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Real singularities and open-book decompositions of the 3-sphere [(*)]

ANNE PICHON [(1)] AND JOSÉ SEADE [(2)]

ABSTRACT. — We study the topology of the real analytic germs $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by $f(z_1, z_2) = z_1^p \overline{z_2} + z_2^q \overline{z_1}$. Such a germ gives rise to a Milnor fibration $\frac{f}{|f|} : \mathbb{S}^3 \setminus L \longrightarrow \mathbb{S}^1$, where L denotes the link of f. We describe topologically this fibration, computing the genus of the fiber and the monodromy. This implies that f is not topologically equivalent to an holomorphic germ, whereas its link L is isotopic to the link of the complex singularity $z_1 z_2 (z_1^{p+1} + z_2^{q+1})$.

RÉSUMÉ. — Nous étudions la topologie des germes analytiques réels $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ définis par $f(z_1, z_2) = z_1^p \overline{z_2} + z_2^q \overline{z_1}$. Un tel germe donne lieu à une fibration à la Milnor $\frac{f}{|f|} : \mathbb{S}^3 \setminus L \longrightarrow \mathbb{S}^1$, L désignant l'entrelacs de f. On décrit topologiquement cette fibration en calculant le genre de la fibre et la monodromie. Ceci implique que f n'est pas topologiquement équivalent à un germe holomorphe, alors que son entrelacs L est isotope à l'entrelacs de la singularité complexe $z_1 z_2 (z_1^{p+1} + z_2^{q+1})$.

Introduction

The study of the topology of isolated complex singularities is closely related to knot theory, and this relation has been long studied by many authors, as for example in [Br], [Mi1], [Du], [EN], [LMW], and many more. The links (knots) that one gets in this way are called algebraic links. By the

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work of Milnor [Mi1], these are fibered links and they give rise to open-book decompositions on the odd-dimensional spheres. More precisely, if

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0),$$

is an analytic germ with an isolated critical point at $0 \in \mathbb{C}^n$, let $L = f^{-1}(0) \cap \mathbb{S}_\varepsilon^{2n-1}$ be the link of the singularity, where $\mathbb{S}_\varepsilon^{2n-1}$ denotes a sufficiently small $(2n-1)$-sphere with radius ε centered at 0 in $\mathbb{C}^n$. Then the map

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon^{2n-1} \setminus L \to \mathbb{S}^1$$

is a C^∞ locally trivial fibration which defines an open-book decomposition of $\mathbb{S}_\varepsilon^{2n-1}$ with binding L. These open-book decompositions have been used for many interesting problems in geometry and topology, as for instance in [La], where Lawson used them to construct a new class of non-singular foliations on the spheres, thus making a break-through in foliations theory. (We refer to [Ra, Ro, Wi], or to Section 1 below, for details about open-book decompositions.) Milnor also proved the following fibration theorem, thus showing that not only the complex analytic germs carry such a beautiful geometric structure: some real singularities do too.

THEOREM.— ([Mi1], 11.2) *Let $f : (\mathcal{U} \subset \mathbb{R}^{n+k}, 0) \longrightarrow (\mathbb{R}^k, 0)$ be the germ of a real analytic function whose jacobian matrix has rank k on an open neighbourhood of 0 in $\mathbb{R}^{n+k}$, except perhaps at 0. Let $\mathbb{S}_\varepsilon^{n+k-1}$ be a small sphere in $\mathbb{R}^{n+k}$, centered at 0, let $L = f^{-1}(0) \cap \mathbb{S}_\varepsilon^{n+k-1}$ and let $N(L)$ be a small tubular neighbourhood of L in $\mathbb{S}_\varepsilon^{n+k-1}$. Then there exists a C^∞ locally trivial fibration $\mathbb{S}_\varepsilon^{n+k-1} \setminus N(L) \longrightarrow \mathbb{S}^{k-1}$.*

There are two "problems" regarding this theorem. Firstly, it is not easy to construct explicit examples of real singularities satisfying these hypothesis; and secondly, they do not necessarily yield to open-book decompositions, because the behaviour of the "pages" (i.e the fibers) near the binding can not be controlled in general. Interesting results regarding this last point have been obtained by [Jq, NR], see also [RSV]. The constructions of [S1] and [S2] provide infinite families of such real analytic functions. In [S2], it is proved that the real analytic maps $f : \mathbb{R}^{2n} \cong C^n \longrightarrow \mathbb{R}^2 \cong C$ defined by

$$f(z_1, ..., z_n) = \lambda_1 z_1^{a_1} \cdot \overline{z}_{\sigma_1} + ... + \lambda_n z_n^{a_n} \cdot \overline{z}_{\sigma_n},$$

satisfy Milnor's condition when the a_i are integers $\geqslant 2$, the λ_i are non-zero complex numbers and $\{\sigma_1, ..., \sigma_n\}$ is any permutation of the set $\{1, ..., n\}$. Moreover, it is proved in that article that for these singularities, the map $\frac{f}{\|f\|} : \mathbb{S}_\varepsilon^{2n-1} \setminus L \to \mathbb{S}^1$ defines an open-book decomposition of the sphere $\mathbb{S}_\epsilon^{2n-1}$ with binding L. The purpose of this work is to determine, in the case $n = 2$, the topology of these open-book decompositions on $\mathbb{S}^3_\epsilon$, i.e. the isotopy class of the binding L, the homeomorphism class of the fibers and the monodromy of the fibration. Since $n = 2$, there are only two types of such singularities, these are :

i) $\lambda_1 z_1^p \cdot \overline{z}_1 + \lambda_2 z_2^q \cdot \overline{z}_2$;

ii) $\lambda_1 z_1^p \cdot \overline{z}_2 + \lambda_2 z_2^q \cdot \overline{z}_1$.

In the first case, it is shown in [RSV] that the homeomorphism defined on the complement of the two axis in $\mathbb{C}^2$ by :

$$(z_1, z_2) \mapsto (|z_1|^{\frac{2}{p-1}} z_1,\ |z_2|^{\frac{2}{q-1}} z_2),$$

extends to a homeomorphism of $\mathbb{C}^2$ and provides a topological equivalence between the singularity $z_1^p \cdot \overline{z}_1 + z_2^q \cdot \overline{z}_2$ and the Pham-Brieskorn singularity $z_1^{p-1} + z_2^{q-1}$. Hence the topology of these singularities is well understood via complex geometry. Thus we only consider here singularities of the type

$$f(z_1, z_2) = \lambda_1 z_1^p \overline{z}_2 + \lambda_2 z_2^q \overline{z}_1 .$$

If p and q are both $\geqslant 2$, then we know from [S2, p. 330] that the $\mathbb{S}^1$-action on $\mathbb{C}^2$ given by:

$$(e^{it}, (z_1, z_2)) \mapsto (e^{is_1 t} z_1, e^{is_2 t} z_2),$$

where $s_1 = \frac{1+q}{pq-1}$ and $s_2 = \frac{1+p}{pq-1}$, leaves invariant the real analytic, 2-dimensional singular surface $\overline{V}_{p,q} = \{\lambda_1 z_1^p \overline{z}_2 + \lambda_2 z_2^q \overline{z}_1 = 0\}$. This action also leaves invariant the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$, thus giving a Seifert decomposition of the 3-sphere, with the Hopf link $\mathbb{S}^3 \cap (\{z_1 z_2 = 0\})$ as the two exceptional fibres. Both of these are components of the link,

$$L = \overline{V}_{p,q} \cap \mathbb{S}^3 ,$$

which is a union of Seifert fibres, so it is a *Seifert link*. In fact these are the links of type 5 in the classification of Seifert links in [EN; 7.3]. These links can also be obtained via complex singularities: they are isotopic to the links in $\mathbb{S}^3$ determined by

$$z_1 z_2 (z_1^{p+1} + z_2^{q+1}) = 0 .$$

So it is natural to ask whether the corresponding Milnor fibrations, and the open-book decompositions on $\mathbb{S}^3$, are equivalent in some sense. We show that this is not the case. In fact we prove (Theorem 1.3) :

1) The open-book fibration provided by $f(z_1, z_2) = \lambda_1 z_1^p \overline{z}_2 + \lambda_2 z_2^q \overline{z}_1$ induces the "negative" orientation around two components of the link. More

precisely, the map $\frac{f}{\|f\|}$ has degree -1 restricted to small meridians of two components of L and degree $+1$ for the other components. (Whereas these degrees are all $+1$ in the holomorphic case.)

