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AN AXIOMATIC TREATMENT OF PAIRS OF ELLIPTIC DIFFERENTIAL EQUATIONS

by Peter A. LOEB

Beginning with a few simple axioms, M. Brelot [6] has developed an axiomatic setting in which many of the classical results of the theory of elliptic differential equations can be established. We shall use Brelot's setting to generalize the results obtained by H. L. Royden in [13]. These results pertain to the classification of open Riemann surfaces and to the existence of an isometric isomorphism from the class of bounded solutions of one elliptic differential equation into the class of bounded solutions of another when a certain « majorizing » relationship exists between the two equations.

The discussion in [13] deals with the solutions of the equation $\Delta u = \mathrm{P}u$ on an open Riemann surface W. In this equation, P is a smooth nonnegative density on W, i.e., an invariant expression which in terms of the local uniformizer $z = x + iy$ has the form $c\,dx\,dy$, the coefficient c being a nonnegative function with continuous first derivatives. The coefficient c depends on the choice of the local uniformizer in such a way that the density P is invariant with respect to that choice.

We say that a majorizing relationship holds between the class of solutions of the equation

$$(1) \qquad \Delta u = \mathrm{Q}u$$

and the class of solutions of the equation

$$(2) \qquad \Delta u = \mathrm{P}u$$

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whenever $P \geqslant Q \geqslant 0$ in the complement of some compact subset A of W. If this inequality holds and ω is an open region in W — A, then a solution h of Equation 1 and a solution k of Equation 2 will satisfy the inequality $h \geqslant k$ in ω if $h \geqslant k \geqslant 0$ on the boundary of ω. It is this latter form of the majorizing relationship which can be expressed in the axiomatic setting. The notation $\mathfrak{H} \geqslant \mathfrak{K}$ will be used for this relationship, where in the above example $\mathfrak{H}$ is the class of solutions of Equation 1 and $\mathfrak{K}$ is the class of solutions of Equation 2.

In Chapter I we discuss the class of solutions of a single elliptic differential equation. By such a class we mean a set of functions $\mathfrak{H}$ which satisfies the three axioms given by Brelot in [6], pp. 61-63. These axioms are stated in Section 1. In Section 2 we give a fourth axiom which is used to establish a strong maximum principle for $\mathfrak{H}$; i.e., if $\mathfrak{H}$ satisfies this axiom, then a nonconstant function in $\mathfrak{H}$ can take neither a nonnegative maximum nor a nonpositive minimum in any open subset of its domain. The other principal results of Chapter I are the solution of the Dirichlet problem (Section 3), the existence of an exhaustion by regular regions for the domain W on which $\mathfrak{H}$ is defined (Section 4), and the classification of $\mathfrak{H}$ with respect to the domain W (Section 5).

In Chapter II we consider pairs of classes $\mathfrak{H}$ and $\mathfrak{K}$ for which the majorizing relationship $\mathfrak{H} \geqslant \mathfrak{K}$ holds. We show in this chapter that there exists an isometric isomorphism which maps the bounded functions of $\mathfrak{K}$ into the bounded functions of $\mathfrak{H}$.

The principal statements in each section are given consecutive decimal numbers, with the digits before the decimal point indicating the section and the digits after the decimal point indicating the order of the statement in the section. Results are usually referred to by number alone. The notation ▌ is used to signify the end of a proof.

I wish to express my deepest thanks to Professor Halsey Royden, who supervised the research presented here. His judgment and his insight have been of tremendous assistance. I am also indebted to Professor Marcel Brelot for many helpful suggestions.

CHAPTER I

PROPERTIES OF A SINGLE HARMONIC CLASS OF FUNCTIONS AND THE ASSOCIATED SUPERHARMONIC AND SUBHARMONIC CLASSES

1. Definition and Basic Properties of a Harmonic Class.

In this section we shall review the immediate consequences of Brelot's axioms for what will be called a harmonic class of functions.

Let W be a locally compact Hausdorff space which is connected and locally connected. We assume that W is not compact. Let $\overline{W}$ denote the Alexandroff one point compactification of W. If A is subset of $\overline{W}$, then by $\overline{A}$ we mean the closure of A relative to $\overline{W}$, and if Ω is an open subset of W, then by $\partial\Omega$ we mean the boundary of Ω relative to $\overline{W}$.

By a region Ω we shall mean a nonempty connected open subset of W. By an inner region or an inner open subset of W we shall mean a region or an open set Ω with $\overline{\Omega} \subset W$.

The functions that we consider on W are extended real-valued functions with the usual lattice ordering $\geqslant$. Given two such functions f and g, we let $f \vee g$ denote the function defined by

$$(f \vee g)(x) = \max\,(f(x),\, g(x))$$

and $f \wedge g$ denote the function defined by

$$(f \wedge g)(x) = \min\,(f(x),\, g(x)).$$

A function f is said to be nonnegative if $f \geqslant \underset{\sim}{0}$ and positive if $f(x) > 0$ for every x in the domain of f. By $f|S$ we mean the restriction of f to a subset S in its domain. If the domain D of g is a proper subset of the domain of f, then we shall often write $f \geqslant g$ instead of $f|D \geqslant g$ and $f + g$ instead of $f|D + g$.

Let f be a function with domain $D \subset \overline{W}$, A a subset of $\overline{W}$, and x_0 a point in $\overline{D \cap A}$. If β is the neighborhood system of x_0 in $\overline{W}$, then by $\limsup_{x \in A,\, x \to x_0} f$ we mean $\inf_{\omega \in \beta}\left(\sup_{x \in \omega \cap D \cap A} f(x)\right)$ and by $\liminf_{x \in A,\, x \to x_0} f$ we mean $\sup_{\omega \in \beta}\left(\inf_{x \in \omega \cap D \cap A} f(x)\right)$. Note that if x_0 is in $D \cap A$ and $f(x_0) \geqslant f(x)$ for all points x in a neighborhood of x_0, then $\limsup_{x \in A,\, x \to x_0} f = f(x_0)$.

Keeping the above definition in mind, we say that an extended real-valued function f with domain D is lower semicontinuous if for every $x_0 \in D$, $-\infty < f(x_0)$ and $\liminf_{x \in D,\, x \to x_0} f = f(x_0)$. By a continuous function we mean a continuous real-valued function. For convenience, the function which is identically equal to the extended real number r will be denoted by $\underset{\sim}{r}$.

An increasing sequence of functions is a sequence $\{f_n\}$ such that $f_{n+1} \geqslant f_n$, and a family of functions directed by increasing order is a non-empty family $\mathcal{F}$ such that for any two functions f_1 and f_2 in $\mathcal{F}$ there is a third function $f \in \mathcal{F}$ such that $f \geqslant f_1 \vee f_2$. The notions of a decreasing sequence of functions and a family of functions directed by decreasing order are similarly defined.

Definition. — *Let $\mathfrak{H}$ be a class of real-valued continuous functions with open domains in* W *such that for each open set* $\Omega \subset W$ *the set* $\mathfrak{H}_\Omega$, *consisting of all functions in* $\mathfrak{H}$ *with domains equal to* Ω, *is a real vector space. An open subset* Ω *of* W *is said to be regular for* $\mathfrak{H}$ *or regular if for every continuous real-valued function* f *defined on* $\partial\Omega$ *there is a unique continuous function* h *defined on* $\overline{\Omega}$ *such that* $h|\partial\Omega = f$, $h|\Omega \in \mathfrak{H}$, *and* $h \geqslant \underset{\sim}{0}$ *if* $f \geqslant \underset{\sim}{0}$. *Moreover, the class* $\mathfrak{H}$ *is called a harmonic class on* W *if it satisfies the following three axioms:*

Axiom I. — *A function* g *with an open domain* $\Omega \subset W$ *is an element of* $\mathfrak{H}$ *if for every point* $x \in \Omega$ *there is a function* $h \in \mathfrak{H}$ *and an open set* ω *with* $x \in \omega \subset \Omega$ *such that* $g|\omega = h|\omega$.

Axiom II. — *There is a base for the topology of* W *such that each set* ω *in the base is a regular inner region.*

Axiom III. — *If $\mathcal{F}$ is a subset of $\mathfrak{H}_\Omega$, where Ω is a region in* W, *and $\mathcal{F}$ is directed by increasing order on Ω, then the upper envelope of $\mathcal{F}$ is either $+\infty$ or a function in $\mathfrak{H}_\Omega$.*

It follows immediately from Axiom I that if h is in $\mathfrak{H}$, then the restriction of h to any nonempty open subset of its domain is again in $\mathfrak{H}$.

Given Axioms I and II, Constantinescu and Cornea ([7], p. 344 and p. 378) have shown that the following axioms are equivalent to Axiom III:

Axiom III_1. — *If Ω is a region in* W *and $\{h_n\}$ is an increasing sequence of functions in $\mathfrak{H}_\Omega$, then either* $\lim h_n = +\infty$ *or* $\lim h_n$ *is in $\mathfrak{H}$.*

Axiom III_2. — *If Ω is a region in* W, A *a compact subset of Ω, and x_0 a point in* A, *then there is a constant* $\mathrm{M} \geqslant 1$ *such that every nonnegative function $h \in \mathfrak{H}_\Omega$ satisfies the inequality*

$$h(x) \leqslant \mathrm{M}h(x_0)$$

at every point x in A.

Given Axiom I, it is easy to show that Axiom III is really a « local axiom »; i.e., $\mathfrak{H}_\Omega$ satisfies the axiom for every open set $\Omega \subset \mathrm{W}$ if $\mathfrak{H}_\Omega$ satisfies the axiom for each open set Ω in a base for the topology of W. Hence for a particular example one only needs to establish the validity of the axiom for the sets in such a base. Similarly, Axioms III_1 and III_2 are local axioms. Also note that if W has a base for its topology consisting of regular open sets, then it has a base consisting of regular inner regions.

Axioms III, III_1, III_2 all have as an immediate consequence the following minimum principle:

Proposition 1.1. — *If Ω is a region in* W *and h is a nonnegative function in $\mathfrak{H}_\Omega$, then either $h(x) > 0$ for every $x \in \Omega$ or $h = 0$.*

Using this minimum principle, we can establish a new criterion for an open set to be regular which is easier to verify than the standard criterion given above.

Proposition 1.2. — *If $\mathfrak{H}$ satisfies Axiom III, III_1 or III_2, then an open set Ω is regular for $\mathfrak{H}$ if*

(1) *for every continuous real-valued function f defined on $\partial\Omega$ there is a continuous function h defined on $\overline{\Omega}$ such that*

$$h|\partial\Omega = f \qquad \textit{and} \qquad h|\Omega \in \mathfrak{H},$$

and

(2) *there is a function h_1 in $\mathfrak{H}_\Omega$ such that* $\inf_{x\in\Omega} h_1(x) > 0$

Proof. — We shall assume that Ω is a region and show that if g is a continuous function on $\overline{\Omega}$ with $g|\Omega \in \mathfrak{H}$ and $g \geqslant \underset{\sim}{0}$ on $\partial\Omega$, then $g \geqslant \underset{\sim}{0}$ in Ω, given $h_1 \in \mathfrak{H}_\Omega$. The proposition follows immediately from this fact.

Assume that g takes a negative value in Ω, and let $\alpha_0 = \inf\{\alpha : \alpha h_1 + g \geqslant \underset{\sim}{0}\}$. Clearly, $\alpha_0 > 0$ and $\alpha_0 h_1 + g \geqslant \underset{\sim}{0}$. For each β such that $0 < \beta < \alpha_0$, let

$$K_\beta = \{x \in \Omega : (\beta h_1 + g)(x) \leqslant 0\}.$$

Each K_β is a compact subset of Ω, and $K_{\beta_1} \subset K_{\beta_2}$ when $\beta_1 > \beta_2$. Therefore there is a point $x_0 \in \cap K_\beta$, and $(\alpha_0 h_1 + g)(x_0) = 0$. By 1.1, $g + \alpha_0 h_1 = \underset{\sim}{0}$. Since this is impossible, we conclude that $g \geqslant \underset{\sim}{0}$. ▌

As an example of a harmonic class of functions we have the C^2-solutions of the elliptic differential equation

$$(1) \qquad \Sigma a_{ik} \frac{\partial^2 h}{\partial x_i \partial x_k} + \Sigma b_i \frac{\partial h}{\partial x_i} + ch = \underset{\sim}{0}$$

on a region in Euclidean n-space R^n, where $\Sigma a_{ik} x_i x_k$ is a positive definite quadratic form and the coefficients of the equation satisfy a local Lipschitz condition. (See Chapter VII of [10].)

Throughout this chapter, $\mathfrak{H}$ will denote an arbitrary harmonic class of functions on W, and throughout the rest of this section Ω will denote an arbitrary set which is regular with respect to H. Let $C(\partial\Omega)$ denote the set of continuous functions on $\partial\Omega$. For each $f \in C(\partial\Omega)$ there is by definition a unique continuous function h with domain $\overline{\Omega}$ such that $h|\partial h = f$ and $h|\Omega$ is in $\mathfrak{H}$. We shall denote $h|\Omega$ by $H(f, \Omega)$ or simply $H(f)$. For each $x \in \Omega$, it is easy to see that $H(f, \Omega)(x)$, as a function of f, is a bounded positive linear functional on $C(\partial\Omega)$. Therefore, there is a finite positive Radon measure

$\rho(x, \Omega)$ defined on $\partial\Omega$ such that $H(f, \Omega)(x) = \int_{\partial\Omega} f\, d\rho(x, \Omega)$ for each $f \in C(\partial\Omega)$.

Definition. — *Let f be an extended real-valued function on $\partial\Omega$. We say that f is integrable with respect to $\mathfrak{H}$ on $\partial\Omega$ or simply that f is integrable if f is integrable with respect to $\rho(x, \Omega)$ for each point $x \in \Omega$. Assume that f is integrable on $\partial\Omega$, and let $H(f, \Omega)$ be the function on Ω which satisfies the equation*

$$H(f, \Omega)(x) = \int_{\partial\Omega} f\, d\rho(x, \Omega)$$

at each point $x \in \Omega$. We call $H(f, \Omega)$ the $\mathfrak{H}$-extension of f in Ω. The symbol $H(f)$ is also used to denote the $\mathfrak{H}$-extension of f.

It is clear that for each x in Ω, $H(f, \Omega)(x)$ is a positive linear functional on the vector space of integrable functions on $\partial\Omega$. Using Axiom III, it can be shown (see [6], p. 65) that a function f on $\partial\Omega$ is integrable if it is integrable with respect to $\rho(x, \Omega)$ for some point x in each component of Ω and that the $\mathfrak{H}$-extension of an integrable function is in $\mathfrak{H}$. In particular, if f is a lower semicontinuous function on $\partial\Omega$ and $\mathcal{G}$ is the family of all continuous functions g on $\partial\Omega$ with $g \leqslant f$, then f is integrable if and only if the upper envelope h of the set $\{H(g, \Omega) : g \in \mathcal{G}\}$ is finite at some point x in each component of Ω. In this case, h is in $\mathfrak{H}$ and $H(f, \Omega) = h$. It follows that if Ω is a regular region and f is an integrable lower semicontinuous function on $\partial\Omega$, then $f(x)$ is not identically equal to $+\infty$ in any open subset of $\partial\Omega$. Moreover, $H(f)$ is positive in Ω if $f \geqslant \underset{\sim}{0}$ and $f \neq \underset{\sim}{0}$.

To conclude this section we consider the behavior of $\mathfrak{H}$-extensions at the boundary of Ω.

