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## BLOCK-DISTRIBUTION IN RANDOM STRINGS

by Peter J. GRABNER

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### 1. Introduction.

We investigate some properties of infinite sequences of independent random variables, which take the values 0 and 1 with probabilities  $p$  and  $q$  respectively (Bernoulli's scheme). It is one of the basic results of probability theory that the limit relation

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N - k : x_n x_{n+1} \dots x_{n+k} = a_1 \dots a_k\}}{N} = \mu_k(A)$$

holds in probability for all blocks  $A = a_1 \dots a_k$  of a given constant length  $k$  ( $\mu_k(A)$  is the  $k$ -fold product measure generated by  $\mu(\{0\}) = p$  and  $\mu(\{1\}) = q$ ). This result can also be naturally imbedded into ergodic theory : consider the infinite product space  $X = \{0, 1\}^{\mathbb{N}}$  equipped with the infinite product measure  $\mu_\infty$  generated by  $\mu$ . Then the shift operator  $S$  (Bernoulli shift) on  $X$  defined by  $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$  is an ergodic transformation on  $X$  (cf. e.g. [Wa]) and the above relation is a consequence of Birkhoff's ergodic theorem.

It is now natural to ask how fast (depending on  $N$ )  $k$  could grow such that this relation persists. In order to answer this question we introduce a

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special notion of discrepancy (cf. [HI], [KN]) :

(1.1)

$$D_N^k(x_1, \dots, x_N) = \max_{A \in \{0,1\}^k} \frac{1}{\sqrt{p^k \mu_k(A)}} \left| \frac{\#\{1 \leq n \leq N-k : x_n x_{n+1} \dots x_{n+k} = a_1 \dots a_k\}}{N} - \mu_k(A) \right|.$$

The following calculations will show that this is a proper measure for the distribution behaviour of the sequence  $x_1, x_2, \dots$ . Note that this definition agrees with the definition in [FKT] for  $p = q = \frac{1}{2}$ .

**DEFINITION.** — A sequence  $x_1, x_2, \dots$  is called  $k(N)$ -distributed with respect to  $\mu$  if

$$\lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0.$$

Our Theorem will show under which conditions almost all sequences are  $k(N)$ -distributed. Without loss of generality assume that  $p \leq q$ . The notation  $\text{lp } n$  is the logarithm to base  $\frac{1}{p}$  :  $\text{lp } n = \log_{\frac{1}{p}} n$ .

**THEOREM.** — Let  $k(N)$  be a non-decreasing sequence of positive integers. Then the following 0-1-law holds

$$\mu_\infty \left( \lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0 \right) = \begin{cases} 1 & \text{if } \text{lp } n - \text{lp } \text{lp } n - k(n) \rightarrow \infty \\ 0 & \text{otherwise.} \end{cases}$$

It clearly follows from Kolmogoroff's 0-1-law or the fact that the set

$$\left\{ \lim_{N \rightarrow \infty} D_N^{k(N)}(x_1, \dots, x_N) = 0 \right\}$$

is invariant under the (ergodic) shift  $S$ , that the only possible values for the above probability are 0 and 1. The proof of this theorem will use bivariate correlation polynomials, which are a generalization of Guibas' and Odlyzko's correlation polynomials in one variable (cf. [GO]). Using these polynomials we are able to compute the probability generating functions of the events we are interested in.

