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AIRTON S. DE MEDEIROS

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Singular foliations and differential p-forms (*)

AIRTON S. DE MEDEIROS (1)

RÉSUMÉ. — Nous introduisons la notion de p-formes différentielles intégrables afin d'étudier la scructure des feuilletages singuliers de codimension p. Nous commençons cette étude dans ce papier en présentant quelques résultats nouveaux.

ABSTRACT. — We introduce the integrable differential p-forms in order to present an adequate analytic object to study the structure of the codimension p singular foliations, and we do start this study throughout the results established in the paper.

0. Introduction

The ideas developed in this article were originally motivated by the study of the structural stability of integrable differential 1-forms (see [1] and [5]). It was very surprising to us to find out that the structural stability of integrable systems of p 1-forms, for $p > 1$, seemed to be a problem of a completely different nature: The singular foliations associated to the structurally stable systems followed an entirely unusual pattern compared to those associated to a single integrable 1-form. Indeed, take for instance $p = n - 1 > 1$. Then, locally, the singular set of such a system (the points where the forms are linearly dependent) is generically a submanifold of codimension two (see Lemma 1.2.1). In particular, the singular foliation determined by the orbits

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(1) Instituto de Matemática, UFRJ, Departamento 05, Ilha do Fundão, Rio de Janeiro, Brazil.
e-mail: airton@impa.br

of a vector field possessing a simple singularity never appears in association with a stable integrable system, contrasting drastically with the case $p = 1$.

This apparent pathology, seen from another point of view, does actually tell us that integrable systems of 1-forms are not the adequate objects to describe the "natural" singular foliations at large. For that reason we were led to introduce the concept of an integrable p-form (Definition 1.2.1). The first convincing evidence that the integrable p-forms were in fact the natural universe from where the integrable 1-forms were taken, was the confirmation that the "Fundamental Lemma for Integrable 1-forms" did actually generalize for integrable p-forms (Proposition 1.3.1). Some other evidences appear along this article (e.g. Remark 1.3.1 and Proposition 4.1).

For conceptual reasons we have introduced the LDS p-forms (Definition 1.2.1). This makes more transparent the algebraic, and the analytic, characters inherent to the various problems under consideration.

It is worth mentioning that there exist singular codimension p plane fields (Remark 3.2.2) which cannot be defined, locally, neither by a system of p 1-forms nor by a set of $n - p$ vector fields.

The uniform singular foliations (Definition 1.3.1) arise by following the natural course of the subject. The relation between the local structure of such foliations and the intrinsic properties of their defining p-forms turns out to be our main concern in the sequel.

In Theorem A we describe completely the foliations $\mathcal{F}$ induced by linear integrable p-forms on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$): Roughly speaking, either $\mathcal{F}$ is the product of the orbits of a linear vector field on $\mathbb{K}^{p+1}$ by a complementary $\mathbb{K}^{n-p-1}$, or $\mathcal{F}$ admits a very simple set of p first integrals, and in this case the foliation is obtained by slicing the level surfaces of a quadratic function in $\mathbb{K}^n$ (one of the p first integrals) by parallel $(n-p+1)$-planes (the leaves of the regular foliation defined by the remaining first integrals). In Corollary 3.2.1 we show that a completely analogous description holds locally, for foliations defined by a holomorphic p-form, around a singularity where the linear part of the form has a singular set of codimension greater than two. This corollary is basically a consequence of a more general result (Theorem C) that characterizes locally, among the holomorphic uniform foliations with singular set of codimension greater than two, those possessing the simple structures described above, around a singularity.

Finally, in §4 we discuss the extension to the real field of the results we have established before concerning holomorphic LDS, and integrable, p-forms.

1. Preliminaries

1.1. Settling the notation

Let M be a differentiable manifold (real or complex). We shall denote by $\Lambda^p(M)$ (resp. $\mathfrak{X}(M)$) the set of differential p-forms (resp. vector fields) on M. If $N \subset M$ is a submanifold and $i\colon N \to M$ is the inclusion map we shall refer to $i^*\omega$ (the pullback of $\omega \in \Lambda^p(M)$ under i) as the restriction of ω to N, normally denoted by $\omega \mid N$. If $X \in \mathfrak{X}(M)$ and N is X-invariant then, $X \mid N$ has obviously an analogous meaning.

The class of differentiability of these objects (deliberately omitted in the notation) will be supposed to be in accordance with the context where they appear, which will be made precise in the text in that very occasion.

When M is a real manifold we shall denote by $C^r(M)$; $r \in \mathbb{Z}^+ \cup \{\infty\}$ (resp. $\mathcal{A}(M)$) the ring of C^r real functions (resp. real analytic functions) in M. In the case M is complex we shall denote by $\mathcal{O}(M)$ the ring of holomorphic functions in M.

If M turns out to be an open subset $U \subset \mathbb{K}^n$; $\mathbb{K} = \mathbb{R}, \mathbb{C}$ then, $\mathfrak{X}(U)$ is a free module generated by the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{K}^n$. Given $X_1, \dots, X_m \in \mathfrak{X}(U)$ we shall denote by $\mathcal{I}(X_1, \dots, X_m)$ the submodule of $\mathfrak{X}(U)$ generated by the X_i; $i = 1, \dots, m$.

Similarly, $\Lambda^1(U)$ is a free module generated by the dual basis $\{dx_1, \dots, dx_n\}$ of $(\mathbb{K}^n)^*$ and for any $p \geqslant 1$, $\Lambda^p(U)$ is generated by the exterior products $dx_{i_1} \wedge \dots \wedge dx_{i_p}$, $1 \leqslant i_1 < \dots < i_p \leqslant n$. Now, if $\alpha_1, \dots, \alpha_m \in \Lambda^1(U)$ we shall denote by $\mathcal{I}(\alpha_1, \dots, \alpha_m)$ the submodule of $\Lambda^1(U)$ generated by these elements. More generally, $\mathcal{I}^p(\alpha_1, \dots, \alpha_m)$ will denote the submodule of $\Lambda^p(U)$ generated by the exterior products $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_p}$; $1 \leqslant i_1 < \dots < i_p \leqslant m$. When it is clear from the context that ω is a p-form we write simply $\omega \in \mathcal{I}(\alpha_1, \dots, \alpha_m)$ to mean that $\omega \in \mathcal{I}^p(\alpha_1, \dots, \alpha_m)$.

