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## Topological tensor products and asymptotic developments<sup>(\*)</sup>

JORGE MOZO-FERNÁNDEZ<sup>(1)</sup>

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**RÉSUMÉ.** — Dans cet article, on étudie la structure topologique des espaces de fonctions avec développement asymptotique fort, dans le cas Poincaré et le cas Gevrey. On démontre qu'ils sont nucléaires, et ils sont le complété du produit tensoriel des espaces à une variable. Nous démontrons aussi que les espaces de séries multisommables à une variable sont nucléaires. Ceci permet de définir ces espaces à plusieurs variables.

**ABSTRACT.** — In this paper, we study the topological structure of the spaces of functions that admit an asymptotic development in several complex variables, in the Poincaré and Gevrey cases. We show that they are nuclear, and they are a completed tensor product of the one variable case. We also show that the spaces of multisummable series in one variable, are nuclear. This allows to extend the definition of these spaces to several variables.

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### 1. Introduction

In [T], Tougeron develops a notion of multisummability in several variables, considering completed tensor products of the sets of multisummable series in one variable. The aim of this paper is to formalize this situation.

More precisely, we study the topological structure of the vector spaces of functions with asymptotic development, in the general and Gevrey cases,

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and of the spaces of summable and multisummable series. We show that all these spaces are nuclear, and that the sets of holomorphic functions in several variables that have asymptotic development, in the sense defined by Majima [M1] and Haraoka [Ha] (for the Gevrey case) are precisely the completed tensor product of the corresponding sets in one variable case.

The spaces of summable and multisummable series being nuclear, the definition of multisummability in several variables as a completed tensor product, as Tougeron does, seems to be the most appropriate, and we think that should be developed. Moreover, this should provide, using the Künneth formula for Fréchet sheaves developed in [G, K], a way of computing the cohomology of asymptotic sheaves in several variables. Nevertheless, this situation does not verify the hypothesis required in the mentioned papers, and in fact, the result is not what we would obtained using a reasonable Künneth formula.

In the paper, if  $U$  is an open set,  $\mathcal{O}(U)$ ,  $C^\infty(U)$  will denote the set of holomorphic functions and of  $C^\infty$  functions, respectively, in  $U$ . If  $K$  is compact,  $C^\infty(K)$  will be the set of  $C^\infty$  functions in the sense of Whitney defined in  $K$ . We shall use multiindex notations. For instance, if  $\mathbf{A} = (A_1, \dots, A_n)$ ,  $\mathbf{N} = (N_1, \dots, N_n)$ ,  $\mathbf{A}^{\mathbf{N}}$  will denote  $A_1^{N_1} \cdots A_n^{N_n}$ , and  $\mathbf{N}! := N_1! \cdots N_n!$ . The number  $N_1 + \cdots + N_n$  is represented by  $|\mathbf{N}|$ .

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## 2. Reminds on topological tensor products and nuclear spaces

### 2.1. Topological tensor products

In this section, we shall recall the main definitions and the properties we shall need about topological tensor products and nuclear spaces. Besides the original work of Grothendieck [G1], these results can be read in [D].

Let  $(E, p)$ ,  $(F, q)$  complex seminormed vector spaces. In  $E \otimes F$  we shall define the following two seminorms:

$$\begin{aligned} (p \otimes_\varepsilon q) &= \sup\{ |(\xi \otimes \eta)(t)| / \xi \in E', \eta \in F', |\xi(x)| \leq p(x), |\eta(y)| \leq q(y) \}, \\ (p \otimes_\pi q)(t) &= \inf\{ \sum_i p(x_i)q(y_i) / t = \sum_i x_i \otimes y_i \}, \end{aligned}$$

where  $E', F'$  are the topological duals of  $E, F$ , respectively. These two seminorms are the lower and the upper bound of all natural seminorms that can be defined on  $E \otimes F$ , as it is explained in [D].

More generally, if  $(E, \{p_i\}_{i \in I})$ ,  $(F, \{q_j\}_{j \in J})$  are locally convex spaces, we define  $E \otimes_\pi F$ ,  $E \otimes_\varepsilon F$  as the topological spaces whose ground set is  $E \otimes F$  and the topology is given by the family  $\{p_i \otimes_\pi q_j\}_{i,j}$  or  $\{p_i \otimes_\varepsilon q_j\}_{i,j}$ . These spaces represent different functors, and so they can be defined by means of universal properties as follows:

(\*) $_\pi$  Given a locally convex space  $G$  and  $f \in B(E, F; G)$  (bilinear and continuous), it exists one and only one  $\bar{f} \in L(E \otimes_\pi F, G)$  such that the diagram

$$\begin{array}{ccc} E \times F & \xrightarrow{f} & G \\ \downarrow & \nearrow \bar{f} & \\ E \otimes_\pi F & & \end{array}$$

commutes.