## 2. Generating Functions.

Throughout this section let  $A = a_1 a_2 \dots a_k$  be a 0-1-string of length  $k$ . We are interested in the cardinalities of the following subsets of the set

$\mathcal{S}_{r,s}$  of strings containing  $r$  digits 0 and  $s$  digits 1 :

(2.1)

$$f_A(r, s) = \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the end}\}$$

$$g_A(r, s) = \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the beginning and at the end}\}$$

$$h_A(r, s) = \#\{B \in \mathcal{S}_{r,s} : B \text{ does not contain } A\}.$$

In order to compute the generating functions of these quantities we introduce the bivariate autocorrelation polynomial  $[AA](z, w)$  :

$$[z^r w^s][AA](z, w) = \begin{cases} 1 & \text{if } a_1 a_2 \dots a_{k-r-s} = a_{r+s+1} a_{r+s+2} \dots a_k \text{ and the} \\ & \text{string } a_1 a_2 \dots a_{r+s} \text{ contains } r \text{ digits 0 and } s \\ & \text{digits 1} \\ 0 & \text{otherwise,} \end{cases}$$

where  $[z^r w^s]P(z, w)$  as usual denotes the coefficient of  $z^r w^s$  in  $P(z, w)$ . We are now ready to formulate

**PROPOSITION 1.** — *The generating functions of the combinatorial expressions (2.1) are given by*

$$F_A(z, w) = \sum_{r,s=0}^{\infty} f_A(r, s) z^r w^s = \frac{z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1 - z - w)[AA](z, w)}$$

$$G_A(z, w) = z^{0(A)} w^{1(A)} + \frac{(z + w - 1) z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1 - z - w)[AA](z, w)}$$

$$H_A(z, w) = \frac{[AA](z, w)}{z^{0(A)} w^{1(A)} + (1 - z - w)[AA](z, w)},$$

where  $0(A)$  and  $1(A)$  denote the number of 0's and 1's in  $A$  respectively.

The proof of this proposition is analogous to the proof of the corresponding results for ordinary generating functions (cf. [GO]).

*Remark 1.* — Obviously these results can be generalized to any finite alphabet.

As in [FKT] we use these functions to compute the probability generating function (p.g.f.) of all strings containing the substring  $A$  exactly  $r$  times :

$$\Phi_A^{(r)}(z) = \frac{z^{-kr}}{\mu_k(A)} F_A(pz, qz)^2 G_A(pz, qz)^{r-1} \text{ for } r \geq 1$$

$$\Phi_A^{(0)}(z) = H_A(pz, qz).$$

Inserting the results of Proposition 1 and setting

$$(2.2) \quad P(z) = \frac{1}{\mu_k(A)} [AA](pz, qz)$$

yields

$$\Phi_A^{(r)}(z) = \frac{z^k \left( (1-z)(P(z) - \frac{1}{\mu_k(A)}) + z^k \right)^{r-1}}{\mu_k(A) \left( (1-z)P(z) + z^k \right)^{r+1}}$$

$$\Phi_A^{(0)}(z) = \frac{P(z)}{(1-z)P(z) + z^k}.$$

### 3. Proof of the Theorem.

We split the proof into two parts; first we show that almost all sequences are  $k(N)$  distributed if  $\text{lp } n - \text{lp } \text{lp } n - k(n) \rightarrow \infty$ . Using our p.g.f. results we can write

(3.1)

$$\mu_\infty(\#\{0 \leq n \leq N - k : x_{n+1} \dots x_{n+k} = a_1 \dots a_k\} = r) = p_A^{(r)}(N) = [z^N] \Phi_A^{(r)}(z) = \frac{1}{2\pi i} \oint_C \Phi_A^{(r)}(z) \frac{dz}{z^{N+1}}.$$

In order to be able to estimate the integral we need information on the the zeros of the polynomial  $(1 - z)P(z) + z^k$ .

LEMMA 1. — *The zero of smallest modulus  $z_0$  of  $(1 - z)P(z) + z^k$  is real and positive and satisfies the estimate*

$$z_0 > 1 + C\mu_k(A)$$

for a positive constant  $C$  only depending on  $p$ .

*Proof.* — As  $F_A(pz, qz)$  is a p.g.f. and  $(1 - z)P(z) + z^k$  is the denominator of this rational function the zero of smallest modulus has to be positive and  $\geq 1$ . Investigation of the derivative shows the existence of the constant  $C$ .