Proposition 1.3. — *Let f be an integrable function on $\partial\Omega$, and let x_1 be any point on $\partial\Omega$. If f is bounded above, then*

$$\limsup_{x \in \Omega,\ x \to x_1} H(f) \leqslant \limsup_{x \in \partial\Omega,\ x \to x_1} f.$$

If f is bounded from below, then

$$\liminf_{x \in \partial\Omega,\ x \to x_1} f \leqslant \liminf_{x \in \Omega,\ x \to x_1} H(f).$$

Proof (from [6]). — Let r be any real number such that $\limsup_{x \in \partial\Omega,\, x \to x_1} f < r$. There is a continuous function Ψ on $\partial\Omega$ such that $\Psi \geqslant f$ and $\Psi(x) < r$ in a neighborhood of x_1. Since $H(f) \leqslant H(\Psi)$, we have

$$\limsup_{x \in \Omega,\, x \to x_1} H(f) \leqslant \lim_{x \in \Omega,\, x \to x_1} H(\Psi) < r.$$

The rest of the proof is clear. ▌

2. The superharmonic and subharmonic classes associated with $\mathfrak{H}$.

In this section we shall review the immediate consequences of Brelot's local definition of the superharmonic and subharmonic classes $\overline{\mathfrak{H}}$ and $\underline{\mathfrak{H}}$ associated with $\mathfrak{H}$. We shall also give a new axiom which is used to establish a maximum principle for $\mathfrak{H}$.

DEFINITION. — *We say that a lower semicontinuous function v with an open domain $\Omega \subset W$ belongs to the class $\overline{\mathfrak{H}}$ if*

(1) *$v(x) < +\infty$ for some point x in each component of Ω, and*

(2) *for every point $x_0 \in \Omega$ such that $v(x_0) < +\infty$ and for every neighborhood ω_0 of x_0 with $\omega_0 \subset \Omega$, there is a regular region ω with $x_0 \in \omega \subset \overline{\omega} \subset \omega_0$ such that v is integrable on $\partial\omega$ and*

$$v(x_0) \geqslant H(v, \omega)(x_0).$$

We say that an upper semicontinuous function u belongs to the class $\underline{\mathfrak{H}}$ if -u belongs to the class $\overline{\mathfrak{H}}$. We call $\overline{\mathfrak{H}}$ the superharmonic class associated with $\mathfrak{H}$ and $\underline{\mathfrak{H}}$ the subharmonic class associated with $\mathfrak{H}$. The symbol $\overline{\mathfrak{H}}_\Omega$ denotes the functions in $\overline{\mathfrak{H}}$ with domain Ω and $\underline{\mathfrak{H}}_\Omega$ denotes the functions in $\underline{\mathfrak{H}}$ with domain Ω. A potential in an open set Ω is a nonnegative function P in $\overline{\mathfrak{H}}_\Omega$ such that if $h \in \mathfrak{H}_\Omega$ and $h \leqslant P$ then $h \leqslant 0$.

The results of this section will be given for the class $\overline{\mathfrak{H}}$; corresponding results hold for $\underline{\mathfrak{H}}$. Clearly, $\mathfrak{H} \subset \overline{\mathfrak{H}} \cap \underline{\mathfrak{H}}$. Let Ω be an open subset of W, v a function in $\overline{\mathfrak{H}}_\Omega$, h a function in

$\mathfrak{H}_\Omega$ and c a nonnegative constant; then $h + v$ and cv are in $\overline{\mathfrak{H}}_\Omega$. Moreover, we have the following generalization of Proposition 1.1:

PROPOSITION 2.1. — *Let Ω be a region in W and let v be a non-negative function in $\overline{\mathfrak{H}}_\Omega$. Then either $v = \underset{\sim}{0}$ or $v(x) > 0$ for all $x \in \Omega$.*

Proof. — Assume that $v \neq \underset{\sim}{0}$ in Ω. Let B be a component of the set $\{x \in \Omega : v(x) > 0\}$. If $B \neq \Omega$, then there is a point x_0 on $\partial B \cap \Omega$, and $v(x_0) = 0$ since v is lower semicontinuous. In this case there is a regular region ω with $x_0 \in \omega \subset \overline{\omega} \subset \Omega$ and $B \not\subset \omega$ such that v is integrable of $\partial\omega$ and

$$v(x_0) \geqslant H(v, \omega)(x_0).$$

Since B is connected, the boundary of ω has a nonempty intersection with B. Therefore, $v|\partial\omega \neq \underset{\sim}{0}$, and thus

$$0 < H(v, \omega)(x_0) \leqslant v(x_0).$$

But $v(x_0) = 0$. It follows from this contradiction that $B = \Omega$. ▌

COROLLARY 2.2. — *Let Ω be a regular open subset of* W *and let v be a lower semicontinuous function defined on $\overline{\Omega}$ such that $v|\partial\Omega \geqslant \underset{\sim}{0}$ and $v|\Omega$ is in $\overline{\mathfrak{H}}_\Omega$. Then $v \geqslant \underset{\sim}{0}$ in Ω.*

Proof. — The proof is essentially the same as the proof of 1.2. ▌

Using the above corollary, we obtain the following theorem which generalizes the theorem on page 72 of [6].

THEOREM 2.3. — *Let Ω be a regular open subset of* W. *Let v be a lower semicontinuous function defined on $\overline{\Omega}$ such that $v|\Omega$ is in $\overline{\mathfrak{H}}$. Then v is integrable on $\partial\Omega$ and $v \geqslant H(v, \Omega)$ in Ω.*

Proof. — If g is a continuous function on $\partial\Omega$ with $g \leqslant v$, then by 2.2, $H(g, \Omega) \leqslant v$ in Ω. The theorem follows from this and the fact that v is finite at some point x in each component of Ω. ▌

COROLLARY 2.4. — *If v is a function in $\overline{\mathfrak{H}}$, then v is not identically equal to $+\infty$ in any open subset of its domain.*

Proof. — Assume that there is a nonempty component B of the interior of the set $v^{-1}[+\infty]$. Then there is a regular region ω the closure of which is contained in the domain of v such that $\partial\omega \cap B \neq \emptyset$ and v is finite at some point $x \in \omega$ (whence $v|\omega \in \overline{\mathfrak{H}}_\omega$). By 2.3, v is integrable on $\partial\omega$ and yet $v = +\infty$ on an open subset of $\partial\omega$. Since this is a contradiction, the corollary follows. ▌

Corollary 2.5. — *If $v \in \overline{\mathfrak{H}}$, then the restriction of v to any nonempty open subset of its domain is again in $\overline{\mathfrak{H}}$. Conversely, if v is a function with open domain $\Omega \subset W$, and if the restriction of v to some neighborhood of each point $x \in \Omega$ is in $\overline{\mathfrak{H}}$, then $v \in \overline{\mathfrak{H}}$.*

Corollary 2.6. $\mathfrak{H} = \overline{\mathfrak{H}} \cap \underline{\mathfrak{H}}$.

If v is a function in $\overline{\mathfrak{H}}$ and Ω is a regular open set the closure of which is contained in the domain of v, then by 2.3 and 2.5, v is integrable on $\partial\Omega$ and $v \geqslant H(v, \Omega)$.

If Ω is any subset of W and v_1 and v_2 are functions in $\overline{\mathfrak{H}}_\Omega$, then it is easy to see that $v_1 \wedge v_2$ is in $\overline{\mathfrak{H}}_\Omega$. Using 2.3 one can show that $v_1 + v_2$ is not identically equal to $+\infty$ in any component of Ω, whence it follows that $v_1 + v_2$ is also in $\overline{\mathfrak{H}}_\Omega$. It also follows from 2.3 that if Ω is a region and $\mathcal{F}$ is a family of functions directed by increasing order on Ω with $\mathcal{F} \subset \overline{\mathfrak{H}}_\Omega$, then the upper enveloppe of $\mathcal{F}$ is either $+\underset{\sim}{\infty}$ or again in $\overline{\mathfrak{H}}_\Omega$. Moreover, we have the following easy to prove consequence of 1.3 and 2.3:

Proposition 2.7. — *Let Ω be an open subset of W and let Ω_0 be a regular open set with $\overline{\Omega}_0 \subset \Omega$. Given a function v in $\overline{\mathfrak{H}}_\Omega$, let v_0 be the function which is equal to v in $\Omega - \Omega_0$ and $H(v, \Omega_0)$ in Ω_0. Then $v \geqslant v_0$ and v_0 is in $\overline{\mathfrak{H}}_\Omega$.*

Using a proof similar to the proof of 1.2, Constantinescu and Cornea ([7], p. 375] have established a generalization of 2.2 which is given below by Proposition 2.9. However, in proving that generalization one should note the following fact:

Lemma 2.8. — *Let $\Omega \subset W$ be an open set on which there exist two functions V and v in $\overline{\mathfrak{H}}_\Omega$. If $B = \{x \in \Omega : v(x) < 0\}$ and $B \neq \emptyset$, then V is finite at some point $x \in B$.*

Proof. — Given $x_0 \in \mathrm{B}$, there is a regular region ω with $x_0 \in \omega \subset \overline{\omega} \subset \Omega$, and

$$\mathrm{H}(v, \omega)(x_0) = \int v \, d\rho(x_0, \omega) \leqslant v(x_0) < 0.$$

Therefore $\rho(x_0, \omega)(\mathrm{B} \cap \partial\omega) > 0$, but since V is integrable on $\partial\omega$, $\rho(x_0, \omega)(\mathrm{V}^{-1}[+\infty] \cap \partial\omega) = 0$. ▌

Proposition 2.9. — *Let $\Omega \subset \mathrm{W}$ be an open set on which there exists a function $\mathrm{V} \in \overline{\mathfrak{H}}_\Omega$ with $\inf\limits_{x \in \Omega} \mathrm{V}(x) > \underset{\sim}{0}$ If v is a function in $\overline{\mathfrak{H}}_\Omega$ such that* lim inf $v \geqslant 0$ *at $\partial\Omega$, then $v \geqslant \underset{\sim}{0}$ in Ω.*

Proof. — Assume that v takes a negative value in Ω and let $\alpha_0 = \inf\{\alpha : \alpha > 0, \alpha\mathrm{V} + v \geqslant \underset{\sim}{0}\}$. By 2.8, $\alpha_0 > 0$. It follows (see proof of 1.2) that $\alpha_0 \mathrm{V} + v = \underset{\sim}{0}$ which is impossible. Therefore, $v \geqslant \underset{\sim}{0}$ in Ω. ▌

Corollary 2.10. — *Let $\Omega \subset \mathrm{W}$ be an open set on which there exists a function $\mathrm{V} \in \overline{\mathfrak{H}}_\Omega$ with $\inf\limits_{x \in \Omega} \mathrm{V}(x) > 0$. Then Ω is regular if for every continuous f on $\partial\Omega$ there is a continuous h on $\overline{\Omega}$ such that $h|\partial\Omega = f$ and $h|\Omega \in \mathfrak{H}$.*

If $\underset{\sim}{1}$ is in $\mathfrak{H}_\mathrm{W}$, we can apply the following consequence of 2.1 to every region $\Omega \subset \mathrm{W}$:

Proposition 2.11. — *Let Ω be a region in* W, *and assume that $\underset{\sim}{1}$ is in $\mathfrak{H}_\Omega$. Let a and b be constants.*

(i) *If v is in $\overline{\mathfrak{H}}_\Omega$ and $v \geqslant \underset{\sim}{a}$, then either $v = \underset{\sim}{a}$ or $v(x) > a$ for every $x \in \Omega$.*

(ii) *If u is in $\underline{\mathfrak{H}}_\Omega$ and $u \leqslant \underset{\sim}{b}$, then either $u = \underset{\sim}{b}$ or $u(x) < b$ for every $x \in \Omega$.*

(iii) *A nonconstant function in $\mathfrak{H}_\Omega$ takes neither a maximum nor a minimum value in Ω.*

The assumption that $\underset{\sim}{1}$ is in $\mathfrak{H}_\mathrm{W}$ is too restrictive for our purposes. We can obtain a result similar to the above proposition for every region $\Omega \subset \mathrm{W}$ by assuming that $\mathfrak{H}$ satisfies the following axiom:

Axiom IV. — *There is a base $\mathcal{O}$ for the topology of* W *such that each set $\omega \in \mathcal{O}$ is a regular inner region with $\underset{\sim}{1} \geqslant \mathrm{H}(\underset{\sim}{1}, \omega)$.*

The assumption that $\mathfrak{H}$ satisfies Axom IV is equivalent to the assumption that $\underset{\sim}{1}$ is in $\overline{\mathfrak{H}}_{\mathrm{W}}$. An example of a harmonic class of functions which satisfies Axiom IV is given by the class of solutions of Equation 1 (following Proposition 1.2) in Section 1 when the coefficient c is nonpositive. (See [8], pp. 326-328.) If $\mathfrak{H}$ satisfies Axiom IV, i.e. if $\underset{\sim}{1} \in \overline{\mathfrak{H}}_{\mathrm{W}}$, then the following theorem is applicable to every region in W. In this case we shall refer to the theorem as the maximum principle for $\mathfrak{H}$.

THEOREM 2.12. — *Let* Ω *be a region in* W, *and assume that* $\underset{\sim}{1}$ *is in* $\overline{\mathfrak{H}}_{\mathrm{W}}$. *Let* a *and* b *be constants with* $a \leqslant 0 \leqslant b$.

(i) *If* v *is in* $\overline{\mathfrak{H}}_{\Omega}$ *and* $v \geqslant \underset{\sim}{a}$, *then either* $v = \underset{\sim}{a}$ *or* $v(x) > a$ *for all* $x \in \Omega$.

(ii) *If* u *is in* $\underline{\mathfrak{H}}_{\Omega}$ *and* $u < \underset{\sim}{b}$, *then either* $u = \underset{\sim}{b}$ *or* $u(x) < b$ *for all* $x \in \Omega$.

(iii) *A nonconstant function in* $\mathfrak{H}_{\Omega}$ *takes neither a nonnegative maximum value nor a nonpositive minimum value in* Ω.

Finally, as an easy to prove consequence of 2.9 we have the following result:

PROPOSITION 2.13. — *Let* Ω *be an open subset of* W *and let* v *be a nonconstant function in* $\overline{\mathfrak{H}}_{\Omega}$. *Let* c *be a real number such that* $c \leqslant \liminf_{x \in \Omega,\, x \to x_0} v$ *for each point* x_0 *on* $\partial\Omega$. *Then* $v(x) > c$ *for every* $x \in \Omega$ *if* $\underset{\sim}{1}$ *is in* $\mathfrak{H}_{\Omega}$ *or if* $\underset{\sim}{1}$ *is in* $\overline{\mathfrak{H}}_{\Omega}$ *and* $c \leqslant 0$.

3. The Dirichlet problem.

Throughout this section, Ω will denote an open subset of W. Brelot [6] has established a criterion for testing the regularity of points on $\partial\Omega \cap \mathrm{W}$ using the assumption that Ω is an inner open set with a positive potential defined on $\overline{\Omega}$. In this section we establish a similar criterion without making this assumption. We do, however, assume that there is a function V in $\overline{\mathfrak{H}}_{\Omega}$ with $\inf_{x \in \Omega} \mathrm{V} > 0$. Thus we may apply 2.9 and 2.10 to Ω.