(\*) $_\varepsilon$  The diagram

$$\begin{array}{ccc} E \otimes_\varepsilon & \xrightarrow{\alpha} & L(F', E) \\ \downarrow \beta & & \downarrow \Psi \\ L(E', F) & \xrightarrow{\varphi} & B(E', F; \mathbb{C}) \end{array}$$

commutes, where  $\alpha$ ,  $\beta$ ,  $\Psi$  and  $\varphi$  are the obvious continuous maps. Moreover, they are isometries for every couple  $p$ ,  $q$  of seminorms in  $E$ ,  $F$ .

We denote  $E \hat{\otimes}_\pi F$ ,  $E \hat{\otimes}_\varepsilon F$  the completions of  $E \otimes_\pi F$ ,  $E \otimes_\varepsilon F$ , respectively. They are again locally convex spaces. The space  $E \hat{\otimes}_\pi F$  verifies the same universal property that  $E \otimes F$ , simply replacing the expression "locally convex space  $G$ " by "locally convex and complete space  $G$ ".

We shall use the following property that relates the behaviour of these constructions with respect to projective limits:

(\*) If  $E = \varprojlim E_i$  and  $E \rightarrow E_i$  has dense image, then

$$E \hat{\otimes}_\pi F \cong \varprojlim E_i \otimes_\pi F, \quad E \hat{\otimes}_\varepsilon F \cong \varprojlim E_i \otimes_\varepsilon F.$$

Some classical, but very important examples, are:

1. If  $X$  is a locally compact space,  $F$  a Fréchet space, then

$$C(X) \hat{\otimes}_\varepsilon F \cong C(X; F).$$

2. If  $U$ ,  $V$  are open sets in  $\mathbb{C}^n$ ,  $\mathbb{C}^m$  respectively, then

$$\mathcal{O}(U) \hat{\otimes}_\varepsilon \mathcal{O}(V) \cong \mathcal{O}(U) \hat{\otimes}_\pi \mathcal{O}(V) \cong \mathcal{O}(U \times V).$$

3. If  $X_1, X_2$  are compact subsets of  $\mathbb{R}^n, \mathbb{R}^m$  respectively, then

$$C^\infty(X_1) \hat{\otimes}_\varepsilon C^\infty(X_2) \cong C^\infty(X_1) \hat{\otimes}_\pi C^\infty(X_2) \cong C^\infty(X_1 \times X_2).$$

## 2.2. Nuclear spaces

The definition of nuclear space is motivated by some of the examples of the preceding section. We recall the definition:

DEFINITION 2.1. — *A linear map  $(E, p) \xrightarrow{\alpha} (F, q)$  between semi-normed spaces is nuclear if “it can be approximated by maps with finitely generated image”. More precisely, there exists  $a_n \in E', f_n \in F$  with*

$$\forall x \in E, \alpha(x) = \sum_{n \geq 1} a_n(x) \cdot f_n$$

and

$$\sum_{n=1}^{\infty} p'(a_n) \cdot q(f_n) < +\infty,$$

where  $p'$  is the seminorm on  $E'$  induced by  $p$  such that if  $\beta \in E'$ ,

$$|\beta(x)| \leq p'(\beta) \cdot p(x).$$

DEFINITION 2.2. — *A locally convex space  $(E, (p_i)_{i \in I})$  is nuclear (where  $(p_i)_{i \in I}$  is a directed family) if*

$$p_i \leq p_j \Rightarrow \hat{E}_j \rightarrow \hat{E}_i$$

*is nuclear, where  $\hat{E}_i$  is the completion of the seminormed space  $(E, p_i)$ .*

Some examples are:

1.  $\mathcal{O}(U)$  and  $C^\infty(U)$  are nuclear ( $U$  open set).
2. If  $X$  is compact,  $C^\infty(X)$  is nuclear.
3. Subspaces, quotients, projective limits and numerable inductive limits of nuclear spaces are nuclear.
4. The sets of convergent series  $\mathbb{C}\{z\}$ , Gevrey series of order  $s$ ,  $\mathbb{C}[[z]]_s$ , and Gevrey series of precised order  $(s, A)$ ,  $\mathbb{C}[[z]]_{s,A}$  (see [R] for details) are nuclear.
5. If  $E$  is nuclear,  $E \hat{\otimes}_\varepsilon F \cong E \hat{\otimes}_\pi F$ .

