ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

OLE E. BARNDORFF-NIELSEN
TINA HVIID RYDBERG

Infinite trees and inverse gaussian random variables

Annales de la faculté des sciences de Toulouse 6e série, tome 8, nº 1 (1999), p. 25-34
<http://www.numdam.org/item?id=AFST_1999_6_8_1_25_0>

Annales de la Faculté des Sciences de Toulouse Vol. VIII, n° 1, 1999

Infinite Trees and Inverse Gaussian Random Variables(*)

OLE E. BARNDORFF-NIELSEN(1) and TINA HVIID RYDBERG(2)

RÉSUMÉ. — On montre que la résistance totale d'un arbre infini peut être modélisée par une variable aléatoire de type gaussienne inverse, en tenant compte aussi du cas où l'arbre n'est pas infini dans toutes les directions et possède des puits de potentiel qui dépendent des trajectoires. On étudie aussi des lois conditionnelles sur les arbres finis.

ABSTRACT. — It is shown that the total resistance of an infinite tree can be modelled as a reciprocal inverse Gaussian random variable, also in cases where the infinite tree is not infinite in all directions and, furthermore, has path dependent potential drops. Finally we study conditional distributions on finite trees.

KEY-WORDS : (reciprocal) inverse Gaussian distributions; Kirchoff-Ohm laws.

1. Introduction

The present paper generalizes results found in O. E. Barndorff-Nielsen and A. E. Koudou [2]. In O. E. Barndorff-Nielsen [1] it was shown that for finite trees with independent inverse or reciprocal inverse Gaussian (IG or RIG) edge resistances, the total resistance is a RIG random variable provided the distributional parameters involved satisfy certain natural conditions. O. E. Barndorff-Nielsen and A. E. Koudou [2] extended the result to infinite trees, but assumed that the potential drop did not depend

(*) Reçu le 10 mars 1998, accepté le 19 juin 1998

(1) Department of Theoretical Statistics, University of Aarhus, Ny Murkegade, DK-8000 Aarhus C (Denmark)
e-mail: atsoebn@mi.aau.dk

(2) Nuffield College, New Road, Oxford OX1 1NF (United Kingdom)
e-mail: tina.rydberg@nuf.ox.ac.uk

on the specific path. We will generalize this result to situations where the potential drop may be path dependent. Furthermore, they assumed that the tree was infinite in all directions. Here we will study trees that can have both finite and infinite parts. Section 2 summarizes the relevant properties of the inverse Gaussian and reciprocal inverse Gaussian distributions and the Kirchoff–Ohm laws, and the extensions of earlier work, indicated above, are established in Section 3.

2. IG, RIG and GIG distributions

The generalized inverse Gaussian distribution $\mathrm{GIG}(\lambda, \delta, \gamma)$ has density function

$$\mathrm{gig}(x; \lambda, \delta, \gamma) = \frac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta\gamma)} x^{\lambda-1} \exp\left(-\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right), \quad x > 0 \qquad (1)$$

where the domain of variation of $(\lambda, \delta, \gamma)$ is given by

$$\begin{cases} \delta \geq 0, & \gamma > 0 \quad \text{if } \lambda > 0 \\ \delta > 0, & \gamma > 0 \quad \text{if } \lambda = 0 \\ \delta > 0, & \gamma \geq 0 \quad \text{if } \lambda < 0 \end{cases}$$

and where K_{λ} denotes the modified Bessel function of the third kind with index λ. In case $\delta = 0$ and $\gamma = 0$ the norming constant in (1) has to be interpreted in terms of the limit of $K_{\lambda}(y)$ for $y \downarrow 0$ (For relevant properties of the Bessel functions see e.g. B. Jørgensen [3]). The GIG distributions possess the property that

$$X \sim \mathrm{GIG}(\lambda, \delta, \gamma) \iff X^{-1} \sim \mathrm{GIG}(-\lambda, \gamma, \delta). \qquad (2)$$

Furthermore, for every constant $a > 0$

$$X \sim \mathrm{GIG}(\lambda, \delta, \gamma) \implies aX \sim \mathrm{GIG}(\lambda, a^{1/2}\delta, a^{-1/2}\gamma). \qquad (3)$$

The most prominent member of the family of GIG distributions is the inverse Gaussian $\mathrm{IG}(\delta, \gamma) = \mathrm{GIG}(-1/2, \delta, \gamma)$ with density function given by

$$\mathrm{ig}(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} \exp\left(-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right).$$

The IG distribution can be given a probabilistic interpretation as the *first* hitting time to the level δ of a Brownian motion with drift γ and diffusion coefficient 1.