DEFINITION. — *Let f be an extended real-valued function on $\partial\Omega$. We denote the set*

$$\left\{ v \in \overline{\mathfrak{H}}_\Omega : \liminf_{x \in \Omega,\, x \to x_0} v \underset{>-\infty}{\geqslant} f(x_0) \quad \textit{for all} \quad x_0 \in \partial\Omega \right\}$$

by the symbol $\mathcal{V}(f, \Omega)$ or simply $\mathcal{V}(f)$, and we denote the set

$$\left\{ u \in \underline{\mathfrak{H}}_\Omega : \limsup_{x \in \Omega,\, x \to x_0} u \underset{<+\infty}{\leqslant} f(x_0) \quad \textit{for all} \quad x_0 \in \partial\Omega \right\}$$

by the symbol $\mathcal{U}(f, \Omega)$ or $\mathcal{U}(f)$. If $\mathcal{V}(f) \neq \varnothing$ and $\mathcal{U}(f) \neq \varnothing$, then we denote the lower envelope of the functions in $\mathcal{V}(f)$ by $\overline{\mathrm{H}}(f, \Omega)$ or simply $\overline{\mathrm{H}}(f)$ and we denote the upper envelope of the functions in $\mathcal{U}(f)$ by $\underline{\mathrm{H}}(f, \Omega)$ or $\underline{\mathrm{H}}(f)$. We shall call $\overline{\mathrm{H}}(f, \Omega)$ the upper $\mathfrak{H}$-extension of f in Ω and $\underline{\mathrm{H}}(f, \Omega)$ the lower $\mathfrak{H}$-extension of f in Ω. If $\overline{\mathrm{H}}(f) = \underline{\mathrm{H}}(f)$, then f is said to be resolutive on $\partial\Omega$.

If f is a function on $\partial\Omega$ with $v \in \mathcal{V}(f)$ and $u \in \mathcal{U}(f)$, then $v - u$ is in $\overline{\mathrm{H}}_\Omega$ and $\liminf\,(v - u) \geqslant 0$ at $\partial\Omega$. By 2.9, $v \geqslant u$, and thus $\overline{\mathrm{H}}(f) \geqslant \underline{\mathrm{H}}(f)$. It is easy to see that

$$\underline{\mathrm{H}}(f) = -\,\overline{\mathrm{H}}(-f).$$

If g is a bounded function on $\partial\Omega$, then since $\exists \mathrm{V} \in \overline{\mathfrak{H}}_\Omega$ with $\inf_{x \in \Omega} \mathrm{V} > 0$, we have $\mathcal{V}(g) \neq \varnothing$ and $\mathcal{U}(g) \neq \varnothing$.

PROPOSITION 3.1. — *Let f be a function on $\partial\Omega$ such that $\mathcal{V}(f) \neq \varnothing$ and $\mathcal{U}(f) \neq \varnothing$. Then $\overline{\mathrm{H}}(f)$ and $\underline{\mathrm{H}}(f)$ are in $\mathfrak{H}_\Omega$.*

Proof. — We need only show that $\overline{\mathrm{H}}(f) \in \mathfrak{H}$. For each $u \in \mathcal{U}(f)$, $u \leqslant \overline{\mathrm{H}}(f)$. Thus $\overline{\mathrm{H}}(f)$ is finite on a dense subset of Ω. If ω is a regular region with $\overline{\omega} \subset \Omega$, then by 2.7 we have $\overline{\mathrm{H}}(f, \Omega)(x) = \inf_{v \in \mathcal{V}(f, \Omega)} \mathrm{H}(v, \omega)(x)$ for each $x \in \omega$. By Axioms I and III, $\overline{\mathrm{H}}(f, \Omega) \in \mathfrak{H}$. ∎

If Ω is regular and f is an integrable function on $\partial\Omega$ with $\mathcal{V}(f) \neq \varnothing$ and $\mathcal{U}(f) \neq \varnothing$, then one can show that

$$\mathrm{H}(f) = \overline{\mathrm{H}}(f) = \underline{\mathrm{H}}(f)$$

(See [6] pp. 84-85). We also have the following result which is due to Brelot:

THEOREM 3.2. (COMPARISON THEOREM). — *Let f be a function on $\partial\Omega$ such that $\mathfrak{V}(f) \neq \emptyset$ and $\mathfrak{U}(f) \neq \emptyset$. Let ω be an open subset of Ω, and let F be the function on $\partial\omega$ such that $F = f$ on $\partial\omega \cap \partial\Omega$ and $F = \overline{H}(f, \Omega)$ on $\partial\omega \cap \Omega$. Then $\overline{H}(f, \Omega) = \overline{H}(F, \omega)$ in ω.*

Proof. — (from [6]) : Given v_0 in $\mathfrak{V}(F, \omega)$, let $v_1 = v_0 \wedge \overline{H}(f, \Omega)$ in ω and $v_1 = \overline{H}(f, \Omega)$ in $\Omega - \omega$. For each

$$v \in \mathfrak{V}(f, \Omega), \qquad v_1 + v - \overline{H}(f, \Omega) \in \mathfrak{V}(f, \Omega),$$

whence $v_1 + v - \overline{H}(f, \Omega) \geqslant \overline{H}(f, \Omega)$. For any $x \in \Omega$ and

$$\forall \varepsilon > 0, \quad \exists v \in \mathfrak{V}(f, \Omega)$$

such that $v(x) - \overline{H}(f, \Omega)(x) < \varepsilon$. Thus $v_0 \geqslant \overline{H}(f, \Omega)$ in ω, whence $\overline{H}(F, \omega) \geqslant \overline{H}(f, \Omega)$. Clearly $\overline{H}(F, \omega) \leqslant \overline{H}(f, \Omega)$. Therefore, $\overline{H}(f, \omega) = \overline{H}(f, \Omega)$ in ω. ▌

If Ω is a regular open set and f is a bounded function on $\partial\Omega$, then a proof similar to the proof of 1.3 shows that for each $x_0 \in \partial\Omega$

(1)
$$\liminf_{x \in \partial\Omega,\, x \to x_0} f \leqslant \liminf_{x \in \Omega,\, x \to x_0} \underline{H}(f) \leqslant \limsup_{x \in \Omega,\, x \to x_0} \overline{H}(f) \leqslant \limsup_{x \in \partial\Omega,\, x \to x_0} f.$$

On the other hand, if Equation 1 is valid for every $x_0 \in \partial\Omega$ and every bounded function on $\partial\Omega$, then for every continuous function f on $\partial\Omega$ we have $\overline{H}(f) = \underline{H}(f)$ and $\lim_{x \in \Omega,\, x \to x_0} \overline{H}(f) = f(x_0)$ at each $x_0 \in \partial\Omega$. Thus in this case Ω is regular and the $\mathfrak{H}$-extension of each continuous function f is equal to $\overline{H}(f)$ and $\underline{H}(f)$. With this in mind we make the following definition :

DEFINITION (1). — *Let x_0 be a point on $\partial\Omega$. We say that x_0 is a regular point for Ω with respect to $\mathfrak{H}$ or simply that x_0 is regular if Equation 1 holds for every bounded function f on $\partial\Omega$.*

The second inequality in Equation 1 is always true since $\underline{H}(f) \leqslant \overline{H}(f)$. The validity of the first inequality in Equation 1

(1) Note that unlike the standard definition of regularity at a point, this definition does not assume the resolutivity of continuous functions on $\partial\Omega$.

follows from the validity of the last inequality since

$$\underline{\mathrm{H}}(f) = -\overline{\mathrm{H}}(-f).$$

Therefore, a point $x_0 \in \partial\Omega$ is a regular point if and only if for every bounded function f on $\partial\Omega$ we have

$$\limsup_{x \in \Omega,\, x \to x_0} \overline{\mathrm{H}}(f) \leqslant \limsup_{x \in \partial\Omega,\, x \to x_0} f.$$

Clearly, Ω is regular if and only if each point $x_0 \in \partial\Omega$ is regular. If Ω_0 is a component of Ω and x_0 is a point on $\partial\Omega_0$, then x_0 is regular for Ω_0 if x_0 is regular for Ω. We next establish a criterion for testing the regularity of points on $\partial\Omega$.

Definition [2]. — *Let x_0 be a point on $\partial\Omega$. By an $\mathfrak{H}$-barrier or a barrier for Ω at x_0 we mean a positive function b in $\mathfrak{H}$ such that b is defined in the intersection of Ω and an open neighborhood of x_0 and* $\lim_{x \in \Omega,\, x \to x_0} b = 0$.

Theorem 3.3. — *Given $x_0 \in \partial\Omega \cap \mathrm{W}$, if there is a barrier b for Ω at x_0 and a function $\mathrm{V}_0 \in \overline{\mathfrak{H}}_\Omega$ bounded in a neighborhood of x_0 with $\inf \mathrm{V}_0 > 0$, then x_0 is a regular point for Ω.*

Proof. — Let f be a bounded function on $\partial\Omega$, and let $c = \limsup_{x \in \partial\Omega,\, x \to x_0} f$. Let ε be any positive constant. Since x_0 is contained in a regular inner region, there is a function $h \in \mathfrak{H}$ defined in a neighborhood of x_0 such that $h(x_0) = c + \varepsilon$. Let ω be a regular inner region containing x_0 such that h is defined on ϖ, b is defined on $\varpi \cap \Omega$, $h \geqslant f$ on $\partial\Omega \cap \varpi$, and V_0 is bounded on $\overline{\omega} \cap \Omega$. Let $\mathrm{F} = f$ on $\partial(\Omega \cap \omega) \cap \partial\Omega$ and $\mathrm{F} = \overline{\mathrm{H}}(f, \Omega)$ on $\partial(\Omega \cap \omega) \cap \Omega = \partial\omega \cap \Omega$; F is bounded. If $\partial\omega \cap \Omega = \emptyset$, then $h \in \mathfrak{V}(\mathrm{F}, \Omega \cap \omega)$, whence

$$\limsup_{x \in \Omega,\, x \to x_0} \overline{\mathrm{H}}(\mathrm{F}, \Omega \cap \omega) \leqslant c + \varepsilon.$$

If $\partial\omega \cap \Omega \neq \emptyset$, let M be a constant $> \sup_{x \in \partial\omega \cap \Omega} (|\mathrm{F}(x)| + |h(x)|)$.

[2] Following Brelot, we have deviated from the classical definition of a barrier by omitting the assumption that $\liminf_{x \in \Omega,\, x \to x_1} b > 0$ for every point $x_1 \neq x_0$ on $\partial\Omega$.

There is a compact subset C of $\partial\omega \cap \Omega$ such that if

$$D = (\partial\omega \cap \Omega) - C \text{ and } \chi_D$$

is the characteristic function of D on $\partial\omega$ then $H(\chi_D, \omega)(x_0) \leqslant \frac{\varepsilon}{M}$. In this case, setting

$$T = h + \frac{M}{\inf\limits_{x \in C} b(x)} b + MH(\chi_D, \omega),$$

we have $T \in \mathfrak{V}(F, \Omega \cap \omega)$ and $\limsup\limits_{x \in \Omega,\, x \to x_0} T \leqslant c + 2\varepsilon$. In either case, therefore, $\limsup\limits_{x \in \Omega,\, x \to x_0} \overline{H}(f, \Omega) \leqslant c + 2\varepsilon$. Since ε is arbitrarily chosen, x_0 is a regular point for Ω. ▌

In establishing a partial converse of the above theorem, we shall need to consider the possibility that a regular inner region may have only one point on its boundary. For example, let W be the real line together with the point $+\infty$ under the order topology, i.e., a base for W consists of intervals of the form $\{x : a < x < b\}$ and $\{x : a < x \leqslant +\infty\}$ where a and b are finite. Let f be in $\mathfrak{H}$ if f is a linear function $\alpha x + \beta$ on an open subset of $W - \{+\infty\}$ or if f is a constant function $\underset{\sim}{c}$ on an open subset of W. It is easy to see that $\mathfrak{H}$ is a harmonic class of functions. Moreover, every inner region which contains the point $+\infty$ is regular and has only one point on its boundary. Nevertheless, we have the following result for the general case:

Proposition 3.4. — *Let Ω be an open subset of* W, *and let x_0 be a regular point for Ω on $\partial\Omega \cap$* W. *Then there is a bounded barrier b for Ω at x_0 with the domain of b equal to Ω if either of the following conditions holds:*

(i) *Ω has at most a countable number of components, and if Ω_0 is any component of Ω, then either $\partial\Omega_0 = \{x_0\}$ or there is a point $x_1 \neq x_0$ on $\partial\Omega_0$ such that x_1 is regular for Ω_0.*

(ii) *There is a countable base for the neighborhood system of x_0 in $\partial\Omega$, and if Ω_0 is any component of Ω, then there is a point $x_1 \neq x_0$ on $\partial\Omega_0$ such that x_1 is regular for Ω_0.*

Proof. — Assume first that Ω is a region. If there is a regular point $x_1 \neq x_0$ on $\partial\Omega$, then by the Urysohn lemma there is a continuous nonnegative function f on $\partial\Omega$ such that $f(x_0) = 0$ and $f(x_1) = 1$. Clearly, $\overline{\mathrm{H}}(f, \Omega)$ is a barrier for Ω at x_0. If $\partial\Omega = \{x_0\}$, then Ω is regular. Let ω be any regular inner region containing x_0 such that $\Omega - \overline{\omega} \neq \emptyset$. Then

$$\partial\omega = (\partial\omega \cap \Omega) \cup (\partial\omega \cap (\mathrm{W} - \overline{\Omega})).$$

Let $c = 2 \sup\limits_{x \in \partial\omega \cap \Omega} \mathrm{H}(\underset{\sim}{1}, \Omega)(x)$. It is easy to see that $\exists d < 0$ such that if $g = c$ on $\partial\omega \cap \Omega$ and $g = d$ on $\partial\omega \cap (\mathrm{W} - \overline{\Omega})$ then $\mathrm{H}(g, \omega)(x_0) = 0$. Let $b = \mathrm{H}(\underset{\sim}{1}, \Omega) \wedge \mathrm{H}(g, \omega)$ in $\omega \cap \Omega$. Then b is a barrier for Ω at x_0.

Assume that Ω satisfies condition (i). In each component Ω_n of Ω there is a positive function $b_n \in \overline{\mathfrak{H}}_{\Omega_n}$ such that $b_n \leqslant \frac{1}{n}$ and $\lim\limits_{x \in \Omega_n,\, x \to x_0} b_n = 0$ if $x_0 \in \partial\Omega_n$. Set $b = b_n$ in each Ω_n. Then b is a barrier for Ω at x_0.

Now assume that Ω satisfies condition (ii). A slight variation of the Urysohn lemma shows that there is a continuous function f on $\partial\Omega$ such that $f(x_0) = 0$ and $f(x) > 0$ for all $x \neq x_0$ on $\partial\Omega$. By 3.2, $\overline{\mathrm{H}}(f, \Omega)$ is positive in Ω. Since $\lim \overline{\mathrm{H}}(f, \Omega) = 0$ at x_0, $\overline{\mathrm{H}}(f, \Omega)$ is a barrier for Ω at x_0. ∎

R.-M. Hervé ([10], p. 443) has shown that if Ω_0 is a component of Ω and $\tilde{x}$ is a point on $\partial\overline{\Omega}_0 \cap \mathrm{W}$, then every neighborhood of $\tilde{x}$ contains at least one point $x_1 \in \partial\Omega_0$ such that x_1 is regular for Ω_0 ([3]).

4. The Existence of an Exhaustion of W by Regular Inner Regions.

Definition. — *An exhaustion of* W *by inner regions is a family* $\mathfrak{R}$ *of inner regions such that if* Ω_1 *and* Ω_2 *are in* $\mathfrak{R}$ *then there is a region* $\Omega \in \mathfrak{R}$ *with* $\Omega_1 \cup \Omega_2 \subset \Omega$, *and* $\mathrm{W} = \bigcup\limits_{\Omega \in \mathfrak{R}} \Omega$.

