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## On the regularity of a free boundary for a nonlinear obstacle problem arising in superconductor modelling<sup>(\*)</sup>

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**ABSTRACT.** — We study the free boundary of solutions to a class of nonlinear obstacle problems. This class of problems contains a particular model derived from the Ginzburg-Landau equation of superconductivity. We consider solutions in a Lipschitz bounded open set  $\Omega$  and prove the regularity of the free boundary when it is close enough to the fixed boundary  $\partial\Omega$ . We also give a result of stability of the free boundary and give a bound on the Hausdorff measure of the free boundary.

**RÉSUMÉ.** — Nous étudions les frontières libres associées à des solutions d'une classe de problèmes de l'obstacle non linéaires. Cette classe de problèmes contient un modèle particulier dérivé des équations de Ginzburg-Landau de la supraconductivité. Nous considérons des solutions dans un ouvert borné  $\Omega$  à bord Lipschitz, et nous prouvons que la frontière libre est régulière lorsque celle-ci est suffisamment proche du bord fixe  $\partial\Omega$ . Nous prouvons aussi un résultat de stabilité de la frontière libre et donnons une borne a priori sur la mesure de Hausdorff de cette frontière libre.

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### 1. Introduction

In this article we are interested in solutions to a nonlinear obstacle problem. This problem is motivated by a work of Chapman, Rubinstein, Schatzman [13] where a model is formally derived from the Ginzburg-Landau theory for a superconductor with a density of vortices in an interior region

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whose boundary is a free boundary. A rigorous derivation of this model has been done by Sandier, Serfaty [23]. See also [4, 12, 25, 24] for some related works on the mathematical analysis of superconductivity. Here we will prove rigorous results on the regularity of the free boundary contained in a Lipschitz domain.

The core of the technical part of this article is an adaptation in the framework of the nonlinear obstacle problem on non-smooth domains of Caffarelli-type techniques [8, 9] originally developed for linear obstacle problems on smooth domains.

The model that we consider in this paper is a nonlinear obstacle problem in a Lipschitz bounded open set  $\Omega \subset \mathbf{R}^n$ . We are interested in the minimization of the energy

$$E(u) = \int_{\Omega} F(|\nabla u|^2) + u^2$$

on the convex set

$$K_{\lambda} = \{u \in H^1(\Omega), \quad u \geq \lambda \quad \text{on } \Omega, \quad u = \lambda_0 \quad \text{on } \partial\Omega\}$$

where  $0 \leq \lambda \leq \lambda_0$  are two constants. We make the following assumption (which implies that the energy  $E$  is strictly convex)

(A0)  $F$  is a  $C^\infty$  convex function satisfying  $F'(0) = 1$  and  $\lim_{q \rightarrow +\infty} F'(q) < +\infty$ .

It is classical that for each  $\lambda$  there exists a unique minimizer  $u_{\lambda}$  of the energy  $E$  on  $K_{\lambda}$ . For such a minimizer the coincidence set is

$$\{u = \lambda\}$$

and the free boundary is

$$\partial\{u = \lambda\}$$

When the free boundary  $\partial\{u = \lambda\}$  is smooth, the solution  $u$  satisfies the following Euler-Lagrange equation

$$\left\{ \begin{array}{l} \operatorname{div} \left( F'(|\nabla u|^2) \nabla u \right) = u \quad \text{on } \Omega \setminus \{u = \lambda\} \\ u = \lambda_0 \quad \text{on } \partial\Omega \\ \left. \begin{array}{l} u = \lambda \\ \frac{\partial u}{\partial n} = 0 \end{array} \right| \quad \text{on } \partial\{u = \lambda\} \end{array} \right.$$

Although there are two boundary conditions on the free boundary, the problem is not overdetermined. These two boundary conditions allow to characterize the free boundary  $\partial\{u = \lambda\}$  which is an unknown in this problem. We refer the reader to the monographs [17, 14, 22] for a presentation of the classical results on the free boundary of the obstacle problem.

### 1.1. Main results

Our main result (for a smooth open set and in the linear case) is the following :

**THEOREM 1.1** (Regularity transfer from the fixed boundary to the free boundary). — *Let us assume that the open set  $\Omega$  is smooth, and that  $F(q) = q$ , then the energy  $E$  has a unique minimizer  $u_\lambda$  on  $K_\lambda$  for all  $\lambda \in [0, \lambda_0]$ . Moreover there exists  $\delta > 0$  such that for all  $\lambda \in (\lambda_0 - \delta, \lambda_0)$ , the free boundary  $\partial\{u_\lambda = \lambda\}$  is a  $C^\infty$   $(n - 1)$ -dimensional manifold homeomorphic to  $\partial\Omega$ .*

Although this result seems very natural, it was an open problem (even in this linear case), that we solve here applying the approach of blow-ups developed by Caffarelli [8] for the regularity of the free boundary of the obstacle problem. Under the assumption that  $\partial\Omega \in C^\infty$ , a nonlinear variant of theorem 1.1 was proved in [5] by A. Bonnet and the author, using the Nash-Moser inverse function theorem in dimension 2. This Nash-Moser approach could work in fact in any dimensions, but it can not be applied to a fixed boundary  $\partial\Omega$  less regular than  $C^\infty$ . On the contrary the approach of Caffarelli [8] allows to deal with non-smooth fixed boundaries  $\partial\Omega$ .

We extend theorem 1.1 to Lipschitz open set  $\Omega$  and for general convex functions  $F$  satisfying assumption (A0). More precisely we make the following two assumptions on the regularity of  $\Omega$ :

(A1) *Exterior sphere condition:*

There exists  $r_0 > 0$  such that for every point  $X_0$  of the boundary  $\partial\Omega$ , there exists a point  $X_1 \in \mathbf{R}^n$ , such that the ball  $B_{r_0}(X_1)$  is included in  $\mathbf{R}^n \setminus \Omega$  and is tangent to  $\partial\Omega$  at  $X_0$ .

