dy. \tag{1.5}$$

Proof. — (1.3) is clear. Since $e^{-t\mathcal{L}}$ is a self-adjoint operator, we have $(e^{-t\mathcal{L}} f, g) = (f, e^{-t\mathcal{L}} g)$, that is, by (1.1),

$$(h_t * f, g) = (f, h_t * g), \qquad f, g \in \mathcal{D}(G)\,. \tag{1.6}$$

It is well-known that for $\varphi \in L^1(G)$

$$(\varphi * f, g) = (f, \varphi^* * g)\,,$$

if one defines

$$\varphi^*(x) := \Delta(x)^{-1}\overline{\varphi(x^{-1})}\,.$$

Thus $h_t = h_t^*$, which yields (1.4).

There remains to establish (1.5). By (1.3) and (1.4), we know that

$$h_{2t}(e) = \int_G h_t(y) h_t(y^{-1})\, dy = \int_G h_t(y) h_t(y) \Delta(y)\, dy = \int_G h_t(y)^2 \Delta(y)\, dy < \infty\,.$$

Replacing y by y^{-1}, we obtain

$$\int_G h_t(y)^2 \Delta(y)\, dy = \int_G h_t(y^{-1})^2 \Delta(y)^{-1}\Delta(y)^{-1} dy\,.$$

In combination with (1.4), this is (1.5). □

The operator $\mathcal{L}$, being self-adjoint, admits a spectral decomposition on $L^2(G)$,

$$\mathcal{L} = \int_0^\infty \lambda\, dE_\lambda\,. \tag{1.7}$$

Associated to $\mathcal{L}$ we consider the following "spherical" partial sum operators $S_R,\ R \geqslant 0$, given by

$$S_R f := \int_0^R dE_\lambda f, \qquad f \in L^2(G)\,. \tag{1.8}$$

Clearly, since $\mathcal{L}$ is right-invariant, S_R is right-invariant and bounded on $L^2(G)$. Denote by χ_A the characteristic function of a set A. From the Schwartz kernel theorem it follows that there exists a unique convolution kernel $K_R \in \mathcal{D}'(G)$ associated to the spectral multiplier $\chi_{[0,R]}$ so that

$$S_R f = K_R * f, \qquad f \in \mathcal{D}(G)\,. \tag{1.9}$$

1.2. The main result

We shall prove the following main result:

THEOREM 1.3. — *Let G and S_R be as above, and assume that $\log(2+\mathcal{L})f \in L^2(G)$. Then $\lim_{R\to\infty} S_R f(x) = f(x)$ for almost every $x \in G$.*

Moreover, for every compact subset K of G there exists a constant $c > 0$ such that

$$\int_K \Big| \sup_{R\geqslant 0} |S_R f(x)| \Big|^2 dx \leqslant c \parallel \log(2+\mathcal{L})f \parallel_2^2 \,. \tag{1.10}$$

The strategy of proof follows the one developed in [4] in the case $G = \mathbb{R}^n$ and $\mathcal{L} = -\Delta$, the standard Laplacian, which in return is based on the following classical result (see, e.g., [1], [9]):

THEOREM 1.4 (Rademacher-Menshov). — *Suppose that (X,μ) is a positive measure space. There is a positive constant c with the following property:*

For each orthogonal subset $\{f_n : n \in \mathbb{N}\}$ in $L^2(X,\mu)$ satisfying

$$\sum_{n=0}^{\infty} (\log(n+2))^2 \parallel f_n \parallel_2^2 < \infty\,, \tag{1.11}$$

the maximal function $F^*(x) := \sup_{N\in\mathbb{N}} |\sum_{n=0}^{N} f_n(x)|$ is in $L^2(X,\mu)$, and

$$\parallel F^* \parallel_2 \leqslant c \left(\sum_{n=0}^{\infty} (\log(n+2))^2 \parallel f_n \parallel_2^2 \right)^{1/2} . \tag{1.12}$$

In particular, when (1.11) *holds, then the series $\sum_{n=0}^{\infty} f_n(x)$ converges almost everywhere on X.*

We also need a Plancherel theorem for functions of the sub-Laplacian. Denote by $L^\infty_{\mathcal{B}}([0,\infty))$ the space of all Borel measurable essentially bounded "multipliers" on $[0,\infty)$. Arguing similarly as for the operators S_R, we see that to any spectral multiplier $\sigma \in L^\infty_{\mathcal{B}}([0,\infty))$ corresponds a unique distribution $K_\sigma \in \mathcal{D}'(G)$ such that

$$\sigma(\mathcal{L})f = K_\sigma * f \quad \text{for all } f \in \mathcal{D}(G)\,.$$

2. Plancherel's theorem for functions of $\mathcal{L}$

In this section we prove a Plancherel theorem which essentially goes back to Hulanicki and Jenkins (see [6], formula (1.8)), at least in the case of a stratified nilpotent Lie group. The latter result had been reproved by Christ in [3], by a different proof. What comes very handy for us about Christ's proof is that it extends in a straight-forward way to arbitrary connected Lie groups.