In order to simplify the reading we shall adopt the following multi-index notation:

Given the ordered set $S_n = \{1, 2, \dots, n\}$, a multi-index I of length $|I| = m$ is any ordered subset of S_n with m elements. The ordered complementary set of I in S_n will be denoted by $\widehat{I}$.

If we are given differential forms α_i; $i \in S_n$ and a multi-index I we set $\alpha_I = \alpha_{i_1} \wedge \dots \wedge \alpha_{i_m}$.

Similarly we introduce the following notation for the interior product of the vector fields X_i; $i \in I$ and the differential form ω, $i(X_{i_1})i(X_{i_2})\ldots i(X_{i_m})\omega = i(X_{i_1},\ldots,X_{i_m})\omega = i(X_I)\omega$.

When $|I| = n-1$ then, $I = \widehat{\{j\}}$ for some $1 \leqslant j \leqslant n$. This justifies the following very useful notation $\alpha_I = \alpha_1\wedge\ldots\wedge\widehat{\alpha_j}\wedge\ldots\wedge\alpha_n = \widehat{\alpha_j}$ specifying the index that is deleted in the product. Analogously, for the interior product, we write $i(X_I)\omega = i(X_1,\ldots,\widehat{X_j},\ldots,X_n)\omega = i(\widehat{X_j})\omega$.

Finally, we recall that a singularity of a differential form ω (resp. a vector field X) is any point $x \in M$ where it vanishes. The set of all singular points will be denoted by $\mathrm{Sing}(\omega)$ (resp. $\mathrm{Sing}(X)$).

When ω (resp. X) is analytic we shall write $\mathrm{codim}(\omega)$ (resp. $\mathrm{codim}(X)$) to refer to the codimension of the analytic subset $\mathrm{Sing}(\omega)$ (resp. $\mathrm{Sing}(X)$).

1.2. The basic definitions and some elementary results

Accordingly as pointed out in the introduction, the integrable systems of p 1-forms, $p > 1$, have shown to be insufficient to generalize adequately some phenomena concerning the single integrable 1-forms, in the sense specified there. The main argument presented to justify this assertion, is an immediate consequence of the following

LEMMA 1.2.1. — *Let $\alpha_1,\ldots,\alpha_p \in \Lambda^1(M)$ and $x_0 \in \mathrm{Sing}(\alpha_1\wedge\ldots\wedge\alpha_p)$. Then, there exist a neighborhood $U(x_0) \subset M$ and $\tilde\alpha_1,\ldots,\tilde\alpha_p \in \Lambda^1(U)$, arbitrarily close to $\alpha_1,\ldots,\alpha_p$ on U, such that $x_0 \in \widetilde{S} = \mathrm{Sing}(\tilde\alpha_1\wedge\ldots\wedge\tilde\alpha_p)$ and $\widetilde{S}$ is a submanifold of codimension $n-p+1$ in some neighborhood of x_0.*

Proof. — By taking into account its local character we may assume that the α_i are restricted to some compact neighborhood U of $x_0 \in \mathbb{K}^n$.

By reordering the indexes we may suppose that $\eta(x_0) = \alpha_1(x_0)\wedge\ldots\wedge\alpha_q(x_0) \neq 0$ $(0 \leqslant q < p)$ and that $\eta(x_0)\wedge\alpha_j(x_0) = 0$ for $j = q+1,\ldots,p$. For $q = 0$ we set $\eta = 1$. Now let $\sigma_{q+1},\ldots,\sigma_{p-1} \in (\mathbb{K}^n)^*$ be such that $\eta(x_0)\wedge\sigma_{q+1}\wedge\ldots\wedge\sigma_{p-1} \neq 0$. Given $\varepsilon > 0$, define $\tilde\alpha_i$, for $i = 1,\ldots,p-1$, by: $\tilde\alpha_i = \alpha_i$ if $i \leqslant q$ and $\tilde\alpha_i = \alpha_i + \varepsilon\sigma_i$ otherwise. It follows that $\tilde\eta(x_0) = \tilde\alpha_1(x_0)\wedge\ldots\wedge\tilde\alpha_{p-1}(x_0) \neq 0$ and that $\tilde\eta(x_0)\wedge\alpha_p(x_0) = 0$. Consequently, $\tilde\eta(x) \neq 0$ in some neighborhood $V(x_0) \subset U$ and therefore the $\tilde\alpha_i$ define a subbundle Σ of the cotangent bundle $T^*(V) \subset T^*(U)$. On the other hand, the set $V \cap \mathrm{Sing}(\tilde\eta\wedge\alpha_p)$, which contains x_0, is nothing but the projection on V of the intersection of Σ with the section Γ of T^*U corresponding to α_p. An evident argument on transversality, produces the perturbation $\tilde\alpha_p$ of α_p, on U, with the desired property. $\square$

Henceforth, throughout this paper, we shall be concerned with differential p-forms $\omega \in \Lambda^p(M)$ whose associate spaces $\mathrm{Ker}(\omega(x))$ of $\omega(x)$ (see [3]) define a distribution (resp. an integrable distribution) of planes of codimension p (also referred to as a plane field of codimension p). These forms are characterized in the following

DEFINITION 1.2.1. — *A differential p-form $\omega \in \Lambda^p(M)$ is said to be locally decomposable off the singular set (LDS) (resp. integrable) if for every $x \in M \backslash \mathrm{Sing}(\omega)$ there exist a neighborhood $V(x) \subset M$ and a system (resp. an integrable system) of 1-forms $\alpha_1, \dots, \alpha_p \in \Lambda^1(V)$ such that $\omega \mid V = \alpha_1 \wedge \dots \wedge \alpha_p$.*

We shall give below (Propositions 1.2.1 and 1.2.2) some other characterizations of the LDS and the integrable p-forms. A very useful one is given in terms of the associate system $\mathrm{Ker}^{\perp}(\omega(x))$ of $\omega(x)$ (see [3]) and the space $\mathcal{E}^*(\omega(x))$, defined more generally as follows: If η is an exterior p-form on a vector space E then, $\mathcal{E}^*(\eta) = \{\alpha \in E^* \mid \alpha \wedge \eta = 0\}$.