2) If we let $k = \gcd(p+1, q+1)$ be the greatest common divisor, $p' = \frac{p+1}{k}$ and $q' = \frac{q+1}{k}$, then the genus of the fibres of $\frac{f}{\|f\|}$ is $\frac{1}{2}k(kp'q' - p' - q' - 1) = \frac{1}{2}(pq - 1 - k)$. (Whereas it equals $\frac{1}{2}k(kp'q' + p' + q' + 1)$ in the holomorphic case.)

3) The monodromy is periodic of order $kp'q' - p' - q' = \frac{1}{k}(pq - 1)$. (Whereas in the holomorphic case, it is periodic of order $kp'q' + p' + q'$.)

This implies (Corollary 3.2) that for each angle $\theta \in [0, \pi[$, the antipodal fibers $f^{-1}(e^{i\theta})$ and $f^{-1}(e^{i(\theta+\pi)})$ are glued together along their common boundary L, forming an oriented surface of genus pq diffeomorphic to the set of points where the real line field (z_2^q, z_1^p) is tangent to $\mathbb{S}^3$.

To obtain these results, we first compute a "topological" resolution $\pi : X \to \mathbb{C}^2$ of the singularity $\{\lambda_1 z_1^p \overline{z}_2 + \lambda_2 z_2^q \overline{z}_1 = 0\}$ via the usual technique for studying complex plane curves, i.e. by performing appropriate blow-ups to get a resolution of the singularity (Sections 2 and 3). The additional problem we have to face is that, since the singularities in question are only real analytic, we have to make a "trick" to transform them by a homeomorphism, in some step of the resolution process, in order to get a "divisor" with normal crossings. In this way we obtain a topological resolution of f, and then a plumbing description of the isotopy class of its link L . Therefore, using the plumbing calculus of [Ne1] we obtain a description of the link L as a Seifert link in $\mathbb{S}^3$. Moreover, the study of the behaviour of $f \circ \pi$ near the branches of the strict transform of f by π enables us to compute explicitely the degrees of the fibration restricted to small meridians of the components of L and establish statement 1).

To obtain 2) and 3) we use a generalization of the method of [Pi] described in Section 1. The idea is that, since L is a Seifert link, the fibres of the open book fibration $\frac{f}{\|f\|}$ are horizontal in the sense of [Wa], i.e. up to isotopy, they are transversal to the Seifert fibres (except in the special cases treated "by hand" in Section 4). Therefore the monodromy of the fibration $\frac{f}{\|f\|}$ is represented by the first return $h : F \to F$ of the Seifert fibres on a fibre F of $\frac{f}{\|f\|}$. This implies that the monodromy is periodic, thus completely classified by the so-called Nielsen graph $\mathcal{G}(h)$ (Section 1). Moreover, $\mathcal{G}(h)$ is completely determined by the data previously computed, namely the Seifert graph of the link L and the degrees of $\frac{f}{\|f\|}$ around the components of L (Proposition 1.2).

In particular, the order N of the monodromy is determined by the Nielsen graph. The projection map $p : F \longrightarrow F/h$ to the space of orbits of h is a N-sheeted cyclic cover over the sphere $\mathbb{S}^2$ with holes (as many as the number of components in the link), ramified at two points, corresponding to the exceptional Seifert fibres. The indices of ramification are encoded in the Nielsen graph. Thus, it is easy to determine the genus of the pages using Hurwitz formula.

In Section 1, we present some results about the classification of horizontal open-book fibrations, restricting the discussion to Seifert links instead of the more general Waldhausen links considered in [Pi]. In section 2 we discuss one particular example among the above real singularities, which illustrates the proof of the result in the general case, given in section 3. Section 4 discusses two special cases of the above singularities, which do not fit within the general framework, thus completing the study of the topology of these singularities, and the corresponding Milnor fibrations, when $n = 2$.

The results of section 3 below, together with [Pi], show that the open-book decompositions on S^3 given by Theorem 3.1 are topologically equivalent to those defined by the singularities:

$$\hat{f}(z_1, z_2) = \bar{z}_1 \bar{z}_2 (z_1^{p+1} + z_2^{q+1}) \,.$$

We notice that this map $\hat{f}$ is a product of the form $\hat{f} = \bar{g} \cdot \phi$, where g and ϕ are both holomorphic maps in $\mathbb{C}^2$ with an isolated critical point at $0 \in \mathbb{C}^2$ and with no common branch. More generally, one can study the singulariies of this type $\hat{f} = \bar{g} \cdot \phi$. This is done in [Pi2].

1. Seifert links and horizontal fibrations

In this section M is a compact oriented 3-dimensional manifold. In the following sections, we use these results taking M to be the sphere $\mathbb{S}^3$.

A link in M means a disjoint, finite union of circles embedded in M. If M is a Seifert manifold, a *Seifert link* L in M is a union of Seifert fibres of some Seifert fibration of M, c.f. [EN; Chapter II]. In the sequel, we avoid considering Seifert links whose complement in M is a solid torus or a product *torus* $\times$ $[0, 1]$. These degenerate cases do not appear among the links of singularities considered in this paper, except in the special cases of Section 4, which are treated "by hand".

Given a Seifert link L, the uniqueness theorem of Waldhausen ([Wa] or [Ja; Theorem VI. 18]), implies that there exists a unique Seifert fibration of M, up to isotopy, for which L is union of Seifert fibres, and the isotopy class of L is characterized by the Seifert graph $G(M, L)$ constructed as follows. The graph $G(M, L)$ has a single vertex. For each component of L (respectively for each exceptional Seifert fibre which is not a component of L), one attaches to the vertex an arrow (respectively a stalk), whose extremity is weighted by the corresponding pair (α, β) of normalized Seifert invariants $(0 \leqslant \beta < \alpha)$. These integers satisfy the equation $\alpha a + \beta b = 0$ in $H_1(N, \mathbb{Z})$, where N is a small tubular neighbourhood of the component of the link (or of the corresponding exceptional fibre), saturated with Seifert fibres, b is an oriented Seifert fibre on ∂N and a is an oriented curve on ∂N such that the intersection $a \cdot b$ is $+1$ on ∂N oriented as the boundary of N. The vertex of the graph is endowed with the two numbers g and e_0, which denote respectively the genus of the base and the rational Euler number e_0 of the Seifert fibration. This number e_0 has important geometric properties and it has been used by several authors. For instance, it is noticed in [Ne2] that a Seifert manifold is the link of a surface singularity if and only if its rational Euler number e_0 is negative. To define e_0 we first recall that if E is an oriented $\mathbb{S}^1$-bundle over an oriented 2-dimensional, compact, connected, manifold B, then its usual Euler class is the primary (i.e. *non-automatically zero*) obstruction for constructing a section of E. This class lives in $H^2(B; \mathbb{Z})$ and it becomes a number when we evaluate it on the orientation class of B. If B has non-empty boundary, then $H^2(B; \mathbb{Z}) \cong 0$, so the bundle is trivial. However, if we fix a choice of a trivialization of E over ∂B, i.e. a section of $\tau : \partial B \to E$, then one has an *Euler class of E relative to τ*, $e(E; \tau) \in H^2(B, \partial B; \mathbb{Z}) \cong \mathbb{Z}$; evaluating $e(E; \tau)$ on the orientation cycle of the pair $(B, \partial B)$ we obtain an integer, which is by definition, the *Euler number* of E *relative to* τ. Now, given an oriented Seifert fibration $\pi : M \to B$ on a 3-manifold M, let us remove from B small, pairwise disjoint, open discs around the points corresponding to the special fibers, and denote by B_0 what is left. Let E be $\pi^{-1}(B_0)$, which is M minus a union of open solid tori. This is an $\mathbb{S}^1$-bundle over B_0. On each boundary torus T_i, one can choose a unique (up to isotopy) oriented curve a which intersects each Seifert fiber in exactly one point and satisfies that $m = \alpha[a] + \beta[b]$, where m is a meridian of T_i, (α, β) are the corresponding reduced Seifert invariants, and $[b]$ is the homology class represented by one Seifert fiber. This curve a determines a section of $E|_{T_i}$. Doing this for each boundary torus we obtain a section of E over ∂B_0. The *Euler number* $e = e(M)$ of the Seifert fibration $\pi : M \to B$ is defined to be the *Euler number* of E *relative to* the given trivialization over ∂B_0. Then the *rational Euler number* of the Seifert fibration, which is the weight of the vertex in the Seifert graph, is defined by:

$$e_0 = e - \sum_{i=1}^{d} \frac{\beta_i}{\alpha_i}\,.$$

For example, figure 1 represents the Seifert graph of the torus link $(2,3)$ obtained from the complex singularity $z_1^2 + z_2^3$. We notice that, by [Ne1], the data of the Seifert graph is equivalent to that of the resolution graph of the singularity $z_1^2 + z_2^3$, which describes the complement of L in S^3 as the 3-manifold obtained by a plumbing process. We refer to [Ne1] for more details and for the relation between the Seifert graph and the plumbing (or resolution) graph.

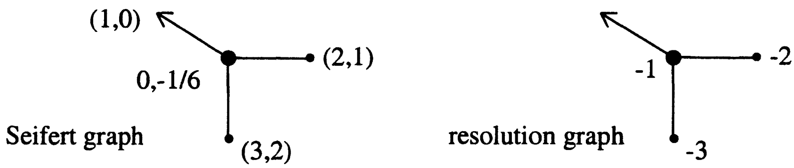


Fig. 1

Let L be a link in a 3-dimensional compact oriented manifold M. An *open-book fibration* of L is a C^∞ locally trivial fibration $\Phi : M \setminus L \longrightarrow \mathbb{S}^1$ which equips M with an open-book decomposition with binding L. In other words, for each component K of the link L, there exists an open tubular neighbourhood $N(K)$ of K in $M \setminus (L \setminus K))$ and a homeomorphism $\tau : \mathbb{S}^1 \times \mathbb{D}^2 \longrightarrow N(K)$ such that for all $(t,z) \in \mathbb{S}^1 \times \mathbb{D}^2$ one has:

$$(\Phi \circ \tau)(t,z) = \frac{z}{|z|}\,.$$

In this case the fibres of Φ are called *the pages*; each page is a 2-dimensional open surface whose closure in M is a compact surface with boundary L. The link L is called the *binding* of the open-book. In the sequel, all the open-book fibrations considered have connected pages.

If (M, L) is a Seifert link and if one has an open-book fibration Φ of L, then such a fibration is said to be *horizontal* if the fibres of Φ are transverse to the Seifert fibres, which are circles. According to [Wa], this transversality is automatically realized, up to isotopy, except in the degenerated cases avoided at the begining of this Section.

Now, let $\Phi : M \setminus L \longrightarrow \mathbb{S}^1$ be an open-book fibration of the link $L \in M$ and let K be a component of L. Let us choose an orientation $\overrightarrow{K}$ of K. Let D be a meridian disk of $N(K)$ oriented so that one has intersection $D \cdot \overrightarrow{K} = +1$ in M, and let us equip its boundary m with the induced orientation. One denotes by $\epsilon(\overrightarrow{K}) \in \{-1, +1\}$ the degree of the restriction of Φ to the oriented meridian $\overrightarrow{m}$. Note that if $-\overrightarrow{K}$ denotes K equipped with the opposite orientation , then $\epsilon(-\overrightarrow{K}) = -\epsilon(\overrightarrow{K})$.

If L is a Seifert link and if Φ is horizontal, then there exist two natural orientations of the link L. The first one, denoted by $\overrightarrow{L}_{flow}$ is obtained as follows: as the Seifert fibres of $M \backslash L$ are transversal to the fibres of Φ, one can orient each Seifert fibre b by a flow which lifts, via Φ, the unit tangent vector field of $\mathbb{S}^1 = \{z \in \mathbb{S}; |z| = 1\}$ compatible with the complex orientation. As the base of the Seifert fibration is orientable, this orientation $\overrightarrow{b}_{flow}$ is the same for each Seifert fibre b of $M \setminus L$ and extends to an orientation $\overrightarrow{L}_{flow}$ of L as union of Seifert fibres.

Let us now equip a fibre F of Φ with its natural orientation, that is $F.\overrightarrow{b}_{flow} = +1$. Then the second natural orientation of L, denoted by $\overrightarrow{L}_{bound}$, is the orientation of L as boundary of F.

It follows from the definitions that $\epsilon(\overrightarrow{K}_{bound}) = +1$ for each component K of L, while $\epsilon(\overrightarrow{K}_{flow})$ can be ± 1. In [Pi], only the open-book fibrations such that $\overrightarrow{L}_{bound} = \overrightarrow{L}_{flow}$, i.e. $\epsilon(\overrightarrow{K}_{flow}) = +1$, are studied.

1.1. Definition

Two horizontal fibrations $\Phi : M \setminus L \longrightarrow \mathbb{S}^1$ and $\Phi' : M' \setminus L' \longrightarrow \mathbb{S}^1$ are *topologically equivalent* if there exist orientation preserving homeomorphisms $H : (M, L) \longrightarrow (M', L')$ and $\rho : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that:

1) $\rho \circ \Phi = \Phi' \circ H|_{(M \setminus L)}$; and,

2) For each component K of L, $\epsilon(\overrightarrow{K}_{flow}) = \epsilon(\overrightarrow{H(K)}_{flow})$.

Notice that these conditions imply that $H(\overrightarrow{L}_{flow}) = \overrightarrow{L'}_{flow}$, i.e. that for each component K of L, the orientation that the flow obtained via Φ induces on K corresponds to the orientation on $H(K)$ given by the flow obtained via Φ'. We remark that the Lemma 4.5 of [Pi] extends easily to the situation we consider here and provides a classification of horizontal fibrations of Seifert links where the $\epsilon(\overrightarrow{K}_{flow})$ are not necessarily $+1$. In fact, [Pi] deals with Waldhausen links, but in the present paper, only Seifert links appear. Thus condition 1 can be replaced by:

1') The fibres of Φ and Φ' are diffeomorphic and their monodromies are conjugated in the mapping-class group of the fibre.