We shall show in this section that there is an exhaustion of W by regular inner regions if there is a positive function in $\overline{\mathfrak{H}}_{\mathrm{W}}$.

([3]) The proof in [10] assumes the existence of a positive potential on $\overline{\Omega}_0$, but that assumption may easily be eliminated. (See the proof of 4.1.)

DEFINITION. — *We shall call a nonempty compact subset* A *of* W *an outer-regular compact set if there exists a barrier for* W — A *at each point* x *on* ∂A.

The first proposition shows that each point $y \in$ W is contained in the interior of an outer-regular compact set. The proofs for this proposition and corollary 4.2 are similar to the proofs of the corresponding results given by R.-M. Hervé in [10] (pp. 439-440). We do not, however, assume the existence of a positive potential on W.

PROPOSITION 4.1. — *Let* D *be a compact subset of* W, *and assume that there is a regular inner region* Ω *containing* D. *Then there is an outer-regular compact set* A *with nonempty interior* A^0 *such that* $D \subset A^0 \subset A \subset \Omega$.

Proof. — Let Ω_1 and Ω_2 be regions such that

$$D \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega.$$

Let $h = H(\underset{\sim}{1}, \Omega)$, and let $\mathcal{G} = \{v \in \overline{\mathfrak{H}}_\Omega : v \geqslant \underset{\sim}{0} \text{ and } v|\overline{\Omega}_1 \geqslant h|\overline{\Omega}_1\}$. Let P be the lower envelope of $\mathcal{G}$ [4]. If $f = 0$ on $\partial\Omega$ and $f = h$ on $\partial\Omega_1$, then $P = \overline{H}(f, \Omega - \overline{\Omega}_1)$ in $\Omega - \overline{\Omega}_1$. Therefore,

$$P|(\Omega - \overline{\Omega}_1)$$

is in $\mathfrak{H}$ and lim P $= 0$ at $\partial\Omega$. If ω is any regular region with $\varpi \subset \Omega$ and if $v \in \mathcal{G}$, then $v \geqslant \overline{H}(v, \omega) = H(v, \omega)$, whence $P \geqslant \overline{H}(P, \omega)$. Since $P(x) > 0$ in $\overline{\Omega}_1$, $P(x) > 0$ in Ω. (See proof of 2.1). Let $\alpha = \min\limits_{x \in \partial\Omega_2} \frac{P(x)}{h(x)}$. Clearly $\alpha > 0$. Setting A equal to the compact set

$$\overline{\Omega}_2 \cup \{x \in \Omega : P(x) \geqslant \alpha h(x)\},$$

we see that the restriction of $\alpha h - P$ to $\Omega - A$ is a barrier for W — A at each point x on ∂A. ▌

COROLLARY 4.2. — *Let* Ω_0 *be a région in* W, *and let* D *be a compact subset of* Ω_0. *Assume that there is a positive function*

[4] In the notation of [6], $P = (\overline{R}_h^{\Omega_1})_\Omega$.

V *in* $\mathfrak{H}_{\Omega_0}$. *Then there is a regular inner region* Ω *with*

$$D \subset \Omega \subset \overline{\Omega} \subset \Omega_0.$$

Proof. — Let Ω_1 and Ω_2 be regions such that

$$D \subset \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_0.$$

It follows from 4.1 that there are a finite number of outer regular compact sets A_i, $i = 1, ..., n$, such that

$$\partial\Omega_2 \subset \bigcup_{i=1}^{n} A_i^\circ \subset \bigcup_{i=1}^{n} A_i \subset W - \overline{\Omega}_1.$$

Let Ω be the component of $\Omega_2 - \bigcup_{i=1}^{n} A_i$ which contains Ω_1. Then by 3.3, Ω is regular. ▌

As a corollary of 4.2 we have the fact that there is an exhaustion of W by regular inner regions if there is a positive function in $\overline{\mathfrak{H}}_W$.

Theorem 4.3. — *Let* A *be a compact subset of* W, *and let* $\mathfrak{R}$ *be the family of all regular inner regions* Ω *which contain* A. *If there is a positive function in* $\overline{\mathfrak{H}}_W$, *then* $\mathfrak{R}$ *is an exhaustion of* W.

Corollary 4.4. — *If there is a positive function in* $\overline{\mathfrak{H}}_W$, *then every compact set is contained in an outer-regular compact set.*

5. $\mathfrak{H}$-measures and the Classification of $\mathfrak{H}$.

Throughout this section we assume that $\mathfrak{H}$ satisfies Axiom IV, that is, that $\underset{\sim}{1}$ is in $\overline{\mathfrak{H}}_W$. Given this assumption, we distinguish between two types of harmonic classes on W. If $\underset{\sim}{1}$ is not in $\mathfrak{H}_W$, then $\mathfrak{H}$ is said to be a hyperbolic harmonic class on W if there are nonzero bounded functions in $\mathfrak{H}_W$, and $\mathfrak{H}$ is said to be a parabolic harmonic class on W if there are no nonzero bounded functions in $\mathfrak{H}_W$. If, however, $\underset{\sim}{1}$ is in $\mathfrak{H}_W$, then there is always a trivial nonzero bounded function in $\mathfrak{H}_W$, namely $\underset{\sim}{1}$. In this case we take an outer-regular compact subset A of W and define $\mathfrak{H}$ to be hyperbolic or parabolic on W depending on whether the upper $\mathfrak{H}$-extension of the function which is equal to $\underset{\sim}{1}$ on ∂W and $\underset{\sim}{0}$ on ∂A is positive in some component of W — A or identically equal to 0. We shall show that this classification of $\mathfrak{H}$ is independent of the choice of A and that it is equivalent to the classification in terms of the existence of a nonzero bounded function in $\mathfrak{H}_W$ when $\underset{\sim}{1} \notin \mathfrak{H}_W$. We shall also show that if $\underset{\sim}{1} \in \mathfrak{H}_W$, then there is a positive potential on W if and only if $\mathfrak{H}$ is hyperbolic.

We first define four functions in $\mathfrak{H}$ which play a special role in defining the classification of $\mathfrak{H}$. They will also be used extensively in Chapter II.

DEFINITION. — *Let* A *be an outer-regular compact subset of* W.

(i) *By the $\mathfrak{H}$-measure for* W, H(W), *we mean the upper $\mathfrak{H}$-extension in* W *of the constant function $\underset{\sim}{1}$ on* ∂W.

(ii) *By the $\mathfrak{H}$-measure for* W — A, H(W — A), *we mean the upper $\mathfrak{H}$-extension in* W — A *of the constant function $\underset{\sim}{1}$ on* ∂(W — A).

(iii) *By the $\mathfrak{H}$-measure of* ∂W *for* W — A, H(∂W, W — A), *we mean the upper $\mathfrak{H}$-extension in* W — A *of the function which is equal to $\underset{\sim}{1}$ on* ∂W *and* $\underset{\sim}{0}$ *on* ∂A.

(iv) *By the $\mathfrak{H}$-measure of* ∂A *for* W — A, H(∂A, W — A), *we mean the lower $\mathfrak{H}$-extension in* W — A *of the function which is equal to $\underset{\sim}{1}$ on* ∂A *and* $\underset{\sim}{0}$ *on* ∂W.

Since A is an outer-regular compact set, H(W — A) and H(∂A, W — A) tend to $\underset{\sim}{1}$ at ∂A and H(∂W, W — A) tends to $\underset{\sim}{0}$ at ∂A. Thus H(W — A) and H(∂A, W — A) are positive in W — A. Moreover, if t is the function on ∂(W — A) such that $t = \underset{\sim}{0}$ on ∂W and $t = \underset{\sim}{1}$ on ∂A, then since

$$\liminf H(\partial A, W - A) \geqslant 0$$

at ∂W we have $H(\partial A, W - A) = \underline{H}(t, W - A) = \overline{H}(t, W - A)$. That is, t is resolutive on ∂(W — A). A similar statement is true for H(W), H(W — A) and H(∂W, W — A).

The following proposition establishes another method of obtaining the four $\mathfrak{H}$-measures defined above:

PROPOSITION 5.1. — *Let* A *be an outer-regular compact set in* W, *and let* $\mathfrak{R}$ *be any exhaustion of* W *consisting of regular inner regions* Ω *which contain* A. *Then*

(i) H(W) *is the lower envelope of the family* $\{F_\Omega : \Omega \in \mathfrak{R}\}$, *where for each* $\Omega \in \mathfrak{R}$, $F_\Omega = H(\underset{\sim}{1}, \Omega)$ *in* Ω *and* $F_\Omega = \underset{\sim}{1}$ *in* $W - \Omega$.

(ii) $H(W - A)$ *is the lower envelope of the family* $\{G_\Omega : \Omega \in \mathfrak{R}\}$, *where for each* $\Omega \in \mathfrak{R}$, $G_\Omega = H(\underset{\sim}{1}, \Omega - A)$ *in* $\Omega - A$ *and* $G_\Omega = \underset{\sim}{1}$ *in* $W - \Omega$.

(iii) $H(\partial W, W - A)$ *is the lower envelope of the family* $\{S_\Omega : \Omega \in \mathfrak{R}\}$, *where if* $s_\Omega = \underset{\sim}{1}$ *on* $\partial\Omega$ *and* $s_\Omega = \underset{\sim}{0}$ *on* ∂A *for each* $\Omega \in \mathfrak{R}$, *then* $S_\Omega = H(s_\Omega, \Omega - A)$ *in* $\Omega - A$ *and* $S_\Omega = \underset{\sim}{1}$ *in* $W - \Omega$.

(iv) $H(\partial A, W - A)$ *is the upper envelope of the family* $\{T_\Omega : \Omega \in \mathfrak{R}\}$, *where if* $t_\Omega = \underset{\sim}{0}$ *on* $\partial\Omega$ *and* $t_\Omega = \underset{\sim}{1}$ *on* ∂A *for each* $\Omega \in \mathfrak{R}$, *then* $T_\Omega = H(t_\Omega, \Omega - A)$ *in* $\Omega - A$ *and* $T_\Omega = \underset{\sim}{0}$ *in* $W - \Omega$.

Proof. — We shall prove (iii); Statements (i), (ii) and (iv) can be proved in a similar fashion. Let s be equal to $\underset{\sim}{1}$ on ∂W and $\underset{\sim}{0}$ on ∂A. For each $\Omega \in \mathfrak{R}$, S_Ω is in $\mathcal{V}(s, W - A)$. Let S be the lower envelope of the functions S_Ω. Since

$$H(\partial W, W - A) = \overline{H}(s, W - A),$$
$$S \geqslant H(\partial W, W - A).$$

On the other hand, the set $\{S_\Omega : \Omega \in \mathfrak{R}, \Omega \supset \Omega_0\}$ is a family of functions directed by decreasing order in $\Omega_0 - A$ for each $\Omega_0 \in \mathfrak{R}$. Hence S is in $\mathfrak{H}$. Furthermore, $\underset{\sim}{0} \leqslant S \leqslant \underset{\sim}{1}$ and S tends to $\underset{\sim}{0}$ at ∂A. Therefore, for every function

$$v \in \mathcal{V}(s, W - A), \qquad v \geqslant S.$$

We therefore have

$$H(\partial W, W - A) = \overline{H}(s, W - A) \geqslant S,$$

and thus $H(\partial W, W - A) = S$. ▌

COROLLARY 5.2. — *Let* A *be an outer-regular compact set in* W. *Then* $H(W - A) = H(\partial A, W - A) + H(\partial W, W - A)$.

Proof. — Let $\mathfrak{R}$ and the functions G_Ω, S_Ω and T_Ω be given as in 5.1. The corollary follows from the fact that for each Ω in $\mathfrak{R}$, $G_\Omega = S_\Omega + T_\Omega$. ▌

If the constant function $\underset{\sim}{1}$ is in $\mathfrak{H}$, then $H(W) = \underset{\sim}{1}$ and $H(W - A) = \underset{\sim}{1}$. Even if $\underset{\sim}{1}$ is not in $\mathfrak{H}$, however, the $\mathfrak{H}$-measures $H(W)$ and $H(W - A)$ have many properties which the function $\underset{\sim}{1}$ would have if it were in $\mathfrak{H}$. These properties are extremal properties in the sense that these $\mathfrak{H}$-measures are the largest or smallest functions which satisfy certain inequalities. We summarize some of the extremal properties of $\mathfrak{H}$-measures in the following proposition. With the exception of Theorem 5.8 and its corollary, the remaining propositions of this section are statements in the axiomatic setting of the corresponding results established for Riemann surfaces by H. L. Royden in [13] (Propositions 1-4, pp. 7-9).

Proposition 5.3. — *Let* A *be an outer-regular compact set in* W, *and let* α *be a nonnegative constant. If* h *is a function in* $\mathfrak{H}_W$, *then*

(i) $$h \leqslant \underset{\sim}{\alpha} \Longrightarrow h \leqslant \alpha H(W);$$

(ii) $$h \leqslant \alpha(\underset{\sim}{1} - H(W)) \Longrightarrow h \leqslant \underset{\sim}{0}.$$

If h *is a function in* $\mathfrak{H}_{W-A}$, *then*

(iii) $h \leqslant \underset{\sim}{\alpha} \Longrightarrow h \leqslant \alpha H(W - A)$;

(iv) $h \leqslant \underset{\sim}{\alpha}$ *and* $\limsup h \leqslant 0$ *at*

$$\partial A \Longrightarrow h \leqslant \alpha H(\partial W, W - A);$$

(v) $h \geqslant \underset{\sim}{0}$ *and* $\liminf h \geqslant \alpha$ *at*

$$\partial A \Longrightarrow h \geqslant \alpha H(\partial A, W - A).$$

Proof. — Statement (i) follows from the fact that if $\alpha^{-1}h \leqslant \underset{\sim}{1}$, then $\alpha^{-1}h \leqslant v$ for each function v in $\mathfrak{V}(\underset{\sim}{1}, W)$, whence

$$a^{-1}\, h \leqslant \overline{H}(\underset{\sim}{1}, W) = H(W).$$

Statements (iii), (iv) and (v) are proved in a similar way. By (i), $h + \alpha H(W) \leqslant \underset{\sim}{\alpha} \Longrightarrow h + \alpha H(W) \leqslant \alpha H(W)$. Therefore, Statement (ii) follows from (i). ▌

We shall use the $\mathfrak{H}$ measure $H(\partial W, W - A)$ to classify $\mathfrak{H}$ on W. To do this, however, we need the following result:

Proposition 5.4. — *If* $H(\partial W, W - A_0) = \underset{\sim}{0}$ *for some outer-regular compact set* A_0 *in* W, *then* $H(\partial W, W - A) = \underset{\sim}{0}$ *for every outer-regular compact set* A *in* W.

Proof. — Let A be any outer-regular compact set in W, and let $D = A \cup A_0$. Then D is also an outer-regular compact set. If φ is any function in $\overline{\mathfrak{H}}_{W-A_0}$ such that $\liminf \varphi \geqslant 1$ at ∂W and $\liminf \varphi \geqslant 0$ at ∂A_0, then $\liminf_{x \in W-D,\, x \to x_0} \varphi \geqslant 0$ at each point x_0 on ∂D. Hence,

$$H(\partial W, W - D) \leqslant H(\partial W, W - A_0) = \underset{\sim}{0},$$

i.e., $H(\partial W, W - D) = \underset{\sim}{0}$.