PROPOSITION 1.2.1. — *The following statements about a p-form $\omega \in \Lambda^p(M)$ are equivalent*

(i) *ω is LDS.*

(ii) *$\omega(x)$ is either of rank p or zero at each $x \in M$.*

(iii) *$i(v_I)\omega \wedge \omega = 0$ for any local frame $\{v_1, \dots, v_n\}$ on M and $|I| = p-1$.*

(iv) *$\mathrm{Ker}^{\perp}(\omega(x)) \subset \mathcal{E}^*(\omega(x))$ for all $x \in M$.*

(v) *$\tau : x \mapsto \tau_x = \mathrm{Ker}(\omega(x))$ is a distribution of planes of codimension p on $M \backslash \mathrm{Sing}(\omega)$.*

PROPOSITION 1.2.2. — *The statements below about a p-form $\omega \in \Lambda^p(M)$ are equivalent*

(i) *ω is integrable.*

(ii) *ω is LDS and $\mathrm{Ker}(d\omega(x) \subset \mathrm{Ker}(\omega(x))$ (or, equivalently, $\mathrm{Ker}^{\perp}(\omega(x)) \subset \mathrm{Ker}^{\perp}(d\omega(x)))$ for all $x \in M \backslash \mathrm{Sing}(d\omega)$.*

(iii) *The distribution τ referred to in (v) of Proposition 1.2.1 is integrable.*

Remark 1.2.1. — These propositions are in fact elementary exercises in exterior algebra. Some immediate, but very useful, consequences are:

(i) The interior product of vector fields by LDS forms, is still an LDS form.

(ii) A closed differential p-form on M is integrable if, and only if, it is LDS.

(iii) If ω is integrable so is $d\omega$.

1.3. Singular Foliations and Plane Fields

The more general definition of singular plane fields and foliations would be the following:

"Let S be a closed subset of the differentiable manifold M. A singular codimension p plane field τ (resp. foliation $\mathcal{F}$) on M, with singular set S, is a pair $\tau = (S, \tau')$ (resp. $\mathcal{F} = (S, \mathcal{F}')$) where τ' (resp. $\mathcal{F}'$) is a codimension p plane field (resp. foliation) on $M \backslash S$."

Of course this general definition is doubtlessly too general. Some regularity around the singular points should be required. For that reason we shall introduce the

DEFINITION 1.3.1. — A codimension p (singular) plane field τ (resp. foliation $\mathcal{F}$) on the differentiable manifold M is said to be uniform if there exists a collection $\{(U_\lambda, \omega_\lambda)\}$ such that:

(i) $\mathcal{U} = \{U_\lambda\}$ is an open covering of M.

(ii) $\omega_\lambda \in \Lambda^p(U_\lambda)$ is LDS (resp. integrable).

(iii) Whenever $V = U_\lambda \cap U_{\lambda'} \neq \phi$ there exists a nonvanishing function g in V such that $\omega_{\lambda'} \mid V = g\, \omega_\lambda \mid V$.

(iv) The restriction of the plane field τ (resp. the foliation $\mathcal{F}$) to each U_λ coincides with the plane field $\tau(\omega_\lambda)$ (resp. foliation $\mathcal{F}(\omega_\lambda)$), induced by ω_λ.

Remark 1.3.1. — In the holomorphic case this regularity condition is "ensured" by the sole assumption that $\mathrm{Sing}(\tau)$ is an analytic subset of codimension greater than one. More precisely: Let τ be a holomorphic singular plane field on the manifold M. If $\mathrm{codim}(\mathrm{Sing}(\tau)) \geqslant 2$ then, there exists a unique uniform plane field $\hat{\tau}$ on M such that $\tau_x = \hat{\tau}_x$ for all $x \in M \backslash \mathrm{Sing}(\tau)$. The proof of this result is entirely analogous to that of singular foliations by curves (see [4], Proposition 1.7).

Henceforward we shall be chiefly concerned with the analysis of the local structure of singular plane fields and foliations around a singular point. This leads naturally to the search of normal forms of the LDS and integrable p-forms around a singularity. The first result in this direction is the Proposition A of [1]. We finish this section with a very simple proof of this result restated here as

PROPOSITION 1.3.1. — *Let $\omega \in \Lambda^p(M)$ be integrable. Then, for every $x \in M \backslash \mathrm{Sing}(d\omega)$ there exists a local system of coordinates around x such that ω reduces to $p+1$ variables.*

Proof. — Clearly we are done if we can show that ω is of constant class $p+1$ on $\widetilde{M} = M \backslash \mathrm{Sing}(d\omega)$ (see [3], Proposition 3.2).

Now, since $d\omega \neq 0$ on $\widetilde{M}$, it follows from Proposition 1.2.2 (iv) that the characteristic system of ω at any point $x \in \widetilde{M}$ is precisely $\mathrm{Ker}^{\perp}(d\omega(x))$ which has, according to Remark 1.2.1 (iii), the constant dimension $p+1$. In other words ω has constant class $p+1$ on $\widetilde{M}$ as desired. □

2. Normal forms of LDS, and integrable, linear p-forms

2.1. A little more notation

Given a vector space E and a subset $X \subset E$ we shall denote by $\langle X \rangle$ the subspace of E spanned by X. If $X = \{v_1, \dots, v_m\}$ we write simply $\langle v_1, \dots, v_m \rangle$.