(A2) *Interior cone condition:*

There exist  $r_0 > 0$  and an angle  $\alpha_0 \in (0, \frac{\pi}{2})$  such that for every point  $X_0$  of the boundary  $\partial\Omega$ , there exists a unit vector  $\nu \in \mathbf{S}^{n-1}$ , such that  $\Omega$

contains the cone

$$\left\{ X \in B_{r_0}(X_0), \quad \left\langle \frac{X - X_0}{|X - X_0|}, \nu \right\rangle \geq \cos \alpha_0 \right\}$$

where  $\langle \cdot, \cdot \rangle$  is the usual scalar product. Theorem 1.1 is a corollary of the following more general result:

**THEOREM 1.2** (Regularity transfer from a Lipschitz fixed boundary). — *Under assumptions (A0)-(A1)-(A2), the energy  $E$  has a unique minimizer  $u_\lambda$  on  $K_\lambda$  for all  $\lambda \in [0, \lambda_0]$ . Moreover there exists  $\delta > 0$  such that for all  $\lambda \in (\lambda_0 - \delta, \lambda_0)$ , the free boundary  $\partial \{u_\lambda = \lambda\}$  is a  $C^\infty$   $(n - 1)$ -dimensional manifold homeomorphic to  $\partial\Omega$ .*

In the application that we have in mind, namely a nonlinear free boundary problem arising in the description of superconductors in dimension two (see Bonnet, Monneau [5], Berestycki, Bonnet, Chapman [2]), the function  $F_0$  is analytic convex but only defined on  $[0, \frac{4}{27})$  by  $F'_0(0) = 1$  and

$$h = (1 - v^2)v \iff v = F'_0(h^2)h$$

Using a  $L^\infty$  control on the gradient of the solution we deduce the following result in this particular case:

**COROLLARY 1.3** (Application to a superconducting model). — *Under assumption (A1)-(A2), with  $F = F_0$ , there exists  $\delta > 0$  such that  $\forall \lambda \in (\lambda_0 - \delta, \lambda_0)$ , there exists a unique solution  $u_\lambda$  minimizer of  $E$  on  $K_\lambda$  satisfying  $\sup_{\overline{\Omega}} |\nabla u_\lambda|^2 < \frac{4}{27}$ ; moreover the free boundary  $\partial \{u_\lambda = \lambda\}$  is a  $C^\infty$   $(n - 1)$ -dimensional manifold homeomorphic to  $\partial\Omega$ .*

Let us mention that part of the methods of [20] could be adapted to this model of superconductivity to get informations on the singularities of the free boundary when  $\lambda < \lambda_0 - \delta$ .

We also prove a result on the perturbation (locally in space) of the free boundary.

**THEOREM 1.4** (Local stability of the free boundary). — *We assume (A0)-(A1)-(A2). Let  $\lambda^* \in (0, \lambda_0)$  be such that there exists a minimizer  $u_{\lambda^*}$  of the energy  $E$  on  $K_{\lambda^*}$  with a free boundary  $\partial \{u_{\lambda^*} = \lambda^*\}$  which is  $C^\infty$  in a compact set  $K^*$  of  $\Omega$ . Then for every smaller compact set  $K \subset\subset K^*$  there exists  $\varepsilon > 0$  such that for every  $\lambda$  satisfying  $|\lambda - \lambda^*| < \varepsilon$ , the unique solution  $u_\lambda$  has a free boundary  $\partial \{u_\lambda = \lambda\}$  which is  $C^\infty$  in  $K$ .*

The proof of this result is based on a geometric criterion for the regularity of the free boundary given by Caffarelli in [8] and on the continuity of the map  $\lambda \mapsto u_\lambda$ . We also refer to the book of Rodrigues [22] for classical results on the global stability of the free boundary.

Finally we give a bound on the Hausdorff measure of the free boundary, generalizing to non-smooth fixed boundaries  $\partial\Omega$ , a result of Brezis, Kinderlehrer [6] based on fine estimates for variational inequalities. Here the proof is an adaptation of the work of Caffarelli [9], developed for linear equations.

**THEOREM 1.5** (Bound on the Hausdorff measure of the free boundary). *Under assumptions (A0)-(A1)-(A2), there exists a constant  $C > 0$  only depending on  $\Omega, \lambda_0, F$  such that for any minimizer  $u_\lambda$  of  $E$  on  $K_\lambda$  with  $\lambda \in [0, \lambda_0]$ , we have*

$$\mathcal{H}^{n-1}(\partial\{u_\lambda = \lambda\}) \leq C$$

## 2. Some known results on blow-up limits

### 2.1. The simple blow-up limit

To prove regularity results on the free boundary, the main tool (first introduced for the obstacle problem by Caffarelli in [10]) is the notion of blow-up.

Let us consider a solution  $u$  to

$$\begin{cases} \Delta u = f \geq 1 & \text{on } \{u > 0\} \cap \Omega \\ u \geq 0 & \text{on } \Omega \text{ and } |D^2 u|_{L^\infty(\Omega)} \leq M \end{cases} \quad (2.1)$$

with  $f \in C^{0,\alpha}(\Omega)$  and  $f(0) = 1$ . We assume that  $X_0$  is a point of the free boundary  $\partial\{u = 0\}$ . Let us consider the following blow-up sequence of functions

$$u^\varepsilon(X) = \frac{u(X_0 + \varepsilon X)}{\varepsilon^2}$$

By assumptions,  $u^\varepsilon(0) = \nabla u^\varepsilon(0) = 0$  and the second derivatives  $|D^2 u^\varepsilon|$  are bounded by a constant independent on  $\varepsilon > 0$ . By Ascoli-Arzelà theorem, up to extraction of a convergent subsequence  $(\varepsilon')$ , we get

$$u^{\varepsilon'} \longrightarrow u^0 \quad \text{uniformly on compact sets of } \mathbf{R}^n$$

This function  $u^0$  is called a blow-up limit of the function  $u$  at the point  $X_0$ .

In any dimensions, the main result for blow-up limits is the following

**THEOREM 2.1** (Caffarelli [10, 8, 11], Weiss [26]; Characterization of a Simple Blow-up Limit). — *The blow-up limit  $u^0$  is unique and only depends on the point  $X_0$  on the free boundary.*

*Moreover either  $X_0$  is a **singular** point and then  $u^0$  is a quadratic form, i.e.*

$$u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q_{X_0} \cdot X \quad \geq 0$$

*where  $Q_{X_0}$  is a symmetric matrix  $n \times n$  such that  $\text{tr } Q_{X_0} = 1$ .*

*Or  $X_0$  is a **regular** point and then there exists a unit vector  $\nu_{X_0} \in \mathbf{S}^{n-1}$  such that*

$$u^0(X) = \frac{1}{2} (\max (< X, \nu_{X_0} >, 0))^2$$

*and the free boundary is a  $C^1$   $(n-1)$ -dimensional manifold in a neighbourhood of  $X_0$ .*

The regularity  $C^1$  can then be improved by Kinderlehrer, Nirenberg results [16], and gives  $C^\infty$  regularity for an obstacle problem where the elliptic operator has  $C^\infty$  coefficients. It is also possible to get similar results with analyticity of the solutions when the coefficients are analytic.