Let (M, L) be a Seifert link and let $\Phi : M \setminus L \longrightarrow \mathbb{S}^1$ be a horizontal fibration with connected fibre $F := \Phi^{-1}(t)$, considered as an oriented surface in M with boundary L. The diffeomorphism $h : F \longrightarrow F$ defined by the first return map on F of the Seifert fibres, oriented by the flow, is a periodic representative of the monodromy of Φ. Then the monodromy of Φ is classified, up to conjugation, by the Nielsen graph $\mathcal{G}(h)$ of h, which is a complete invariant defined in [Pi] from the work of Nielsen [Ni]. Let us recall briefly the construction of this graph. Given an oriented suface F and an orientation-preserving periodic diffeomorphism $\tau : F \longrightarrow F$ with order N, the projection $p : F \longrightarrow O$ onto the space of orbits of τ, is a N-sheeted cyclic cover, branched over a finite number of points $P_1, \dots, P_{d'}$, called the exceptional orbits. Let $D_1, \dots, D_{d'}$ be disjoint open discs in O such that $P_i \in D_i$ for all $i = 1, \dots, d'$; let us set $C_i = \partial D_i$ for all $i = 1, \dots, d'$, and $\check{O} = O \setminus \coprod_{i=1}^{d'} D_i$. Denote also by C_i, $i = d'+1, \dots, d'+d$, the boundary components of O. To each exceptional orbit P_i, $i = 1, \dots, d'$, and to each boundary component C_i, $i = d'+1, \dots, d'+d$, of O one associates a triple $(m_i, \lambda_i, \sigma_i)$ defined as follows. Let us endow O and $\check{O}$ with the orientations induced, via p, from that on F, and let us equip each C_i, $i = 1, \dots, d'+d$, with the orientation opposite to that of the boundary of $\check{O}$. The integer m_i is the number of connected components of $p^{-1}(C_i)$; then define λ_i by $\lambda_i m_i = N$, and σ_i is the integer modulo λ_i defined by $\rho([C_i]) = m_i \sigma_i$, where $\rho : H_1(\check{O}, \mathbb{Z}) \longrightarrow \mathbb{Z}/N\mathbb{Z}$ is the homomorphism associated to the N-sheeted cyclic cover $p_{|p^{-1}(\check{O})}$. Then the Nielsen graph $\mathcal{G}(\tau)$ consists of a single vertex, weighted by the genus g_o of the quotient surface O and by N, to which d' *stalks* and d *boundary-stalks* are attached, representing respectively the exceptional orbits and the boundary components of O. The extremity of each stalk or boundary-stalk is equipped with the corresponding triple $(m_i, \lambda_i, \sigma_i)$.

Let $\Phi : M \setminus L \longrightarrow \mathbb{S}^1$ be a horizontal fibration, with connected fibre F, of a Seifert link (M, L), and let $h : F \to F$ be the periodic representative of the monodromy of Φ obtained by the first return on F of the Seifert fibres. The following is an extension of ([Pi], Lemme 4.4) that follows immediately from the arguments used to prove that lemma in [Pi]. It asserts that the Nielsen graph of Φ is determined by the Seifert invariants of (M, L) and by the weights $\epsilon(\overrightarrow{K}_{flow})$.

1.2. Proposition

i) If $L = \coprod_{i=1}^{d} K_i$ and if $G(M, L)$ is the graph represented on figure 2, with $\epsilon_i = \epsilon(\overrightarrow{K_i}_{flow})$, then the order of h is given by

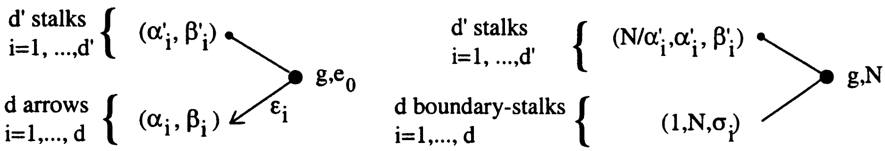
$$N = -\frac{1}{e_0} \sum_{i=1}^{d} \frac{\epsilon_i}{\alpha_i},$$

ii) The points in the orbits space F/h that correspond to the exceptional *orbits are the intersection points of the fiber F with the exceptional fibres of the Seifert fibration. Moreover, the exceptional orbit corresponding to the exceptional Seifert indexed by i carries the triple $(N/\alpha_i, \alpha_i, \beta_i)$.*

iii) The boundary component of F/h corresponding to the component K_i of L carries the triple $(1, N, \sigma_i)$, where σ_i, modulo N, is given by the equality:

$$\alpha_i \sigma_i - N\beta_i + \epsilon_i = 0\,.$$

The Nielsen graph $\mathcal{G}(h)$ is the graph on the right in figure 2.



Seifert graph G(S³,L) Nielsen graph of h

Fig. 2

In particular the projection $\pi : F \to F/h$ provides a description of the fibre F as a N-sheeted cyclic cover over the surface with genus g, branched over d' points with branching indices $\alpha_i, i = 1, \ldots, d'$ and N for the d remaining points. Thus we know the topology of the fiber F: it has d boundary components and its genus is obtained from the Hurwitz formula :

$$g_F = \frac{1}{2}\left(2Ng + (N-1)(d-2) + Nd' - \sum_{i=1}^{d'} \frac{N}{\alpha'_i}\right) \tag{1.3}$$

2. An Example

As a motivation for the following section, we study here the topology of the singularity

$$f(z_1, z_2) \;=\; z_2^3 \cdot \overline{z}_1 \;+\; z_1^5 \cdot \overline{z}_2\,.$$

Following the method used for studying the topology of complex plane curves (see for instance [EN, LMW]), let us try to describe the topology of the link L by performing a composition $\pi : X \longrightarrow \mathbb{C}^2$ of a finite number of blow-ups of points, starting with the blow-up of the origin in $\mathbb{C}^2$. We will see that we need to consider also a homeomorphism θ in such a way that the total transform $(f \circ \pi)^{-1}(0)$ has normal crossings. One then identifies the 3-sphere S^3_ϵ with the boundary of a small semi-algebraic tubular neighbourhood W of the exceptional divisor $E = \pi^{-1}(0)$ and L with the intersection of the strict transform of f with the boundary ∂W of W.

Let $\pi_1 : X_1 \to \mathbb{C}^2$ be the blow-up of $0_{\mathbb{C}^2}$ and set $\Psi_1 = f \circ \pi_1$. In the first coordinate chart of X_1, i.e. over $\mathbb{C}^2 \setminus \{z_2 = 0\}$, the blow-up is given by: $z_1 \mapsto z_1 z_2$ and $z_2 \mapsto z_2$. The exceptional divisor E_1 has equation $z_2 = 0$ and one has:

$$\Psi_1(z_1, z_2) = (f \circ \pi_1)(z_1, z_2) = f(z_1 z_2, z_2) = z_2^3 \overline{z}_1 \overline{z}_2 + z_1^5 z_2^5 \overline{z}_2 = z_2^3 \overline{z}_2 (\overline{z}_1 + z_1^5 z_2^2).$$

In the total transform $\Psi_1(z_1, z_2) = 0$, the factor $z_2^3 \overline{z}_2$ corresponds to the equation of $E_1 := \{z_2 = 0\}$, while $(\overline{z}_1 + z_1^5 z_2^2) = 0$ is the equation of a smooth branch S_1 of the strict transform of f, namely the complex curve with equation $z_1 = 0$.

In the second chart of X_1, i.e. over $\mathbb{C}^2 \setminus \{z_1 = 0\}$, the blow-up is given by $z_1 \mapsto z_1$ and $z_2 \mapsto z_1 z_2$; E_1 has equation $z_1 = 0$ and

$$(f \circ \pi_1)(z_1, z_2) \;=\; z_1^3 z_2^3 \overline{z}_1 + z_1^5 \overline{z}_1 \overline{z}_2 \;=\; z_1^3 \overline{z}_1 (z_2^3 + z_1^2 \overline{z}_2) \;.$$

We notice that $z_2^3 + z_1^2 \overline{z}_2 = 0$ is the equation of a singular real surface S, so we have to resolve this singularity.