When we refer to $\mathbb{K}^m \subset \mathbb{K}^n$ it is to be understood that some multi-index I of length m does exist such that $\mathbb{K}^m = \langle e_{i_1}, \dots, e_{i_m} \rangle$. We shall indicate this by writing $\mathbb{K}^m(x_{i_1}, \dots, x_{i_m})$ or $\mathbb{K}^m(x_I)$

Any p-form ω on $\mathbb{K}^m \subset \mathbb{K}^n$ may be thought of as a p-form on $\mathbb{K}^n$ by simply considering the form $\pi^*(\omega)$, where $\pi: \mathbb{K}^n \to \mathbb{K}^m$ is the projection associated with the well-defined decomposition $\mathbb{K}^n = \mathbb{K}^m \oplus \mathbb{K}^{n-m}$. We shall make no distinction between the forms ω and $\pi^*\omega$ as far as ω has to be treated as a form on $\mathbb{K}^n$.

Finally, given a multi-index I, we define $\omega \mid (dx_I) = \omega \mid (dx_{i_1}, \dots, dx_{i_m})$ by: $(\omega \mid (dx_I))(x) = i^*(\omega(x))$ where i denotes the inclusion map $i: \mathbb{K}^m \to \mathbb{K}^n$.

Remark. — It happens very often, especially along the proofs, that the signal preceding some terms present in the formulae, does not indeed affect the conclusions. When this is the case we simply neglect the signal and write $\pm$ before each one of those terms.

2.2. The main result for linear forms

We shall consider LDS (resp. integrable) p-forms in $\Lambda^p(\mathbb{K}^n)$ whose coefficients are $\mathbb{K}$-linear functions. Our purpose is to describe completely these sets of forms, accordingly as precisely stated in

THEOREM A. — *Let ω be a linear LDS (resp. integrable) p-form on $\mathbb{K}^n$. Then, there exists a linear change of coordinates such that ω reduces to one of the following normal forms:*

(i) *$\omega = \alpha \wedge dx_1 \wedge \ldots \wedge dx_{p-1}$ for some linear 1-form α on $\mathbb{K}^n$ (resp. $\omega = df \wedge dx_1 \wedge \ldots \wedge dx_{p-1}$ for some quadratic function f in $\mathbb{K}^n$)*

(ii) *$\omega = \omega \mid (dx_1, \ldots, dx_{p+1})$ i.e. ω reduces to $p+1$ differentials (resp. $\omega = \omega \mid \mathbb{K}^{p+1}(x_1, \ldots, x_{p+1})$ i.e. ω reduces to $p+1$ variables).*

Proof. — We shall consider first the case where ω is only LDS.

Let $x_0 \in \mathbb{K}^n \backslash \mathrm{Sing}(\omega)$ and let $\alpha_1, \ldots, \alpha_p$ be as in Definition 1.2.1. Since ω is linear we must have $\omega = \omega'_{x_0}$, where ω'_{x_0} is the derivative of ω at x_0. By computing this derivative we find $\omega = \omega'_{x_0} = (\alpha_1 \wedge \ldots \wedge \alpha_p)'_{x_0} = \sum\limits_{i=1}^{p} \alpha_1(x_0) \wedge \ldots \wedge \alpha'_i(x_0) \wedge \ldots \wedge \alpha_p(x_0)$.

Now we set $(-1)^{i-1} \alpha'_i(x_0) = \pi_i$, which are linear 1-forms, and observe that the constant 1-forms $\alpha_i(x_0)$ are linearly independent once $\omega(x_0) \neq 0$. Then, there exist a linear change of coordinates such that $\omega = \sum\limits_{i=1}^{p} \pi_i \wedge \widehat{dx_i}$.

On the other hand, we notice that the π_i may be chosen to have the particular form $\pi_i = \ell_i \, dx_i + \bar{\pi}_i$, where $\bar{\pi}_i \in \mathcal{I}(dx_{p+1}, \ldots, dx_n)$, and consequently ω may be written in the form

$$\omega = \ell \, dx_I + \sum_{i=1}^{p} \bar{\pi}_i \wedge \widehat{dx_i}$$

where $I = \{1, \ldots, p\}$ and ℓ is some linear function in $\mathbb{K}^n$.

In particular, for any $j \in I$ we have $i(\widehat{e_j})\omega = \pm \ell \, dx_j \pm \bar{\pi}_j$. The condition $i(\widehat{e_j})\omega \wedge \omega = 0$ of Proposition 1.2.1 (iii), implies that $\sum\limits_{i=1}^{p} \bar{\pi}_j \wedge \bar{\pi}_i \wedge \widehat{dx_i} = 0$ and then, $\bar{\pi}_j \wedge \bar{\pi}_i = 0 \;\forall\, i, j \in I$.

If all the $\bar{\pi}_i = 0$, ω is clearly decomposable. For that reason we may suppose that $\bar{\pi}_1 \neq 0$. It follows then, from the equations $\bar{\pi}_1 \wedge \bar{\pi}_i = 0$ and the lemma of division for 1-forms [2], that we have exactly the following possibilities:

(a) There exist a constant 1-form σ and linear functions $\bar{\ell}_i$ such that $\bar{\pi}_i = \bar{\ell}_i\,\sigma$.

(b) There exist constants $c_i \in \mathbb{K}$ such that $\bar{\pi}_i = c_i\bar{\pi}_1\ \forall i \in I$. Clearly $c_1 = 1$.

In the first case ω reduces to $p+1$ differentials. In fact $\omega = \ell\, dx_I + \sum\limits_{i=1}^{p} \bar{\ell}_i\, \sigma \wedge \widehat{dx_i}$ and since $\sigma \in \langle dx_{p+1}, \ldots, dx_n\rangle$ there exists a linear change of coordinates such that $\omega \in \mathcal{I}(dx_1, \ldots, dx_{p+1})$.

Finally, in case (b), ω decomposes for we have

$$\omega = \ell\, dx_I + \bar{\pi}_1 \wedge \sum_{i=1}^{p} c_i\, \widehat{dx_i} = (\ell\, dx_1 + \bar{\pi}_1) \wedge \sum_{i=1}^{p} c_i\, \widehat{dx_i}$$

and since the second factor is a constant $(p-1)$-form on $\mathbb{K}^p(x_1, \ldots, x_p)$, it decomposes. The linear change of coordinates reducing ω to the normal form (i) is evident. This of course finishes the proof of the theorem for LDS p-forms.