Let us detail an important example of nuclear space. If  $K$  is compact, denote  $C_{\{M_p\}}^\infty(K)$  the set of Whitney  $C^\infty$  functions on  $K$  such that

$$|D^N f| \leq C \cdot \mathbf{A}^N \cdot M_{|N|}$$

for certain  $M, \mathbf{A} > 0$ , where  $\{M_p\}_{p=0}^\infty$  is a sequence of positive numbers. For a fixed  $A$ , denote  $C_{\{M_p\}, A}^\infty(K)$  the corresponding space. Hence,

$$C_{\{M_p\}}^\infty(K) = \varinjlim C_{\{M_p\}, A}^\infty(K).$$

If  $U$  is an open set, denote

$$C_{\{M_p\}}^\infty(U) = \varprojlim C_{\{M_p\}}^\infty(K),$$

where the limit runs over all compacts  $K \subseteq U$ .

In [Ko], it is shown:

**THEOREM 2.1** *If  $\{M_p\}_p$  satisfies the condition*

$$M_{p+1} \leq C \cdot H^p \cdot M_p, \text{ for certain } C, H > 0,$$

*then the spaces  $C_{\{M_p\}, A}^\infty(K)$ ,  $C_{\{M_p\}}^\infty(K)$  and  $C_{\{M_p\}}^\infty(U)$  are nuclear.*

### 3. Asymptotic developments

**3.1.** The notion of asymptotic development in one variable is due to Poincaré. If  $V$  is a (open) sector in  $\mathbb{C}$ , a function  $f \in \mathcal{O}(V)$  has  $\hat{f} = \sum_{n \geq 0} a_n z^n$  as an asymptotic development at the origin (and we shall write  $f \sim \hat{f}$ ) if, for every proper subsector  $W < V$  and  $N \in \mathbb{N}$ , we have

$$z \in W \Rightarrow \left| f(z) - \sum_{n < N} a_n z^n \right| \leq C(W, N) \cdot |z|^N.$$

This notion is generalized by Majima [M1, M2] to several variables. If  $V$  is a polysector (product of sectors) in  $\mathbb{C}^n$ , consider families

$$\mathcal{F} = \{f_{\alpha_J}(z_{J^c}) \in \mathcal{O}(V_{J^c}) \mid \emptyset \neq J \subseteq \{1, \dots, n\}, \alpha_J \in \mathbb{N}^J\}.$$

A holomorphic function  $f \in \mathcal{O}(V)$  has  $\mathcal{F}$  as an asymptotic development at the origin ( $f \sim \mathcal{F}$ ) if, for every proper subpolysector  $W < V$  and  $\mathbf{N} \in \mathbb{N}^n$  we have

$$z \in W \Rightarrow |f(z) - \text{App}_{\mathbf{N}}(\mathcal{F})(z)| < C(W, \mathbf{N}) \cdot |z^{\mathbf{N}}|,$$

where the approximating function  $\text{App}_{\mathbf{N}}(\mathcal{F})$  is defined as

$$\text{App}_{\mathbf{N}}(\mathcal{F})(z) = \sum_{\emptyset \neq J \subseteq \{1, \dots, n\}} \sum_{\alpha_J < \mathbf{N}_J} f_{\alpha_J}(z_{J^c}) \cdot z_J^{\alpha_J}.$$

Denote  $\mathcal{A}(V)$  the set of holomorphic functions with asymptotic development in  $V$ . This generalizes the one variable case, taking  $\mathcal{F} = \{a_n\}_{n \geq 0}$ . The asymptotic development, if it exists, it is unique. Moreover, it behaves well with respect to the usual operations on functions (sums, products, derivatives,...).

An important kind of asymptotic development are the so-called “of Gevrey type”. If  $\mathbf{s} = (s_1, \dots, s_n) \in (\mathbb{R}_{\geq 0})^n$ , we shall say that  $f \in \mathcal{O}(V)$  has the family  $\mathcal{F}$  as  $\mathbf{s}$ -Gevrey asymptotic development if, in the definition,  $C(W, \mathbf{N})$  can be chosen as

$$C_W \cdot \mathbf{A}_W^{\mathbf{N}} \cdot \mathbf{N}!^{\mathbf{s}}.$$

Denote  $\mathcal{A}_{\mathbf{s}}(V) \subseteq \mathcal{A}(V)$  the set of functions with  $\mathbf{s}$ -Gevrey asymptotic development. For further details and properties of asymptotic developments, see [M1, M2, Mo].

### 3.2. Characterization

The following characterization of  $\mathcal{A}(V)$  and  $\mathcal{A}_{\mathbf{s}}(V)$  are important, and will be used in the sequel:

**THEOREM 3.1** ([Z, HE]). — *If  $f \in \mathcal{O}(V)$ , the following are equivalent:*

1.  $f \in \mathcal{A}(V)$ .
2. If  $W < V$ ,  $\mathbf{N} \in \mathbb{N}^n$ ,  $|D^{\mathbf{N}}f|_W$  is bounded.
3. If  $W < V$ , there exists  $F \in C^\infty(\mathbb{C}^n)$  such that  $F|_W = f|_W$ .