## 2.2. More general blow-up limits

We now recall a result which characterizes the limits of some more general blow-up sequences where the origin moves with the scaling.

**LEMMA 2.2** (General Blow-up Limits, [8]). — *Let*

$$u^\varepsilon(X) = \frac{u_\varepsilon(X_\varepsilon + \varepsilon X)}{\varepsilon^2}$$

*where  $u_\varepsilon$  is a sequence of solutions to*

$$\begin{cases} \Delta u_\varepsilon = f_\varepsilon \geq 1 & \text{on } \{u_\varepsilon > 0\} \cap \Omega_\varepsilon \\ u_\varepsilon \geq 0 & \text{on } \Omega_\varepsilon \text{ and } |D^2 u_\varepsilon|_{L^\infty(\Omega_\varepsilon)} \leq M \end{cases}$$

*with  $|f_\varepsilon|_{C^{0,\alpha}(\Omega_\varepsilon)} \leq M$ . We assume that  $u_\varepsilon(X_\varepsilon) = 0$  and that  $\frac{1}{\varepsilon}d(X_\varepsilon, \partial\Omega_\varepsilon) \geq r > 0$  as  $\varepsilon \rightarrow 0$ . Then up to extraction of a convergent subsequence  $(\varepsilon')$ , we get*

$$u^{\varepsilon'} \longrightarrow u^0 \quad \text{uniformly on compact sets of } \Omega^0$$

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for some open set  $\Omega^0$  and where  $u^0$  is convex and satisfies

$$\begin{cases} \Delta u^0 = f_0(0) \geq 1 & \text{on } \{u^0 > 0\} \cap \Omega^0 \\ u^0 \geq 0 & \text{on } \Omega^0 \quad \text{and} \quad |D^2 u^0|_{L^\infty(\Omega^0)} \leq M \end{cases}$$

Moreover either

i) the interior of the coincidence set of the blow-up limit is empty:

$$\{u^0 = 0\}^0 = \emptyset$$

Or

ii) the interior of the coincidence set of the blow-up limit satisfies

$$\{u^0 = 0\}^0 \neq \emptyset$$

and 0 is a regular point for  $u^0$  and also for all  $u^{\varepsilon'}$  with  $\varepsilon'$  small enough.

Another useful result is the following nondegeneracy property of the solution:

**LEMMA 2.3** (Nondegeneracy, [8]). — Let  $u$  be a solution to problem (2.1) and  $0 \in \overline{\{u > 0\}}$ . If  $B_r(0) \subset \Omega$ , then

$$\sup_{B_r(0)} (u(X) - u(0)) \geq \frac{r^2}{2n}$$

*Proof of lemma 2.3.* — Apply the maximum principle to  $w(X) = u(X) - u(0) - \frac{1}{2n}|X|^2$  in  $B_r(0) \cap \{u > 0\}$ .

### 3. A bound on the second derivatives

In this section we will prove the following result

**PROPOSITION 3.1** (Control near the fixed boundary  $\partial\Omega$ ). — Under the assumptions of theorem 1.2, let us define  $\varepsilon = \sqrt{2 \left( \frac{\lambda_0 - \lambda}{\lambda} \right)}$ . Then there exist constants  $C, c > 0$  such that for all  $\lambda \in [0, \lambda_0]$  we have

$$u_\lambda(X) - \lambda \geq c\varepsilon^2 \quad \text{on } \{X \in \Omega, \quad \text{dist}(X, \partial\Omega) \leq c\varepsilon\} \quad (3.1)$$

$$|\nabla u_\lambda(X)| \leq C\varepsilon \quad \text{on } \Omega \quad (3.2)$$



and for all  $\delta \in (0, 1]$

$$|D^2 u_\lambda(X)| \leq C/\delta^2 \quad \text{on} \quad \{X \in \Omega, \quad \text{dist}(X, \partial\Omega) \geq c\varepsilon\delta\} \quad (3.3)$$

Moreover we have

$$\text{div}(F'(|\nabla u_\lambda|^2)\nabla u_\lambda) = u_\lambda \, 1_{\{u_\lambda > \lambda\}} \quad \text{on} \quad \Omega$$

where for the function  $u_\lambda \geq \lambda$  we define

$$1_{\{u_\lambda > \lambda\}}(X) = \begin{cases} 1 & \text{if } u_\lambda(X) > \lambda \\ 0 & \text{if } u_\lambda(X) = \lambda \end{cases}$$

*Remark 3.2.* — For a smooth  $\Omega$ , some  $L^\infty$  bounds on the second derivatives are given in [6] for fixed  $\lambda$ . Here we need to precise the dependence of the constants as  $\lambda$  goes to  $\lambda_0$ . The exterior sphere condition gives a control (3.1) from below on  $u_\lambda$ , and with the help of Harnack inequality we get the  $L^\infty$  bounds (3.3) on the second derivatives up to the case  $\lambda = \lambda_0$ . Because the fixed boundary  $\partial\Omega$  is not smooth here, the bound (3.3) on the second derivatives goes to infinity when the point reaches the fixed boundary  $\partial\Omega$  (case  $\delta = 0$ ).

We consider the minimizer  $u_\lambda$  of the convex energy

$$E(u) = \int_{\Omega} F(|\nabla u|^2) + u^2$$

on the convex set

$$K_\lambda = \{u \in H^1(\Omega), \quad u \geq \lambda \quad \text{on} \quad \Omega, \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega\}$$

We first prove that the minimizer  $u_\lambda$  satisfies the following Euler-Lagrange equation

LEMMA 3.3 (Euler-Lagrange equation). —

$$\text{div}(F'(|\nabla u_\lambda|^2)\nabla u_\lambda) = u_\lambda \, 1_{\{u_\lambda > \lambda\}} \quad \text{on} \quad \Omega$$

Although this result seems natural, we do not know any references where it is proved (except in the linear case). We give a complete proof below.