Since $H(\partial W, W - A) \leqslant \underset{\sim}{1}$ and $\lim H(\partial W, W - A) = 0$ at ∂A, $H(\partial W, W - A)(x) < 1$ for each $x \in W - A$.

$$\text{Let } m = \sup_{x \in D - A} H(\partial W, W - A)(x).$$

Clearly, $m < 1$. We shall show that $H(\partial W, W - A) \leqslant \underset{\sim}{m}$ in $W - A$.

If φ is any function in $\overline{\mathfrak{H}}_{W-D}$ such that $\liminf \varphi \geqslant 1$ at ∂W and $\liminf \varphi \geqslant 0$ at ∂D, then $H(\partial W, W - A) \leqslant \underset{\sim}{m} + (1 - m)\varphi$ in $W - D$. Therefore,

$$H(\partial W, W - A) \leqslant \underset{\sim}{m} + (1 - m)\, H(\partial W, W - D) = \underset{\sim}{m}$$

in $W - D$. It follows that $H(\partial W, W - A) \leqslant \underset{\sim}{m}$ in $W - A$. By Part (iv) of Proposition 5.3 we have

$$H(\partial W, W - A) \leqslant m H(\partial W, W - A).$$

But $m < 1$. Therefore $H(\partial W, W - A) = \underset{\sim}{0}$. ▌

Definition. — *Let* A *be an outer-regular compact set in* W. *The harmonic class* $\mathfrak{H}$ *is said to be hyperbolic on* W *if*

$$H(\partial W, W - A) \neq \underset{\sim}{0}$$

and parabolic on W *if* $H(\partial W, W - A) = \underset{\sim}{0}$.

As we noted before, we cannot classify $\mathfrak{H}$ on W in terms of H(W) being positive or equal to $\underset{\sim}{0}$ because $H(W) = \underset{\sim}{1}$ whenever $\underset{\sim}{1} \in \mathfrak{H}$. We do, however, have the following proposition for the case that $\underset{\sim}{1}$ is not in $\mathfrak{H}_W$:

PROPOSITION 5.5. — *The following statements are equivalent if the constant function* $\underset{\sim}{1}$ *is not in* $\mathfrak{H}_W$:

(i) $\mathfrak{H}$ *is parabolic on* W.

(ii) $H(W) = \underset{\sim}{0}$.

(iii) *There is no nonzero bounded function in* $\mathfrak{H}_W$.

(iv) $\underset{\sim}{1}$ *is a potential.*

Proof. — We show first that (i) $\Longleftrightarrow$ (ii), Let A be any outer-regular compact set in W, and assume that $H(\partial W, W - A) = \underset{\sim}{0}$. Since $H(W)(x) < 1$ for all points x in W, the maximum m of H(W) on A is less than 1. Let v be any function in $\overline{\mathfrak{H}}_{W-A}$ such that $\liminf v \geqslant 1$ at ∂W and $\liminf v \geqslant 0$ at ∂A. Then, $H(W) \leqslant \underset{\sim}{m} + (1 - m)v$ in $W - A$, whence

$$H(W) \leqslant \underset{\sim}{m} + (1 - m)\, H(\partial W, W - A) = \underset{\sim}{m}$$

in $W - A$. Since we also have $H(W) \leqslant \underset{\sim}{m}$ in A, it follows from part (i) of Proposition 5.3 that $H(W) \leqslant mH(W)$. But $m < 1$. Therefore $H(W) = \underset{\sim}{0}$.

Now assume that $H(W) = \underset{\sim}{0}$, and let A be any outer-regular compact set in W. If v is a function in $\mathfrak{V}(\underset{\sim}{1}, W)$, then $\liminf v \geqslant 0$ at ∂A. Hence

$$H(\partial W,\ W - A) \leqslant H(W) = \underset{\sim}{0}.$$

Thus (i) $\Longleftrightarrow$ (ii).

If h is a function in $\mathfrak{H}_W$ such that $-\underset{\sim}{M} \leqslant h \leqslant \underset{\sim}{M}$ for some constant M, then $|h| \leqslant MH(W)$. Thus we have (ii) $\Longrightarrow$ (*iii*). Clearly, (iii) $\Longrightarrow$ (*ii*). By part (i) of Proposition 5.3, H(W) is the greatest minorant of $\underset{\sim}{1}$ in $\mathfrak{H}_W$. Thus (ii) $\Longleftrightarrow$ (iv). ▌

The next proposition is an extension of the maximum principle for the case that $\mathfrak{H}$ is parabolic on W.

PROPOSITION 5.6. — *Let* Ω *be a region in* W *such that* $\overline{\Omega} \neq \overline{W}$, *and let* $C = \partial\Omega \cap W$. *Let* $h \in \mathfrak{H}_\Omega$ *be a function which is bounded from above, and let* α *be a nonnegative constant such that*

$$\limsup_{x \in \Omega,\ x \to x_0} h \leqslant \alpha$$

at each point $x_0 \in C$. *Then* $h \leqslant \underset{\sim}{\alpha}$ *in* Ω *if* $\mathfrak{H}$ *is parabolic on* W.

Proof. — By 4.1 there exists an outer-regular compact set A with $A \subset W - \overline{\Omega}$. The fact that $\mathfrak{H}$ is parabolic on W implies that $H(\partial W, W - A) = \underset{\sim}{0}$. Let M be an upper bound for h. Let v be any function in $\overline{\mathfrak{H}}_{W-A}$ such that $\liminf v \geqslant 1$ at ∂W and $\liminf v \geqslant 0$ at ∂A. Then

$$\liminf_{x \in \Omega,\ x \to x_0} (Mv + \underset{\sim}{\alpha} - h) \geqslant 0$$

at each point x_0 on $\partial\Omega$. Hence, $Mv \geqslant h - \underset{\sim}{\alpha}$ in Ω, and thus

$$h - \underset{\sim}{\alpha} \leqslant MH(\partial W, W - A) = \underset{\sim}{0}.$$

Thus $h \leqslant \underset{\sim}{\alpha}$ in Ω. ▮

We next establish an extremal property of the $\mathfrak{H}$-measure $H(\partial A, W - A)$ for the case that H is hyperbolic on W.

Proposition 5.7. — *Let* A *be an outer-regular compact set in* W, *and let* h *be a function in* $\mathfrak{H}_W$ *such that*

$$h \leqslant MH(\partial A, W - A) \quad in \quad W - A$$

for some positive constant M. *Then* $h \leqslant \underset{\sim}{0}$ *in* W *if* $\mathfrak{H}$ *is hyperbolic on* W.

Proof. — Let $m = \max\left(0, \sup_{x \in A} h(x)\right)$. Without loss of generality we may assume that $m \leqslant M$. Now

$$MH(\partial A, W - A) - h \geqslant \underset{\sim}{0}$$

in $W - A$ and $\liminf (MH(\partial A, W - A) - h) \geqslant M - m$ at ∂A. Therefore,

$$MH(\partial A, W - A) - h \geqslant (M - m)\, H(\partial A, W - A)$$

by Proposition 5.3, Part (vi). Hence

$$h \leqslant mH\,(\partial A, W - A)$$

in $W - A$ and $h \leqslant \underset{\sim}{m}$ in W.

Assume that $m \neq 0$. By 5.2,

$$H(W - A) = H(\partial W, W - A) + H(\partial A, W - A),$$

and by assumption $H(\partial W, W - A) \neq \underset{\sim}{0}$. Therefore,

$$H(\partial A, W - A) \neq \underset{\sim}{1}.$$

Since $h \leqslant m\mathrm{H}(\partial \mathrm{A}, \mathrm{W}-\mathrm{A})$ in $\mathrm{W}-\mathrm{A}$, $h \neq \underset{\sim}{m}$ in $\mathrm{W}-\mathrm{A}$. Thus $h(x) < m$ for all x in W and in particular for all x in A. The restriction of h to A takes its maximum value on A and that maximum value is less then m. But then $m = 0$. It follows from this contradiction that $m = 0$ and $h \leqslant \underset{\sim}{0}$ in W. ▌

If $\underset{\sim}{1} \notin \mathfrak{H}_{\mathrm{W}}$, then it follows from Part (ii) of Proposition 5.3 that $\underset{\sim}{1} - \mathrm{H(W)}$ is a positive potential on W. On the other hand we have the following result for the case that $\underset{\sim}{1}$ is in $\mathfrak{H}_{\mathrm{W}}$:

Theorem 5.8. — *Assume that $\underset{\sim}{1}$ is in $\mathfrak{H}_{\mathrm{W}}$. Then there is a positive potential on* W *if and only if* $\mathfrak{H}$ *is hyperbolic on* W.

Proof. — Let A be an outer-regular compact set. If $\mathfrak{H}$ is hyperbolic on W and P is the positive function in $\overline{\mathfrak{H}}_{\mathrm{W}}$ such that $\mathrm{P} = \underset{\sim}{1}$ in A and $\mathrm{P} = \mathrm{H}(\partial \mathrm{A}, \mathrm{W}-\mathrm{A})$ in $\mathrm{W}-\mathrm{A}$, then by 5.7, P is a potential on W.

Now assume that $\mathfrak{H}$ is parabolic on W. Also assume that there is a positive potential F on W. Since $\underset{\sim}{1} \in \mathfrak{H}_{\mathrm{W}}$, we may assume that F is bounded; e. g., replace F with $\mathrm{F} \wedge \underset{\sim}{1}$. Let $f = \mathrm{F}$ on $\partial\mathrm{A}$ and $f = \underset{\sim}{0}$ on $\partial\mathrm{W}$. Clearly, $\mathrm{F} \geqslant \overline{\mathrm{H}}(f, \mathrm{W}-\mathrm{A})$ in $\mathrm{W}-\mathrm{A}$. Let $\alpha = \min_{x \in \mathrm{A}} \mathrm{F}(x)$; then $\alpha > 0$. At each $x_0 \in \partial\mathrm{A}$ we have

$$\liminf_{x \in \mathrm{W}-\mathrm{A},\, x \to x_0} \overline{\mathrm{H}}(f, \mathrm{W}-\mathrm{A}) \geqslant \liminf_{x \in \partial\mathrm{A},\, x \to x_0} f \geqslant \alpha.$$

Thus by Part (v) of Proposition 5.3,

$$\overline{\mathrm{H}}(f, \mathrm{W}-\mathrm{A}) \geqslant \alpha \mathrm{H}(\partial\mathrm{A}, \mathrm{W}-\mathrm{A}).$$

But since $\mathrm{H}(\partial\mathrm{W}, \mathrm{W}-\mathrm{A}) = \underset{\sim}{0}$, $\mathrm{H}(\partial\mathrm{A}, \mathrm{W}-\mathrm{A}) = \underset{\sim}{1}$ by 5.2. Therefore $\mathrm{F} \geqslant \underset{\sim}{\alpha}$ in W which is impossible. Thus there is no positive potential if $\mathfrak{H}$ is parabolic on W. ▌

Corollary 5.9. — *Assume that* $\mathfrak{H}$ *is parabolic on* W *and that $\underset{\sim}{1} \in \mathfrak{H}_{\mathrm{W}}$. Then every lower bounded function in $\overline{\mathfrak{H}}_{\mathrm{W}}$ and every upper bounded function in $\underline{\mathfrak{H}}_{\mathrm{W}}$ is a multiple of $\underset{\sim}{1}$.*

Proof. — We need only prove the corollary for a positive function V in $\overline{\mathfrak{H}}_{\mathrm{W}}$. It is easy to see that V has a greatest nonnegative minorant h in $\mathfrak{H}_{\mathrm{W}}$. Since $\mathrm{V} - h$ is not a positive potential, $\mathrm{V} = h$. Moreover, we have shown that for every $\beta > 0$, $\underset{\sim}{1} \wedge \beta\mathrm{V}$ is in $\mathfrak{H}_{\mathrm{W}}$. It follows that V is a multiple of $\underset{\sim}{1}$. ▌

CHAPTER II

PROPERTIES OF PAIRS OF HARMONIC CLASSES ON W

6. The Harmonic Class $V^{-1}H$.

Throughout this section, $\mathfrak{H}$ will denote a harmonic class of functions on W. Let V be a positive continuous function on W. We denote the set of quotients $\{h/V : h \in \mathfrak{H}\}$ by the symbol $V^{-1}\mathfrak{H}$ and the set $\{h/V : h \in \mathfrak{H}_\Omega\}$ by $V^{-1}\mathfrak{H}_\Omega$. It is well known that $V^{-1}\mathfrak{H}$ is a harmonic class on W. Note that $\mathfrak{H} = V(V^{-1}\mathfrak{H})$. If Ω is an inner open subset of W, then Ω is regular for $\mathfrak{H}$ if and only if Ω is regular for $V^{-1}\mathfrak{H}$, Furthermore, if f is a function on $\partial\Omega$ and Ω is regular for $\mathfrak{H}$ and $V^{-1}\mathfrak{H}$, then f is integrable with respect to $\mathfrak{H}$ if and only if f is integrable with respect to $V^{-1}\,\mathfrak{H}$. In this case the $V^{-1}\mathfrak{H}$-extension of f is equal to $V^{-1}H(Vf,\ \Omega)$.

If φ is in $\overline{\mathfrak{H}}$ and ω is a regular inner region with ϖ contained in the domain of φ, then $V^{-1}\,\varphi$ is greater than or equal to the $V^{-1}\mathfrak{H}$-extension of $V^{-1}\varphi$ in ω since

$$V^{-1}\varphi \geqslant V^{-1}\ H(\varphi,\ \omega) = V^{-1}H(V(V^{-1}\varphi),\ \omega).$$

It follows that the superharmonic class associated with $V^{-1}\mathfrak{H}$ is the set $V^{-1}\overline{\mathfrak{H}} = \{V^{-1}\varphi : \varphi \in \overline{\mathfrak{H}}\}$ and the subharmonic class associated with $V^{-1}\mathfrak{H}$ is the set $V^{-1}\underline{\mathfrak{H}} = \{V^{-1}u : u \in \underline{\mathfrak{H}}\}$.

Let Ω be an inner open subset of W. Clearly, for any bounded function f on $\partial\Omega$, the upper and lower $V^{-1}\mathfrak{H}$-extensions of f in Ω are equal to $V^{-1}\overline{H}(fV,\ \Omega)$ and $V^{-1}\underline{H}(fV,\ \Omega)$ respectively. Moreover, if x_0 is a point on $\partial\Omega$ and b is an $\mathfrak{H}$-barrier for Ω at x_0, then $V^{-1}b$ is a $V^{-1}\mathfrak{H}$-barrier for Ω at x_0. It follows that a compact subset of W is outer-regular with respect to $\mathfrak{H}$ if and only if it is outer-regular with respect to $V^{-1}\mathfrak{H}$.

The constant function $\underset{\sim}{1}$ is in $V^{-1}\mathfrak{H}$ if and only if V is in $\mathfrak{H}$.

Moreover $\underset{\sim}{1}$ is in $V^{-1}\overline{\mathfrak{H}}$ if and only if V is in $\overline{\mathfrak{H}}$. Thus we have the following result:

THEOREM 6.1. — *Let* V *be a positive continuous function on* W. *Then* $V^{-1}\mathfrak{H}$ *satisfies Axiom IV if and only if* V *is in* $\overline{\mathfrak{H}}$.

Given 6.1, we can use the function V to generalize the classification of $\mathfrak{H}$ defined in Section 5.

DEFINITION. — *If* V *is in* $\overline{\mathfrak{H}}_W$, *then* $\mathfrak{H}$ *is called* V*-parabolic if* $V^{-1}\mathfrak{H}$ *is parabolic and* $\mathfrak{H}$ *is called* V*-hyperbolic if* $V^{-1}\mathfrak{H}$ *is hyperbolic. A function* $h \in H$ *is called* V*-bounded if there is a constant* M *such that* $|h| \leqslant MV$.