Let now

$$k(n) = \text{lp } n - \text{lp } \text{lp } n - \text{lp } \psi(n),$$

where  $\psi(n) \rightarrow \infty$ . We need estimates for the probability that the number of occurrences  $Z_N(A)$  of a block  $A$  deviates too far from the mean value :

(3.2)

$$L_N(\delta_A) = \mu_\infty(Z_N(A) < N\mu_k(A)(1 - \delta_A)) \quad \text{and}$$

$$U_N(\delta_A) = \mu_\infty(Z_N(A) > N\mu_k(A)(1 + \delta_A)).$$

These probabilities are sums of the  $p_A^{(r)}(N)$  defined in (3.1) :

$$(3.3) \quad \begin{aligned} L_N(\delta_A) &= \sum_{r < N\mu_k(A)(1-\delta_A)} p_A^{(r)}(N) \quad \text{and} \\ U_N(\delta_A) &= \sum_{r > N\mu_k(A)(1+\delta_A)} p_A^{(r)}(N). \end{aligned}$$

We will use the integral representation (3.1) to estimate these quantities.

For convenience we now introduce some notations

$$(3.4) \quad \begin{aligned} Q(z) &= (1-z)P(z) + z^k \\ a(z) &= \frac{z^k}{Q(z)^2}, \quad b(z) = 1 + \frac{z-1}{\mu_k(A)Q(z)}. \end{aligned}$$

This gives

$$\Phi_A^{(r)}(z) = \frac{1}{\mu_k(A)} a(z)b(z)^{r-1}$$

for  $r \geq 1$ . Observe further that

$$(3.5) \quad \begin{aligned} a(1 \pm \varepsilon) &= 1 + O\left(\frac{1}{\mu_k(A)}\varepsilon\right) \\ b(1 \pm \varepsilon) &= 1 \pm \frac{\varepsilon}{\mu_k(A)} + O\left(\frac{\varepsilon^2}{\mu_k(A)^2}\right) \\ b^j(1 \pm \varepsilon) &= \exp\left(\pm \frac{\varepsilon j}{\mu_k(A)} + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right)\right) \\ (1 \pm \varepsilon)^{-n} &= \exp(\mp n\varepsilon + O(n\varepsilon^2)). \end{aligned}$$

We can now write

$$U_N(\delta_A) = \frac{1}{2\pi i} \oint_C \frac{1}{\mu_k(A)} a(z) \frac{b^j(z)}{1-b(z)} \frac{dz}{z^{N+1}},$$

where  $j = [N\mu_k(A)(1+\delta_A)]$ . As all the power series involved have positive coefficients and because of Lemma 1 we can estimate

$$U_N(\delta_A) \leq \frac{1}{\mu_k(A)} a(1-\varepsilon) \frac{b^j(1-\varepsilon)}{1-b(1-\varepsilon)} (1-\varepsilon)^{-N}$$

for every positive  $\varepsilon < C\mu_k(A)$ . Using (3.5) yields

$$U_N(\delta_A) \leq \frac{1}{\varepsilon} \frac{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)}{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)} \exp\left(\left(N - \frac{j}{\mu_k(A)}\right)\varepsilon + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right) + O(N\varepsilon^2)\right).$$

Inserting  $\varepsilon = \left(\mu_k(A) \frac{\text{lp } N}{N}\right)^{\frac{1}{2}}$  into the above inequality yields

$$(3.6) \quad U_N(\delta_A) \leq \exp(-\delta_A (N\mu_k(A) \text{lp } N)^{\frac{1}{2}} + C_1 \log N).$$

In the same way we treat the lower tail. Let now  $j = \lfloor N\mu_k(A)(1-\delta_A) \rfloor$ . Thus we obtain

$$L_N(\delta_A) = \frac{1}{2\pi i} \oint_C \left( \frac{P(z)}{Q(z)} + \frac{a(z)}{\mu_k(A)} \frac{b^j(z) - 1}{b(z) - 1} \right) \frac{dz}{z^{N+1}}.$$