*Proof of lemma 3.3.* — Let

$$(s)^+ = \begin{cases} s & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

Then the minimization of  $E$  on  $K_\lambda$  is equivalent to the minimization of the convex energy

$$E_\lambda(u) = \int_{\Omega} F(|\nabla u|^2) + \left( (u - \lambda)^+ + \lambda \right)^2$$

on the convex set

$$K = \{u \in H^1(\Omega), \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega\}$$

Because  $u_\lambda$  is the minimizer of  $E_\lambda$  on  $K$ , we have for every  $\varphi \in C_0^\infty(\Omega)$  and  $t \in [0, 1]$ :

$$E_\lambda(u_\lambda + t\varphi) \geq E_\lambda(u_\lambda)$$

Then Lebesgue's dominated convergence theorem gives

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0} \left( \frac{E_\lambda(u_\lambda + t\varphi) - E_\lambda(u_\lambda)}{t} \right) \\ &= \int_{\Omega} 2F'(|\nabla u_\lambda|^2) \nabla u_\lambda \nabla \varphi \\ &\quad + 2 u_\lambda (\varphi \operatorname{sgn}^+(u_\lambda - \lambda) + \varphi^+ (1 - \operatorname{sgn}^+(u_\lambda - \lambda))) \end{aligned}$$

where

$$\operatorname{sgn}^+(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

Considering  $\varphi$  and  $-\varphi$  we get that  $\operatorname{div}(F'(|\nabla u_\lambda|^2) \nabla u_\lambda) \in L^\infty(\Omega)$ . Using the regularity theory for elliptic equations (see [21]) we deduce that  $u \in C_{loc}^{1,\alpha}(\Omega)$ . Consequently  $\{u_\lambda > \lambda\}$  is an open set and the Euler-Lagrange equation is satisfied on this open set. Furthermore a classical argument using the nondegeneracy lemma 2.3 proves that the Lebesgue measure of the free boundary  $\partial\{u_\lambda = \lambda\}$  is zero. This implies the full Euler-Lagrange equation. This ends the proof of lemma 3.3.

Let us recall that when  $\Omega$  is smooth, there exists a constant  $C_0 > 0$  such that for each  $\lambda \in [0, \lambda_0]$  we have the following properties (see Brézis, Kinderlehrer [6]):

(H1)

$$|\nabla u_\lambda(X)| \leq C_0 \quad \text{on} \quad \Omega$$

(H2)

$$u \in C_{loc}^{1,1}(\Omega)$$

In a first case we will prove proposition 3.1 assuming (H1)-(H2), and in a second case we will justify these assumptions.

**Case A: we assume (H1)-(H2) and that  $\partial\Omega$  is smooth**

**Step 1: proof of (3.1)**

We will build a subsolution  $u_0$  such that (for some point  $X_\varepsilon$  which will be made precise below)

$$\frac{u_\lambda(X) - \lambda}{\lambda} \geq \varepsilon^2 u_0 \left( \frac{|X - X_\varepsilon|}{\varepsilon} \right) \quad \text{for} \quad \frac{|X - X_\varepsilon|}{\varepsilon} \in [r_0, r_0 + \tau_0] \quad (3.4)$$

with  $\varepsilon = \sqrt{2 \left( \frac{\lambda_0 - \lambda}{\lambda} \right)}$ .

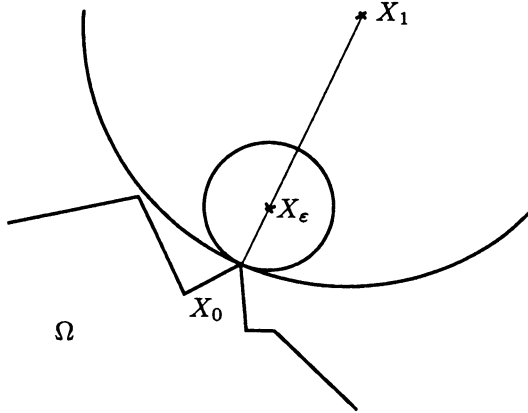


Figure 1. — Construction of a subsolution outside the ball  $B_{|X_0 - X_\varepsilon|}(X_\varepsilon)$

For some  $\tau_0 > 0$ , we consider a solution  $u_0$  of

$$\begin{cases} \Delta u_0 = \mu > 1 & \text{on } B_{r_0 + \tau_0}(0) \setminus B_{r_0}(0) \\ u_0 = \frac{1}{2} & \text{on } \partial B_{r_0}(0) \\ u_0 = 0 & \text{on } \partial B_{r_0 + \tau_0}(0) \end{cases}$$

By symmetry we have  $u_0(X) = u_0(|X|)$ . Let us recall that for each point  $X_0 \in \partial\Omega$ , there exists  $X_1 \in \mathbf{R}^n$ , such that  $B_{r_0}(X_1)$  is included in  $\mathbf{R}^n \setminus \Omega$  and

is tangent to  $\partial\Omega$  at  $X_0$ . Now considering the function  $u_\lambda$  at a scale close to the fixed boundary  $\partial\Omega$  we introduce the point  $X_\varepsilon = X_0 + \varepsilon(X_1 - X_0)$  and the following function (see figure 1)

$$w^\varepsilon(X) = \frac{u_\lambda(X_\varepsilon + \varepsilon X) - \lambda}{\lambda\varepsilon^2}$$

which satisfies on  $\frac{\Omega - X_\varepsilon}{\varepsilon}$ :

$$\begin{cases} A_\varepsilon(w^\varepsilon) \leq 1 \\ 0 \leq w^\varepsilon \leq \frac{1}{2} \end{cases}$$

where the quasilinear elliptic partial differential operator  $A_\varepsilon$  is defined in (4.1).

Moreover for a good choice of  $\mu > 1, \tau_0 > 0$ , we have on  $B_{r_0+\tau_0}(0) \setminus B_{r_0}(0)$ :

$$\begin{cases} A_\varepsilon(u_0) \geq 1 \\ 0 \leq u_0 \leq \frac{1}{2} \end{cases}$$

Then by the Maximum Principle (see Berestycki, Nirenberg [3]), we can slide  $u_0$  below  $w^\varepsilon$  and we get

$$w^\varepsilon \geq u_0 \quad \text{on} \quad B_{r_0+\tau_0}(0) \setminus B_{r_0}(0)$$

This is equivalent to (3.4) whose we deduce (3.1). This ends the proof of step 1.

## Step 2: proof of (3.2): estimate on the gradient : $|\nabla u_\lambda| \leq \lambda\varepsilon|u'_0(r_0)|$

We first remark that a straightforward consequence of step 1 is that

$$\limsup_{X \rightarrow \partial\Omega} \left( \frac{\lambda_0 - u_\lambda}{\text{dist}(X, \partial\Omega)} \right) \leq \lambda\varepsilon|u'_0(r_0)|$$

From the fact that  $u = \text{constant} = \lambda_0$  on  $\partial\Omega$ , we deduce that  $|\nabla u| \leq \lambda\varepsilon|u'_0(r_0)|$  on  $\partial\Omega$ . Now the estimate on the gradient comes from the fact that the gradient is maximal on the boundary  $\partial\Omega$ . For the convenience of the reader we recall this classical argument.