As a consequence of 5.1 we have the following criterion for determining the classification of $V^{-1}\mathfrak{H}$.

PROPOSITION 6.2. — *Let* V *be a positive continuous function in* $\overline{\mathfrak{H}}_W$. *Let* A *be an outer-regular compact set in* W *and let* $\mathfrak{R}$ *be an exhaustion of* W *consisting of regular inner regions* Ω *which contain* A. *Let* S *be the lower envelope of the family* $\{S_\Omega : \Omega \in \mathfrak{R}\}$, *where if* $s_\Omega = V$ *on* $\partial\Omega$ *and* $s_\Omega = \underset{\sim}{0}$ *on* ∂A *for each* $\Omega \in \mathfrak{R}$, *then* $S_\Omega = H(s_\Omega, \Omega - A)$ *in* $\Omega - A$ *and* $S_\Omega = V$ *in* $W - \Omega$. *Then* $\mathfrak{H}$ *is* V*-parabolic if and only if* $S = \underset{\sim}{0}$.

If V is a positive continuous function on W, then a function P is a potential for $\mathfrak{H}$ if and only if $V^{-1}P$ is a potential for $V^{-1}\mathfrak{H}$. Given a positive continuous function V in $\overline{\mathfrak{H}}_W$, we shall show that $V^{-1}\mathfrak{H}$ is parabolic if and only if V is a potential or there is no potential for $\mathfrak{H}$ on W (in which case $V \in \mathfrak{H}$).

PROPOSITION 6.3. — *Let* V *be a positive continuous function in* $\overline{\mathfrak{H}}_W$, *but assume that* V *is not in* $\mathfrak{H}_W$. *Then* $V^{-1}\mathfrak{H}$ *is parabolic on* W *if and only if* V *is a potential for* $\mathfrak{H}$.

Proof. — Since $\underset{\sim}{1} = V^{-1}V$ is not in $V^{-1}\mathfrak{H}_W$, $V^{-1}\mathfrak{H}$ is parabolic if and only if $\underset{\sim}{1}$ is a potential for $V^{-1}\mathfrak{H}$, i.e., if and only if V is a potential for $\mathfrak{H}$. ▮

PROPOSITION 6.4. — *Assume that there is a positive function in* $\overline{\mathfrak{H}}_W$. *Then the following statements are equivalent:*

(i) *If* V *is any positive function in* $\overline{\mathfrak{H}}_W$, *then* V *is in* $\mathfrak{H}_W$ *and every function* v in $\overline{\mathfrak{H}}_W$ *with* $v \geqslant mV$ *for some constant* m *is a multiple of* V.

(ii) *There is no positive potential for* $\mathfrak{H}$ *on* W.

(iii) $\mathfrak{H}$ *is* V*-parabolic for some positive* V *in* $\mathfrak{H}_W$.

(iv) *There is at least one positive function in* $\mathfrak{H}_W$, *and* $\mathfrak{H}$ *is* V*-parabolic for every positive* V *in* $\mathfrak{H}_W$.

Proof. — Letting $m = 0$ we see that (i) $\Longrightarrow$ (ii). Given (ii), we see that every positive $V \in \overline{\mathfrak{H}}_W$ is equal to its greatest minorant in $\mathfrak{H}_W$. Thus by 5.9, (ii) $\Longrightarrow$ (i). The rest of the proof follows from 5.8. ▌

If V_1 and V_2 are positive functions in $\overline{\mathfrak{H}}_W$ such that $V_1 \leqslant MV_2$ for some constant M, then V_1 is a potential for $\mathfrak{H}$ if V_2 is a potential for $\mathfrak{H}$. Thus we have the following consequence of 6.3 and 6.4:

PROPOSITION 6.5. — *Let* V_1 *and* V_2 *be positive continuous functions in* $\overline{\mathfrak{H}}_W$. *If* $V_1 \leqslant MV_2$ *for some constant* M, *then* $\mathfrak{H}$ *is* V_1*-parabolic if* $\mathfrak{H}$ *is* V_2*-parabolic. If* $mV_1 \leqslant V_2$ *for some positive constant* m, *then* $\mathfrak{H}$ *is* V_2*-hyperbolic if* $\mathfrak{H}$ *is* V_1*-hyperbolic.*

COROLLARY 6.6. — *Let* V_1 *and* V_2 *be positive continuous functions in* $\overline{\mathfrak{H}}_W$ *such that* $V_1 = V_2$ *in the complement of some compact subset* A *of* W. *Then* $\mathfrak{H}$ *is* V_1*-parabolic if and only if* H *is* V_2*-parabolic.*

COROLLARY 6.7. — *Let* V *be a positive continuous function in* $\overline{\mathfrak{H}}_W$, *and assume that* $\underset{\sim}{1}$ *is also in* $\overline{\mathfrak{H}}_W$. *If* $V \leqslant \underset{\sim}{M}$ *for some positive constant* M *and if* $\mathfrak{H}$ *is parabolic on* W, *then* $V^{-1}\,\mathfrak{H}$ *is parabolic on* W. *If* $\underset{\sim}{m} \leqslant V$ *for some positive constant* m, *and if* $\mathfrak{H}$ *is hyperbolic on* W, *then* $V^{-1}\,\mathfrak{H}$ *is hyperbolic on* W.

Brelot ([6], pp. 94-95) has shown that if there is a positive function in $\overline{\mathfrak{H}}_W$ then there is a positive continuous function in $\overline{\mathfrak{H}}_W$. Using this fact, we establish as an application of 5.8. the following result of Constantinescu and Cornea ([7], p. 381.)

THEOREM 6.8. — *If there is a positive function in* $\overline{\mathfrak{H}}_W$, *then there is a positive potential in any region* Ω *such that* $W - \overline{\Omega} \neq \emptyset$.

Proof. — We may assume that $\underset{\sim}{1} \in \overline{\mathfrak{H}}_W$. Let A be an outer-regular compact set in Ω. Let

$$\mathcal{G} = \{ v \in \overline{\mathfrak{H}}_{W-A} : \liminf v \geqslant 0 \text{ at } \partial A, \\ \lim v = 1 \text{ at } \partial\Omega \text{ and } v|W - \Omega = \underset{\sim}{1} \}.$$

Let G be the lower envelope of $\mathcal{G}$. It is easy to see that

$$G|\Omega - A = H(\partial\Omega, \Omega - A).$$

Moreover, for any regular region ω with $\bar{\omega} \subset W - A$ we have $G \geqslant \overline{H}(G, \omega)$. But there is one such region ω such that

$$\partial\omega \cap (W - \overline{\Omega}) \neq \emptyset \qquad \text{and} \qquad \omega \cap (\Omega - A) \neq \emptyset.$$

Thus G is positive in some component of $\Omega - A$. It follows that $\mathfrak{H}$ is hyperbolic on Ω and so there is a positive potential in Ω. ▌

7. Comparable Harmonic Classes.

In this section we shall consider pairs of harmonic classes $\mathfrak{H}$ and $\mathfrak{K}$ where $\overline{\mathfrak{K}}$ contains all positive functions of $\mathfrak{H}$ which have domains in the complement of some fixed compact subset of W.

Let $\mathcal{F}$ be any set of functions with open domains in W and let W_0 be an open subset of W. Then $\mathcal{F}|W_0$ will denote the set $\{f|(W_0 \cap \Omega_f) : f \in \mathcal{F}, \Omega_f \text{ is the domain of } f\}$. We shall usually take W_0 to be the complement $\tilde{A}$ of a compact subset A of W. Moreover, $\mathcal{F}^+$ will denote the nonnegative functions in $\mathcal{F}$ and $\mathcal{F}^-$ will denote the nonpositive functions in $\mathcal{F}$.

Definition. — *Let $\mathfrak{H}$ and $\mathfrak{K}$ be harmonic classes on* W. *We say that $\mathfrak{H}$ majorizes $\mathfrak{K}$ or that $\mathfrak{K}$ is majorized by $\mathfrak{H}$ if there is a compact subset* A *of* W *such that* $\mathfrak{H}^+|\tilde{A} \subset \overline{\mathfrak{K}}$.

We do not exclude the possibility that A is an empty set. We write $\mathfrak{H} \geqslant \mathfrak{K}$ when $\mathfrak{H}$ majorizes $\mathfrak{K}$, and we call $\mathfrak{H}$ and $\mathfrak{K}$ comparable harmonic classes. The set A is called an excluded set for the ordered pair $(\mathfrak{H}, \mathfrak{K})$.

An example of a comparable pair of harmonic classes is given by the solutions on a region W in R^n of the elliptic differential equation

$$(1) \qquad \Sigma a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \Sigma b_i \frac{\partial u}{\partial x_i} = Qu$$

and the solutions on W of the equation

$$(2) \qquad \Sigma a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \Sigma b_i \frac{\partial u}{\partial x_i} = Pu$$

where

(i) the coefficients of (1) and (2) satisfy a local Lipschitz condition,

(ii) the left side of (2) is the same as the left side of (1)

(iii) $\Sigma a_{ij}x_i x_j$ is a positive definite quadratic form.

(iv) $\mathrm{P} \geqslant \mathrm{Q}$ and $\mathrm{P} \geqslant \underset{\sim}{0}$.

If h is a solution of (1) and k is a solution of (2) and if

$$h \geqslant k \geqslant \underset{\sim}{0}$$

on the boundary of an inner region ω, then $h \geqslant k$ in ω. For if not, and if ω_0 is the subset of ω in which $k(x) > h(x)$, then since $k - h$ takes a maximum value in each component of ω_0 and

$$\Sigma a_{ij} \frac{\partial^2(k-h)}{\partial x_i \, \partial x_j} + \Sigma b_i \frac{\partial(k-h)}{\partial x_i} = \mathrm{P}k - \mathrm{Q}h \geqslant (\mathrm{P}-\mathrm{Q})h \geqslant \underset{\sim}{0}$$

in ω_0, $k - h$ is constant in each component of ω_0. (See [8], p. 326). But then $k - h = \underset{\sim}{0}$ in ω_0, which means that ω_0 is empty. Thus the class of solutions of (1) majorizes the class of solutions of (2). If we have $\mathrm{P} \geqslant \mathrm{Q}$ and $\mathrm{P} \geqslant \underset{\sim}{0}$ only in the complement of some compact subset A of W, then A is an excluded set for the pair of solution classes of (1) and (2).

Throughout this section, $\mathfrak{H}$ and $\mathfrak{K}$ will be comparable harmonic classes with $\mathfrak{H} \geqslant \mathfrak{K}$ and A will be an excluded set for the pair $(\mathfrak{H}, \mathfrak{K})$. The relationship $\mathfrak{H}^+|\tilde{\mathrm{A}} \subset \overline{\mathfrak{K}}$ implies several other useful relationships between the superharmonic and subharmonic classes associated with $\mathfrak{H}$ and the corresponding classes associated with $\mathfrak{K}$. These relationships are listed below in Proposition 7.2, but first we need the following result:

Proposition 7.1. — *There is a base $\mathcal{B}$ for the topology of* W — A *such that each set ω in $\mathcal{B}$ is an inner region which is regular for both $\mathfrak{H}$ and $\mathfrak{K}$.*

Proof. — By Axiom II, there is a base $\mathcal{B}$ for the topology of W — A consisting of inner regions which are regular for $\mathfrak{H}$. Let ω be an element of $\mathcal{B}$. Without loss of generality, we may assume that there is a region Ω which is regular for $\mathfrak{K}$ such that $\bar{\omega} \subset \Omega$. Let k be a positive function in $\mathfrak{K}_\Omega$. If there is only one point x_0 on $\partial\omega$, then for any number α we have

$$\lim_{x \in \omega,\ x \to x_0} \frac{\alpha}{k(x_0)} k = \alpha.$$

Thus in this case ω is regular for $\mathfrak{K}$. If there are at least two points on $\partial\omega$, then for each $x_0 \in \partial\omega$ there is a continuous nonnegative function f on $\partial\omega$ such that $f(x_0) = 0$ but $f \neq \underset{\sim}{0}$. Clearly, $\mathrm{H}(f, \omega)$ is a $\mathfrak{K}$-barrier for ω at x_0. By 3.3, ω is regular for $\mathfrak{K}$. ▌

Proposition 7.2. — *We have the following consequences of the fact that* $\mathfrak{H}^+|\tilde{\mathrm{A}} \subset \overline{\mathfrak{K}}$:

$$\text{(i)}\ \overline{\mathfrak{H}}^+|\tilde{\mathrm{A}} \subset \overline{\mathfrak{K}}; \qquad \text{(iii)}\ \underline{\mathfrak{K}}^+|\tilde{\mathrm{A}} \subset \underline{\mathfrak{H}};$$
$$\text{(ii)}\ \underline{\mathfrak{H}}^-|\tilde{\mathrm{A}} \subset \underline{\mathfrak{K}}; \qquad \text{(iv)}\ \overline{\mathfrak{K}}^-|\tilde{\mathrm{A}} \subset \overline{\overline{\mathfrak{H}}}.$$

Proof. — Let ν be any function in $\overline{\mathfrak{H}}^+|\tilde{\mathrm{A}}$. To prove Statement (i) we must show that ν is in $\overline{\mathfrak{K}}$. Let ω be an inner region such that ϖ is contained in the domain of ν and ω is regular for $\mathfrak{H}$ and $\mathfrak{K}$. We must show that $\nu \geqslant \mathrm{K}(\nu, \omega)$ where $\mathrm{K}(\nu, \omega)$ is the $\mathfrak{K}$-extension of ν in ω. Let f be any continuous function on $\partial\omega$ such that $\nu \geqslant f$. Then $\nu \geqslant \mathrm{H}(f, \omega) \geqslant \mathrm{K}(f, \omega)$, whence $\nu \geqslant \mathrm{K}(\nu, \omega)$. Thus ν is in $\overline{\mathfrak{H}}$. A similar proof establishes Statement (iii). Statement (ii) follows from (i), and Statement (iv) follows from (iii). ▌

If we call $\mathfrak{H}$ and $\mathfrak{K}$ equivalent harmonic classes whenever there is a compact set $\mathrm{A} \subset \mathrm{W}$ such that $\mathfrak{H}|\tilde{\mathrm{A}} = \mathfrak{K}|\tilde{\mathrm{A}}$, then it will follow from the corollary to the next proposition that the relation $\geqslant$ defines a partial ordering in the set of equivalence classes of harmonic classes on W.

Proposition 7.3. — *Let* W_0 *be an open subset of* $\mathrm{W} - \mathrm{A}$. *If there is a positive function* F *which is in both* $\mathfrak{H}_{\mathrm{W}_0}$ *and* $\mathfrak{K}_{\mathrm{W}_0}$, *then* $\mathfrak{H}|\mathrm{W}_0 = \mathfrak{K}|\mathrm{W}_0$.