We can now estimate

$$L_N(\delta_A) \leq \frac{P(1+\varepsilon)}{Q(1+\varepsilon)}(1+\varepsilon)^{-N} + \frac{1}{\mu_k(A)} j b^j (1+\varepsilon) a(1+\varepsilon) (1+\varepsilon)^{-N}.$$

Using the same value for  $\varepsilon$  as above yields

$$(3.7) \quad L_N(\delta_A) \leq \exp\left(-\delta_A (N\mu_k(A) \operatorname{lp} N)^{\frac{1}{2}} + C_2 \log N\right).$$

Combining this with (3.6) yields

$$(3.8) \quad \begin{aligned} \mu_\infty \left( \left| \frac{Z_N(A)}{N} - \mu_k(A) \right| > \delta_A \mu_k(A) \right) \\ \leq \exp(-\delta_A (N\mu_k(A) \operatorname{lp} N)^{\frac{1}{2}} + C_3 \log N). \end{aligned}$$

Let now  $\delta_A = \delta \left( \frac{p^k}{\mu_k(A)} \right)^{\frac{1}{2}}$  and observe that  $p^k = \frac{\operatorname{lp} N}{N} \psi(N)$ .

Therefore we have

$$(3.9) \quad \begin{aligned} \mu_\infty(D_N^{k(N)}(\omega) > \delta) &\leq 2^{k(N)} \exp(-\delta \psi(N)^{\frac{1}{2}} \operatorname{lp} N + C_3 \log N) \\ &\leq \exp(-\delta \psi(N)^{\frac{1}{2}} \operatorname{lp} N + C' \log N). \end{aligned}$$

We now choose  $\delta$  as a function of  $N$

$$\delta = \psi(N)^{-\frac{1}{4}}$$

and observe that

$$\sum_{N=1}^{\infty} \exp(-\psi(N)^{\frac{1}{4}} \operatorname{lp} N + C' \log N) < \infty.$$

Thus by the Borel-Cantelli lemma (cf. [Fe]), we obtain the first part of our Theorem.

We now have to prove that almost no series are  $k(N)$ -distributed if  $\operatorname{lp} n - \operatorname{lp} \operatorname{lp} n - k(n) \not\rightarrow \infty$  (we confine ourselves to the case  $p < \frac{1}{2}$ , because the case  $p = \frac{1}{2}$  has been treated by Grill [Gr]). We introduce a set  $\mathcal{A}$  of strings of length  $k$ , which have only trivial autocorrelation and do not overlap each other :

$$\mathcal{A} = \left\{ \underbrace{0 \dots 0}_l \underbrace{A}_{l+d(k)-2} \underbrace{1 \dots 1}_l \right\},$$

where  $l = \left\{ \frac{k}{3} \right\} + 1$  and  $d(k) = k \bmod 3$ . We need the p.g.f.  $\varphi(z)$  of all strings not containing an element of  $\mathcal{A}$ . This function satisfies the equations

$$\begin{aligned} \varphi(z) + \varphi_{A_1}(z) + \dots + \varphi_{A_m}(z) &= z\varphi(z) + 1 \\ \varphi_{A_1}(z) &= z^k \mu_k(A_1)\varphi(z) \\ &\dots \\ \varphi_{A_m}(z) &= z^k \mu_k(A_m)\varphi(z), \end{aligned}$$

where  $A_1, \dots, A_m$  are the elements of  $\mathcal{A}$  and  $\phi_{A_l}(z) (l = 1, \dots, m)$  is the p.g.f. of the blocks ending with  $A_l$  but containing no further occurrence of any element of  $\mathcal{A}$ . Solving these equations yields

$$(3.10) \quad \varphi(z) \frac{1}{1 - z + \mu_k(\mathcal{A})z^k}.$$

Note that the simplicity of these equations comes from the trivial overlap structure of the elements of  $\mathcal{A}$ .