Using the relation in (2) leads from $\mathrm{IG}(\delta,\gamma)$ to the reciprocal inverse Gaussian $\mathrm{RIG}(\delta,\gamma) = \mathrm{GIG}(1/2,\delta,\gamma)$ distribution with density function

$$\mathrm{rig}(x;\delta,\gamma) = \frac{\gamma}{\sqrt{2\pi}} e^{\gamma\delta} x^{-1/2} \exp\left(-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right) .$$

Like the IG distribution the RIG distribution can be given a probabilistic interpretation. In fact the RIG distribution is the distribution of the *last* hitting time to the level δ of a Brownian motion with drift γ and diffusion coefficient 1 (cf. P. Vallois [4]). These hitting time interpretations are of direct relevance for the type of models for resistances on trees considered in the next section, see [2].

The gamma (Γ) distribution is also in the family of GIG distributions. It is the special case where $\lambda > 0$ and $\delta = 0$, i.e. $\mathrm{GIG}(\lambda, 0, \gamma) = \Gamma(\lambda, \gamma^2/2)$, and the density is then given by

$$\gamma\left(x;\lambda,\frac{\gamma^2}{2}\right) = \frac{(\gamma^2/2)^\lambda}{\Gamma(\lambda^2)} x^{\lambda-1} \exp\left(-\frac{1}{2}\gamma^2 x\right) .$$

We have the following well known convolution properties of GIG random variables:

1) $\Gamma\left(\lambda_1,\frac{\gamma^2}{2}\right) * \Gamma\left(\lambda_2,\frac{\gamma^2}{2}\right) = \Gamma\left(\lambda_1+\lambda_2,\frac{\gamma^2}{2}\right)$,

2) $\mathrm{IG}(\delta_1,\gamma) * \mathrm{IG}(\delta_2,\gamma) = \mathrm{IG}(\delta_1+\delta_2,\gamma)$,

3) $\mathrm{IG}(\delta_1,\gamma) * \mathrm{RIG}(\delta_2,\gamma) = \mathrm{RIG}(\delta_1+\delta_2,\gamma)$,

4) $\mathrm{GIG}(-\lambda,\delta,\gamma) * \Gamma\left(\lambda,\frac{\gamma^2}{2}\right) = \mathrm{GIG}(\lambda,\delta,\gamma)$ for every $\lambda > 0$.

Note that the properties 2) and 3) are immediate consequences of the hitting time interpretations.

By the Kirchoff–Ohm laws, if two networks with resistances R and R' are connected sequentially the total resistance is $R + R'$ while if they are connected in parallel the overall resistance is

$$(R^{-1} + R'^{-1})^{-1} . \tag{4}$$

3. Infinite trees

Let $T = (\mathcal{V}, \mathcal{E}, s)$ be an infinite rooted tree, i.e. a connected oriented acyclic graph, with root s, set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$. If $v \in \mathcal{V}\backslash\{s\}$ let $\zeta(v)$ denote the vertex preceding v according to the order on the tree. A path is a sequence $p = (v_1, \ldots, v_n, \ldots)$ such that $v_n = \zeta(v_{n+1})\ \forall\, n$. We define an *infinite ray* $\pi = (s, v_1, \ldots, v_n, \ldots)$ as an infinite path starting at s and a *finite ray* $\pi = (s, v_1, \ldots, v_n)$ as a finite path starting at s and such that there does not exist $v \in \mathcal{V}\backslash\pi$ for which $v_n = \zeta(v)$. Let

$$\partial T = \{\text{infinite rays}\} \cup \{\text{finite rays}\}$$

denote the boundary of T.

Let φ be a potential function on T, i.e. φ is a function $\varphi : \mathcal{V} \mapsto [0, \infty)$, suppose that φ is non increasing according to the natural order on T, and let

$$\psi(e) = \varphi(\zeta(v)) - \varphi(v)$$

be the drop in potential along the edge $e \in \mathcal{E}$, $e = \{\zeta(v), v\}$. We assume that φ is zero at ∂T, i.e. if $\pi = (s, v_1, \ldots, v_n)$ is a finite ray then $\varphi(v_n) = 0$ and if $\pi = (s, v_1, \ldots, v_n, \ldots)$ is an infinite ray then $\varphi(v_n) \downarrow 0$ as $n \to \infty$.

For $v \in \mathcal{V}$, let V_v denote the subset of the boundary which can be reached from a path starting in v and let T_v be the subtree with root v. We identify V_v and ∂T_v.

Let γ be a deterministic measure on ∂T, satisfying

$$\gamma(\partial T_v) > 0 \quad \forall\, v \in \mathcal{V}.$$

Furthermore, let M be a random measure on ∂T, such that

- if A and B are disjoint, then $M(A)$ and $M(B)$ are independent;
- the distribution of $M(A)$ is $\mathrm{IG}(\gamma(A), 0)$ for any Borel subset of ∂T.