One branch of S as equation $z_2 = 0$. Unfortunately, the presence of both z_2 and $\overline{z}_2$ in the equation of S does not allow one to factorize neither z_2 nor $\overline{z}_2$. Therefore it is useless trying to separate the branch $z_2 = 0$ from the other branches of S by performing additional blow-ups. Indeed, let us try an additional blow-up $\pi' : X' \to X$. In the second chart we have,

$$(\Psi_1 \circ \pi')(z_1, z_2) = \Psi_1(z_1, z_1 z_2) = z_1^5 \overline{z}_1 (z_2^3 z_1 + \overline{z}_1 \overline{z}_2) \,,$$

and the factor $(z_2^3 z_1 + \overline{z}_1 \overline{z}_2)$ still involves z_2 and $\overline{z}_2$ with the same exponents as before, so this singularity can not be resolved by blow-ups.

Thus we start again with the term $z_2^3 + z_1^2 \overline{z}_2$, defining S, and we make a trick: we compose π_1 with a orientation-preserving homeomorphism θ : $X_1 \to X_1$ in order to replace S by a complex plane curve. So we start with the term $(z_2^3 + z_1^2 \overline{z}_2)$ and we write it as $\overline{z}_2 ((\frac{z_2}{|z_2|^{\frac{1}{2}}})^4 + z_1^2)$; now define the map $\theta : X_1 \to X_1$ by:

$$\theta(z_1, z_2) = (z_1, \frac{z_2}{|z_2|^{\frac{1}{2}}})$$

in the second chart. This is well defined away from the two complex lines transverse to E_1 with equations $z_2 = 0$ and $z_2 = \infty$ and it extends in the obvious way to a homeomorphism from X_1 to X_1. This homeomorphism coincides with the identity map on the two lines and it is a diffeomorphism on their complement. In the second chart, the inverse map is $\theta^{-1}(z_1, z_2) = (z_1, z_2|z_2|)$, and the image of S by θ has equation:

$$\overline{z}_2|z_2|({z_2}^4 + {z_1}^2) = 0,$$

or equivalently $\overline{z}_2({z_2}^4 + {z_1}^2) = 0$. The term ${z_2}^4 + {z_1}^2$ defines a complex analytic plane curve, which can be resolved by a finite sequence of blow-ups of points in the usual way. Let $\pi_2 : X_2 \longrightarrow X_1$ be the blow-up of the point $(z_1, z_2) = (0, 0)$ of X_1. In the second chart, the exceptional divisor $E_2 = \pi_2^{-1}(0, 0)$ has equation $z_1 = 0$ and the strict transform of $\theta(S)$ by π_2 has equation:

$$\overline{z}_2\,|z_2|\,(z_1^2 z_2^4 + 1) = 0.$$

The factor $\overline{z}_2$ corresponds to a smooth branch S_2 of the strict transform of f by $f \circ \pi_1 \circ \theta^{-1} \circ \pi_2$. The term $z_1^2 z_2^4 + 1 = 0$ does not intersect the exceptional divisor, so it has no contribution for the topology of L. In the first chart E_2 has equation $z_2 = 0$ and the strict transform of $\theta(S)$ by π_2 has equation

$${z_2}^2 + {z_1}^2 = 0,$$

which corresponds to the equation of two transverse smooth complex curves S_3 and S_4, which are separated by performing the blow-up $\pi_3 : X_3 \to X_2$ of their common point.

Therefore, if we let $\pi = \pi_3 \circ \pi_2 \circ \theta \circ \pi_1$, then the total transform $(f \circ \pi)^{-1}(0)$ has normal crossings and the strict transform $\overline{\pi^{-1}(f^{-1}(0) \setminus \{0\})}$ consists of the four smooth curves $S_i, i = 1, \ldots, 4$. The configuration of the divisor $(f \circ \pi)^{-1}(0)$ is represented on figure 3, each irreducible compact component E_j being weighted by its self intersection in X.

Let us now identify $\pi^{-1}(\mathbb{S}^3_\epsilon)$ with a small tubular neighborhood W of $\pi^{-1}(0)$ in X obtained by a plumbing process. The link $L = f^{-1}(0) \cap \mathbb{S}^3_\epsilon$ is, up to isotopy, the intersection of $S_1 \cup S_2 \cup S_3 \cup S_4$ with the boundary of W. Therefore L has four components $K_i = S_i \cap \partial W, i = 1, \ldots, 4$, and its isotopy class is encoded in the dual plumbing graph Γ of the divisor $(f \circ \pi)^{-1}(0)$, also represented on figure 3. As this graph has a single *rupture vertex* (i.e. a vertex with more than three incident edges or arrows), then the link L is a Seifert link, as we already known from [S2]. By using the plumbing calculus of [Ne1], one computes from Γ the Seifert graph $G(\mathbb{S}^3, L)$, also represented on figure 3.

Before computing the degrees $\epsilon(\overrightarrow{K}_{i\,flow})$, let us introduce some definitions and make some remarks.

Let C be an irreducible component of the total transform $(f \circ \pi)^{-1}(0)$. As in [LMW; 1.3.2], define a *curvette* of C as a smooth complex curve in X intersecting transversally C at a smooth point of $(f \circ \pi)^{-1}(0)$. One defines the multiplicity $m(C)$ of f along C as the degree of the restriction of f to a curvette of C.

Remember that in a neighbourhood of S_1 in X, $f \circ \pi$ has the following local expression :

$$(f \circ \pi)(z_1, z_2) = (f \circ \pi_1)(z_1, z_2) = z_2^3 \overline{z}_2 (\overline{z}_1 + z_1^5 z_2^2),$$

where $z_2 = 0$ is the equation of E_1 and where $\overline{z}_1 + z_1^5 z_2^2 = 0$ is that of S_1. Therefore $m(E_1)$ is the degree of the map $z_2 \mapsto z_2^3 \overline{z}_2$, that is $m(E_1) = 3 - 1 = +2$, and $m(S_1)$ is the degree of $z_1 \mapsto \overline{z}_1$, so $m(S_1) = -1$. Similarly, $m(S_2) = -1$, and $m(S_3) = m(S_4) = +1$ as S_3 and S_4 appear through holomorphic factors in the local expression of $f \circ \pi$.

Now, one remarks that, as in the usual resolution of complex plane curves, the multiplicity of a compact irreducible component of the divisor created by the blow-up of a point P is the sum of the multiplicities of the components of the total transform passing through P. Therefore $m(E_2) = m(E_1) + m(S_2) + m(S_3) + m(S_4) = 3$, and $m(E_3) = m(E_1) + m(E_2) + m(S_3) + m(S_4) = 7$.

At this step, we can compute already the order of the periodic monodromy h of $\frac{f}{\|f\|}$. Indeed, by definition, the periodic monodromy is the diffeomorphism of first return of a Seifert fibre of on a fibre of $\frac{f\circ\pi}{\|f\circ\pi\|}$. Therefore, its degree is equal, up to sign, to the degree of $\frac{f\circ\pi}{\|f\circ\pi\|}$ restricted to a regular Seifert fibre of (S_3, L) disjoint from L. But one of the main ideas of the plumbing calculus is that a regular Seifert fibre of (S_3, L) is, up to isotopy, the intersection with ∂W of a curvette γ of the rupture component of the exceptional divisor (i.e. that which corresponds to the rupture vertex of the dual resolution graph). Then the order N of h is, up to sign, the degree of the restriction of f to a curvette γ of E_3, i.e. $N = m(E_3) = 7$.