Similarly, for the Gevrey case we have:

**THEOREM 3.2.** — *If  $f \in \mathcal{O}(V)$ , the following are equivalent:*

1.  $f \in \mathcal{A}_{\mathbf{s}}(V)$ .
2. If  $W < V$ ,  $\mathbf{N} \in \mathbb{N}^n$ , there exists  $C_W, \mathbf{A}_W > 0$  such that  $|D^{\mathbf{N}}f|_W \leq C_W \cdot \mathbf{A}_W^{\mathbf{N}} \cdot \mathbf{N}!^{\mathbf{s}+1}$ .
3. If  $W < V$ , there exists  $F \in C^\infty(\mathbb{C}^n)$  such that  $F|_W = f|_W$  and  $|D^{\mathbf{N}}F| \leq C \cdot \mathbf{A}^{\mathbf{N}} \cdot \mathbf{N}!^{\mathbf{s}+1}$ .

### 3.3. Topological structure of the spaces of asymptotically developable functions

The space  $\mathcal{A}(V)$  can be provided with a Fréchet space structure. If  $W < V$  and  $N \in \mathbb{N}^n$ , the map

$$\begin{array}{ccc} \mathcal{A}(V) & \xrightarrow{p_{W,N}} & \mathbb{R}_{\geq 0} \\ f & \longmapsto & \sup\{|D^N f(z)| \mid z \in W\} \end{array}$$

is a seminorm, and the set  $\{p_{W,N}\}_{W,N}$  gives to  $\mathcal{A}(V)$  a structure of locally convex, Hausdorff space. One can show, by standard techniques of complex analysis [He] that it is complete and so, a Fréchet space, as the topology can be generated by a numerable family of seminorms.

Alternatively, the family of seminorms

$$q_{W,N}(f) = \sup \left\{ \frac{1}{|z^N|} \cdot |f(z) - \text{App}_N(TA(f))(z)| \mid z \in W \right\}$$

gives to  $\mathcal{A}(V)$  the same Fréchet space structure, as we have bounds

$$\begin{aligned} q_{W,N}(f) &\leq \frac{1}{N!} \cdot p_{W,N}(f), \\ p_{W,N}(f) &\leq N! \cdot \left( \frac{1+\varepsilon}{\varepsilon} \right)^{|N|} \cdot q_{W',N}, \end{aligned}$$

where  $W < W' < V$ , and for  $z \in W$ ,  $\bar{B}(0, \varepsilon|z|) \subseteq W'$ .

In the same way that in complex analysis it is shown that  $\mathcal{O}(U \times V) \cong \mathcal{O}(U; \mathcal{O}(V))$  as Fréchet spaces, one can show that there is an isomorphism  $\mathcal{A}(V_1 \times V_2) \cong \mathcal{A}(V_1; \mathcal{A}(V_2))$ , for every pair of polysectors  $V_1, V_2$ .

In the Gevrey case, there is also a locally convex space structure. If  $W$  is a compact polysector and  $A > 0$ , denote

$$\mathcal{R}(W) = C^\infty(W) \cap \mathcal{O}(\overset{\circ}{W})$$

and

$$\begin{aligned} \mathcal{R}_{s,A} &= \{f \in \mathcal{R}(W) \mid \text{there exists } C > 0 \text{ such that } |D^N f| \leq C \cdot A^N \cdot N!^{s+1}\}, \\ \mathcal{R}_s(W) &= \{f \in \mathcal{R}(W) \mid \text{there exist } C, A > 0 \text{ such that } |D^N f| \leq C \cdot A^N \cdot N!^{s+1}\}. \end{aligned}$$

The space  $\mathcal{R}_{s,A}$  is a Banach space with the norm

$$p_{W,A}(f) = \sup \left\{ \frac{1}{A^N N!^{s+1}} \cdot |D^N f(z)| \mid z \in W \right\}$$

and we have

$$\mathcal{A}_s(V) = \varprojlim_{W < V} \mathcal{R}_s(W) = \varprojlim_{W < V} \varinjlim_{\mathbf{A}} \mathcal{R}_{s,\mathbf{A}}(W).$$

So, in  $\mathcal{A}_s(V)$  there is a locally convex structure but it is not a Fréchet space.