For  $u = u_\lambda$ , we have

$$a_{ij}(\nabla u) u_{ij} = u \quad \text{on} \quad \Omega \setminus \{u = \lambda\}$$

where  $a_{ij}(p) = F'(|p|^2)\delta_{ij} + 2F''(|p|^2)p_i p_j$ . Let us take  $v = \partial_\xi u$  where  $\xi \in \mathbf{S}^{n-1}$ . Then

$$a_{ij}v_{ij} + b_k v_k = v \quad \text{on} \quad \Omega \setminus \{u = \lambda\}$$

where  $b_k = (a_{ij})'_{p_k} \cdot u_{ij}$ . The Maximum Principle implies that  $v = \partial_\xi u$  is maximal on  $\partial\Omega \cup \partial\{u = \lambda\}$ . Taking all directions  $\xi \in \mathbf{S}^{n-1}$  we deduce that  $|\nabla u|$  is maximal on  $\partial\Omega$ , because  $\nabla u = 0$  on  $\partial\{u = \lambda\}$ . This ends the proof of step 2.

### Step 3: proof of (3.3)

Let

$$w(X) = \frac{u_\lambda(\varepsilon X) - \lambda}{\lambda \varepsilon^2}$$

Then

$$\begin{cases} A_\varepsilon(w) = 1 & \text{on} \quad \{w > 0\} \\ 0 \leq w \leq \frac{1}{2} \end{cases}$$

where the operator  $A_\varepsilon$  is defined in (4.1). Let  $Y_0 \in \frac{\Omega}{\varepsilon}$  such that  $\text{dist}(Y_0, \frac{\partial\Omega}{\varepsilon}) \geq c$ . We will prove a bound on  $|D^2 w(Y_0)|$ . To this end we will apply the method of Alt and Phillips [1], using the following Harnack inequality of Krylov, Safonov for non-divergence operator (a similar Harnack inequality for divergence operator is also applicable, see Gilbarg, Trudinger [15]):

**THEOREM 3.4** (Harnack inequality for non-divergence operators; [7]). — *If*

$$\begin{cases} a_{ij}v_{ij} = f & \text{on} \quad B_1 \subset \mathbf{R}^n \\ v \geq 0 & \text{on} \quad B_1 \end{cases}$$

and for the matrix  $a = (a_{ij})$

$$0 < c_0 \leq a \leq C_0$$

then there exists a constant  $C = C(n, C_0, c_0) > 0$  such that

$$\sup_{B_{\frac{1}{2}}} v \leq C \left( \inf_{B_{\frac{1}{2}}} v + |f|_{L^n(B_1)} \right)$$

We will also use the following interior estimate:

On the regularity of a free boundary for a nonlinear obstacle problem

THEOREM 3.5 (Interior estimate,[15]). — *Let us assume that*

$$a_{ij}v_{ij} + cv = f \quad \text{on} \quad B_r \subset \mathbf{R}^n$$

*and for the matrix*  $a = (a_{ij})$

$$0 < c_0 \leq a$$

*If for some*  $\alpha \in (0, 1)$  *there exists a constant*  $C_0 > 0$  *such that*

$$|a_{ij}|_{L^\infty(B_r)} + r^\alpha [a_{ij}]_{\alpha; B_r} + r^2 |c|_{L^\infty(B_r)} + r^{2+\alpha} [c]_{\alpha; B_r} \leq C_0$$

*where*  $[\cdot]_{\alpha; B_r}$  *is defined by*

$$[g]_{\alpha; B_r} = \sup_{x, y \in B_r, x \neq y} \left( \frac{|g(x) - g(y)|}{|x - y|^\alpha} \right)$$

*Then*

$$r^2 |D^2 v|_{L^\infty(B_{\frac{r}{2}})} \leq C (|v|_{L^\infty(B_r)} + r^2 |f|_{L^\infty(B_r)} + r^{2+\alpha} [f]_{\alpha; B_r})$$

*for some constant*  $C = C(n, \alpha, C_0, c_0) > 0$ .

Let  $w_r(X) = w(Y_0 + rX)$ . Applying Harnack inequality theorem 3.4 to  $w_r$  we get

$$\sup_{B_{\frac{r}{2}}(Y_0)} w \leq C \left( \inf_{B_{\frac{r}{2}}(Y_0)} w + r^2 \right) \quad (3.5)$$

Let

$$\rho = \sqrt{\frac{w(Y_0)}{2C}}$$

i) Case  $\rho < c\delta$ .

Then  $Y_0$  is close to  $\{w = 0\}$  and  $\rho$  can be arbitrarily small. We apply Harnack inequality (3.5) with  $r = \rho$  and we get

$$0 < w(Y_0) \leq \sup_{B_{\frac{\rho}{2}}(Y_0)} w \leq 2C \inf_{B_{\frac{\rho}{2}}(Y_0)} w$$

Let us remark that we have (see theorem 6.1, p. 281 of Ladyshenskaya, Ural'tseva [18])

$$[w]_{\alpha; B_1} \leq C$$

where the constant  $C$  has the following dependence

$$C = C(n, \alpha, |w|_{L^\infty(B_2)}, F, \lambda_0, r_0) > 0.$$

Then applying theorem 3.5, we deduce that

$$r^2 |D^2 w|_{L^\infty(B_{\frac{r}{2}}(Y_0))} \leq C (|w|_{L^\infty(B_r(Y_0))} + r^2)$$

With the choice  $r = \rho$ , this implies

$$|D^2 w(Y_0)| \leq C$$

ii) case  $\rho \geq c\delta$ .

We apply the previous interior estimate with  $r = c\delta$ . Using the fact that  $|w| \leq \frac{1}{2}$ , we find

$$|D^2 w(Y_0)| \leq C/\delta^2$$

iii) Conclusion :

$$|D^2 u_\lambda| \leq C/\delta^2 \quad \text{on} \quad \{X \in \Omega, \quad \text{dist}(X, \partial\Omega) \geq c\varepsilon\delta\}$$

i.e. (3.3) is proved.