Because of this simple overlap structure it is easy to see that

$$(3.11) \quad \begin{aligned} \phi_{j_1 \dots j_m}(z) &= \frac{(j_1 + \dots + j_m)!}{j_1! \dots j_m!} \mu_k(A_1)^{j_1} \dots \mu_k(A_m)^{j_m} z^{k(j_1 + \dots + j_m)} \varphi(z)^{j_1 + \dots + j_m + 1} \end{aligned}$$

is the p.g.f. of all blocks containing  $A_l$  exactly  $j_l$  times ( $l = 1 \dots m$ ). As in the first part of the proof we use

$$(3.12) \quad \begin{aligned} M_N(\delta) &= \mu_\infty (|Z_N(A_l) - N\mu_k(A_l)| \leq N\mu_k(A_l)\delta_{A_l}, l = 1 \dots m) \\ &= \frac{1}{2\pi i} \oint_C \sum_{\substack{|j_l - N\mu_k(A_l)| \leq N\mu_k(A_l)\delta_{A_l} \\ l=1, \dots, m}} \varphi_{j_1 \dots j_m}(z) \frac{dz}{z^{N+1}}, \end{aligned}$$

where  $\delta_{A_l} = \delta \left( \frac{p^k}{\mu_k(A_l)} \right)^{\frac{1}{2}}$ .

We want to treat (3.12) exactly like the corresponding expressions in the first part of the proof. For this purpose we need information on the zeros of the polynomial  $1 - z + \mu_k(\mathcal{A})z^k$ .

**LEMMA 2.** — *The zero of smallest modulus  $z_0$  of  $1 - z + \mu_k(\mathcal{A})z^k$  is real and satisfies*

$$z_0 > 1 + \mu_k(\mathcal{A}).$$

*Proof.* — The proof of the first statement is as in the proof of Lemma 1. For the proof of the inequality insert  $z = 1 + \mu_k(\mathcal{A})$  into the polynomial.



Observe now that

$$\frac{1}{2\pi i} \oint_C \varphi(z)^{J+1} \frac{dz}{z^{N-kJ+1}} \leq \varphi(1 + \varepsilon)^{J+1} (1 + \varepsilon)^{kJ-N}$$

for  $\varepsilon \leq \mu_k(\mathcal{A})$ . Inserting  $\varepsilon = \mu_k(\mathcal{A}) - \frac{J}{N}$  and performing similar calculations as in the first part of the proof yields

(3.13)

$$\begin{aligned} & \frac{1}{2\pi i} \oint_C \varphi(z)^{J+1} \frac{dz}{z^{N-kJ+1}} \\ & \leq \frac{1}{\mu_k(\mathcal{A})^{J+1}} \exp\left(-\frac{(J - N\mu_k(\mathcal{A}))^2}{2N\mu_k(\mathcal{A})} + O(k\mu_k(\mathcal{A})^2N)\right). \end{aligned}$$

Let now  $n = N\mu_k(\mathcal{A})$ ,  $J = j_1 + \dots + j_m$  and  $p_l = \frac{\mu_k(A_l)}{\mu_k(\mathcal{A})}$  and insert

(3.13) into (3.12) to obtain

(3.14)

$$\begin{aligned} & M_N(\delta) \\ & \leq \frac{1}{\mu_k(\mathcal{A})} \sum_{\substack{|j_l - np_l| \leq np_l \delta_{A_l} \\ l=1, \dots, m}} \frac{J!}{j_1! \dots j_m!} \prod_{l=1}^m p_l^{j_l} \exp\left(-\frac{(J-n)^2}{2n} + O(k\mu_k(\mathcal{A})n)\right), \end{aligned}$$

where  $\sum_{l=1}^m p_l = 1$ . Thus we have arrived at an expression that we can treat by the normal approximation of the multinomial distribution.