Proof. — Let h be in $\mathfrak{H}|\mathrm{W}_0$, and let ω be an inner region such that ϖ is contained in the domain of h. Let $\hat{h}$ be the restriction of h to ω and $\hat{\mathrm{F}}$ be the restriction of F to ω. By the compactness of ϖ, there is a positive constant m such that $\hat{h} + m\hat{\mathrm{F}} \geqslant \underset{\sim}{0}$. Since $\hat{\mathrm{F}}$ is in $\mathfrak{H}$, $\hat{h} + m\hat{\mathrm{F}}$ is in $\overline{\mathfrak{K}}$, and since

$\hat{F}$ is in $\mathfrak{K}$, $\hat{h}$ is in $\overline{\mathfrak{K}}$. There is also a positive constant M such that $\hat{h} - M\hat{F} \leqslant \underset{\sim}{0}$. Since $\hat{F}$ is in $\mathfrak{H}$, $\hat{h} - M\hat{F}$ is in $\underline{\mathfrak{K}}$, and since $\hat{F}$ is in $\mathfrak{K}$, $\hat{h}$ is in $\mathfrak{K}$. Thus the restriction of h to any inner region in its domain is in $\overline{\mathfrak{K}} \cap \underline{\mathfrak{K}} = \mathfrak{K}$. By Axiom I, h is in $\mathfrak{K}$. Similarly, if $k \in \mathfrak{K}|W_0$, then $k \in \mathfrak{H}$. Thus $\mathfrak{H}|W_0 = |\mathfrak{K}W_0$ ▌

Corollary 7.4. — *Let* $\mathfrak{H}$, $\mathfrak{K}$, *and* L *be harmonic classes on* W. *If* $\mathfrak{H} \geqslant \mathfrak{K}$ *and* $\mathfrak{K} \geqslant L$, *then* $\mathfrak{H} \geqslant L$. *If* $\mathfrak{H} \geqslant \mathfrak{K}$ *and* $\mathfrak{K} \geqslant \mathfrak{H}$ *and if* A *is an excluded set for both the ordered pair* $(\mathfrak{H}, \mathfrak{K})$ *and the ordered pair* $(\mathfrak{K}, \mathfrak{H})$, *then* $\mathfrak{H}|\tilde{A} = \mathfrak{K}|\tilde{A}$.

Proof. — The first statement follows from 7.2. To prove the second statement, let ω be any regular inner region in W — A. By Axiom I it is sufficient to show that $\mathfrak{H}|\omega = \mathfrak{K}|\omega$. Now $H(\underset{\sim}{1}, \omega)$ is a positive function in $\mathfrak{H}_\omega$. Since $\mathfrak{H} \geqslant \mathfrak{K}$, $H(\underset{\sim}{1}, \omega)$ is in $\overline{\mathfrak{K}}$, and since $\mathfrak{K} \geqslant \mathfrak{H}$, $H(\underset{\sim}{1}, \omega)$ is in $\underline{\mathfrak{K}}$. Thus $H(\underset{\sim}{1}, \omega)$ is in $\mathfrak{K}_\omega$. By 7.3, $\mathfrak{H}|\omega = \mathfrak{K}|\omega$. ▌

The next proposition shows that the relation $\mathfrak{H} \geqslant \mathfrak{K}$ is preserved under division by a positive continuous function V.

Proposition 7.5. — *Let* V_1 *and* V_2 *be positive continuous functions on* W *such that* $V_1 = V_2$ *in the complement of some compact subset* D *of* W. *Then*

$$\mathfrak{H} \geqslant \mathfrak{K} \Longrightarrow V_1^{-1}\,\mathfrak{H} \geqslant V_2^{-1}\,\mathfrak{K}.$$

Proof. — We may assume that D is contained in an excluded set A. Let $V_1^{-1}h$ be a positive function in $V_1^{-1}\mathfrak{H}|\tilde{A}$. Then h is in $\mathfrak{H}^+|\tilde{A}$, whence h is in $\overline{\mathfrak{K}}$. Therefore, $V_1^{-1}h$ is in $V_2^{-1}\overline{\mathfrak{K}}$, which is the superharmonic class associated with $V_2^{-1}\mathfrak{K}$. Thus $V_1^{-1}\mathfrak{H}^+|\tilde{A} \subset V_2^{-1}\overline{\mathfrak{K}}$ ▌

In the following theorem we use the results of Section 3 to describe the relationship between open sets which are regular for $\mathfrak{H}$ and open sets which are regular for $\mathfrak{K}$. Recall that in establishing the results of Section 3 for an open set Ω and a harmonic class $\mathfrak{H}$, we assumed the existence of a function $V \in \overline{\mathfrak{H}}_\Omega$ with $\inf_{x \in \Omega} V(x) > 0$.

Theorem 7.6. — *Let* Ω *be an open subset of* W *with* $\partial\Omega \subset \overline{W} - A$, *and let* x_0 *be a point on* $\partial\Omega \cap W$. *Assume either* *that Ω has at most a countable number of components or that there is a countable base for the neighborhood system of x_0 in $\partial\Omega$ and there is at least one point different from x_0 on the boundary of each component of Ω. If Ω is regular for $\mathfrak{H}$, then there is a $\mathfrak{K}$-barrier for Ω at x_0. If Ω is regular for $\mathfrak{K}$ and if there is a positive continuous function* V *defined on $\overline{\Omega}$ such that* $V|\Omega \in \overline{\mathfrak{K}}$ *and* $V|\Omega - A \in \overline{\mathfrak{H}}$, *then there is an $\mathfrak{H}$-barrier for Ω at x_0.*

Proof. — The first statement follows from 3.4 and 7.2. To prove the second statement we assume first that Ω is a region. If $\partial\Omega - \{x_0\} = \emptyset$, let ω be an inner region which is regular for $\mathfrak{H}$ such that $\overline{\omega} \subset W - A$, $x_0 \in \omega$ and $\Omega - \overline{\omega} \neq \emptyset$. Let $c = 2 \sup\limits_{x \in \partial\omega \cap \Omega} V(x)$. It is easy to see that $\exists d < 0$ such that if $g = c$ on $\partial\omega \cap \Omega$ and $g = d$ on $\partial\omega \cap W - \overline{\Omega}$ then

$$H(g, \omega)(x_0) = 0.$$

Let $b = V$ in $\Omega - \omega$ and $b = V \wedge H(g, \omega)$ in $\omega \cap \Omega$. Then $b|\Omega - A$ is an $\mathfrak{H}$-barrier for Ω at x_0.

If $\partial\Omega - \{x_0\} \neq \emptyset$, then there is a continuous nonnegative function f on $\partial\Omega$ such that $f(x_0) = 0$ but $f \neq \underset{\sim}{0}$, and $f \leqslant V$. Since $V + K(f - V, \Omega)$ is in $\overline{\mathfrak{K}}$, $V + K(f - V, \Omega) \geqslant \underset{\sim}{0}$ by 2.2. By 7.2, $K(f - V, \Omega)|\Omega - A$ is in $\overline{\mathfrak{H}}$. Thus the restriction of $V + K(f - V, \Omega)$ to $\Omega - A$ is an $\mathfrak{H}$-barrier for Ω at x_0.

The rest of the proof for the case that Ω has at most a countable number of components is similar to the corresponding proof of 3.4. Now assume that there is a countable base for the neighborhood system of x_0 in $\partial\Omega$ and that there is at least one point other than x_0 on the boundary of each component of Ω. Then there is a continuous function f on $\partial\Omega$ such that $f(x_0) = 0$, $f \leqslant V$ and $f(x) > 0$ for every $x \neq x_0$ on $\partial\Omega$. The function $[V + K(f - V, \Omega)]|\Omega - A$ is an $\mathfrak{H}$-barrier for Ω at x_0. ▌

Corollary 7.7. — *Let Ω be an inner open subset of* W *with* $\partial\Omega \subset W - A$. *Assume either that Ω has at most a countable number of components or that $\partial\Omega$ satisfies the first axiom of countability in the relative topology and there are at least two points on the boundary of each component of Ω. Also assume that there* *is a positive continuous function* V *defined on* $\overline{\Omega}$ *such that* $V|\Omega$ *is in both* $\overline{\mathfrak{H}}$ *and* $\overline{\mathfrak{K}}$. *Then* Ω *is regular for* $\mathfrak{H}$ *if and only if* Ω *is regular for* $\mathfrak{K}$.

The last theorem of this section establishes the fact that if $\underset{\sim}{1} \in \overline{\mathfrak{H}}_W$ and $\underset{\sim}{1} \in \overline{\mathfrak{K}}_W$, then $\mathfrak{K}$ is parabolic on W if $\mathfrak{H}$ is. We can show that the converse is not true by considering the open unit circle $W = \{z : |z| < 1\}$, and taking $\mathfrak{H}$ to be the set of solutions of the equation $\Delta u = \underset{\sim}{0}$ on W and $\mathfrak{K}$ to be the set of solutions of the equation $\Delta u = 4(1 + |z|^2)(1 - |z|^2)^{-2} u$ on W. It is well known that $\mathfrak{H}$ is hyperbolic on W. If

$$\Omega_n = \left\{ z;\ |z| < 1 - \frac{1}{n} \right\}$$

then $\{\Omega_n\}$ is an exhaustion of W and the $\mathfrak{K}$-extension of $\underset{\sim}{1}$ in Ω_n, $K(1, \Omega_n)$, equals

$$\frac{2}{n}\left(1 - \frac{1}{2n}\right)\frac{1}{(1 - |z|^2)}.$$

Since $\lim_{n\to\infty} K(\underset{\sim}{1}, \Omega_n) = \underset{\sim}{0}$, $\mathfrak{K}$ is parabolic on W by 5.1 and 5.5.

THEOREM 7.8. — *If* $\underset{\sim}{1} \in \overline{\mathfrak{H}}_W$ *and* $\underset{\sim}{1} \in \overline{\mathfrak{K}}_W$ *then* $\mathfrak{K}$ *is parabolic on* W *if* $\mathfrak{H}$ *is parabolic on* W.

Proof. — By 4.4, we may assume that the excluded set A is outer-regular with respect to $\mathfrak{H}$. It follows that A is also outer-regular with respect to $\mathfrak{K}$. Let v be any function in $\overline{\mathfrak{H}}_{W-A}$ such that $\liminf v \geqslant 0$ at ∂A and $\liminf v \geqslant 1$ at ∂W. Since v is in $\overline{\mathfrak{K}}$, $v \geqslant K(\partial W, W - A)$, whence

$$H(\partial W,\ W - A) \geqslant K(\partial W,\ W - A).$$

If $\mathfrak{H}$ is parabolic on W, then $H(\partial W, W - A) = \underset{\sim}{0}$ and $\mathfrak{K}$ is parabolic on W. ▌

COROLLARY 7.9. — *Let* V_1 *and* V_2 *be positive continuous functions on* W *such that* $V_1 \in \overline{\mathfrak{H}}$, $V_2 \in \overline{\mathfrak{K}}$ *and* $V_1 = V_2$ *in the complement of some compact subset* D *of* W. *Then* $\mathfrak{K}$ *is* V_2-*parabolic if* $\mathfrak{H}$ *is* V_1-*parabolic.*

Proof. — Clearly, $\underset{\sim}{1} \in V_1^{-1}\mathfrak{H}$ and $\underset{\sim}{1} \in V_2^{-1}\mathfrak{K}$. By Proposition 7.5, $V_1^{-1}\mathfrak{H} \geqslant V_2^{-1}\mathfrak{K}$. Therefore, $V_2^{-1}\mathfrak{K}$ is parabolic if $V_1^{-1}\mathfrak{H}$ is parabolic. ▌

COROLLARY 7.10. — *Let* V_1 *be a positive continuous potential for* $\mathfrak{H}$ *on* W, *and let* V_2 *be a positive continuous function in* $\overline{\mathfrak{K}}_W$ *such that* $V_2 = V_1$ *in the complement of some compact subset* D *of* W. *Then either* V_2 *is a potential for* $\mathfrak{K}$ *on* W *or there is no potential for* $\mathfrak{K}$ *on* W *in which case* $V_2 \in \mathfrak{K}$ *and*

$$\mathfrak{H}|\tilde{A} \cap \tilde{D} = \mathfrak{K}|\tilde{A} \cap \tilde{D}.$$

Proof. — By 6.3, $V_1^{-1}\mathfrak{H}$ is parabolic, so by 7.9, $V_2^{-1}\mathfrak{K}$ is parabolic on W. Therefore, either V_2 is a potential for $\mathfrak{K}$ or there is no potential for $\mathfrak{K}$ on W and $V_2 \in \mathfrak{K}$. In the latter case, $V_2|\tilde{A} \cap \tilde{D} \in \mathfrak{H}$, whence by 7.3, $\mathfrak{H}|\tilde{A} \cap \tilde{D} = \mathfrak{K}|\tilde{A} \cap \tilde{D}$. ▌

8. The Bounded Functions in Comparable Harmonic Classes.

If Ω is an open subset of W and $\mathfrak{H}$ is harmonic on W, then the set of all bounded functions in $\mathfrak{H}_\Omega$ forms a Banach space with $||h|| = \sup_{x \in \Omega} |h(x)|$. We denote this by $\mathcal{B}\mathfrak{H}_\Omega$.

If $\mathfrak{K}$ is the set of solutions of $\Delta u = \underset{\sim}{0}$ in the plane $\mathbf{R}^2$, then $\mathcal{B}\mathfrak{K}_W$ consists of all multiples of $\underset{\sim}{1}$. On the other hand if $\mathfrak{H}$ is the set of solutions of the equation $\Delta u = \mathrm{P}u$ in $\mathbf{R}^2$, where $\mathrm{P}(z) = 0$ for $|z| \geqslant 1$ and $\mathrm{P}(z) = e^{(|z|-1)^{-1}}$ for $|z| < 1$, then $\mathcal{B}\mathfrak{H}_W = \{\underset{\sim}{0}\}$ even though $\mathfrak{H} \geqslant \mathfrak{K}$. If, however $\mathfrak{H}$ and $\mathfrak{K}$ are harmonic classes which satisfy Axiom IV such that $\mathfrak{H} \geqslant \mathfrak{K}$, and if $\mathfrak{H}$ is hyperbolic on W, then we can show in the general case that there is an isometric isomorphism of $\mathcal{B}\mathfrak{K}_W$ onto a subspace of $\mathcal{B}\mathfrak{H}_W$. We do not proceed to this result directly, however, because in general we must work in the complement of some compact set. Thus we shall first establish Proposition 8.1. This proposition and Theorem 8.3 and its corollary are statements in the axiomatic setting of the corresponding results given by H. L. Royden in [13], pp. 10-15.

PROPOSITION 8.1. — *Let* $\mathfrak{H}$ *and* $\mathfrak{K}$ *be harmonic classes which satisfy Axiom IV such that* $\mathfrak{H} \geqslant \mathfrak{K}$, *and let* A *be an excluded compact set for the pair* $(\mathfrak{H}, \mathfrak{K})$. *Assume that* A *is outer-regular* *with respect to both* $\mathfrak{H}$ *and* $\mathfrak{K}$. *Let* $\mathcal{B}_0\mathfrak{K}_{W-A}$ *denote the subspace of* $\mathcal{B}\mathfrak{K}_{W-A}$ *consisting of all functions of* $\mathcal{B}\mathfrak{K}_{W-A}$ *which vanish at* ∂A. *Then there is an isometric isomorphism of* $\mathcal{B}_0\mathfrak{K}_{W-A}$ *onto a subspace of* $\mathcal{B}\mathfrak{H}_W$. *If also* $\mathfrak{K} \geqslant \mathfrak{H}$ *and* A *is an excluded set for the pair* $(\mathfrak{K}, \mathfrak{H})$, *then* $\mathcal{B}_0\mathfrak{K}_{W-A}$ *and* $\mathcal{B}\mathfrak{H}_W$ *are isometric if there is a positive potential for* $\mathfrak{H}$ *on* W.