### Case B: justification of (H1)-(H2)

Here we consider a general Lipschitz bounded open set  $\Omega$  satisfying assumptions (A1), (A2) of theorem 1.2. We can mollify this open set  $\Omega$  such that it gives a bigger and smooth open set  $\Omega^\eta$  where  $\eta$  is the mollification parameter such that  $\Omega^\eta = \Omega$  for  $\eta = 0$ . This smooth open set  $\Omega^\eta$  still satisfies assumptions (A1), (A2) uniformly in  $\eta$  small enough. We can in particular consider the minimizer  $u_\lambda^\eta$  of the energy

$$E^\eta(u) = \int_{\Omega^\eta} F(|\nabla u|^2) + u^2$$

on the convex set

$$K_\lambda^\eta = \{u \in H^1(\Omega^\eta), \quad u \geq \lambda \quad \text{on} \quad \Omega^\eta, \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega^\eta\}$$

This minimizer  $u_\lambda^\eta$  satisfies (H1)-(H2), and then (3.1),(3.2),(3.3).

Taking the limit  $\eta \rightarrow 0$ , we can extract (by Ascoli-Arzelà theorem) a convergent subsequence  $u_\lambda^\eta \rightarrow u$  such that  $u$  still satisfies (3.1),(3.2),(3.3). We have the

LEMMA 3.6. — *The limit  $u$  is the minimizer  $u_\lambda$  of the energy  $E$  on  $K_\lambda$ .*

This ends the proof of proposition 3.1.

*Proof of lemma 3.6.* — Let us recall that by (3.2),  $u_\lambda^\eta$  is bounded in  $W^{1,\infty}$  uniformly in  $\eta$  small enough. Let

$$\tilde{u}_\lambda = \begin{cases} \lambda_0 & \text{on } \Omega^\eta \setminus \Omega \\ u_\lambda & \text{on } \Omega \end{cases}$$

By construction, we have

$$E^\eta(\tilde{u}_\lambda) \geq E^\eta(u_\lambda^\eta)$$

At the limit  $\eta = 0$ , we get

$$E(u_\lambda) \geq E(u)$$

The uniqueness of the minimizer  $u_\lambda$  proves that  $u = u_\lambda$ . This ends the proof of the lemma 3.6.

#### 4. Regularity of the free boundary near $\partial\Omega$ : proof of theorem 1.2

We will prove theorem 1.2, thanks to Caffarelli result (lemma 2.2) applied to a particular blow-up sequence.

**Case**  $F(q) = q$

If theorem 1.2 is false, then there exist a sequence of reals  $\varepsilon_n = \sqrt{2 \left( \frac{\lambda_0 - \lambda^n}{\lambda^n} \right)} \rightarrow 0$  and a sequence of singular points  $X_{\lambda^n} \in \partial \{u_{\lambda^n} = \lambda^n\}$ . Because of proposition 3.1, we have  $\text{dist}(X_{\lambda^n}, \partial\Omega) > c\varepsilon_n$ . Then we define

$$w^{\varepsilon_n}(X) = \frac{u_{\lambda^n}(X_{\lambda^n} + \varepsilon_n X) - \lambda^n}{\lambda^n \varepsilon_n^2}$$

We have

$$\begin{cases} \Delta w^{\varepsilon_n} = 1 + \varepsilon_n^2 w^{\varepsilon_n} & \text{on } \{w^{\varepsilon_n} > 0\} \\ 0 \leq w^{\varepsilon_n} \leq \frac{1}{2} \end{cases}$$

Now from proposition 3.1 we have the following  $L^\infty$  bound on the second derivatives:

$$|D^2 w^{\varepsilon_n}(X)| \leq C \quad \text{for } \text{dist}(X_{\lambda^n} + \varepsilon_n X, \partial\Omega) \geq c\varepsilon_n$$

Consequently from lemma 2.2, there exists a subsequence which converges to a convex function  $w^0$  defined on  $\Omega_0$ , where  $\Omega_0$  is the limit of the sets



$\frac{1}{\varepsilon_n} (\Omega - X_{\lambda^n})$  (for an extracted subsequence). Moreover  $w^0$  satisfies

$$\begin{cases} \Delta w^0 = 1 & \text{on } \{w^0 > 0\} \\ 0 \leq w^0 \leq \frac{1}{2} & \text{and } |D^2 w^0(X)| \leq C \text{ for } \text{dist}(X, \partial\Omega_0) \geq c \end{cases}$$

Because  $\Omega$  satisfies an interior cone condition (A2),  $\Omega_0$  inherits the same property. Moreover because we have made a blow-up close to the fixed boundary  $\partial\Omega$ , we deduce that  $\Omega_0$  contains an infinite cone  $\mathcal{C}_0$  with a non-empty interior. Now we have two cases (see lemma 2.2):

i) the interior of the coincidence set of the blow-up limit is empty, and then the closure  $\overline{\{w^0 > 0\}}$  contains the cone  $\mathcal{C}_0$ . It is then sufficient to take a ball  $B_r \subset \mathcal{C}_0$  with  $r$  large enough such that (by the nondegeneracy lemma 2.3)

$$\sup_{B_r} w^0 \geq \frac{r^2}{2n}$$

which is in contradiction with  $0 \leq w^0 \leq \frac{1}{2}$ .

ii) the interior of the coincidence set of the blow-up limit is not empty, and then 0 is a regular point for  $w^0$ , and also a regular point for  $w^{\varepsilon'_n}$  for  $\varepsilon'_n$  small enough. This means that  $X_{\lambda^n}$  are regular points for  $u_{\lambda^n}$ . Contradiction.

### Case $F$ general

In this case we introduce the operator (for  $\varepsilon = \sqrt{2 \left( \frac{\lambda_0 - \lambda}{\lambda} \right)}$ )

$$A_\varepsilon(w) = a \left( \left( \frac{\lambda_0}{1 + \frac{\varepsilon^2}{2}} \right) \varepsilon \nabla w \right) D^2 w - \varepsilon^2 w \quad (4.1)$$

where  $a(p) = F'(p^2)Id + 2F''(p^2)p \otimes p$ . Then we have

$$\begin{cases} A_{\varepsilon_n}(w^{\varepsilon_n}) = 1 & \text{on } \{w^{\varepsilon_n} > 0\} \\ 0 \leq w^{\varepsilon_n} \leq \frac{1}{2} \end{cases}$$

A generalization of previous Caffarelli results to more general linear elliptic operators

$$L = \alpha_{ij} \partial_{ij} + \beta_i \partial_i + \gamma$$

is available in [8]. This allows us to get similar results in the same way for our general case.

This ends the proof of theorem 1.2.

## 5. Stability: proof of theorem 1.4

In this section we will prove theorem 1.4 on stability. A similar result is already known in the linear case (see for instance the book of Rodrigues [22] for general results of stability). In our case we use the approach of Caffarelli [8].