PROPOSITION 2.1. — *There exists a unique sigma finite positive Borel measure ω on $[0,\infty)$ such that the following holds:*

If $\sigma \in L^\infty_{\mathcal{B}}([0,\infty))$, then $K_\sigma \in L^2(G)$ if and only if

$$\int_0^\infty |\sigma(\lambda)|^2 d\omega(\lambda) < \infty\,.$$

Moreover, then

$$\| K_\sigma \|_2^2 = \int_0^\infty |\sigma(\lambda)|^2 d\omega(\lambda)\,.$$

Proof. — The proof follows very closely the proof of Proposition 3 in ([3], pp. 79-80): Observe first that if $\sigma, \tau \in L^\infty_{\mathcal{B}}([0,b])$ with $K_\tau \in L^2(G)$, then

$$\sigma(\mathcal{L})K_\tau = K_{\sigma\tau} = K_{\tau\sigma}\,. \tag{2.1}$$

This can be shown as in [3]. Next, fix $b>0$. Following Hulanicki and Stein (see [5]), we conclude with (2.1) that

$$K_\sigma \in L^2(G) \quad \text{for every} \quad \sigma \in L^\infty_{\mathcal{B}}([0,b])\,.$$

Indeed, since $\sigma(\lambda) = (\sigma \exp)(\lambda)e^{-\lambda}$, we have $K_\sigma = (\sigma\exp)(\mathcal{L})h_1 \in L^2(G)$, since $h_1 \in L^2(G)$.

In particular $\Gamma := K_\chi \in L^2(G)$, where $\chi := \chi_{[0,b]}$. For $t>0$, set

$$\phi_t := e^{-t\mathcal{L}}\Gamma = K_{\chi\exp(-t\,\cdot)} = \chi(\mathcal{L})h_t\,.$$

Consider any $\sigma \in L^\infty_{\mathcal{B}}([0,b])$. As in [3], by means of (2.1), one easily shows that

$$(\phi_t, |\sigma|^2(\mathcal{L})\phi_t) = \int_0^\infty e^{-2t\lambda}|\sigma|^2(\lambda)\,d(\Gamma, E_\lambda\Gamma) \to \int_0^\infty |\sigma|^2(\lambda)\,d(\Gamma, E_\lambda\Gamma)$$

as $t \to 0$, where $d(\Gamma, E_\lambda\Gamma)$ is the positive, finite Borel measure that assigns to each Borel set $B \subset [0,+\infty[$ the mass $(\Gamma, \chi_B(\mathcal{L})\Gamma)$.

On the other hand, since $\phi_t = \chi(\mathcal{L})h_t$, one easily shows that

$$(\phi_t, |\sigma|^2(\mathcal{L})\phi_t) = \| \sigma(\mathcal{L})h_t \|_2^2\,.$$

And, by (2.1),

$$\sigma(\mathcal{L})h_t = K_{\sigma\exp(-t\,\cdot)} = e^{-t\mathcal{L}}K_\sigma\,,$$

so that $\sigma(\mathcal{L})h_t \to K_\sigma$ in $L^2(G)$ as $t \to 0$, by spectral theory. We conclude that

$$\| K_\sigma \|_2^2 = \int_0^b |\sigma|^2(\lambda)\, d(\Gamma, E_\lambda \Gamma)$$

for every $\sigma \in L^\infty_{\mathcal{B}}([0,b])$.

At most one Borel measure on $[0,b]$ can have this property, and b is arbitrary, so there exists a unique sigma-finite Borel measure ω on $[0,\infty)$ satisfying

$$\| K_\sigma \|_2^2 = \int_0^\infty |\sigma(\lambda)|^2\, d\omega(\lambda)$$

for all σ with compact support on $[0,+\infty)$.