Assume that  $N$  runs through a subsequence of  $\mathbb{N}$  such that

$$\text{lp } N - \text{lp } N - k(N) \rightarrow \limsup_{N \rightarrow \infty} (\text{lp } N - \text{lp } N - k(N)) = \text{lp } C < \infty.$$

It will suffice to prove our theorem for the case that  $\limsup \text{lp } N - \text{lp } N - k(N) > -\infty$ , such that  $0 < C < \infty$ . Observe now that  $N\mu_k(A_l)\delta_{A_l} = \delta \sqrt{CN\mu_k(A_l) \text{lp } N}$  and use Stirling's formula

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \leq n! \leq \frac{11}{10} \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

to obtain

(3.15)

$$\begin{aligned}
 M_N(\delta) &\leq \frac{11}{10} \frac{\exp(O(k\mu_k(\mathcal{A})n))}{\mu_k(\mathcal{A})\sqrt{2\pi}^{m-1}} \sum_{|j_i - np_i| \leq \delta \sqrt{CN\mu_k(A_i) \log N}} \frac{\sqrt{J}}{\sqrt{j_1 \cdots j_m}} \\
 &\quad \times \frac{J^J}{j_1^{j_1} \cdots j_m^{j_m}} \prod_{l=1}^m p_l^{j_l} \exp\left(-\frac{(J-n)^2}{2n}\right) \\
 &= \frac{11}{10} \frac{\exp(O(k\mu_k(\mathcal{A})n))}{n^{m-12} \mu_k(\mathcal{A}) (2\pi)^{\frac{m-1}{2}} \sqrt{p_1 \cdots p_m}} \sum_{\substack{|x_l| \leq \delta_{A_l} \\ l=1, \dots, m}} \frac{\sqrt{1+\eta}}{\sqrt{(1+x_1) \cdots (1+x_m)}} \\
 &\quad \times \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2 + O(n\eta^3) + O\left(n \sum_{l=1}^m p_l x_l^3\right)\right),
 \end{aligned}$$

where  $j_l = np_l(1+x_l)$  and  $J = n(1+\eta)$ . The terms in the last exponential come from  $(1+x)^{1+x} = \exp\left(x + \frac{x^2}{2} + O(x^3)\right)$  for  $x \rightarrow 0$  and the observation that  $j_1 + \cdots + j_m = J$  transforms to  $\sum p_l x_l = \eta$ . In the following we will use  $p < \frac{1}{2}$  which yields  $\delta_{A_l} \rightarrow 0$  for our choice of  $\mathcal{A}$  (in the case  $p = \frac{1}{2}$  we have  $\delta_{A_l} = \delta$  and the following arguments cannot be used).

Inserting the definition of  $\delta_{A_l}$  yields the estimate

(3.16)

$$\begin{aligned}
 \left| n \sum_{l=1}^m p_l x_l^3 \right| &\leq \frac{(\log N)^{\frac{3}{2}}}{\sqrt{n}} \sum_{l=1}^m \frac{1}{\sqrt{p_l}} \\
 &= O\left(N^{-\frac{1}{3} + \frac{1}{6} \frac{\log q}{\log p} + \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)} (\log N)^{\frac{4}{3} - \frac{1}{6} \frac{\log q}{\log p} - \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)}\right)
 \end{aligned}$$

and a similar estimate holds for  $n\eta^3$ . Using an exponential estimate yields

$$\begin{aligned}
 &\frac{\sqrt{1+\eta}}{\sqrt{(1+x_1) \cdots (1+x_m)}} \\
 &= \exp\left(O\left(N^{-\frac{1}{3} + \frac{1}{6} \frac{\log q}{\log p} + \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)} (\log N)^{\frac{4}{3} - \frac{1}{6} \frac{\log q}{\log p} - \frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)}\right)\right).
 \end{aligned}$$

Inserting these inequalities into (3.15) and setting  $\alpha = \frac{1}{6} \frac{\log q}{\log p} +$

