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UNFOLDINGS OF FOLIATIONS WITH MULTIFORM FIRST INTEGRALS

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In this note we study unfoldings of codim 1 local foliations $F = (\omega)$ generated by germs ω of the form

$$\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$$

for some germs f_i of holomorphic functions and complex numbers λ_i , generalizing the situation considered in [10].

For such a foliation F satisfying some side conditions, we determine the set $U(F)$ of equivalence classes of first order unfoldings ((1.7) Proposition) and give explicitly a universal unfolding of F ((1.11) Theorem) as an application of the versality theorem in [7]. In section 2, it is shown that the unfolding theory for $F = (\omega)$, $\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$ is equivalent to the unfolding theory for the "multiform function" $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$. In section 3, we consider foliations with holomorphic or meromorphic first integrals. In either case, it turns out that the given generator ω is of the form considered in section 1. Thus, under the conditions of (1.11) Theorem, such a foliation has a universal unfolding (Theorems (3.4) and (3.10)). If the conditions are not satisfied, then the space $U(F)$ may have obstructed elements ((3.6) Example).

This work was inspired by the extension theory of Cerveau and Moussu for forms with holomorphic integrating factors [1,4]. An unfolding is certainly an extension and, by the implicit function

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theorem, an extension can be thought of as an unfolding. Also a morphism in the unfolding theory is a morphism in the extension theory. However, the converse is not true in general. Thus a versal unfolding is a versal extension but not vice versa. In [1] and [4], it is proved that a germ ω of the form in section 1 of this note (or more, generally, ω with holomorphic integrating factor f , i.e., $d\left(\frac{\omega}{f}\right) = 0$ for some f in Θ) has a mini-versal extension.

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$$1. \text{ Unfoldings of } \omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}.$$

Let ${}_n\Theta$ (or simply Θ) denote the ring of germs of holomorphic functions at the origin 0 in $\mathbf{C}^n = \{(z_1, \dots, z_n)\}$ and let ${}_n\Omega$ (or simply Ω) denote the Θ -module of germs of holomorphic 1-forms at 0. For an element ω in Ω , we denote by $S(\omega)$ (the germ at 0 of) the set of zeros of ω and call it the singular set of ω .

Let ω be an element in ${}_n\Omega$ of the form

$$\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i},$$

where f_i are germs in Θ and λ_i are complex numbers. If we set $F_i = f_1 \dots \hat{f}_i \dots f_p$ (omit f_i) for each $i = 1, \dots, p$, we may write $\omega = \sum_{i=1}^p \lambda_i F_i df_i$. Note that ω is integrable; $d\omega \wedge \omega = 0$.

By regrouping the f_i 's, if necessary, we may always assume that

$$(1.1) \quad \lambda_i \neq \lambda_j (\neq 0), \quad \text{if } i \neq j.$$

In what follows we also assume that $\text{codim } S(\omega) \geq 2$, which implies that

$$(1.2) \quad \text{each } f_i \text{ is reduced, i.e., for any non-unit } g \text{ in } \Theta, f_i \text{ is not divisible by } g^2,$$

and that

$$(1.3) \quad f_i \text{ and } f_j \text{ are relatively prime, if } i \neq j.$$

Let F be the codim 1 local foliation at 0 in $\mathbf{C}^n$ generated

by ω as above ([6] 4, [7] 1, [8]). The set $U(F)$ of equivalence classes of first order unfoldings of F is given by ([6] 6, [7] 1).

$$U(F) = I(\omega) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right),$$

where $I(\omega)$ is an ideal in $\mathcal{O}$ defined by

$$I(\omega) = \{h \in \mathcal{O} \mid h d\omega = \eta \wedge \omega \text{ for some } \eta \in \Omega\}$$

and $\left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$ is the ideal generated by

$$\sum_{i=1}^p \lambda_i F_i \frac{\partial f_i}{\partial z_1}, \dots, \sum_{i=1}^p \lambda_i F_i \frac{\partial f_i}{\partial z_n}.$$

For a q -tuple of integers $i_1, \dots, i_q$ with $1 \leq i_1 < \dots < i_q \leq p$, let $I(i_1, \dots, i_q)$ denote the ideal in $\mathcal{O}$ generated by

$$f_{i_2} \dots f_{i_q}, \dots, f_{i_1} \dots \hat{f}_{i_j} \dots f_{i_q} \text{ (omit } f_{i_j}), \dots, f_{i_1} \dots f_{i_{q-1}}.$$

Note that $I(1, \dots, p) = (F_1, \dots, F_p)$ (the ideal generated by $F_1, \dots, F_p$). We denote by $ht I$ the height of an ideal I in $\mathcal{O}$.