The claim then follows for arbitrary $\sigma \in L^\infty_{\mathcal{B}}([0,+\infty))$ by standard approximation arguments. $\square$

Corollary 2.2. — *The following identities hold true:*

$$\begin{aligned} \| h_t \|_2^2 &= \int_0^\infty e^{-2t\lambda} d\omega(\lambda) \quad \text{for every} \quad t > 0\,; \\ \| K_R \|_2^2 &= \int_0^R d\omega(\lambda)\,. \end{aligned}$$

3. On inverse Laplace transforms

Observe next that, by Proposition 1.2,

$$h_{2t}(e) = \int_G h_t(y)^2\, dy\,,$$

so that, as a function of t,

$$h_t(e) = \int_0^\infty e^{-t\lambda}\, d\omega(\lambda), \quad t > 0\,,$$

is the Laplace-transform of the measure ω.

Lemma 3.1. — *Let ω be a positive sigma finite Borel measure on $[0,\infty)$, and denote by*

$$\Omega(t) := \int_0^\infty e^{-t\lambda}\, d\omega(\lambda), \quad t > 0\,,$$

its Laplace transform. Moreover, put

$$W(R) := \int_0^R d\omega(\lambda) = \omega([0,R])\,.$$

(i) *If there exist constants $a, C \geqslant 0$ such that*

$$\Omega(t) \leqslant Ct^{-a} \quad \textit{for } t \ll 1, \tag{3.1}$$

then there is a constant C_1 such that

$$W(R) \leqslant C_1 R^a \quad \text{for } R \gg 1. \tag{3.2}$$

(ii) *If in addition to* (3.1), *also an estimate from below*

$$\Omega(t) \geqslant C_0 t^{-a}, \quad t \ll 1\,, \tag{3.3}$$

holds for some $C_0 > 0$, then there is a constant $C_2 > 0$ such that W also satisfies an inverse inequality

$$W(R) \geqslant C_2 R^a \quad \text{for } R \gg 1.$$

Proof. — Let $R > 0$. An integration by parts shows that

$$\Omega(t) \geqslant \int_0^R e^{-t\lambda}\, d\omega(\lambda) = e^{-tR} W(R) + t \int_0^R e^{-t\lambda} W(\lambda)\, d\lambda\,.$$

Since $\omega \geqslant 0$, this implies, by (3.1),

$$e^{-tR} W(R) \leqslant Ct^{-a} \quad \text{for } t \ll 1\,,$$

i.e.,

$$W(R) \leqslant Ct^{-a} e^{tR}, \quad \text{for every} \quad t \ll 1\,.$$

The right-hand side is minimal for $t = \frac{a}{R}$. Thus, for R sufficiently large, we obtain

$$W(R) \leqslant C\left(\frac{a}{R}\right)^{-a} e^a = C_1 R^a\,,$$

with $C_1 = Ca^{-a}e^a$. This proves (i).

To prove (ii), assume that (3.1) and (3.3) hold. Then, by (3.3), for $0 < t \ll 1$,

$$C_0 \leqslant \int_0^R t^a e^{-t\lambda}\, d\omega(\lambda) + \int_R^\infty t^a e^{-t\lambda}\, d\omega(\lambda) = I + II\,. \tag{3.4}$$

Again, an integration by parts yields

$$II = t^a e^{-t\lambda} W(\lambda)\Big|_R^\infty + \int_R^\infty t^{a+1} e^{-t\lambda} W(\lambda)\, d\lambda.$$

Because of (3.2), the boundary term at ∞ vanishes, and thus

$$II \leqslant \int_R^\infty t^{a+1} e^{-t\lambda} W(\lambda)\, d\lambda\,.$$

Using again (3.2), for R sufficiently large, we obtain

$$II \leqslant C_1 \int_R^\infty t^{a+1} e^{-t\lambda} \lambda^a\, d\lambda = C_1 \int_{Rt}^\infty \lambda^a e^{-\lambda}\, d\lambda\,.$$

Choose $A > 0$ so large that $C_1 \int_A^\infty \lambda^a e^{-\lambda} d\lambda \leqslant C_0/2$, and put $t = \frac{A}{R}$. Then we have $II \leqslant C_0/2$, so that, by (3.4),

$$\frac{C_0}{2} \leqslant I = \int_0^R \left(\frac{A}{R}\right)^a e^{-\frac{A}{R}\lambda}\, d\omega(\lambda) \leqslant A^a R^{-a} W(R)\,,$$

hence $W(R) \geqslant (\frac{C_0}{2} A^{-a}) R^a$. □

Corollary 3.2. — *Let ω be the "Plancherel measure" from Proposition 2.1, and let $W(R) := \int_0^R d\omega(\lambda)$, $R \geqslant 0$. Then there is a constant $C_1 > 0$ such that*

$$W(R) \leqslant C_1 R^\alpha, \quad R \gg 1\,,$$

where $\alpha = \alpha(G)$ is as in (1.2).