Let us now compute the degree $\epsilon(\overrightarrow{K_1}_{flow})$. Remember again that in a neighbourhood of S_1,

$$(f \circ \pi)(z_1, z_2) = z_2^3 \overline{z}_2 (\overline{z}_1 + z_1^5 z_2^2).$$

In this local chart, $W = \{(z_1, z_2)\ ;\ |z_2| \leqslant \eta\}$, where $\eta << 1$. Thus,

$$K_1 = S_1 \cap \partial W = \{(z_1, z_2)\ ;\ z_1 = 0,\ |z_2| = \eta\}.$$

Let $\overrightarrow{K_1}_{\mathbb{C}}$ be the knot K_1 oriented as the boundary of the complex curve $S_1 \cap W$. Let us compute first $\epsilon(\overrightarrow{K_1}_{\mathbb{C}})$. By definition, it is the degree of the restriction of $(f \circ \pi)/|f \circ \pi|$ to a small meridian of K_1, which is nothing but the degree of the restriction of $f \circ \pi$ to a curvette of S_1. Therefore $\epsilon(\overrightarrow{K_1}_{\mathbb{C}}) = m(S_1) = -1$.

Let us now compare the orientations $\overrightarrow{K_1}_{\mathbb{C}}$ and $\overrightarrow{K_1}_{flow}$. Let γ be a curvette of E_3 and let us orient the Seifert fibre $b = \partial W \cap \gamma$ as the boundary of the complex curve $W \cap \gamma$. As $m(E_3) = 7$ is positive, $\overrightarrow{b}_{\mathbb{C}} = \overrightarrow{b}_{flow}$. Moreover, by plumbing calculus, one knows that the orientation of $\overrightarrow{K_1}_{\mathbb{C}}$ as a Seifert fibre is comppatible with that of $\overrightarrow{b}_{\mathbb{C}}$. Therefore, $\overrightarrow{K_1}_{\mathbb{C}} = \overrightarrow{K_1}_{flow}$, and then, $\epsilon(\overrightarrow{K_1}_{flow}) = m(S_1) = -1$.

Similarly, $\epsilon(\overrightarrow{K_2}_{flow}) = m(S_2) = -1$; $\epsilon(\overrightarrow{K_3}_{flow}) = m(S_3) = +1$ and $\epsilon(\overrightarrow{K_4}_{flow}) = m(S_4) = +1$.

Using the Proposition of Section 1, one computes the Nielsen graph $\mathcal{G}(h)$ of the monodromy of $\frac{f}{\|f\|}$ from the Seifert graph $G(\mathbb{S}^3, L)$ and from the degrees $\epsilon(\overrightarrow{K_i}_{flow})$, $i = 1, \ldots, 4$. This is represented on figure 3. In particular, one can recover that the order of the monodromy is 7 from the formula i) of Proposition 1.2.

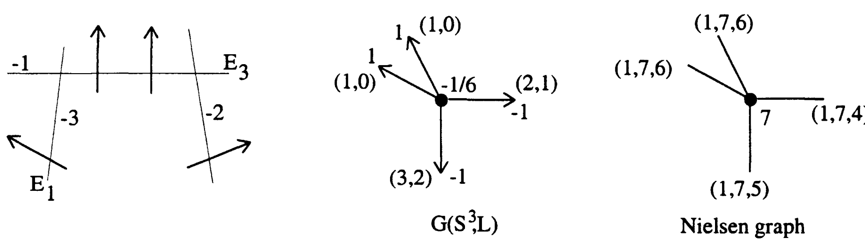


Fig. 3

3. The general case

The arguments of the previous section generalize to the following result:

3.1. Theorem

Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ *be the real analytic germ defined by*

$$f(z_1, z_2) = z_1^p \overline{z}_2 + z_2^q \overline{z}_1,$$

with p *and* q *integers,* $p \geqslant q \geqslant 2$. *Then:*

i) The link $L = f^{-1}(0) \cap \mathbb{S}^3$ is a Seifert link with $k+2$ components, where $k = \gcd(p+1, q+1)$. Two of these components are the Hopf link $(\{z_1 z_2 = 0\}) \cap S^3$, while the others are a torus link of type $(p+1, q+1)$.

ii) The degree $\epsilon(\overrightarrow{K}_{flow})$ equals -1 for the two components of the Hopf link, and $+1$ for each of the k remaining components.

iii) The monodromy of the fibration $\frac{f}{\|f\|} : \mathbb{S}^3 \setminus L \longrightarrow S^1$ has a periodic representative h whose order is $N = kp'q' - p' - q' = \frac{1}{k}(pq-1)$, where $p' = \frac{p+1}{k}$ and $q' = \frac{q+1}{k}$.

iv) Each fiber $F_\theta = (\frac{f}{\|f\|})^{-1}(e^{i\theta})$ has genus $\frac{1}{2}k(N-1) = \frac{1}{2}(pq-1-k)$.

v) The plumbing graph of $(\mathbb{S}^3, L)$, the Seifert graph $G(\mathbb{S}^3, L)$ and the Nielsen graph of h are represented on figure 4, where $p'\sigma_1 - N\beta_1 - 1 = 0$ and $q'\sigma_2 - N\beta_2 - 1 = 0$.

We remark that we stated the theorem above considering the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ for simplicity, but one can replace this by any sphere centered at 0, of arbitrary positive radius, by [S2; Proposition 2.1]. Also, that same result gives us an explicit representative of the monodromy of this fibration: this is given by the map $(z_1, z_2) \mapsto (e^{\frac{2\pi i(q+1)}{pq-1}} z_1 \, , e^{\frac{2\pi i(p+1)}{pq-1}} z_2)$.

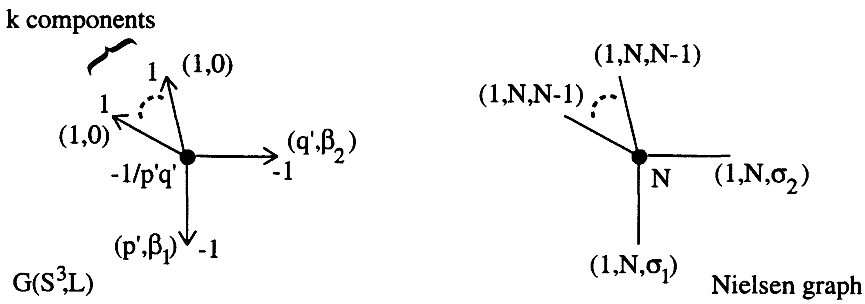


Fig. 4

Proof.— Let $\pi_1 : X_1 \to \mathbb{C}^2$ be the blow-up of $0_{\mathbb{C}^2}$. In the first chart, $C^2 \setminus \{z_2 = 0\}$, one has:

$$(f \circ \pi_1)(z_1, z_2) = f(z_1 z_2, z_2) = z_2^q \overline{z}_2 (\overline{z}_1 + z_1^p z_2^{p-q}),$$

and $\overline{z}_1 + z_1^p z_2^{p-q} = 0$ is the equation of a smooth branch S_1 of the strict transform of f, namely the complex curve with equation $z_1 = 0$. Its multiplicity is the degree of the map $z_1 \mapsto \overline{z}_1$, then $m(S_1) = -1$. In the second chart one has,

$$(f \circ \pi_1)(z_1, z_2) = z_1^q \overline{z}_1 \, (z_2^q + z_1^{p-q} \overline{z}_2) \,,$$

and, just as in Section 2, it is useless to keep making additional blow-ups to resolve the singularity:

$$S = \{ z_2^q + z_1^{p-q} \overline{z}_2 = 0 \} \,,$$

due to the presence of both z_2 and $\overline{z}_2$ in the equation. Thus we make the same trick we did before. Let us write

$$z_1^q \overline{z}_1 \, (z_2^q + z_1^{p-q} \overline{z}_2) = z_1^q \overline{z}_1 \overline{z}_2 \, \big((\frac{z_2}{|z_2|^{\frac{2}{q+1}}})^{q+1} + z_1^{p-q} \big) \,.$$