The integrable case actually reduces to closed forms only. In fact, Proposition 1.3.1 furnishes a linear change of coordinates reducing ω to $p+1$ variables.

Now suppose $\omega = \omega \mid (dx_1, \ldots, dx_{p+1})$ is closed and let $p+1 < j \leqslant n$ be arbitrary. By computing the Lie derivative $Le_j\, \omega$ we find $Le_j\, \omega = d(i(e_j)\omega) + i(e_j)d\omega = 0$. Hence, ω is independent of x_j and it follows that ω is a form on $\mathbb{K}^{p+1}(x_1, \ldots, x_{p+1})$ as desired.

Finally, if $\omega = \alpha \wedge dx_I$ is closed, there exist constant 1-forms σ_i such that $d\alpha = \sum\limits_{i=1}^{p-1} \sigma_i \wedge dx_i$. This shows that $d\alpha = d\beta$ for some linear 1-form $\beta \in \mathcal{I}(dx_1, \ldots, dx_{p-1})$, consequently $\alpha = \beta + df$ for some quadratic function f, and clearly $\alpha \wedge dx_I = df \wedge dx_I$. $\square$

An interesting consequence of Theorem A is the following

COROLLARY 2.2.1. — *A linear LDS p-form ω on $\mathbb{K}^n$ is decomposable if, and only if,* $\operatorname{codim}_E(\omega \mid E) \leqslant 2$ *for every $(p+1)$-dimensional subspace $E \subset \mathbb{K}^n$.*

Proof. — The necessity of the above condition is well known in a much more general situation (see Lemma 3.1.2).

Conversely, suppose ω nondecomposable be such that $\operatorname{codim}_E(\omega \mid E) \leqslant 2$ for all $(p+1)$-dimensional subspace $E \subset \mathbb{K}^n$. It follows from the theorem that ω reduces to a form in $\mathcal{I}(dx_1, \dots, dx_{p+1})$. Hence, $\omega = i(A)dx_I$; $I = \{1, \dots, p+1\}$, where the linear vector field $A \in \mathcal{I}(e_1, \dots, e_{p+1})$ and may, therefore, be regarded as a linear transformation $A \colon \mathbb{K}^n \to \mathbb{K}^{p+1}$.

On the other hand, $\omega \mid E = i(A \mid E)dx_I$ for all $(p+1)$-dimensional subspaces $E \subset \mathbb{K}^n$ that turn out to be the graph of some linear transformation $L \colon \mathbb{K}^{p+1} \to \mathbb{K}^{n-p-1}$, and then, by the hypothesis, $\operatorname{rank}(A \mid E) \leqslant 2$ for all those subspaces. But this implies that $\operatorname{rank}(A) \leqslant 2$ for, by elementary arguments, there always exists such a subspace E with the property that $A(E) = \operatorname{Im}(A)$. Thus, in a suitable coordinate system, not shuffling $\mathbb{K}^{p+1}$ and $\mathbb{K}^{n-p-1}$, $A \in \mathcal{I}(e_1, e_2)$ and $\omega = c(i(A)dx_1 \wedge dx_2) \wedge dx_3 \wedge \dots \wedge dx_{p+1}$, for some constant $c \in \mathbb{K}$. A contradiction. □

3. Holomorphic plane fields and foliations

3.1. Some auxiliary results

The results we present below, yet of a very elementary nature, are fundamental tools and will be referred to very often in the sequel.

LEMMA 3.1.1 (B-lemma). — *Let U be an open subset of $\mathbb{C}^n$ and $\omega, \tilde{\omega} \in \Lambda^p(U)$. Suppose that*

(i) *ω is LDS and* $\operatorname{codim}(\omega) \geqslant 2$.

(ii) *$\mathcal{E}^*(\omega) \subset \mathcal{E}^*(\tilde{\omega})$, where $\mathcal{E}^*(\omega) = \{\alpha \in \Lambda^1(U) \mid \alpha \wedge \omega = 0\}$.*

Then, there exists $g \in \mathcal{O}(U)$ such that $\tilde{\omega} = g\omega$.

Proof. — Let $x \in U \backslash \operatorname{Sing}(\omega)$ and write $\omega(x) = \alpha_1 \wedge \dots \wedge \alpha_p$. Then, by hypothesis (ii), $\alpha_i \wedge \tilde{\omega}(x) = 0$, once $\alpha_i \in \langle i(e_I)\omega(x); |I| = p-1 \rangle$ and $i(e_I)\omega \in \mathcal{E}^*(\omega)$ (Proposition 1.2.1 (iii)). The division property of the α_i implies the existence of $g(x) \in \mathbb{C}$ such that $\tilde{\omega}(x) = g(x)\omega(x)$. Since $g(x)$ is unique the result follows from (i) and the Hartogs' extension theorem. □

Remark 3.1.1. — The condition $\mathcal{E}^*(\omega) \subset \mathcal{E}^*(\tilde{\omega})$ has the following useful characterization: Let $0 \neq \omega \in \Lambda^p(U)$ be LDS and $\tilde{\omega} \in \Lambda^q(U)$ be arbitrary. Then, $\mathcal{E}^*(\omega) \subset \mathcal{E}^*(\tilde{\omega})$ if, and only if, $i(e_I)\omega \wedge \tilde{\omega} = 0$ for all I such that $|I| = p-1$ (i.e. $\operatorname{Ker}^{\perp}(\omega(x)) \subset \mathcal{E}^*(\tilde{\omega}(x))\ \forall x \in U$).

LEMMA 3.1.2. — *Let* $\omega = \alpha_1 \wedge \ldots \wedge \alpha_p \in \Lambda^p(U)$ *be such that* $\mathrm{Sing}(\omega) \neq \phi$. *Then,* $\mathrm{codim}(\omega) \leqslant n - p + 1$.