Proof. — By Corollary 2.2, Proposition 1.2 and Proposition 1.1 we have

$$\int_0^\infty e^{-2t\lambda} d\omega(\lambda) = \| h_t \|_2^2 = h_{2t}(e) \leqslant c(2t)^{-\alpha}\,,$$

if $0 < t < 1/2$. The estimate therefore follows from Lemma 3.1 (i). □

Remarks 3.3. — a) The results in [8] even show that $h_t(e) \sim t^{-\alpha}$, $0 < t < 1$. Lemma 3.1 (i)(ii) therefore implies that in fact $W(R) \sim R^\alpha$ as $R \to \infty$. In particular,

$$\lim_{R\to\infty} W(R) = +\infty. \tag{3.5}$$

The latter fact can also be derived easily from Proposition 2.1:

Suppose that $\sup_{R>0} W(R) < \infty$. Then $\omega([0,\infty)) < \infty$, so that, by Proposition 2.1, the multiplier $\sigma = 1$ is represented by a convolution kernel $K_1 \in L^2(G)$. This means that $\delta_e = K_1 \in L^2(G)$. But this cannot happen on a connected Lie group.

b) We also note that $W(R) = \omega([0,R])$ is increasing and right-continuous in R.

4. Proof of the main theorem

By the spectral theorem, the space of functions $f \in L^2(G)$ such that $S_R f = f$ for some $R > 0$ is dense in $L^2(G)$, and for these functions obviously $\lim_{R\to\infty} S_R f(x) = f(x)$. We therefore only have to prove the estimate (1.10).

To this end, we choose a sequence $0 =: R_0 < R_1 < R_2 < \ldots$ recursively as follows:

$$R_{k+1} := \sup\{R \geqslant R_k : W(R) < W(R_k) + 1\}, \quad k \geqslant 0.$$

By Remark 3.3 a), this recursion leads to an infinite sequence $\{R_k\}_{k\in\mathbb{N}}$ tending to infinity. Moreover, since $W(R) = \omega([0,R])$ is increasing and right-continuous in R, we have

$$W(R_{k+1}) \geqslant W(R_k) + 1, \tag{4.1}$$

for every k.

By (4.1) and Corollary 3.2, we have

$$k \leqslant W(R_k) \leqslant C_1 R_k^{\alpha}\,.$$

This easily implies

$$\log(3+k) \leqslant c\log(2+R_k), \quad k \geqslant 1\,. \tag{4.2}$$

for some constant $c = c(\alpha)$.

For $k \in \mathbb{N}$ we let $\psi_k := \chi_{(R_k,R_{k+1}]}$ denote the characteristic function of the interval $(R_k, R_{k+1}]$, and define the pairwise orthogonal projections P_k on $L^2(G)$ by $P_0 := S_0 = \chi_{\{0\}}(\mathcal{L})$, and

$$P_k := S_{R_k} - S_{R_{k-1}} = \psi_{k-1}(\mathcal{L}), \quad k \geqslant 1\,.$$

Then

$$S_{R_n} = \sum_{k=0}^{n} P_k.$$

Assume that $\log(2+\mathcal{L})f \in L^2(G)$. Clearly, the functions $P_k f$, $k \in \mathbb{N}$, are pairwise orthogonal. Moreover, by (4.2), for $k \geqslant 1$

$$\begin{aligned}
(\log(2+k))^2 \parallel P_k f \parallel_2^2 &= \int_{(R_{k-1},R_k]} (\log(3+k-1))^2 d(f, E_\lambda f) \\
&\leqslant c^2 \int_{(R_{k-1},R_k]} (\log(2+R_{k-1}))^2\, d(f, E_\lambda f) \\
&\leqslant c^2 \int_{(R_{k-1},R_k]} (\log(2+\lambda))^2\, d(f, E_\lambda f),
\end{aligned}$$

and clearly $(\log(2+0))^2 \parallel P_0 f \parallel_2^2 = c' \int_{\{0\}} (\log(2+\lambda))^2\, d(f, E_\lambda f)$. Therefore

$$\begin{aligned}
\sum_{k=0}^{\infty} (\log(2+k))^2 \parallel P_k f \parallel_2^2 &\leqslant C_1 \int_0^{\infty} (\log(2+\lambda))^2\, d(f, E_\lambda f) \\
&= C_1 \parallel \log(2+\mathcal{L}) f \parallel_2^2 < \infty\,.
\end{aligned}$$