Alternatively, as before, one can take seminorms

$$q_{W,\mathbf{N}}(f) = \sup \left\{ \frac{1}{|z^{\mathbf{N}}| \mathbf{N}!^s \mathbf{A}^{\mathbf{N}}} \cdot |f(z) - \text{App}_{\mathbf{N}}(TA(f))(z)| \mid z \in W \right\}$$

in  $\mathcal{R}_{s,\mathbf{A}}(W)$ . The structure they define in  $\mathcal{A}_s(V)$  is the same we have defined by means of  $\{p_{W,\mathbf{N}}\}_{W,\mathbf{N}}$ .

### 3.4. Nuclearity

Firstly, we shall treat the case of  $\mathcal{A}(V)$ .

**THEOREM 3.3.** —  $\mathcal{A}(V)$  is nuclear. If  $V_1, V_2$  are polysectors,  $\mathcal{A}(V_1) \hat{\otimes} \mathcal{A}(V_2) \cong \mathcal{A}(V_1 \times V_2)$ .

*Proof.* — The space  $\mathcal{R}(W)$  is a subspace of  $C^\infty(W)$ , and so, it is nuclear (here  $W$  is a compact polysector). By theorem 3.1,

$$\mathcal{A}(V) = \varprojlim_{W < V} \mathcal{R}(W),$$

hence,  $\mathcal{A}(V)$  is nuclear.

For the second statement, we first remark that  $\mathcal{A}(V)$  is dense in  $\mathcal{R}(W)$ . In fact, as  $\overset{\circ}{W}$  is a product of simply connected open sets in  $\mathbb{C}$ , it is a Runge domain, so every holomorphic function can be approached by polynomials.

Now, as  $\mathcal{R}(W) \subseteq C^\infty(W)$ , we have

$$\mathcal{R}(W_1) \hat{\otimes} \mathcal{R}(W_2) \subseteq C^\infty(W_1 \times W_2).$$

If  $\mathcal{P}_{n,W}$  is the vector space  $\mathbb{C}[X_1, \dots, X_n]$  with the topology given by the seminorms

$$p_{W,\mathbf{N}}(f) = \sup\{|D^{\mathbf{N}}f(x)| \mid x \in W\},$$

then  $\mathcal{P}_{n,W}$  is a subspace of  $\mathcal{R}(W)$  and in fact  $\mathcal{R}(W)$  is the closure of  $\mathcal{P}_{n,W}$ . So:

$$\mathcal{R}(W_1 \times W_2) = \mathcal{P}_{n_1,W_1} \hat{\otimes} \mathcal{P}_{n_2,W_2} \subseteq \mathcal{R}(W_1) \hat{\otimes} \mathcal{R}(W_2) \subseteq C^\infty(W_1 \times W_2).$$

It is clear that  $\mathcal{R}(W_1) \hat{\otimes} \mathcal{R}(W_2) \subseteq \mathcal{R}(W_1 \times W_2)$ . This can be seen looking at the explicit form of the elements of the  $\pi$ -completed tensor product: an element of  $E \otimes_{\pi} F$  can be written as

$$\sum_{i=1}^{\infty} e_i \otimes f_i,$$

where

$$\sum_{i=1}^{\infty} p(e_i)q(f_i) < +\infty$$

for seminorms  $p, q$  in  $E, F$  respectively. So:

$$\mathcal{A}(V_1) \hat{\otimes} \mathcal{A}(V_2) = \varprojlim_{W_1 < V_1} \mathcal{R}(W_1) \hat{\otimes} \varprojlim_{W_2 < V_2} \mathcal{R}(W_2) \cong \varprojlim \mathcal{R}(W_1 \times W_2) = \mathcal{A}(V_1 \times V_2).$$

□

Consider now the Gevrey case. The space  $\mathcal{R}_s(W)$  is nuclear, as it is a subspace of  $C_{\{M_p\}}^{\infty}(W)$  with  $M_p = p!^{|s|+1}$ , and this sequence satisfies the condition

$$M_{p+1} \leq C \cdot H^p \cdot M_p$$

for some  $C, H > 0$ .

The same argument as before proves that  $\mathcal{A}_s(V)$  is nuclear. We also have

$$\mathcal{R}_{s_1}(W_1) \hat{\otimes} \mathcal{R}_{s_2}(W_2) \cong \mathcal{R}_{(s_1, s_2)}(W_1 \times W_2),$$

where  $(s_1, s_2)$  denotes the concatenation of  $s_1$  and  $s_2$ . So, it follows that

$$\mathcal{A}_{s_1}(V_1) \hat{\otimes} \mathcal{A}_{s_2}(V_2) \cong \mathcal{A}_{(s_1, s_2)}(V_1 \times V_2).$$