We now compose π_1 with the homeomorphism $\theta : X_1 \to X_1$ defined in the second chart by

$$\theta(z_1, z_2) = (z_1, \frac{z_2}{|z_2|^{\frac{2}{q+1}}}),$$

out of the two complex lines $z_2 = 0$ and $z_2 = \infty$; this extends as the identity map on these two lines. In the second chart, the inverse map is $\theta^{-1}(z_1, z_2) = (z_1, z_2 |z_2|^{\frac{2}{q-1}})$, and $\theta(S)$ has equation:

$$\overline{z}_2 |z_2|^{\frac{2}{q-1}} ({z_2}^{q+1} + {z_1}^{p-q}) = 0 \,,$$

or equivalently $\overline{z}_2 ({z_2}^{q+1} + {z_1}^{p-q}) = 0$. This is the equation of a complex plane curve, so it can be resolved by a finite sequence $\pi' : X \longrightarrow X_1$ of blow-ups of points by the classical way. As in Section 2, after one blow-up the term $\overline{z}_2$ gives rise to a smooth branch S_2 transverse to the new component of the exceptional divisor which has multiplicty $m(S_2) = -1$. After a finite sequence of additional blow-ups one obtains $k := \gcd(p+1, q+1)$ other branches $S_3, \ldots, S_{k+2}$, all transverse to the same component C of the exceptional divisor. To identify the link in $\mathbb{S}^3$ defined by these k components of L we observe that in the chart $\{z_1 \neq 0\}$ of the blow-up X_1, the equation $z_2^{q+1} + z_1^{p-q} = 0$ is the equation of the strict transform of the holomorphic curve $z_2^{q+1} + z_1^{p+1} = 0$, whose link is well known to be a torus link of type $k(\frac{p+1}{k}, \frac{q+1}{k})$, by [Br] (see also [Mil]). Therefore, identifying $\pi^{-1}(\mathbb{S}^3_\epsilon)$ with a tubular neighborhood W of the divisor $\pi^{-1}(0)$, the link L has $k+2$ components $K_i = S_i \cap \partial W$, two of them, K_1 and K_2 consisting the Hopf link, and the remaining components $K_3, \ldots, K_{k+2}$ consisting a torus link of type $k(\frac{p+1}{k}, \frac{q+1}{k})$.

This determines the resolution (plumbing) graph of the singularity. Again, using classical computations of Seifert invariants and the plumbing calculus of [Ne1], one obtains from this the Seifert graph $G(\mathbb{S}^3, L)$: there are two exceptional Seifert fibres, which are K_1 and K_2, with Seifert invariants respectively $\alpha_1 = (p+1)/k = p'$ and $\alpha_2 = (q+1)/k = q'$. As the ambient space is the 3-sphere, then the rational Euler class of the Seifert fibration is $e_0 = -\frac{1}{p'q'}$ and the two classes $\beta_1 p'$ and $\beta_2 q'$ are related by $p'\beta 2 + q'\beta_1 = 1$ (see [JN] for more details).

As in Section 2, one obtains the weights $\epsilon(\overrightarrow{K_i}_{flow}), i = 1, \ldots, k+2$ by computing the multiplicities $m(S_i), i = 1, \ldots, k+2$ of the branches of the strict tranform of f by π. These are $\epsilon(\overrightarrow{K_1}_{flow}) = \epsilon(\overrightarrow{K_2}_{flow}) = -1$, and $\epsilon(\overrightarrow{K_i}_{flow}) = +1, i = 3, \ldots, k+2$

As in Section 2, we can already compute the order N of the periodic monodromy of $f/|f|$ by computing the multiplicity of the rupture component of the exceptional divisor. In the complex case $z_1 z_2 (z_1^{k+1} + z_2^{k+1})$ this multiplicity equals $kp'q' + p' + q'$, the term $kp'q'$ coming from the k branches of $z_1^{k+1} + z_2^{k+1}$, and the terms p' and q' from the two branches z_1 and z_2. In our real case, the computations of the multiplicities following the sequence of blow-ups are the same as in the complex case, except that the multiplicities of the branches corresponding to the Hopf link (i.e. the branches z_1 and z_2) are counted negatively. Therefore, $N = kp'q' - p' - q'$. Using Section 1, one obtains the Nielsen graph and the genus of the fiber. In particular, one can recover the order N of the monodromy from the formula i) of Proposition 1.2. We notice that in this case, the corresponding Seifert decomposition of the 3-sphere has only two exceptional fibers (the Hopf link), and both of them are components of L. Hence, in the Nielsen graph of the monodromy, all the stalks are boundary-stalks. □

In order to state the following result we notice that the function $f(z_1, z_2) = z_1^p \bar{z}_2 + z_2^q \bar{z}_1$ in Theorem 1 can be regarded as the Hermitian product of the vector fields $\xi = (z_2^q, z_1^p)$ and $\iota = (z_1, z_2)$.

3.2. Corollary

With the hypothesis and notation of Theorem 3.1, each pair of antipodal fibers F_θ and $F_{\theta+\pi}$ is naturally glued together along the link L forming a smooth real analytic surface S_θ in $\mathbb{S}^3$. The genus of S_θ is $p \cdot q$, the Poincaré-Hopf index at 0 of the vector field ξ , and S_θ is diffeomorphic to the set of points where the real line field spanned by ξ is tangent to $\mathbb{S}^3$.

Proof. — That the fibers F_θ and $F_{\theta+\pi}$ are naturally glued together along the link L forming a smooth real analytic surface S_θ in $\mathbb{S}^3$ is proved in [S2], where it is also shown that this surface is diffeomorphic to the set of points where the real line field spanned by ξ is tangent to $\mathbb{S}^3$. That the genus of this surface is pq follows from Theorem 1, which says that each fibre F_θ has genus $\frac{1}{2}(pq-1-k)$ and it has $k+2$ boundary components. When we glue two such fibers along their boundary to get S_θ, we create $k+1$ handles. Hence the genus of S_θ is:

$$g(S_\theta) = 2g(F_\theta) + (k+1) = p \cdot q\,,$$

as stated. Finally, that this number is the Poincaré-Hopf index of the vector field ξ follows from the fact that this vector field is holomorphic, so its index at 0 equals the dimension of the vector space $\mathcal{O}_{\mathbb{C}^2,0}/(z_2^q, z_1^p)$. $\square$

4. The special cases

In this section we study the special cases when, at least, one of the two exponents p and q is 1. There are two essentially different possibilities: $p = q = 1$, and $p > q = 1$.

As before, we let $\pi_1 : X_1 \longrightarrow \mathbb{C}^2$ be the blow-up of 0_{C^2} and let us identify S^3_ϵ with the boundary of a small semi-algebraic neighborhood W of the exceptional divisor $E_1 = \pi_1^{-1}(0)$ in X_1, with equation say $|z_2| \leqslant \eta$, $(0 < \eta << 1)$ in the first chart, and $|z_1| \leqslant \eta$ in the second chart. Let us equip ∂W with the Hopf fibration $\partial W \longrightarrow E_1$, given by $(z_1, z_2) \longmapsto z_1$ in the first chart.

a) Consider first the case: $f(z_1,z_2) = \lambda_1 z_1 \cdot \overline{z}_2 + \lambda_2 z_2 \cdot \overline{z}_1$. By [RSV], if $|\lambda_1| = |\lambda_2|$, then f is not a submersion in a punctured neighborhood of $0 \in \mathbb{C}^2$ and the corresponding link $L = f^{-1}(0) \cap \mathbb{S}^3_\varepsilon$ is not fibered. In fact, this link is not even a 1-dimensional manifold. Indeed, without lost of generality, one can assume that $\lambda_1 = \lambda_2 = 1$; in the first coordinate chart of X_1,

$$(f \circ \pi_1)(z_1,z_2) = f(z_1 z_2, z_2) = z_2 \overline{z}_2(\overline{z}_1 + z_1).$$

Then the strict transform of f by π_1 has equation $\overline{z}_1 + z_1 = 0$ in this chart, and by symmetry, $\overline{z}_2 + z_2 = 0$ in the second chart. Therefore it is a real 3-dimensional manifold transverse to the boundary of W and its intersection with ∂W is homeomorphic to L. Then L is a real surface embedded in ∂W, namely an unknotted torus saturated by Hopf fibres.