(1.4) LEMMA. — Suppose $ht(f_i, f_j, f_k) = 3$ if i, j, k are distinct and f_i, f_j, f_k are non-units. Then we have

$$I(i_1, \dots, i_q) = \bigcap_{\{j_1, \dots, j_{q-1}\} \subset \{i_1, \dots, i_q\}} I(j_1, \dots, j_{q-1})$$

for $q \geq 3$.

Proof. — Without loss of generality, we may assume that $(i_1, \dots, i_q) = (1, \dots, q)$. Obviously, the left hand side in the above equality is in the right hand side. Take any element h in the right hand side. We set $F'_{ij} = f_1 \dots \hat{f}_i \dots \hat{f}_j \dots f_q$ (omit f_i and f_j) for each pair of distinct indexes i, j and

$$F'_{ijk} = f_1 \dots \hat{f}_i \dots \hat{f}_j \dots \hat{f}_k \dots f_q$$

for each triple of distinct indexes i, j, k . Then we may write

$$(1.5) \quad h = \sum_{i \neq j} a_{ij} F'_{ij}, \quad a_{ij} \in \mathcal{O},$$

for each $j = 1, \dots, q$. Now we show that a_{ij} is in the ideal (f_i, f_j) for each i, j with $i \neq j$, which would imply that h is in $I(1, \dots, q)$.

This is obviously true if f_i or f_j is a unit. Thus we assume that f_i and f_j are non-units. If k is an index different from i or j , we have, from (1.5),

$$F'_{ijk}(a_{ij}f_k - a_{ik}f_j) = \left(\sum_{\ell \neq i, k} a_{\ell k} F'_{i\ell k} - \sum_{m \neq i, j} a_{mj} F'_{imj} \right) f_i.$$

By our assumption, f_i and F'_{ijk} are relatively prime. Hence

$$a_{ij}f_k - a_{ik}f_j = af_i$$

for some a in $\mathcal{O}$. Thus $a_{ij}f_k$ is in (f_i, f_j) . If f_k is a unit, then a_{ij} is in (f_i, f_j) . If f_k is a non-unit, then by our assumption $ht(f_i, f_j, f_k) = 3$. Hence a_{ij} is in (f_i, f_j) . Q.E.D.

(1.6) COROLLARY. — *Under the assumption of (1.4) Lemma,*

$$(F_1, \dots, F_p) = \bigcap_{i \neq j} (f_i, f_j).$$

(1.7) PROPOSITION. — *If the assumption of (1.4) Lemma is satisfied and if $df_1 \wedge \dots \wedge df_p \neq 0$, then we have $I(\omega) = (F_1, \dots, F_p)$, thus*

$$U(F) = (F_1, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right).$$

Proof. — If we set $F_{ij} = f_1 \dots \hat{f}_i \dots \hat{f}_j \dots f_p$ for $i \neq j$, we have

$$d\omega = \sum_{1 \leq i < j \leq p} (\lambda_j - \lambda_i) F_{ij} df_i \wedge df_j.$$

From this we see easily that

$$\lambda_i F_i d\omega = \sum_{i \neq j} (\lambda_i - \lambda_j) F_{ij} df_j \wedge \omega,$$

which shows that $(F_1, \dots, F_p) \subset I(\omega)$. Conversely, take any element h in $I(\omega)$. Thus

$$(1.8) \quad h d\omega = \eta \wedge \omega$$

for some η in Ω . Let U be a small neighborhood of 0 on which the germs $f_1, \dots, f_p$, h and η have representatives and let S be the set of zeros of $df_1 \wedge \dots \wedge df_p$ in U . By our assumption, the set S is an analytic set of $\text{codim} \geq 1$. As in the proof of [10] (2.1) Lemma, from (1.8), we may write

$$(1.9) \quad \eta = \sum_{i=1}^p \phi_i df_i,$$

$$(1.10) \quad (\lambda_j - \lambda_i) h = \lambda_j \phi_i f_i - \lambda_i \phi_j f_j$$