### 3.5. Summability

The notion of summability is only well established in the one variable case. If  $d$  is a direction issued from the origin, we shall say that a formal series

$$\hat{f} = \sum_{n \geq 0} a_n \cdot z^n$$

is  $s$ -summable in the direction  $d$  if there exists a sector  $V$  bisected by  $d$ , of opening greater than  $s\pi$ , and  $f \in \mathcal{A}_s(V)$  with  $f \sim_s \hat{f}$ . If  $\mathbb{C}\{z\}_{s,d}$  is the set of  $s$ -summable power series in direction  $d$ , there is an isomorphism

$$\mathbb{C}\{z\}_{s,d} \cong \varprojlim \mathcal{A}_s(V)$$

where the inductive limit of the second term runs over all sectors bisected by  $d$ , whose opening is greater than  $s\pi$ . The isomorphism is well defined because the function  $f \in \mathcal{A}_s(V)$  such that  $f \sim_s \hat{f}$  is unique (and is called the  $s$ -sum of  $\hat{f}$  in  $V$ ).

This gives to  $\mathbb{C}\{z\}_{s,d}$  a locally convex space structure.

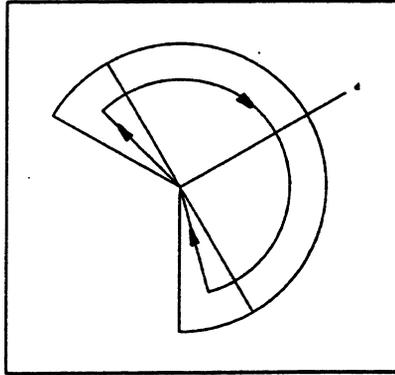
Let us recall briefly Borel-Laplace transform. If  $\hat{f} = \sum_{n \geq 0} a_n z^{n+1}$ , its formal 1-Borel transform is

$$\hat{\mathcal{B}}_1 \hat{f} = \sum_{n \geq 0} \frac{a_n}{n!} z^n,$$

and the inverse map,  $\hat{\mathcal{L}}_1$  is the formal 1-Laplace transform. If  $V$  is a sector of opening greater than  $\pi$  bisected by  $d$ , and  $f \in \mathcal{O}(V)$ , its 1-Borel transform in direction  $d$  is

$$\mathcal{B}_1 f = \frac{1}{2\pi i} \cdot \int_{\gamma} f(u) \cdot \exp\left(\frac{z}{u}\right) \cdot d(u^{-1}),$$

where the path of integration is drawn in the following picture



$\mathcal{B}_1 f$  is a holomorphic function for the values of  $z$  where the integral is defined. The 1-Laplace transform in the direction  $d$  of a function  $f$  defined in a sector  $V$ , of infinite radius, bisected by  $d$ , is

$$\mathcal{L}_1 f = \int_d f(u) \cdot \exp\left(-\frac{z}{u}\right) \cdot du,$$

with the same observation as before about the domain of definition.

It is “classic” that  $\hat{f} \in \mathbb{C}\{z\}_{1,d}$  if and only if  $\hat{\mathcal{B}}_1 \hat{f}$  is convergent, and can be extended to an infinite sector  $V$  bisected by  $d$ , with exponential growing of order 1, i.e.,

$$|\hat{\mathcal{B}}_1 \hat{f}(z)| \leq C_1 \cdot \exp(C_2 |z|).$$

In that case,  $\mathcal{L}_1(\hat{\mathcal{B}}_1 \hat{f}(z))$  is defined, and holomorphic in  $V'$ , sector bisected by  $d$ , and of opening greater than  $\pi$ . Moreover, it is the 1-sum of  $\hat{f}$  in  $V$ .

According to this, we can define the locally convex structure of  $\mathbb{C}\{z\}_{s,d}$  in a different way. Let  $K_{R,W}$  be the union of the closed ball  $\bar{B}(0,R)$  and the closed sector (of infinite radius)  $W$ . The subset  $\mathcal{C}_{R,W,A}$  of  $C^\infty(K_{R,W}) \cap \mathcal{O}(K_{R,W})$  is given by the functions  $f$  such that

$$|f(z)| \leq C \cdot e^{A|z|}$$

when  $z \in W$ . It is a locally convex space with the seminorms

$$p_{R,W,A}(f) = \sup\{|f(z)| \mid |z| \leq R\} + \sup\left\{\frac{|f(z)|}{e^{A|z|}} \mid z \in W\right\}.$$

The formal 1-Borel transform  $\hat{\mathcal{B}}_1$  defines an isomorphism

$$\mathbb{C}\{z\}_{1,d} \cong \varinjlim \mathcal{C}_{R,W,A}$$

where  $R \rightarrow 0$ ,  $W$  varies among the sectors bisected by  $d$ , and  $A > 0$ .