If we define the discrete maximal operator $\mathcal{M}$ by

$$\mathcal{M}f(x) := \sup_{n\geqslant 0} \Big|\sum_{k=0}^{n} P_k f(x)\Big| = \sup_{n\geqslant 0} |S_{R_n} f(x)|\,,$$

the Rademacher-Menshov theorem thus implies that

$$\| \mathcal{M}f \|_2 \leqslant C \, \| \log(2+\mathcal{L})f \|_2 \,, \tag{4.3}$$

where the constant C is independent of f.

We can dominate the maximal function over arbitrary $R \geqslant 0$ by the maximal function over the sequence $(R_n)_{n=0}^{\infty}$ plus a remainder:

$$\sup_{R \geqslant 0} |S_R f(x)| \leqslant \mathcal{M}f(x) + \sup_{n \geqslant 0} \left(\sup_{R_n \leqslant r < R_{n+1}} |S_r f(x) - S_{R_n} f(x)| \right).$$

To control the remainder term, fix a compact subset K in G.

Fix $n \in \mathbb{N}$, and assume that $R_n \leqslant r < R_{n+1}$. By our definition of R_n, then $W(r) < W(R_n) + 1$. Moreover, $S_r f - S_{R_n} f = \eta(\mathcal{L})f = K_\eta * f$, if we put $\eta := \chi_{(R_n, r]}$. Proposition 2.1 then implies that

$$\| K_\eta \|_2^2 = \int_0^\infty \eta^2(\lambda)\, d\omega(\lambda) = \omega((R_n, r]) = W(r) - W(R_n) < 1,$$

i.e.,

$$\| K_\eta \|_2 \leqslant 1 \,.$$

Moreover, since $\eta = \eta\psi_n$, we have

$$K_\eta * f = K_\eta * (\psi_n(\mathcal{L})f) = K_\eta * (P_{n+1} f) \,.$$

By Cauchy-Schwarz' inequality, we therefore obtain

$$\begin{aligned} |S_r f(x) - S_{R_n} f(x)| &= \left| \int_G K_\eta(y) P_{n+1} f(y^{-1}x)\, dy \right| \\ &= \left| \int_G K_\eta(xy) P_{n+1} f(y^{-1})\, dy \right| \\ &= \left| \int_G K_\eta(xy^{-1}) \Delta(y)^{-1} P_{n+1} f(y)\, dy \right| \\ &\leqslant A_\eta(x) \, \| P_{n+1} f \|_2 \,, \end{aligned}$$

where

$$\begin{aligned} A_\eta^2(x) &:= \int_G |K_\eta(xy^{-1})\Delta(y)^{-1}|^2\, dy \\ &= \Delta(x)^{-1} \int_G |K_\eta(y^{-1})\Delta(y)^{-1}|^2\, dy \\ &= \Delta(x)^{-1} \, \| K_\eta^* \|_2^2 \;. \end{aligned}$$

But, since the operator $\eta(\mathcal{L})$ is self-adjoint, we have $K_\eta^* = K_\eta$, so that

$$A_\eta(x) = \Delta(x)^{-1/2} \, \| K_\eta \|_2 \leqslant \Delta(x)^{-1/2} \,,$$

hence

$$|S_r f(x) - S_{R_n} f(x)| \leqslant \Delta(x)^{-1/2} \parallel P_{n+1} f \parallel_2 \ .$$

Since K is compact and $\Delta^{-1/2}$ is continuous, we thus get for $x \in K$

$$|S_r f(x) - S_{R_n} f(x)| \leqslant C_K \parallel P_{n+1} f \parallel_2 ,$$

hence

$$\begin{aligned} &\sup_{n\geqslant 0}\left(\sup_{R_n\leqslant r<R_{n+1}} |S_r f(x) - S_{R_n} f(x)|\right) \\ \leqslant \quad & C_K \left(\sum_{n=0}^{\infty} \parallel P_{n+1} f \parallel_2^2\right)^{1/2} \leqslant C_K \parallel f \parallel_2 , \end{aligned}$$

for every $x \in K$. We thus obtain

$$\sup_{R\geqslant 0} |S_R f(x)| \leqslant \mathcal{M} f(x) + C_K \parallel f \parallel_2$$

for every $x \in K$. In combination with (4.3) this implies estimate (1.10), which completes the proof of Theorem 1.3. □

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