for some holomorphic functions $\phi_1, \dots, \phi_p$ on $U - S$. Now we show that ϕ_i can be extended to holomorphic functions on U . From (1.9) and (1.10), we have

$$\phi_i \omega = \lambda_i F_i \eta + h \sum_{j \neq i} (\lambda_j - \lambda_i) F_{ij} df_j$$

for each $i = 1, \dots, p$. Since the right hand side is holomorphic in U , this shows that ϕ_i is holomorphic in $U - S(\omega)$. Therefore, by the assumption that $\text{codim } S(\omega) \geq 2$, ϕ_i can be extended to a holomorphic function on U . Thus from (1.10) and (1.6) Corollary, we see that h is in $(F_1, \dots, F_p)$. Q.E.D.

For an element h in $\mathcal{O}$, we denote the corresponding element in $\mathcal{O} / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$ by $[h]$. The following result follows from (1.7) Proposition and the versality theorem in [7] (cf. the proof of [10] (2.4) Theorem).

(1.11) THEOREM. — *Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbb{C}^n$ generated by a germ ω of the form*

$$\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$$

for some f_i in $\mathcal{O}$ and λ_i in $\mathbb{C}$. Suppose the conditions (a) $\lambda_i \neq \lambda_j$ ($\neq 0$) for $i \neq j$, (b) $\text{codim } S(\omega) \geq 2$, (c) $ht(f_i, f_j, f_k) = 3$ for $i \neq j \neq k \neq i$ such that f_i, f_j, f_k are non-units, and (d) $df_1 \wedge \dots \wedge df_p \neq 0$ are satisfied. If the dimension of the $\mathbb{C}$ -vector space $(F_1, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$, $F_i = f_1 \dots \hat{f}_i \dots f_p$, is finite, then F has a universal unfolding. In fact, if

$$\left[\sum_{i=1}^p \lambda_i u_i^{(1)} F_i \right], \dots, \left[\sum_{i=1}^p \lambda_i u_i^{(m)} F_i \right], \quad u_i^{(j)} \in \mathcal{O},$$

is a $\mathbb{C}$ -basis of $(F_1, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$, then the unfolding

$\mathcal{F} = (\tilde{\omega})$ of F with parameter space $\mathbf{C}^m = \{(t_1, \dots, t_m)\}$ generated by $\tilde{\omega} = \tilde{f}_1 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$, where $\tilde{f}_i$ are germs in $_{n+m}\mathcal{O}$ given by

$$\tilde{f}_i = f_i + \sum_{k=1}^m u_i^{(k)} t_k, \text{ is universal.}$$

(1.12) COROLLARY (Cerveau-Lins Neto [1] Th. E₅, [2] Prop. 6, see also [9] (3.2) Th.). — If $F = (\omega)$ is the codim 1 local foliation at 0 in $\mathbf{C}^n = \{(z_1, \dots, z_n)\}$ generated by $\omega = z_1 \dots z_n \sum_{i=1}^n \lambda_i \frac{dz_i}{z_i}$ for some λ_i in $\mathbf{C}$ with $\lambda_i \neq \lambda_j \neq 0$ ($i \neq j$), then every unfolding of F is trivial, in fact $U(F) = 0$.

Proof. — We have

$$(F_1, \dots, F_n) = \left(\sum_{i=1}^n \lambda_i F_i \partial f_i \right) = (z_1 \dots \hat{z}_i \dots z_n).$$

Hence $U(F) = 0$.

(1.13) Remark. — The universal unfolding given in (1.11) Theorem is infinitesimally versal. However, if the conditions in (1.11) are not satisfied, $U(F)$ may have obstructed elements (see (3.6) Example).

(1.14) Remark. — Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbf{C}^n$ generated by a germ ω of the form

$$\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}, \quad \lambda_i \neq \lambda_j \neq 0 \quad (i \neq j),$$

with $\text{codim } S(\omega) \geq 2$ and let $\mathcal{F}$ be an unfolding of F with parameter space $\mathbf{C}^q$. Then by a result of Cerveau and Moussu ([1] 4^e Partie, Th. C₄, [4]), we have that

(1.15) $\mathcal{F}$ has a generator $\tilde{\omega}$ of the form

$$\tilde{\omega} = \tilde{f}_1 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}, \quad \tilde{f}_i \in_{n+q}\mathcal{O}.$$

Moreover, if ω has no meromorphic first integrals (Sec. 3), then we may assume that ([1] 2^e Partie, Ch. I, Prop. 1.5, [3])

$$(1.16) \quad \tilde{f}_i(z, 0) = f_i(z), \quad i = 1, \dots, p.$$

The facts (1.15) and (1.16) also follow from (1.11) Theorem in case the conditions in (1.11) are satisfied.