Analogously, by ramification, this can be done for  $\hat{f} \in \mathbb{C}\{z\}_{s,d}$ . The  $s$ -sum of  $\hat{f}$  can be obtained as the  $k$ -Laplace transform of  $\hat{\mathcal{B}}_k \hat{f}$ , formal  $k$ -Borel transform (here  $k = 1/s$ ). Further details can be seen, e.g., in [B].

A series  $\hat{f} \in \mathbb{C}[[z]]_s$  is called  $s$ -summable if it is summable in all directions but a finite number. Denote  $\mathbb{C}\{z\}_s^{-d}$  the set of series that are summable in all directions but (perhaps)  $d$ . Let  $V$  be a sector of opening  $\frac{\pi}{s} + 2\pi$ , bisected by  $d$ , in the Riemann surface of the logarithm, and  $V_R = \{z \in V \mid |z| < R\}$ . Using Borel-Laplace transform one can see that

$$\mathbb{C}\{z\}_s^{-d} \cong \varinjlim_{R \rightarrow 0} \mathcal{A}_s(V_R).$$

If  $\mathbb{C}\{z\}_s^{-d_1 \dots -d_r}$  denotes the  $\mathbb{C}$ -algebra of  $s$ -Gevrey series summable in all directions but (perhaps)  $d_1, \dots, d_r$ , the set

$$\{\mathbb{C}\{z\}_s^{-d_1 \dots -d_r}\}$$

defines a direct system by the inclusions

$$\mathbb{C}\{z\}_s^{-d_1 \cdots -d_r} \hookrightarrow \mathbb{C}\{z\}_s^{-d_1 \cdots -d_{r+1}}$$

and

$$\mathbb{C}\{z\}_s \cong \varprojlim \mathbb{C}\{z\}_s^{-d_1 \cdots -d_r}.$$

So we have

**THEOREM 3.4.** —  $\mathbb{C}\{z\}_{s,d}$  and  $\mathbb{C}\{z\}_s$  are nuclear.

In fact, all the above constructions are projective limits or numerable inductive limits of nuclear spaces, and so, they are also nuclear.

### 3.6. Multisummability

There are several definitions of multisummable power series in the literature, but all of them agree. If  $0 < s_1 < s_2 < \cdots < s_q < +\infty$ ,  $k_i = 1/s_i$ , and  $d_1, \dots, d_q$  are directions “close enough” (i.e., according to [B],  $|d_j - d_{j-1}| \leq \frac{\pi}{2\kappa_j}$ ,  $2 \leq j \leq q$ , where  $1/\kappa_j = s_j - s_{j-1}$ ), a series  $\hat{f} \in \mathbb{C}[[z]]_{s_q}$  is  $\mathbf{k} = (k_1, \dots, k_q)$ -multisummable in the multidirection  $\mathbf{d} = (d_1, \dots, d_q)$  if and only if  $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_q$ , where  $\hat{f}_1 \in \mathbb{C}\{z\}_{s_i, d_i}$ . This is equivalent to say that

$$f := \mathcal{L}_{k_1, d_1} \circ \mathcal{A}_{k_1, k_2} \circ \cdots \circ \mathcal{A}_{k_{q-1}, k_q} \circ \hat{\mathcal{B}}_{k_q} \hat{f}$$

is well defined, where  $\mathcal{A}_{k_j, k_{j+1}}$  are the acceleration operators introduced by Écalle, that can be defined by the property

$$\mathcal{L}_k \circ \mathcal{A}_{k, k'} = \mathcal{L}_{k'}$$

if  $k > k'$  (in appropriate domains, see [B] for details). The function  $f$  we have obtained is called the *multisum* of  $f$  in the multidirection  $\mathbf{d}$ . This approach, by means of the acceleration operators, is more useful to our purposes. The functions

$$g_j := \mathcal{A}_{k_j, k_{j+1}} \circ \cdots \circ \mathcal{A}_{k_{q-1}, k_q} \circ \hat{\mathcal{B}}_{k_q} \hat{f}, \quad j = 1, \dots, q$$

are defined in a sector of bisecting direction  $d_j$ , and have exponential growing of order  $k_j$  there, so a family of seminorms taking into account this increasing can be defined for each  $j$ , as in 3.5. Also,  $\hat{\mathcal{B}}_{k_q} \hat{f}$  is convergent, and another family of seminorms reflects this property. Collecting all these seminorms, a locally convex structure may be defined in  $\mathbb{C}\{z\}_{\mathbf{k}, \mathbf{d}}$  as before.