Proof. — Accordingly as explicitly stated in [6] this result is a consequence of a classical lemma on "bordering determinants" but, unfortunately, no precise reference to that lemma is pointed out by the author. Anyway, the result follows immediately from the very proof of Lemma 1.2.1 and the semicontinuity of the codimension of an analytic subset.

It is worth noticing that the last step carried out in the proof of Lemma 1.2.1 (the perturbation of α_p), for this specific purpose, is actually unnecessary.

Another proof of this result may be found in [8], Theorem 2. □

3.2. The main result on the local structure of plane fields

It follows from Corollary 2.3.1 that a nondecomposable linear p-form restricted to a suitable $(p+1)$-dimensional subspace has a singular set of codimension greater than two. This motivates the investigation of the local structure of holomorphic LDS p-forms around a singularity where the linear part satisfies this condition. In Theorem B we treat this problem in a more general context.

Since we are in fact interested in the local properties of p-forms around a singularity, it is more adequate to consider henceforth, germs of functions, p-forms, and vector fields, at the origin of $\mathbb{C}^n$. These sets of germs will be respectively denoted by $\mathcal{O}(n)$, $\Lambda^p(n)$ and $\mathfrak{X}(n)$.

THEOREM B. — *Let* $\omega \in \Lambda^p(n)$, $p \geqslant 2$, *be LDS and let* $I = \{1, \ldots, p-1\}$ *and* $I' = \{1, \ldots, p+1\}$. *Then,*

(i) *If* $\mathrm{codim}(i(e_I)\omega) \geqslant 3$, *the plane field* $\tau(\omega)$ *is given by a system of p 1-forms. More precisely, there exist* $\alpha_1, \ldots, \alpha_{p-1} \in \Lambda^1(n)$ *such that* $\omega = \alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p-1}$, *where* $\alpha_0 = \pm i(e_I)\omega$, $\alpha_i = dx_i + \tilde{\alpha}_i$ *for all* $i \in I$ *and* $\tilde{\alpha}_i \in \mathcal{I}(dx_p, \ldots, dx_n)$.

(ii) *If* $\mathrm{codim}(\omega \mid (dx_{I'})) \geqslant 3$, *the plane field* $\tau(\omega)$ *is given by a set of* $n-p$ *vector fields. More precisely, there exist* $X_1, \ldots, X_{n-p-1} \in \mathfrak{X}(n)$ *such that* $\omega = i(X_0, X_1, \ldots, X_{n-p-1}) dx_1 \wedge \ldots \wedge dx_n$, *where* X_0 *satisfies* $\omega \mid (dx_{I'}) = i(X_0) dx_{I'}$, $X_i = e_{p+i+1} + \widetilde{X}_i$ *for* $i = 1, \ldots, n-p-1$, *and* $\widetilde{X}_i \in \mathcal{I}(e_1, \ldots, e_{p+1})$.

Proof. — At the first place we observe that the star of Hodge establishes a "dual" relation connecting parts (i) and (ii) of the theorem. Namely, ω satisfies the hypothesis in (i) if, and only if, $*\omega$ satisfies the hypothesis in (ii). Thus, part (ii) of the theorem reduces trivially to part (i).

Before we proceed to the proof of part (i) we point out that the form α_0 referred to in the theorem turns out to be the only 1-form in $\mathcal{I}(dx_p, \ldots, dx_n)$ such that $\omega = \alpha_0 \wedge dx_I + \eta$, where $i(e_I)\eta = 0$.

The proof will be carried out by induction on $p \geqslant 2$.

For $p = 2$ we have $\omega = \alpha_0 \wedge dx_1 + \eta$, as remarked above, and $\alpha_0 \wedge \eta = 0$ according to Proposition 1.2.1 (iii). Since $\operatorname{codim}(\alpha_0) \geqslant 3$ it follows from the division lemma [2] that $\eta = \alpha_0 \wedge \tilde{\alpha}_1$ for some $\tilde{\alpha}_1 \in \mathcal{I}(\widehat{dx_1})$ and then $\omega = \alpha_0 \wedge (dx_1 + \tilde{\alpha}_1)$ as desired.

Now let $\omega \in \Lambda^{p+1}(n)$ satisfy the hypothesis of the theorem and write $\omega = \omega' \wedge dx_p + \tilde{\omega}$, where $\omega', \tilde{\omega} \in \mathcal{I}(\widehat{dx_p})$. Clearly $\omega' = \pm i(e_p)\omega$ and therefore it is LDS (Remark 1.2.1 (i)). Furthermore, ω' satisfies the hypothesis in (i), for $i(e_I)\omega' = i(e_1, \ldots, e_p)\omega$.

Then, by the induction hypothesis, there exist $\alpha_i = dx_i + \tilde{\alpha}_i\,, \quad i \in I$, such that $\omega' = \alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p-1}$ and $\tilde{\alpha}_i \in \mathcal{I}(dx_p, \ldots, dx_n)$. Since ω' is independent of dx_p, the $\tilde{\alpha}_i$ do actually lie in $\mathcal{I}(dx_{p+1}, \ldots, dx_n)$.

On the other hand, since $i(e_J)\omega' = \pm i(e_J)i(e_p)\omega$, it follows from Proposition 1.2.1 (iii) that $i(e_J)\omega' \wedge \tilde{\omega} = 0$ for all J with $|J| = p - 1$. Then, by Remark 3.1.1, we conclude that $\alpha_k \wedge \tilde{\omega} = 0, \;\; k = 0, 1, \ldots, p-1$. Since $\{\alpha_1, \ldots, \alpha_{p-1}, dx_p, \ldots, dx_n\}$ is a local coframe and $\tilde{\omega}$ is independent of dx_p, we find that $\tilde{\omega} = \theta \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p-1}$ for some 2-form $\theta \in \mathcal{I}(dx_{p+1}, \ldots, dx_n)$.