(1.17) *Remark.* — If a foliation F is generated by a germ ω of the form $\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, then F has a generator of a similar form such that each function germ involved in the expression is a non-unit.

2. Multiform functions.

A germ of multiform function at 0 in $\mathbf{C}^n$ is an expression $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ for some germs f_i in ${}_n\Theta$ and non-zero complex numbers λ_i . Two multiform functions $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ and $g_1^{\mu_1} \dots g_q^{\mu_q}$ are equal if they are equal as germs of multivalued functions, i.e., $f_1^{\lambda_1} \dots f_p^{\lambda_p} g_1^{-\mu_1} \dots g_q^{-\mu_q} = 1$. Let $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ be a multiform function. By regrouping the factors of the f_i 's, if necessary, we may always assume that the conditions (1.1), (1.2) and (1.3) are satisfied. Then the expression $f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is uniquely determined up to the order of the f_i 's and units of Θ . The critical set $C(f)$ of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is defined to be the singular set $S(\omega)$ of the 1-form $\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$. In this section, we consider only multiform functions f with $\text{codim } C(f) \geq 2$.

An *unfolding* of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ is a germ $\tilde{f}$ of multiform function at 0 in $\mathbf{C}^n \times \mathbf{C}^m = \{(z, t)\}$ which can be written as $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ for $\tilde{f}_i$ in ${}_{n+m}\Theta$ with $\tilde{f}_i(z, 0) = f_i(z)$, $i = 1, \dots, p$. We call $\mathbf{C}^m$ the parameter space of $\tilde{f}$.

(2.1) DEFINITION. — Let $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ and $g = g_1^{\lambda_1} \dots g_p^{\lambda_p}$ be two unfoldings of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ with parameter spaces $\mathbf{C}^m$ and $\mathbf{C}^q$, respectively. A morphism from g to $\tilde{f}$ consists of germs of holomorphic maps $\Phi : (\mathbf{C}^n \times \mathbf{C}^q, 0) \longrightarrow (\mathbf{C}^n \times \mathbf{C}^m, 0)$ and $\phi : (\mathbf{C}^q, 0) \longrightarrow (\mathbf{C}^m, 0)$ such that

(a) the diagram

$$\begin{array}{ccc} (\mathbf{C}^n \times \mathbf{C}^q, 0) & \xrightarrow{\Phi} & (\mathbf{C}^n \times \mathbf{C}^m, 0) \\ \downarrow & & \downarrow \\ (\mathbf{C}^q, 0) & \xrightarrow{\phi} & (\mathbf{C}^m, 0) \end{array}$$

is commutative, where the vertical maps are the projections,

$$(b) \quad \Phi(z, 0) = (z, 0) \quad \text{and}$$

$$(c) \quad g = \Phi^* \tilde{f}, \text{ i.e., } g_1^{\lambda_1} \dots g_p^{\lambda_p} = (\Phi^* \tilde{f}_1)^{\lambda_1} \dots (\Phi^* \tilde{f}_p)^{\lambda_p}.$$

(2.3) DEFINITION. — An unfolding $\tilde{f}$ of f is versal if for any unfolding g of f , there is a morphism from g to $\tilde{f}$.

Note that if $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ is an unfolding of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$, then $\mathfrak{F} = (\tilde{\omega})$, $\tilde{\omega} = \tilde{f}_1 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}$, is an unfolding of $F = (\omega)$, $\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, with the same parameter space as that of $\tilde{f}$. For the definition of morphisms for unfoldings of foliations, see [10] (1.2) Definition.