*Proof of theorem 1.4.* — Let us assume that the theorem is false. Then for a compact set  $\mathcal{K} \subset \subset \mathcal{K}^*$  we can find a sequence  $(\lambda^n)_n$  such that  $\lambda^n \rightarrow \lambda^*$  and a sequence of singular points  $(X_{\lambda^n})_n$  of the free boundaries  $\partial \{u_{\lambda^n} = \lambda^n\} \cap \mathcal{K}$ . Up to extract a subsequence we can assume

$$X_{\lambda^{n'}} \longrightarrow X_{\lambda^*} \in \{u_{\lambda^*} = \lambda^*\} \cap \mathcal{K}$$

where we have used the continuity of the map

$$\lambda \longmapsto u_\lambda$$

The continuity of this map is a consequence of the  $L^\infty$  bound on the gradient of  $u_\lambda$  uniformly in  $\lambda$  (see (3.2)). This continuity easily follows by a classical argument from Ascoli-Arzelà theorem, and the uniqueness of the solutions  $u_\lambda$  for each  $\lambda$ .

Let us recall that for  $\varepsilon = \sqrt{2 \left( \frac{\lambda_0 - \lambda}{\lambda} \right)}$  we have (the operator  $A_\varepsilon$  is defined in (4.1))

$$A_\varepsilon(w_\lambda) = 1 \quad \text{on} \quad \{w_\lambda > 0\}$$

where for some point  $X_\lambda \in \Omega$ :

$$w_\lambda(X) = \frac{u_\lambda(X_\lambda + \varepsilon X) - \lambda}{\lambda \varepsilon^2}$$

Using the adaptation of the nondegeneracy lemma 2.3 (see Caffarelli [8]) for general linear elliptic operators, we get the existence of a constant  $c_0 > 0$  such that

$$\sup_{B_r(X_{\lambda^n})} (u_{\lambda^n}(X) - \lambda^n) \geq c_0 r^2$$

Then at the limit we get

$$\sup_{B_r(X_{\lambda^*})} (u_{\lambda^*}(X) - \lambda^*) \geq c_0 r^2$$

which proves that  $X_{\lambda^*} \in \partial \{u_{\lambda^*} = \lambda^*\}$ . In particular because  $X_{\lambda^*}$  is a regular point for  $u_{\lambda^*}$ , i.e. 0 is a regular point for  $w_{\lambda^*}$ , we get that the blow-up sequence

$$w_{\lambda^*}^\delta(X) = \frac{w_{\lambda^*}(\delta X)}{\delta^2}$$

converges (up to extraction of a subsequence) to a blow-up limit of regular type (see theorem 2.1; for an extension to general linear elliptic operators, see Caffarelli [8]):

$$w_{\lambda^*}^0(X) = \frac{1}{2} (\max(< X, \nu_{X_{\lambda^*}} >, 0))^2$$

We realize that the origin 0 is obviously a regular point of  $w_{\lambda^*}^0$ . Finally we can consider the other blow-up sequence:

$$w_{\lambda^n}^{\delta^n}(X) = \frac{w_{\lambda^n}(\delta^n X)}{(\delta^n)^2}$$

Because for  $\delta^n = \delta$  fixed and  $\lambda^n \rightarrow \lambda^*$ , this sequence of functions converges to  $w_{\lambda^*}^\delta$ , we see that we can choose a sequence  $(\delta^n)_n$  slowly decreasing to zero such that

$$w_{\lambda^n}^{\delta^n} \longrightarrow w_{\lambda^*}^0$$

Then applying an adaptation of lemma 2.2 (see Caffarelli [8]) still true for general linear elliptic operators, we deduce from the fact that 0 is a regular point for the blow-up limit of  $w_{\lambda^n}^{\delta^n}$ , that 0 is also a regular point for  $w_{\lambda^n}^{\delta^n}$  for  $n$  large enough. This means that  $X_{\lambda^n}$  is a regular point for  $u_{\lambda^n}$ . Contradiction. This ends the proof of theorem 1.4.

## 6. Hausdorff measure of the free boundary: proof of theorem 1.5

In this section we give the proof of theorem 1.5, which is an adaptation of a method of Caffarelli presented in the linear case in [9, 19]. We perform the proof in two steps.

### Step 1

For the function  $u = u_\lambda$ , let

$$O^\eta = \{X \in \Omega, \quad |\nabla u(X)| < \eta \quad \text{and} \quad u(X) > \lambda\}$$

For a function  $u \geq \lambda$ , we note

$$1_{\{u > \lambda\}}(X) = \begin{cases} 1 & \text{if } u(X) > \lambda \\ 0 & \text{if } u(X) = \lambda \end{cases}$$

LEMMA 6.1 (Estimate in a neighbourhood of the free boundary). — *If*

$$\left\{ \begin{array}{ll} \nabla \cdot (F'(|\nabla u|^2) \nabla u) = u \, 1_{\{u > \lambda\}} & \text{on } \Omega \\ u \geq \lambda > 0 & \text{on } \partial\Omega \\ |D^2 u(X)| \leq M & \text{on } \{X \in \Omega, \, \text{dist}(X, \partial\Omega) \geq c\varepsilon\} \end{array} \right.$$

then for all compact  $\mathcal{K} \subset \{X \in \Omega, \, \text{dist}(X, \partial\Omega) \geq c\varepsilon\}$  such that  $\partial\mathcal{K}$  is  $C^1$ , there is a constant  $C = C(M)$ , such that

$$|O^\eta \cap \mathcal{K}| \leq \eta C \lambda^{-2} (|\mathcal{K}| + \mathcal{H}^{n-1}(\partial\mathcal{K}))$$

where  $|\mathcal{K}|$  is the volume of  $\mathcal{K}$  and  $\mathcal{H}^{n-1}(\partial\mathcal{K})$  is the  $(n-1)$  dimensional Hausdorff measure of its perimeter.

Remark 6.2 (The Hausdorff measure). — Let us recall the definition of the Hausdorff measure. If  $U$  is a set, let

$$\text{diam}(U) = \sup_{X, X' \in U} |X' - X|$$

Then for  $s \geq 0$  and a set  $A$  let

$$\mathcal{H}_\delta^s(A) = c_s \inf_{\{U^i\}_i, A \subset \bigcup_i U^i, \text{diam } U^i \leq \delta} \sum_i (\text{diam } U^i)^s$$

which is a nondecreasing function of  $\delta$ . Then the  $s$ -dimensional Hausdorff measure is

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A)$$

The constant  $c_s$  must be chosen such that the Hausdorff measure coincides with the Lebesgue measure of  $\mathbf{R}^s$  if  $s \in \mathbf{N}$ .