Proof. — By Part (iv) of Proposition 5.3, $\mathcal{B}_0\mathfrak{K}_{W-A} = \{\underset{\sim}{0}\}$ if $\mathfrak{K}$ is parabolic on W. If $\mathfrak{K} \geqslant \mathfrak{H}$ and $\mathfrak{K}$ is parabolic on W, then $\mathfrak{H}$ is parabolic on W, and thus $\mathcal{B}\mathfrak{H}_W = \{\underset{\sim}{0}\}$ unless there is no positive potential for $\mathfrak{H}$ on W, in which case $\underset{\sim}{1} \in \mathfrak{H}_W$. Therefore, the proposition holds if $\mathfrak{K}$ is parabolic on W. We assume for the remainder of the proof that $\mathfrak{K}$ is hyperbolic on W.

Let $\mathcal{R}$ be the set of all regular inner regions Ω in W such that $A \subset \Omega$; $\mathcal{R}$ is an exhaustion of W. Let k be a nonnegative function in $\mathcal{B}_0\mathfrak{K}_{W-A}$. For each $\Omega \in \mathcal{R}$ we set h_Ω equal to k in $W - \Omega$ and $H(k, \Omega)$ in Ω. By 2.3 we have $h_\Omega \geqslant k$, and consequently $h_{\Omega_2} \geqslant h_{\Omega_1}$ if $\Omega_2 \supset \Omega_1$. Thus the family $\{h_\Omega : \Omega \in R\}$ is directed by increasing order on W, and for each $\Omega \in R$ we have $h_\Omega \leqslant \|k\|$. Let πk be the upper envelope of the functions h_Ω. It follows from Axiom III that πk is in $\mathcal{B}\mathfrak{H}_W$. We thus define a mapping π from the nonnegative functions of $\mathcal{B}_0\mathfrak{K}_{W-A}$ into $\mathcal{B}\mathfrak{H}_W$. If k and $\hat{k}$ are nonnegative functions in $\mathcal{B}_0\mathfrak{K}_{W-A}$ and if a is a nonnegative constant, then we have

$$\pi k + \pi\hat{k} = \pi[k + \hat{k}], \qquad \pi a k = a\pi k,$$

and

$$\underset{\sim}{0} \leqslant k \leqslant \pi k \leqslant \|k\|. \tag{1}$$

Moreover, if h is any function in $\mathcal{B}\mathfrak{H}_W$, then

$$\underset{\sim}{0} \leqslant k \leqslant h \Longrightarrow \pi k \leqslant h. \tag{2}$$

Since $K(\partial W, W - A) \leqslant \underset{\sim}{1}$, it follows from 5.3 and Equation (1) that

$$K(\partial W, W - A) \leqslant \pi k(\partial W, W - A) \leqslant H(W) \leqslant \underset{\sim}{1}.$$

If k is an arbitrary element of $\mathcal{B}_0\mathfrak{K}_{W-A}$, then

$$k + \|k\|\, K(\partial W, W - A) \geqslant \underset{\sim}{0}$$

by 5.3. Hence any function in $\mathfrak{B}_0\mathfrak{K}_{W-A}$ can be expressed as the difference of two positive functions in $\mathfrak{B}_0\mathfrak{K}_{W-A}$. If

$$k_1 - k_2 = k_3 - k_4,$$

then $k_1 + k_4 = k_3 + k_2$ and $\pi k_1 + \pi k_4 = \pi k_3 + \pi k_2$, whence $\pi k_1 - \pi k_2 = \pi k_3 - \pi k_4$. Therefore, we can extend the definition of π to all of $\mathfrak{B}_0\mathfrak{K}_{W-A}$ by setting $\pi(k_1 - k_2) = \pi k_1 - \pi k_2$ for each pair of positive functions k_1 and k_2 in $\mathfrak{B}_0\mathfrak{K}_{W-A}$. With this definition, π is a linear mapping of $\mathfrak{B}_0\mathfrak{K}_{W-A}$ into $\mathfrak{B}\mathfrak{H}_W$.

For any k in $\mathfrak{B}_0\mathfrak{K}_{W-A}$ we have

$$\begin{aligned} \underset{\sim}{0} \leqslant \pi k + \|k\|\, \pi K(\partial W, W - A) &= \pi[k + \|k\|\, K(\partial W, W - A)] \\ &\leqslant \pi[2\|k\|\, K(\partial W, W - A)] \\ &\leqslant 2\|k\|\, \pi K(\partial W, W - A). \end{aligned}$$

Therefore,

$$|\pi k| \leqslant \|k\|\, \pi K(\partial W, W - A) \leqslant \|k\|,$$

and thus π does not increase norms.

We now proceed to define a linear mapping ρ of $\mathfrak{B}\mathfrak{H}_W$ into $\mathfrak{B}_0\mathfrak{K}_{W-A}$. Let h be a nonnegative function in $\mathfrak{B}\mathfrak{H}_W$. For each $\Omega \in \mathfrak{R}$ let f_Ω be equal to h on $\partial\Omega$ and $\underset{\sim}{0}$ on ∂A. Let k_Ω be equal to $K(f_\Omega, \Omega - A)$ in $\Omega - A$ and equal to h in $W - \Omega$. If Ω_1 and Ω_2 are in $\mathfrak{R}$ and $\Omega_2 \supset \Omega_1$, then $h \geqslant k_{\Omega_1} \geqslant k_{\Omega_2} \geqslant \underset{\sim}{0}$. Let ρh be the lower envelope of the functions k_Ω; ρh is in $\mathfrak{B}_0\mathfrak{K}_{W-A}$. Moreover

$$\underset{\sim}{0} \leqslant \rho h \leqslant h, \tag{3}$$

while if k is any positive function in $\mathfrak{B}_0\mathfrak{H}_{W-A}$, we have

$$k \leqslant h \Longrightarrow k \leqslant \rho h. \tag{4}$$

For arbitrary functions h in $\mathfrak{B}\mathfrak{H}_W$ we have $h + \|h\|\, H(W) \geqslant \underset{\sim}{0}$. Therefore we may, as above, extend ρ to a linear mapping on all of $\mathfrak{B}\mathfrak{H}_W$ into $\mathfrak{B}_0\mathfrak{K}_{W-A}$ by setting $\rho[h_1 - h_2] = \rho h_1 - \rho h_2$ for any pair of positive functions h_1 and h_2 in $\mathfrak{B}\mathfrak{H}_W$. By 5.3 and the linearity of ρ we have

$$\begin{aligned} \underset{\sim}{0} \leqslant \rho h + \|h\|\rho H(W) &= \rho[h + \|h\| H(W)] \\ &\leqslant \rho[2\|h\| H(W)] \\ &\leqslant 2\|h\|\rho H(W). \end{aligned}$$

Hence
$$|\rho h| \leqslant \|h\|\rho H(W) \leqslant \|h\|,$$

and so ρ does not increase norms.

For any positive function k in $\mathcal{B}_0\mathfrak{K}_{W-A}$ we have $\pi k \geqslant k$ by (1) and hence by (4) we must have $\rho\pi k \geqslant k$. Thus

$$K(\partial W, W - A) \leqslant \rho\pi K(\partial W, W - A) \leqslant \underset{\sim}{1},$$

whence by 5.3 we have $K(\partial W, W - A) = \rho\pi K(\partial W, W - A)$. Consequently, for an arbitrary k in $\mathcal{B}_0\mathfrak{K}_{W-A}$ we have

$$\begin{aligned} k + \|k\|K(\partial W, W - A) &\leqslant \rho\pi[k + \|k\|K(\partial W, W - A)] \\ &\leqslant \rho\pi k + \|K\|K(\partial W, W - A). \end{aligned}$$

Hence $\rho\pi k \geqslant k$ for all k in $\mathcal{B}_0\mathfrak{K}_{W-A}$. Replacing k by $-k$ we obtain $\rho\pi k \leqslant k$, and so $\rho\pi k = k$. Since neither π nor ρ increase norms, we see that π is an isometric isomorphism of $\mathcal{B}_0\mathfrak{K}_{W-A}$ onto a subspace of $\mathcal{B}\mathfrak{H}_W$, and this establishes the first part of the proposition.

If it is also true that $\mathfrak{K} \geqslant \mathfrak{H}$ and A is an excluded set for $(\mathfrak{K}, \mathfrak{H})$, then $H(\partial W, W - A) = K(\partial W, W - A)$, and so by (1), $H(\partial W, W - A) \leqslant \pi K(\partial W, W - A)$. We then have

$$\begin{aligned} \underset{\sim}{0} \leqslant H(W) - \pi K(\partial W, W - A) &\leqslant H(W) - H(\partial W, W - A) \\ &\leqslant H(W - A) - H(\partial W, W - A) = H(\partial A, W - A) \end{aligned}$$

by 5.2. By 5.7 and 7.8, $H(W) = \pi K(\partial W, W - A)$. Therefore,

$$\pi\rho H(W) = \pi(\rho\pi K(\partial W, W - A)) = \pi K(\partial W, W - A) = H(W).$$

By (2) and (3) we have for each h in $\mathcal{B}\mathfrak{H}_W$

$$h + \|h\|H(W) \geqslant \pi\rho[h + \|h\|H(W)] = \pi\rho h + \|h\|H(W),$$

whence $h \geqslant \pi\rho h$ for every h in $\mathcal{B}\mathfrak{H}_W$. Thus $\pi\rho h = h$, and we see that in this case π is an isometry of $\mathcal{B}_0\mathfrak{K}_{W-A}$ onto $\mathcal{B}\mathfrak{H}_W$. ▌

Corollary 8.2. — *Let $\mathfrak{H}$ be a harmonic class which satisfies Axiom IV, and assume that there is a positive potential for $\mathfrak{H}$ on* W. *Let* A *be a compact subset of* W *such that* A *is outer-regular with respect to $\mathfrak{H}$, and let $\mathcal{B}_0\mathfrak{H}_{W-A}$ denote the functions in $\mathcal{B}\mathfrak{H}_{W-A}$ which vanish at* ∂A. *Then there is an isometric isomorphism of $\mathcal{B}_0\mathfrak{H}_{W-A}$ onto $\mathcal{B}\mathfrak{H}_W$.*

Theorem 8.3. — *Let $\mathfrak{H}$ and $\mathfrak{K}$ be harmonic classes which satisfy Axiom IV such that $\mathfrak{H} \geqslant \mathfrak{K}$, and assume that $\mathfrak{H}$ is hyper-*

bolic on W. *Then there is an isometric isomorphism of* $\mathcal{B}\mathfrak{K}_W$ *onto a subspace of* $\mathcal{B}\mathfrak{H}_W$. *If* $\mathfrak{K} \geqslant \mathfrak{H}$ *as well, then* $\mathcal{B}\mathfrak{K}_W$ *and* $\mathcal{B}\mathfrak{H}_W$ *are isometric.*

Proof. — If $\mathfrak{K}$ is parabolic on W, then either $\mathcal{B}\mathfrak{K}_W = \{\underset{\sim}{0}\}$ or $\mathcal{B}\mathfrak{K}_W$ consists of all multiples of $\underset{\sim}{1}$. In the latter case, the isometry is defined by mapping $\underset{\sim}{1}$ onto H(W), for since

$$H(W) = \overline{H}(\underset{\sim}{1}, W) = \underline{H}(\underset{\sim}{1}, W),$$

we have $\sup_{x \in W} H(W)(x) = 1$. We shall assume for the remainder of the proof that $\mathfrak{K}$ is hyperbolic on W.

Let A be an excluded set for $(\mathfrak{H}, \mathfrak{K})$ and assume that A is outer-regular with respect to both $\mathfrak{H}$ and $\mathfrak{K}$. If $\mathfrak{K} \geqslant \mathfrak{H}$, assume that A is also an excluded set for $(\mathfrak{K}, \mathfrak{H})$. By 8.2 there is an isometric isomorphism π of $\mathcal{B}\mathfrak{K}_W$ onto $\mathcal{B}_0\mathfrak{K}_{W-A}$. By 8.1 there is an isometric isomorphism λ of $\mathcal{B}_0\mathfrak{K}_{W-A}$ onto a subspace of $\mathcal{B}\mathfrak{H}_W$, and λ maps $\mathcal{B}_0\mathfrak{K}_{W-A}$ onto $\mathcal{B}\mathfrak{H}_W$ if $\mathfrak{K} \geqslant \mathfrak{H}$. The map $\lambda \circ \pi$ is the desired isomorphism. ∎

Corollary 8.4. — *If* $\mathfrak{H}$ *and* $\mathfrak{K}$ *are harmonic classes which satisfy Axiom IV such that* $\mathfrak{H} \geqslant \mathfrak{K}$, *and if* $\underset{\sim}{1}$ *is in* $\mathfrak{H}_W$, *then the first two of the following statements are equivalent and imply the third and fourth:*

(i) *There are at least two linearly independent functions in* $\mathcal{B}\mathfrak{K}_W$.

(ii) *There is a function in* $\mathcal{B}\mathfrak{K}_W$ *which assumes both positive and negative values.*

(iii) *There is a nonconstant function in* $\mathcal{B}\mathfrak{H}_W$.

(iv) *There is a positive potential for* $\mathfrak{H}$ *on* W.

Proof. — If k_1 and k_2 are linearly independent functions, we can always choose constants α and β so that $\alpha k_1 + \beta k_2$ assumes both positive and negative values. On the other hand, if k is a function in $\mathcal{B}\mathfrak{K}_W$ which assumes both positive and negative values, it is not linearly dependent on K(W). Hence we have at least two linearly independent elements of $\mathcal{B}\mathfrak{K}_W$. Thus we see that the first two statements are equivalent and imply that dim $\mathcal{B}\mathfrak{K}_W \geqslant 2$. But by 8.3 this implies that dim $\mathcal{B}\mathfrak{H}_W \geqslant 2$, and so there must be one nonconstant function in $\mathcal{B}\mathfrak{H}_W$ and a positive potential for $\mathfrak{H}$ on W. ∎

Let $\mathfrak{H}$ be a harmonic class of functions on W and let V be a positive continuous function in $\overline{\mathfrak{H}}_W$. We denote the set of V-bounded function in $\mathfrak{H}_W$ by $\mathcal{B}(V)\mathfrak{H}_W$. The space $\mathcal{B}(V)\mathfrak{H}_W$ is a Banach space with the norm

$$||h||_V = \sup_{x \in W} V^{-1} h(x).$$

Clearly, the mapping $h \to V^{-1} h$ is an isometric isomorphism of $\mathcal{B}(V)\mathfrak{H}_W$ onto $\mathcal{B}V^{-1}\mathfrak{H}_W$, where $\mathcal{B}V^{-1}\mathfrak{H}_W$ is the set of bounded functions in $V^{-1}\mathfrak{H}_W$. Therefore, we have the following consequence of 6.1, 7.5 and 8.3.

THEOREM 8.5. — *Let $\mathfrak{H}$ and $\mathfrak{K}$ be harmonic classes with $\mathfrak{H} \geqslant \mathfrak{K}$. Let V_1 and V_2 be positive continuous functions on* W *such that $V_1 \in \overline{\mathfrak{H}}$, $V_2 \in \overline{\mathfrak{K}}$ and $V_1 = V_2$ in the complement of some compact subset* D *of* W. *Assume that $\mathfrak{H}$ is V_1-hyperbolic on* W. *Then there is an isometric isomorphism of $\mathcal{B}(V_2)\mathfrak{K}_W$ onto a subspace of $\mathcal{B}(V_1)\mathfrak{H}_W$. If $\mathfrak{K} \geqslant \mathfrak{H}$ as well, then $\mathcal{B}(V_2)\mathfrak{K}_W$ and $\mathcal{B}(V_1)\mathfrak{H}_W$ are isometric.*

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