$\frac{1}{3} \text{lp}\left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)$  yields

$$\begin{aligned}
 M_N(\delta) &\leq \frac{\exp(O(N^{-\frac{1}{3} + \alpha} (\log N)^{\frac{4}{3} - \alpha})) n^{\frac{m+1}{2}} \sqrt{p_1 \cdots p_m}}{\mu_k(\mathcal{A}) (2\pi)^{\frac{m-1}{2}}} \\
 &\quad \times \sum_{|x_l| \leq \delta_{A_l}} \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2\right) \frac{1}{(np_1) \cdots (np_m)}.
 \end{aligned}$$

The sum in the last line can be interpreted as a lower Riemann sum for the integral

$$\int_{|x_l| \leq \delta_{A_l}} \exp\left(-\frac{n}{2} \sum_{l=1}^m p_l x_l^2\right) dx_1 \cdots dx_m$$

using the lattice

$$\left\{ (x_1, \dots, x_m) \mid x_l = \frac{j_l}{np_l} - 1, \quad |x_l| \leq \delta_{A_l}, \quad l = 1, \dots, m \right\}.$$

Thus we obtain

$$(3.17) \quad M_N(\delta) \leq \exp(O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha} + O(\log N))) (\Phi(\delta \sqrt{C \log N}))^m,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-\frac{t^2}{2}} dt \sim 1 - \sqrt{\frac{2}{\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}$$

for  $x \rightarrow \infty$ . Therefore we can estimate

$$(3.18) \quad M_N(\delta) \leq \exp\left(-\sqrt{\frac{2}{\pi}} \frac{m}{\delta \sqrt{C \log N} e^{\frac{1}{2} \delta^2 C \log N}} + O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha} + O(\log N))\right).$$

Observe now

$$(3.19) \quad \begin{aligned} m &\asymp \left(\frac{N}{\log N}\right)^{\frac{1}{3} \log 2} \\ \mu_k(\mathcal{A}) &\asymp \left(\frac{\log N}{N}\right)^{-\frac{1}{3} \log pq} \\ n &\asymp N^{1 + \frac{1}{3} \log pq} (\log N)^{-\frac{1}{3} \log pq}. \end{aligned}$$

Inserting these estimates into (3.18) yields

$$M_N(\delta) \leq \exp\left(-D \frac{N^{\frac{1}{3} \log 2 - \frac{1}{2} \delta^2 C \log e}}{\delta (\log N)^{\frac{1}{2} + \frac{1}{3} \log 2}} + O(N^{-\frac{1}{3} + \alpha}(\log N)^{\frac{4}{3} - \alpha} + O(\log N))\right),$$

where  $D > 0$  is a constant implied by (3.19). The right hand side tends to 0 for sufficiently small  $\delta > 0$ , because  $\frac{1}{3} \log 2 > -\frac{1}{3} + \alpha$  holds for  $p < \frac{1}{2}$ .

Note that

$$\mu_\infty(D_N^{k(N)}(\omega) < \delta) \leq M_N(\delta).$$

Thus the proof is complete. □

*Remark 2.* — Modifying (1.1) one can also investigate discrepancies

$$D_N^{k,\phi}(\omega) = \max_{A \in \{0,1\}^k} \sqrt{\frac{\phi(k)}{\mu_k(A)} \left| \frac{\#\{1 \leq n \leq N-k : x_n x_{n+1} \dots x_{n+k} = A\}}{N} - \mu_k(A) \right|},$$

where  $\phi$  is a monotonically increasing function. Then the same calculations as above yield

$$\mu_\infty \left( \lim_{N \rightarrow \infty} D_N^{k(N),\phi}(\omega) = 0 \right) = \begin{cases} 1 & \text{if } \lim_{N \rightarrow \infty} \frac{N\phi(k(N))}{\log N} = \infty \\ 0 & \text{otherwise.} \end{cases}$$

This answers a question posed by Flajolet, Kirschenhofer and Tichy [FKT], Remark 2.

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