A direction  $d_j$  is singular of level  $k_j$  for a series  $\hat{f}$  if directions  $d_{j+1}, \dots, d_q$  can be chosen such that

$$f_l = \mathcal{A}_{k_l, k_{l+1}} \circ \dots \circ \mathcal{A}_{k_{q-1}, k_q} \circ \hat{B}_{k_q} \hat{f}$$

are well defined ( $l \geq j$ ) but  $\mathcal{A}_{k_{j-1}, k_j} f_j$  is not. If, given  $k_1 > \dots > k_q$ ,  $\hat{f}$  has a finite number of singular directions at each level,  $\hat{f}$  is  $\mathbf{k}$ -multisummable.  $\mathbb{C}\{z\}_{\mathbf{k}}$  is the set of  $\mathbf{k}$ -multisummable power series.

The same reasonings as in the summable case allows us to show that  $\mathbb{C}\{z\}_{\mathbf{k}, \mathbf{d}}$ ,  $\mathbb{C}\{z\}_{\mathbf{k}}$  are nuclear locally convex spaces.

#### 4. Conclusion

The properties of nuclearity we have shown for the sheaves of summable and multisummable series allow us to define similar notions in several variables. As all the considered spaces are nuclear, the completed tensor product

$$\mathbb{C}\{z_1\}_{s_1} \hat{\otimes} \mathbb{C}\{z_2\}_{s_2}$$

can be taken as a definition of  $(s_1, s_2)$ -summable series in two variables ( $s_1$ -summable with respect to  $z_1$ ,  $s_2$ -summable with respect to  $z_2$ ). The same can be done in order to define multisummable power series in several variables.

It would be interesting to find an appropriate version of Künneth formula in order to make the computations of the cohomology of asymptotic sheaves (see [M1, M2, Ha, Z, Mo]) easier. Forgetting the radius of the sectors, a sheaf  $\mathcal{A}$  (functions with asymptotic development) can be defined over  $\mathbb{S}^1$ . Let  $\mathcal{A}_0$  be the subsheaf of  $\mathcal{A}$  of functions with null asymptotic development. These are Fréchet sheaves. In two variables, the completed tensor product  $\mathcal{A} \hat{\otimes} \mathcal{A}$  is the sheaf over  $\mathbb{T}^2$  of the functions with asymptotic development, and  $\mathcal{A}_0 \hat{\otimes} \mathcal{A}_0$  the functions  $f$  with asymptotic development such that all the functions of the family  $TA(f)$  are zero. Malgrange and Sibuya showed that  $H^1(\mathbb{S}^1; \mathcal{A}_0) \cong \mathbb{C}[[z]]/\mathbb{C}\{z\}$ . In [K], a Künneth formula is developed to compute  $H^n(X \times Y; \mathcal{F} \hat{\otimes} \mathcal{G})$ , where  $\mathcal{F}, \mathcal{G}$  are Fréchet sheaves, and  $X, Y$  second countable paracompact Hausdorff spaces. The precise result is

**THEOREM 4.1.** — *Under the hypothesis:*

1. *There are arbitrarily fine coverings  $\mathcal{U}, \mathcal{V}$  of  $X, Y$  respectively, such that  $H^p(\mathcal{U}; \mathcal{F}), H^q(\mathcal{V}; \mathcal{G})$  are Hausdorff ( $p, q \leq n$ ).*
2.  *$\mathcal{F}$  (or  $\mathcal{G}$ ) is nuclear.*

Then,

$$(*) \quad H^n(X \times Y; \mathcal{F} \hat{\otimes} \mathcal{G}) \cong \prod_{p+q=n} H^p(X; \mathcal{F}) \hat{\otimes} H^q(Y; \mathcal{G}).$$

Unfortunately, this result does not fit our situation, in order to compute  $H^1(\mathbf{T}^2; \mathcal{A}_0)$ , because the first hypothesis is not fulfilled: the first cohomology group  $H^1(\mathbf{S}^1; \mathcal{A}_0) \cong \mathbb{C}[[z]]/\mathbb{C}\{z\}$  is not Hausdorff, as  $\mathbb{C}\{z\}$  is not closed in the Fréchet topology of  $\mathbb{C}[[z]]$ . In fact, it is known that

$$H^1(\mathbf{T}^2; \mathcal{A}_0) \cong \hat{\mathbb{C}}\{z_1, z_2\}/\mathbb{C}\{z_1, z_2\},$$

where  $\hat{\mathbb{C}}\{z_1, z_2\}$  denotes the set of series  $\sum_{i,j} a_{ij} z_1^i \cdot z_2^j$  such that all the one variable series  $\sum_i a_{ij} z_2^j$ ,  $\sum_i a_{ij} z_1^i$  are convergent with the same radius of convergence, and the second member of the equivalence (\*) in theorem 4.1 would be zero, as  $H^0(\mathbf{S}^1; \mathcal{A}_0) = 0$ . So, Künneth formula seems not to be applicable in this situation.

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