(2.4) LEMMA. — Let $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ and $g = g_1^{\lambda_1} \dots g_p^{\lambda_p}$ be two unfoldings of $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ with parameter spaces $\mathbb{C}^m$ and $\mathbb{C}^l$, respectively. A pair (Φ, ϕ) of germs of holomorphic maps $\Phi : (\mathbb{C}^n \times \mathbb{C}^l, 0) \longrightarrow (\mathbb{C}^n \times \mathbb{C}^m, 0)$ and $\phi : (\mathbb{C}^l, 0) \longrightarrow (\mathbb{C}^m, 0)$ is a morphism from g to $\tilde{f}$ if and only if it is a morphism from

$$\mathfrak{F}' = (\theta), \quad \theta = g_1 \dots g_p \sum_{i=1}^p \lambda_i \frac{dg_i}{g_i},$$

to

$$\mathfrak{F} = (\tilde{\omega}), \quad \tilde{\omega} = \tilde{f}_1 \dots \tilde{f}_p \sum_{i=1}^p \lambda_i \frac{d\tilde{f}_i}{\tilde{f}_i}.$$

Proof. — We first note that if $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ and

$$\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i},$$

we may write $d \log f = \frac{df}{f} = \frac{1}{f_1 \dots f_p} \omega$. Suppose (Φ, ϕ) is a morphism from g to $\tilde{f}$. Then we have

$$(2.5) \quad \chi \cdot \theta = \Phi^* \tilde{\omega},$$

where $\chi = \frac{\Phi^* \tilde{f}_1 \dots \Phi^* \tilde{f}_p}{g_1 \dots g_p}$. Since the right hand side of (2.5) is holomorphic and $\text{codim } S(\theta) \geq 2$, we see that χ is in $_{n+l}\mathcal{O}$.

Moreover, since $\tilde{f}_i(z, 0) = g_i(z, 0) = f_i(z)$ and $\Phi(z, 0) = (z, 0)$, we have $\chi(z, 0) = 1$. Hence (Φ, ϕ) is a morphism from $\mathcal{S}'$ to $\mathcal{S}$.

Conversely, suppose (Φ, ϕ) is a morphism from $\mathcal{S}'$ to $\mathcal{S}$. Then there is a germ χ in ${}_{n+l}\mathcal{O}$ with $\chi(z, 0) = 1$ satisfying $\chi \cdot \theta = \Phi^* \tilde{\omega}$.

Now we prove that χ is equal to $\frac{\Phi^* \tilde{f}_1 \cdots \Phi^* \tilde{f}_p}{g_1 \cdots g_p}$. Once this is

done, we have $d \log g = d \log \Phi^* f$. Since the restrictions of g and $\Phi^* \tilde{f}$ to $\mathbf{C}^n \times \{0\}$ are both equal to f , we get $g = \Phi^* \tilde{f}$, which shows that (Φ, ϕ) is a morphism from g to $\tilde{f}$. Let $s = (s_1, \dots, s_l)$ be coordinates on $\mathbf{C}^l$. In general, for an element $\tilde{h}$ in ${}_{n+l}\mathcal{O}$, consider the power series expansion of $\tilde{h}$ in s ; $\tilde{h}(z, s) = \sum_{|\nu| \geq 0} h^{(\nu)}(z) s^\nu$, where ν denotes an l -tuple $(\nu_1, \dots, \nu_l)$ of non-negative integers, $|\nu| = \nu_1 + \dots + \nu_l$, $s^\nu = s_1^{\nu_1} \cdots s_l^{\nu_l}$ and $h^{(\nu)}$ are germs in ${}_n\mathcal{O}$. If $h^{(0)} \neq 0$, $(0) = (0, \dots, 0)$, then for each ν , there is a germ $\phi^{(\nu)}$ of meromorphic function at 0 in $\mathbf{C}^n$ such that

$$\sum_{\lambda + \mu = \nu} h^{(\lambda)} \phi^{(\mu)} = \begin{cases} 1 \dots |\lambda| = 0, \\ 0 \dots |\lambda| > 0. \end{cases}$$

Thus we have an expression $\frac{1}{\tilde{h}} = \sum_{|\nu| \geq 0} \phi^{(\nu)} s^\nu$. If we set

$$\rho = \chi \cdot \frac{g_1 \cdots g_p}{\Phi^* \tilde{f}_1 \cdots \Phi^* \tilde{f}_p},$$

we may write $\rho(z, s) = \sum_{|\nu| \geq 0} \rho^{(\nu)}(z) s^\nu$,

where $\rho^{(\nu)}$ are germs of meromorphic functions at 0 in $\mathbf{C}^n$ with $\rho^{(0)} = 1$. For our purpose, it suffices to show that $\rho^{(\nu)} = 0$ if $|\nu| > 0$. We may also write