Finally, the condition $\alpha_0 \wedge \tilde{\omega} = 0$ says that $\alpha_0 \wedge \theta = 0$, once $i(e_I)(\alpha_0 \wedge \tilde{\omega}) = \pm\alpha_0 \wedge \theta$. The division property of α_0 assures that $\theta = \alpha_0 \wedge \beta$ for some $\beta \in \mathcal{I}(dx_{p+1}, \ldots, dx_n)$ and then, $\omega = \alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p-1} \wedge (dx_p + \tilde{\alpha}_p)$, where $\tilde{\alpha}_p = \pm\beta$, which finishes the proof. □

Remark 3.2.1. — The hypothesis in Theorem B are normalized, in the sense that any previous change of coordinates is allowed. In particular if for some germ of a $(p+1)$-dimensional submanifold $i\colon S \to (\mathbb{C}^n, 0)$ one has $\operatorname{codim}(i^*\omega) \geqslant 3$ then, ω reduces to normal form (ii). In particular the existence of transversal embedded planes, which holds for any 1-form ([7], Theorem 2), is in general false for LDS p-forms if $p \geqslant 2$.

Finally, we close this section with the following

COROLLARY 3.2.1. — *Let $\omega \in \Lambda^p(n)$ be LDS and $\omega(0) = 0$. Suppose that the linear part $J_0^1(\omega)$ of ω at 0 is nondecomposable (resp. codim $(J_0^1(\omega)) \geqslant 3$). Then, ω reduces to the normal form (ii) (resp. (i) or (ii)) of Theorem B.*

Proof. — It is easily seen from Proposition 1.2.1 (iii) that the first nonzero jet at the origin of an LDS $\omega \in \Lambda^p(n)$ is as well LDS. Then, in the first case, it follows from Corollary 2.2.1 that for some $(p+1)$-dimensional subspace $E \subset \mathbb{C}^n$ we must have $\operatorname{codim}_E(J_0^1(\omega) \mid E) \geqslant 3$ and consequently $\operatorname{codim}_E(\omega \mid E) \geqslant 3$. Now the first assertion follows from Remark 3.2.1 above.

By taking into account the "duality" induced by the star of Hodge, referred to in the beginning of the proof of Theorem B, and by following the above reasoning we obtain immediately the second assertion. $\square$

Remark 3.2.2. — The normal forms furnished by Theorem B are far from describing the general local structure of LDS p-forms. In fact, the plane field induced on $\mathbb{C}^4$ by the LDS 2-form $\omega = x_3^2\, dx_1 \wedge dx_2 + x_1^2\, dx_1 \wedge dx_3 + (x_1x_2 + x_3x_4)dx_2 \wedge dx_3 + x_4^2\, dx_3 \wedge dx_4 + x_2^2\, dx_2 \wedge dx_4 + (x_1x_2 - x_3x_4)dx_1 \wedge dx_4$ has an isolated singularity at 0 and, therefore, cannot be of the type described in the theorem, in virtue of Lemma 3.1.2.

3.3. The main result on the local structure of foliations

In this section we shall basically improve the normal forms furnished by Theorem B, under the additional hypothesis that ω is integrable.

A very important role is played, in the establishment of these normal forms, by the following elementary results related to the reduction of variables:

LEMMA 3.3.1. — *Let $\omega \in \Lambda^p(n)$ be integrable and* $\operatorname{codim}(\omega) \geqslant 2$. *If $X \in \mathfrak{X}(n)$ is such that $i(X)\omega = 0$ then, $L_X\omega = h\omega$ for some $h \in \mathcal{O}(n)$. In particular if $X = e_k$ then, $\omega = g(\omega \mid \mathbb{C}^{n-1}(\widehat{x_k}))$, where $g \in \mathcal{O}(n)$ and $g(0) = 1$.*

Proof. — Since ω is integrable $i(e_I)\omega \wedge d\omega = 0\ \forall I;\ |I| = p-1$. Consequently, it follows from the hypothesis that $i(e_I)\omega \wedge i(X)d\omega = 0$ and then, by the B-lemma, there exists $h \in \mathcal{O}(n)$ such that $i(X)d\omega = h\omega$ or equivalently $L_X\omega = h\omega$ as desired. Clearly the differential equation $L_{e_k}\omega = h\omega$ means exactly that $\omega = g(\omega \mid \mathbb{C}^{n-1}(\widehat{x_k}))$ as stated. $\square$

LEMMA 3.3.2. — *Let $\omega \in \Lambda^p(n)$ be integrable and $d\omega \neq 0$. If $X \in \mathfrak{X}(n)$ is such that $i(X)d\omega = 0$ then, $L_X\omega = 0$. In particular if $X = e_k$ then, $\omega = \omega \mid \mathbb{C}^{n-1}(\widehat{x_k})$.*

Proof. — In view of the hypothesis, we deduce from the relations $i(e_I)\omega \wedge d\omega = 0$, that $(i(X)i(e_I)\omega)d\omega = 0$ and then $i(e_I)(i(X)\omega) = 0 \ \forall I$, $|I| = p-1$. In other words $i(X)\omega = 0$. Hence $L_X\omega = 0$ and the proof is finished. □

THEOREM C. — *Let $\omega \in \Lambda^p(n)$, $p \geqslant 2$, be integrable and let $I = \{1, \ldots, p-1\}$ and $I' = \{1, \ldots, p+1\}$. Then,*

(i) *If* $\operatorname{codim}(i(e_I)\omega) \geqslant 3$, *there exists a holomorphic change of coordinates such that $\omega = g\, df \wedge dx_1 \wedge \ldots \wedge dx_{p-1}$ where $f, g \in \mathcal{O}(n)$ and $g(0) = 1$.*

(ii) *If* $\operatorname{codim}(\omega \mid (dx_{I'})) \geqslant 3$, *there exists a holomorphic change of coordinates such that $\omega = g\omega \mid \mathbb{C}^{p+1}(x_1, \ldots, x_{p+1})$ where $g \in \mathcal{O}(n)$ and $g(0) = 1$.*

Proof. — The hypothesis (i) and (ii) above are exactly the same of Theorem B. We may therefore assume that ω is either of the form (i) or (ii) of that theorem.