*Proof of lemma 6.1.* — Because  $F' \in C^{1,1}$ , we have  $h_i \in C^{0,1}$  where

$$h_i = \begin{cases} -\eta & \text{if } F' \nabla_i u \leq -\eta \\ F' \cdot \nabla_i u & \text{if } |F' \nabla_i u| \leq \eta \\ \eta & \text{if } F' \nabla_i u \geq \eta \end{cases} \quad (6.1)$$

We note  $X_i$  the vector field defined by  $X_i = \nabla_i(F' \nabla u) \in L^\infty$ . Then the Stokes formula gives :

$$\int_{\mathcal{K}} \nabla h_i \cdot X_i = \int_{\partial\mathcal{K}} h_i(X_i \cdot n) - \int_{\mathcal{K}} h_i(\nabla \cdot X_i) \quad (6.2)$$

But  $\nabla \cdot X_i = \nabla_i(\nabla \cdot (F' \nabla u)) = \nabla_i u$  on  $\{u > \lambda\}$ , and  $h_i = 0$  on  $\{u = \lambda\}$ .  
Then

$$\int_{O^n \cap \mathcal{K}} \nabla(F' \nabla_i u) \cdot \nabla_i(F' \nabla u) \leq \eta C(M)(|\mathcal{K}| + \mathcal{H}^{n-1}(\partial \mathcal{K})) \quad (6.3)$$

But

$$\nabla(F' \nabla_i u) \cdot \nabla_i(F' \nabla u) = [\nabla_i(F' \nabla_i u)]^2 + \sum_{k \neq i} [F' D_{ik}^2 u]^2 + O(|\nabla u|^2)$$

and

$$\left| \int_{O^n \cap \mathcal{K}} O(|\nabla u|^2) \right| \leq \eta C(M) |\mathcal{K}|$$

Making the sum  $\sum_i$ , we get

$$\int_{O^n \cap \mathcal{K}} \sum_i (\nabla_i(F' \nabla_i u))^2 \leq \eta C(M)(|\mathcal{K}| + \mathcal{H}^{n-1}(\partial \mathcal{K})) \quad (6.4)$$

But

$$\sum_i (\nabla_i(F' \nabla_i u))^2 \geq \left( \frac{\nabla \cdot (F' \nabla u)}{2} \right)^2 \geq \frac{u^2}{4} \geq \frac{\lambda^2}{4}$$

and then we get the expected result.

## Step 2

The Hausdorff measure is bounded from above by:

$$\mathcal{H}^{n-1}(\Gamma) \leq \lim_{\eta \rightarrow 0} \inf_{\{B_\eta(Y_i)\}} \frac{1}{\eta} \sum_i |B_\eta(Y_i)| \quad (6.5)$$

where  $\Gamma = \partial \{u = \lambda\}$  is the free boundary, and where  $\{B_\eta(Y_i)\}_i$  is a covering of  $\Gamma$  by balls of center  $Y_i$  on  $\Gamma$  and of radius  $\eta$ .

From proposition 3.1, we know that

$$u(X) - \lambda \geq c\varepsilon^2 \quad \text{while} \quad \text{dist}(X, \partial \Omega) < c\varepsilon \quad \text{where} \quad \varepsilon = \sqrt{2 \left( \frac{\lambda_0 - \lambda}{\lambda} \right)}$$

which in particular implies

$$\text{dist}(\{u = \lambda\}, \partial \Omega) \geq c\varepsilon$$

Now starting from a point  $Y_i$  on  $\partial\{u = \lambda\}$  we have from (3.3)

$$u(X) - \lambda \leq \frac{1}{2}C|X - Y_i|^2 \quad \text{while} \quad \text{dist}(X, \partial\Omega) \geq c\varepsilon$$

Therefore we get

$$\text{dist}(B_\eta(Y_i), \partial\Omega) \geq c\varepsilon \quad \text{while} \quad \frac{1}{2}C\eta^2 < c\varepsilon^2$$

i.e. for  $\eta$  small enough. Then for such  $\eta$  we have

$$B_\eta(Y_i) \cap \{u > \lambda\} \subset B_\eta(Y_i) \cap \{u > \lambda, |\nabla u| \leq C\eta\} \subset B_\eta(Y_i) \cap O^{C\eta}$$

From the nondegeneracy lemma 2.3, we deduce the existence of a real  $\gamma \in (0, 1)$  such that

$$|B_\eta(Y_i) \cap \{u > \lambda\}| \geq \gamma |B_\eta(Y_i)|$$

As a consequence we get

$$|B_\eta(Y_i)| \leq \gamma^{-1} |B_\eta(Y_i) \cap O^{C\eta}|$$

Thus

$$\begin{aligned} \eta^{-1} \sum_i |B_\eta(Y_i)| &\leq \eta^{-1} \gamma^{-1} \sum_i |B_\eta(Y_i) \cap O^{C\eta}| \\ &\leq \eta^{-1} \gamma^{-1} \int_\Omega \sum_i 1_{B_\eta(Y_i)} 1_{O^{C\eta}} \\ &\leq \eta^{-1} \gamma^{-1} \sup(\sum_i 1_{B_\eta(Y_i)}) \int_\Omega 1_{O^{C\eta}} \\ &\leq \eta^{-1} \gamma^{-1} \sup(\sum_i 1_{B_\eta(Y_i)}) |O^{C\eta}| \\ &\leq \gamma^{-1} C_n C' \lambda^{-2} (|\mathcal{K}_\varepsilon| + \mathcal{H}^{n-1}(\partial\mathcal{K}_\varepsilon)) \end{aligned}$$

where we have used the fact that we can always use locally finite recovering  $\{B_\eta(Y_i)\}_i$  such that  $\sum_i 1_{B_\eta(Y_i)} \leq C_n$  where the constant only depends on the dimension  $n$ . On the other hand we have applied lemma 6.1 introducing a smooth compact set  $\mathcal{K}_\varepsilon$  such that

$$\mathcal{K}_\varepsilon \subset \{X \in \Omega, \quad 2c\varepsilon \geq \text{dist}(X, \partial\Omega) \geq c\varepsilon\}$$

In fact  $\mathcal{K}_\varepsilon$  can be seen as a smooth approximation of  $\partial\Omega$ . Consequently we get

$$\mathcal{H}^{n-1}(\Gamma) \leq C$$

where the constant  $C$  only depends on  $\Omega$ ,  $\lambda_0$  and  $F$ , and is uniform with respect to  $\lambda \in [0, \lambda_0]$ . This proves theorem 1.5.

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