$$d \log \Phi^* \tilde{f} = \sum_{|\nu| \geq 0} \alpha^{(\nu)} s^\nu + \sum_{k=1}^l \sum_{|\nu| \geq 0} \nu_k F^{(\nu)}(z) s^{\nu-1k} ds_k,$$

$$d \log g = \sum_{|\nu| \geq 0} \beta^{(\nu)} s^\nu + \sum_{k=1}^l \sum_{|\nu| \geq 0} \nu_k G^{(\nu)}(z) s^{\nu-1k} ds_k,$$

where 1_k denotes the ℓ -tuple with 1 in the k -th component and 0 in the others, the addition and subtraction of two ℓ -tuples are done componentwise, $\alpha^{(\nu)}$ and $\beta^{(\nu)}$ are germs of meromorphic 1-forms and $F^{(\nu)}$ and $G^{(\nu)}$ are germs of meromorphic functions at 0 in $\mathbf{C}^n$. Note that $\alpha^{(0)} = \beta^{(0)}$. Since $d \log \Phi^* \tilde{f}$ and $d \log g$ are both closed forms, we have

$$(2.6) \quad dF^{(\nu)} = \alpha^{(\nu)} \quad \text{and} \quad dG^{(\nu)} = \beta^{(\nu)}.$$

On the other hand, from $\rho d \log g = d \log \Phi^* \tilde{f}$, we have

$$(2.7) \quad \alpha^{(\nu)} = \sum_{\lambda + \mu = \nu} \rho^{(\lambda)} \beta^{(\mu)} \quad \text{and} \quad \nu_k F^{(\nu)} = \sum_{\lambda + \mu = \nu} \mu_k \rho^{(\lambda)} G^{(\mu)}$$

for all ν . From (2.6) and (2.7), it is not difficult to show that $\rho^{(\nu)} = 0$ for $|\nu| > 0$. Q.E.D.

In view of (1.14) Remark and (2.4) Lemma, the unfolding theory for multiform functions $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ satisfying (1.1), (1.2), (1.3) and $\text{codim } C(f) \geq 2$ (as well as other conditions described in (1.14)) is equivalent to the unfolding theory for foliations $F = (\omega)$ with $\text{codim } S(F) \geq 2$ generated by germs ω of the form $\omega = f_1 \dots f_p \sum_{i=1}^p \lambda_i \frac{df_i}{f_i}$, $\lambda_i \neq \lambda_j \neq 0$ ($i \neq j$). In particular, from (1.11) Theorem, we have the following

(2.8) THEOREM. — Let $f = f_1^{\lambda_1} \dots f_p^{\lambda_p}$ be a germ of multiform function at 0 in $\mathbf{C}^n$ satisfying (1.1), (1.2), (1.3), $\text{codim } C(f) \geq 2$ and the conditions (c) and (d) in (1.11) Theorem. If

$$\dim_{\mathbf{C}} (F_1, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right), \quad F_i = f_1 \dots \hat{f}_i \dots f_p,$$

is finite, then f has a versal unfolding. In fact if $\tilde{f}_i$ are the germs in (1.11), then the unfolding $\tilde{f} = \tilde{f}_1^{\lambda_1} \dots \tilde{f}_p^{\lambda_p}$ of f is versal.

3. Foliations with holomorphic or meromorphic first integrals.

The following application of the results in section 1 was pointed out by K. Saito. First we observe the following

(3.1) LEMMA. — *Let f be a germ in $\mathcal{O}$ with $f(0) = 0$ and let g be a reduced germ in $\mathcal{O}$. If $df = g\theta$ for some θ in Ω , then f is divisible by g^2 .*

Proof. — From the condition, we see that f vanishes on the zero set of g . Hence g divides f ; $f = f'g$ for some f' in $\mathcal{O}$. Then we have $df = gdf' + f'dg$. Thus f' must be also divisible by g . Q.E.D.