If $|\lambda_1| \neq |\lambda_2|$ the situation is more interesting. In this case, and in the case b) below, [RSV] implies that $\frac{f}{\|f\|}$ gives rise to a Milnor fibration on the complement of a regular neighbourhood $N(L)$ of L in the 3-sphere, but the methods used in that article do not describe the behaviour of $\frac{f}{\|f\|}$ near the link L. In particular, it could not be decided in [RSV] whether or not the map $\frac{f}{\|f\|}$ defines an open-book decomposition of $\mathbb{S}^3_\varepsilon$.

Over the first chart $\{z_2 \neq 0\}$ one has, $(f\circ\pi_1)(z_1, z_2) = z_2\overline{z}_2(\lambda_1 z_1+\lambda_2\overline{z}_1)$. This gives one branch S_1 of the strict transform with equation $z_1 = 0$. In the second chart, one obtains by the same way a second branch S_2 with equation $z_2 = 0$. Therefore the link L is the Hopf link. Moreover, in the first chart,

$$\left(\frac{f}{\|f\|}\circ\pi_1\right)(z_1, z_2) = \frac{\lambda_1 z_1 + \lambda_2\overline{z}_1}{|\lambda_1\overline{z}_1 + \lambda_2 z_1|}.$$

Then for $\alpha \in S^1$, the inverse image by π_1 of the fiber $(\frac{f}{\|f\|})^{-1}(\alpha)$ is an annulus with boundary L, which is saturated with the fibres of the Hopf fibration corresponding to $\arg(z_1) = \theta$, where θ is a constant depending only on α. Therefore $\frac{f}{\|f\|}$ is a fibration which gives rise to an open-book decomposition of $\mathbb{S}^3$ with binding the Hopf link L and whose pages are annuli.

In the present case, it is not possible to study the monodromy using Section 1 because the Seifert fibres are not transversal to the fibers of $\frac{f\circ\pi_1}{|f\circ\pi_1|}$. For the same reason, the orientations $\overrightarrow{K_i}_{flow}$ are not defined.

However, take $\partial W = \{(z_1, z_2); |z_2| = \eta\}$ (expressed in the first chart), where $\eta \in \mathbb{R}^*$ is fixed. Then $L = (S_1 \cup S_2) \cap \partial W$, and the map

$$\Phi : \partial W \setminus L \longrightarrow]0, +\infty[\times\mathbb{S}^1_\eta$$

defined by $\Phi(z_1, z_2) = (|z_1|, z_2)$ is a locally trivial fibration whose fibres are circles, each of them intersecting transversally each fibre of Φ at a single point. Therefore the monodromy of $\frac{f}{|f|}$ is the identity map.

Finally, without lost of generality one can assume that $|\lambda_2| > |\lambda_1|$. Then, $m(S_1) = -1$, i.e. $\epsilon(\overrightarrow{K_1}_{\mathbb{C}}) = -1$, and $m(S_2) = +1$, i.e. $\epsilon(\overrightarrow{K_2}_{\mathbb{C}}) = +1$ if we endow these knots with orientation as in Section 2.

b) Consider now the case $f(z_1, z_2) = \lambda_1 z_1^p\overline{z}_2+\lambda_2 z_2\overline{z}_1$, $p \geqslant 2$. In this case [RSV] shows that one has an associated Milnor fibration on the complement of a tubular neighbourhood $N(L)$ of the link, provided we restrict to spheres of radius less than $|\lambda_2|/|\lambda_1|$. Let us describe the behaviour of $\frac{f}{|f|}$ on $N(L)$ and show that this map indeed gives an open book decomposition of $\mathbb{S}^3$. For this, we first describe the link of this singularity using blow-ups. In the first chart of the blow-up X_1 one has,

$$(f \circ \pi_1)(z_1, z_2) = z_2\overline{z}_2(\lambda_2\overline{z}_1 + \lambda_1 z_1^p z_2^{p-1}),$$

and in the second one,

$$(f \circ \pi_1)(z_1, z_2) = z_1\bar{z}_1(\lambda_2 z_2 + \lambda_1 z_1^{p-1}\bar{z}_2)\,.$$

Then the strict transform of f by π_1 has two branches S_1 and S_2, S_1 having equation $\lambda_2\bar{z}_1 + \lambda_1 z_1^p z_2^{p-1} = 0$ in the first chart, i.e. $z_1 = 0$, and S_2, $\lambda_2 z_2 + z_1^p z_1^{p-1}\bar{z}_2 = 0$ in the second chart, i.e. $z_2 = 0$. Let us set $K_1 = S_1 \cap \partial W$ and $K_2 = S_2 \cap \partial W$. Therefore $L = K_1 \cup K_2$ is the Hopf link. Let us describe the behaviour of the fibers of $\frac{f}{|f|}$ near the link L.

Let T_1 be a small solid torus, regular neighbourhood of L_1 in ∂W. Say

$$T_1 = \{(z_1, z_2) \mid |z_1| \leqslant \eta', |z_2| = \eta\}\,,$$

expressed in the first chart, where $0 < \eta' << \eta << 1$ and where $\lambda_1\eta^{p-1} < \lambda_2$. Let us study the restriction of $(\frac{f}{\|f\|} \circ \pi_1)$ to a meridian disk D of T_1, with equation say $z_2 = w_2$, where w_2 is fixed on $\mathbb{S}^1_\eta$. Then, for all $(z_1, w_2) \in D$,

$$\left(\frac{f}{\|f\|} \circ \pi_1\right)(z_1, w_2) = \frac{\lambda_2\overline{z_1} + (\lambda_1 w_2^{p-1})z_1^{p-1}}{|\lambda_2\overline{z_1} + (\lambda_1 w_2^{p-1})z_1^{p-1}|}$$

Then for any $\alpha \in S^1$ the intersection of $(\frac{f}{\|f\|} \circ \pi_1)^{-1}(\alpha)$ with D is a radial arc with equation $arg(z_1) = \theta$, where θ depends on α and w_2. Thus the restriction of $\frac{f}{\|f\|} \circ \pi_1$ to T_1 is the projection of an open-book fibration. Moreover, as $\eta' < 1$ and $\lambda_1\eta^{p-1} < \lambda_2$, the degree of f restricted to the boundary of D is that of $z_1 \mapsto \lambda_2\overline{z_1}$, then $\epsilon(\overrightarrow{K_{1\mathbb{C}}}) = -1$. By the same arguments, one obtains that the restriction of $(\frac{f}{\|f\|} \circ \pi_1)$ to a small solid torus T_2, regular neighbourhood of L_1 in ∂W, is also open-book, and that $\epsilon(\overrightarrow{K_{2\mathbb{C}}}) = +1$. Finally, the degree of $f \circ \pi_1$ restricted to a curvette of E_1 is zero. Therefore, the fibres of $\frac{f}{\|f\|} \circ \pi_1$ are vertical in the sense of Waldhausen. This means that each of them is, up to isotopy, union of Seifert fibres. Then they are annuli and the monodromy of $\frac{f}{\|f\|}$ is the identity map by the same arguments as in case a).

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