In the case (i) the result is a straightforward consequence of the Theorem of Frobenius with Singularities [6]. As a matter of fact, $\omega = \alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_{p-1}$, the system $\{\alpha_0, \ldots, \alpha_{p-1}\}$ is integrable and $\operatorname{codim}(\alpha_0 \wedge \ldots \wedge \alpha_{p-1}) \geqslant 3$. Then, there exists a set $\{f_0, \ldots, f_{p-1}\}$ of first integrals, such that $f_i(0) = 0$, $\alpha_i = \sum_{j=0}^{p-1} a_{ij}\, df_j$ and $\det(a_{ij}(0)) \neq 0$. Hence, $\omega = \alpha_0 \wedge \ldots \wedge \alpha_{p-1} = \det(a_{ij}) df_0 \wedge \ldots \wedge df_{p-1}$. On the other hand, $\alpha_1 \wedge \ldots \wedge \alpha_{p-1} \in \mathcal{I}(df_0 \wedge \ldots \wedge \widehat{df}_j \wedge \ldots \wedge df_{p-1}; j = 0, \ldots, p-1)$ and since $i(e_I)\alpha_1 \wedge \ldots \wedge \alpha_{p-1} = \pm 1$ it follows that some of the $df_0 \wedge \ldots \wedge \widehat{df}_j \wedge \ldots \wedge df_{p-1}$ does not vanish at the origin. This makes clear the desired change of coordinates reducing ω to the form $\omega = g\, df \wedge dx_1 \wedge \ldots \wedge dx_{p-1}$.

In the case (ii) we consider initially the holomorphic change of coordinates $f_1 \colon (\mathbb{C}^n, 0) \hookleftarrow$ satisfying:

(a) $f_1 \circ i = i$, where $i \colon \mathbb{C}^{n-1}(x_1, \ldots, x_{n-1}) \to \mathbb{C}^n$ is the inclusion map.

(b) $f_1^* X_{n-p-1} = e_n$.

This is attained by conjugating the flows of e_n and X_{n-p-1} starting from the common cross section $\mathbb{C}^{n-1}$.

Clearly $f_1^* \omega \in \mathcal{I}(dx_1, \ldots, dx_{n-1})$ because of (b), and then by Lemma 3.3.1, there exists $g_1 \in \mathcal{O}(n)$ with $g_1(0) = 1$ such that $f_1^*\omega = g_1(f_1^* \omega) \mid \mathbb{C}^{n-1}$. On the other hand we conclude from (a) that $(f_1^* \omega) \mid \mathbb{C}^{n-1} = \omega \mid \mathbb{C}^{n-1}$, and since $\mathbb{C}^{n-1}$ is X_i-invariant for $i = 0, \ldots, n-p-2$, the result follows by an obvious recurrence procedure. □

In order to get further information about the local structure of an integrable p-form it is very useful to look at its exterior derivative. A concrete example of this situation is furnished by the integrable 1-forms which, as one can see, are completely absent in the results established before. In this direction we have the following direct consequence of Theorem C (ii) and Lemma 3.3.2:

COROLLARY 3.3.1. — *Let $\omega \in \Lambda^p(n)$, $p \geqslant 1$, be integrable and suppose that* $\operatorname{codim}(d\omega \mid (dx_1, \ldots, dx_{p+2})) \geqslant 3$. *Then, ω reduces to the form $\omega \mid \mathbb{C}^{p+2}(x_1, \ldots, x_{p+2})$ by means of a holomorphic change of coordinates.*

4. Final comments

Firstly, we shall discuss briefly the validity of the results established in the foregoing sections, in the context of real analytic, and class C^r, p-forms.

At the first place we notice that Theorem B holds, if the hypothesis on $\operatorname{codim}(i(e_I)\omega)$ in part (i) (resp. $\operatorname{codim}(\omega \mid dx_{I'})$ in part (ii)) is replaced by the requirement that the 1-form $i(e_I)\omega$ (resp. $*(\omega \mid dx_{I'})$ on $\mathbb{C}^{p+1}(x_{I'})$) has the property of division with respect to 2-forms.

As a matter of fact, the only doubtful point in the proof of Theorem B would be the reference to Remark 3.1.1. However, it is easily seen that this remark holds under the sole assumption that ω is not a torsion element of $\Lambda^p(U)$, which is readily verified if $i(e_I)\omega$ satisfies the above requirement.

On the other hand, the existence of an R-sequence ($R = \mathcal{A}(U), C^r(U)$) of length three (see [9] p.234), among the coefficients of $i(e_I)\omega$, is sufficient to guarantee the required division property. (This follows at once from the same arguments utilized in [2]). In particular, a real version of Corollary 3.2.1 is promptly available.

The part (ii) of Theorem C holds, as well, under the same assumptions described above. In fact, the only suspicious point in its proof would be the veracity of Lemma 3.3.1, which is, by its turn, a direct consequence of the B-lemma. And, it is not difficult to show that the B-lemma holds if ω satisfies the hypothesis under consideration.

Part (i) of Theorem C involves a more delicate question concerning the existence of first integrals and, as far as we know, there is no general result, in the real field, analogous to the Theorem of Frobenius with Singularities of [6]. Clearly, in the analytic case, we may eventually apply Theorem C (i), by looking at the complexification of ω. For example, the integrable version of the second assertion in Corollary 3.2.1, for real analytic forms, may be established by following this procedure.

Finally, we would like to mention that, for holomorphic foliations of the complex projective space $\mathbb{C}P(n)$, we have the following general result, whose proof is completely analogous to the classical codimension one case:

PROPOSITION 4.1. — *Let $\mathcal{F}$ be a singular foliation of $\mathbb{C}P(n)$ such that* $\operatorname{codim}(\operatorname{Sing}(\mathcal{F})) \geqslant 2$, *and let $\pi\colon \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P(n)$ be the canonical projection. Then, the singular foliation $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ of $\mathbb{C}^{n+1}$ is given by a single homogeneous integrable form on $\mathbb{C}^{n+1}$.*

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