Similarly we have

(3.2) LEMMA. — *Let f be a germ in $\mathcal{O}$ with $f(0) = 0$ and let g be a germ in $\mathcal{O}$ of the form $g = f_1^{k_1} \dots f_r^{k_r}$ for some germs f_i in $\mathcal{O}$ and positive integers k_i such that (a) f_i are reduced, and (b) f_i and f_j are relatively prime if $i \neq j$. If $df = g\theta$ for some θ in Ω , then f is divisible by $f_1^{k_1+1} \dots f_r^{k_r+1}$.*

Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbb{C}^n$ with $\text{codim } S(\omega) \geq 2$. Suppose ω has a holomorphic first integral f , i.e., $\omega \wedge df = 0$ for some f in $\mathcal{O}$ ([5] p. 470). Without loss of generality, we may always assume that $f(0) = 0$. Since $\text{codim } S(\omega) \geq 2$, we may write $df = g\omega$ for some g in $\mathcal{O}$. If g is a unit in $\mathcal{O}$, $F = (\omega) = (df)$ is a Haefliger foliation and unfoldings of F are well understood [7,10]. We may write $g = f_1^{k_1} \dots f_r^{k_r}$, where k_i are positive integers with $k_i \neq k_j$ for $i \neq j$ and f_i are (non-constant) germs in $\mathcal{O}$ satisfying the conditions (a) and (b) in (3.2) Lemma. Then, from (3.2) Lemma, we have $f = f_1^{k_1+1} \dots f_r^{k_r+1} f_{r+1}$ for some f_{r+1} in $\mathcal{O}$. By computing df , we have

$$(3.3) \quad \omega = f_1 \dots f_{r+1} \sum_{i=1}^{r+1} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \dots 1 & 1 \leq i \leq r, \\ 1 \dots i = r+1 & . \end{cases}$$

Note that, since $\text{codim } S(\omega) \geq 2$, f_{r+1} is reduced and that f_{r+1} and f_i are relatively prime for $i = 1, \dots, r$. Let $p = r$ and replace λ_i by $f_{r+1} \lambda_i$ if f_{r+1} is a constant and let $p = r+1$ otherwise. Then from (1.11) Theorem, we have

(3.4) THEOREM. — *Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbb{C}^n$ with $\text{codim } S(F) \geq 2$. If $\omega \wedge df = 0$ for some f in $\mathcal{O}$, then ω can be written as (3.3). Moreover, if (a) $ht(f_i, f_j, f_k) = 3$*

for distinct indexes $i, j, k = 1, \dots, p$ such that f_i, f_j, f_k are non-units, (b) $df_1 \wedge \dots \wedge df_p \neq 0$ and (c)

$$\dim_{\mathbf{C}}(F_1, \dots, F_p) / \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right), \quad F_i = f_1 \dots \hat{f}_i \dots f_p,$$

is finite, then F has a universal unfolding. In fact, a universal unfolding is constructed explicitly as in (1.11) Theorem.

(3.5) *Example.* — Let $F = (\omega)$ be the foliation at 0 in $\mathbf{C}^2 = \{(x, y)\}$ generated by

$$\omega = y(3x + 2y^2) dx + 2x(x + 2y^2) dy.$$

For $f = x^2 y^2 (x + y^2)$ and $g = xy$, we have $df = g\omega$. Letting $f_1 = F_2 = xy$, $f_2 = F_1 = x + y^2$, $\lambda_1 = 2$ and $\lambda_2 = 1$, we see that the complex vector space

$$(F_1, F_2) / \left(\sum_{i=1}^2 \lambda_i F_i \partial f_i \right) = (x + y^2, xy) / (y(3x + 2y^2), x(x + 2y^2))$$

is three dimensional and we may choose $[x + y^2] = \left[\frac{1}{2} \lambda_1 F_1 \right]$,

$[xy] = [\lambda_2 F_2]$ and $[x^2] = \left[\frac{1}{2} \lambda_1 x F_1 - \lambda_2 y F_2 \right]$ as its basis. Thus by (3.4) Theorem, we see that the unfolding $\mathfrak{F} = (\tilde{\omega})$ of F with parameter space $\mathbf{C}^3 = \{(t_1, t_2, t_3)\}$ given by

$$\tilde{\omega} = 2\tilde{f}_2 d\tilde{f}_1 + \tilde{f}_1 d\tilde{f}_2,$$

$$\tilde{f}_1 = xy + \frac{1}{2} t_1 + \frac{1}{2} x t_3, \quad \tilde{f}_2 = x + y^2 + t_2 - y t_3$$

is universal. Note that $d\tilde{f} = \tilde{g}\tilde{\omega}$ for $\tilde{f} = \tilde{f}_1^2 \tilde{f}_2$ and $\tilde{g} = \tilde{f}_1$.

Here is an example of $F = (\omega)$ with a holomorphic first integral which has obstructed elements in $U(F)$.

(3.6) *Example.* — Let $F = (\omega)$ be the foliation at 0 in $\mathbf{C}^2 = \{(x, y)\}$ generated by

$$\omega = y(3x + 2y) dx + x(3x + 4y) dy.$$

For $f = x^2 y^3 (x + y)$ and $g = x^2 y^3$, we have $df = g\omega$. Thus in the previous situation, we have $f_1 = x$, $f_2 = y$, $f_3 = x + y$, $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 1$. Note that $ht(f_1, f_2, f_3) = 2$. If we set $h = 3x + 4y$, then $hd\omega = \eta \wedge \omega$ for $\eta = 3dx$. Hence $[h]$ is in $U(F)$ and $\mathfrak{F}^{(1)} = (\tilde{\omega})$,

$$\tilde{\omega} = y(3x + 2y) dx + (3x^2 + 4xy + t) dy + (3x + 4y) dt$$

is a first order unfolding of F corresponding to $[h]$. However, it is not difficult to show that there is no unfolding corresponding to $[h]$.

Next we consider a foliation $F = (\omega)$ ($\text{codim } S(\omega) \geq 2$) with a meromorphic first integral, i.e., we suppose that $\omega \wedge d\left(\frac{f}{g}\right) = 0$ for some relatively prime germs f and g in $\mathcal{O}$. In what follows we assume that g is reduced. Since $\text{codim } S(\omega) \geq 2$, we may write

$$(3.7) \quad gdf - fdg = h\omega$$

or

$$(3.8) \quad d\left(\frac{f}{g}\right) = \frac{h}{g^2} \omega$$

for some h in $\mathcal{O}$. Note that if h is a unit, F is generated by $gdf - fdg$ and unfoldings of such an F are well understood [10]. Since f and g are relatively prime and g is reduced, from (3.7), we see that g and h are relatively prime. Thus by (3.8), $\frac{f}{g} = c$ is a constant on the zero set of h . If we write $h = f_1^{k_1} \dots f_r^{k_r}$, where k_i are positive integers with $k_i \neq k_j$ for $i \neq j$ and f_i are non-constant germs in $\mathcal{O}$ satisfying the conditions (a) and (b) in (3.2) Lemma, then we have $f - gc = f_1^{k_1+1} \dots f_r^{k_r+1} f_{r+2}$ for some f_{r+2} in $\mathcal{O}$. We set $f_{r+1} = g$. By computing $d\left(\frac{f}{g}\right)$, we have

$$(3.9) \quad \omega = f_1 \dots f_{r+2} \sum_{i=1}^{r+2} \lambda_i \frac{df_i}{f_i}, \quad \lambda_i = \begin{cases} k_i + 1 \dots 1 & 1 \leq i \leq r, \\ -1 \dots i = r + 1, \\ 1 \dots i = r + 2. \end{cases}$$

Note that, since $\text{codim } S(\omega) \geq 2$, f_{r+2} is also reduced and that f_i and f_j are relatively prime for distinct indexes i, j with $1 \leq i, j \leq r + 2$. Let $p = r + 1$ and replace λ_i by $f_{r+2} \lambda_i$ if f_{r+2} is a constant and let $p = r + 2$ otherwise. Then from (1.11) Theorem, we have

(3.10) THEOREM. — Let $F = (\omega)$ be a codim 1 local foliation at 0 in $\mathbb{C}^n$ with $\text{codim } S(F) \geq 2$. Suppose $\omega \wedge d\left(\frac{f}{g}\right) = 0$ for some f and g in $\mathcal{O}$ such that f and g are relatively prime and that g is reduced.

Then ω can be written as (3.9). If (a) $ht(f_i, f_j, f_k) = 3$ for distinct indexes $i, j, k = 1, \dots, p$ such that f_i, f_j, f_k are non-units, (b) $df_1 \wedge \dots \wedge df_p \neq 0$ and (c) $\dim_{\mathbf{C}}(F_1, \dots, F_p) \Big/ \left(\sum_{i=1}^p \lambda_i F_i \partial f_i \right)$, $F_i = f_1 \dots \hat{f}_i \dots f_p$, is finite, then F has a universal unfolding.

In fact, a universal unfolding is constructed as in (1.11) Theorem.

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