The existence of such measures follows simply from the particular case discussed in Proposition 5.1 of [2].

We equip the edges $e \in \mathcal{E}$, $e = \{\zeta(v), v\}$, with inverse Gaussian random variables X_e such that the X_e's are mutually independent and independent of M, and

$$X_e \sim \mathrm{IG}(\psi(e), \gamma(\partial T_v)).$$

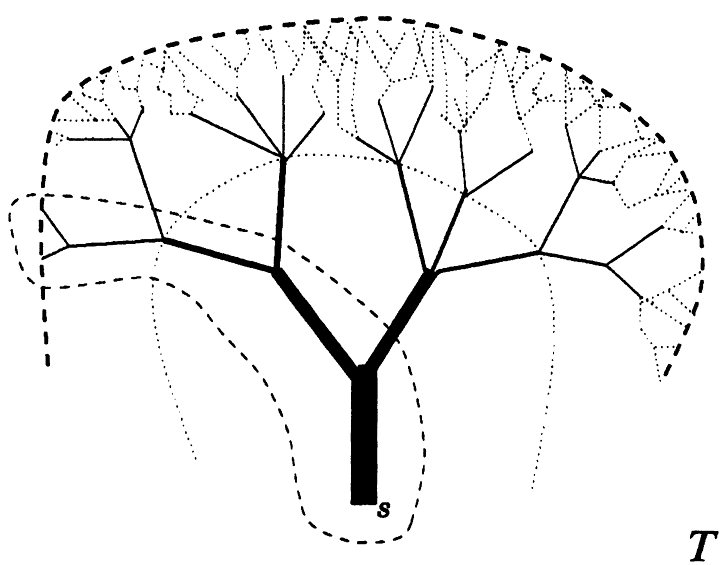

Fig. 1 Example of a tree which consists of a finite (dashed line) and an infinite part. The boundary of the tree is the thick dashed line. The dotted line delimits the tree $T^{(3)}$ of height 3, s denotes the root of the tree.

The influence of the boundary on the resistance will be modelled by means of the random measure M. In particular, if $\pi = (s, v_1, \ldots, v_n)$ is a finite ray we equip π (or v_n) with the gamma random variable

$$W_{v_n} = \left(M(\partial T_{v_n})\right)^{-1} \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\gamma(\partial T_{v_n})^2\right) .$$

The tree T equipped with these random resistances we denote by $\boldsymbol{T}$.

It is our goal to give a natural definition of the resistance of the whole tree $\boldsymbol{T}$ and to show that with this definition we have

$$R(\boldsymbol{T}) \sim \mathrm{RIG}(\varphi(s), \gamma(\partial T)) .$$

In order to do this we define the following auxiliary variables. Let $\boldsymbol{T}^{(n)}$ denote the tree of height n, see Figure 1. At height n we define three other trees $\widetilde{\boldsymbol{T}}^{(n)}$, $\widetilde{\boldsymbol{T}}_c^{(n)}$ and $\breve{\boldsymbol{T}}^{(n)}$, see Figures 2 and 3:

$\widetilde{\boldsymbol{T}}^{(n)}$: Take $\boldsymbol{T}^{(n)}$ and to each end vertex $v \in \partial T^{(n)} \backslash \partial T$ associate the gamma random variable $W_v = \left(M(\partial T_v)\right)^{-1}$ with distribution $\Gamma(1/2, (1/2)\gamma(\partial T_v)^2)$. Such a random variable has already been associated to all finite rays.

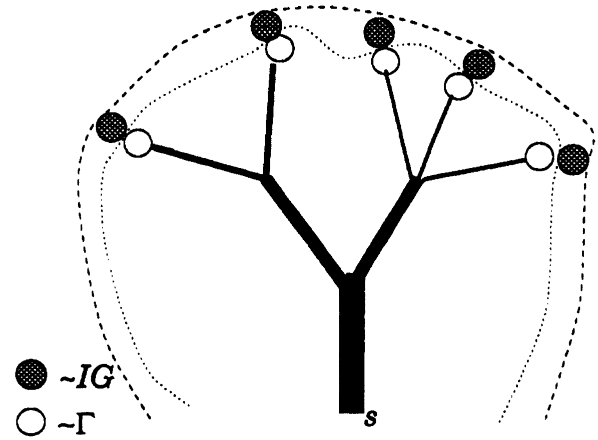

Fig. 2 $T^{(3)}$ from Figure 1 and the added random variables.
The dotted line delineates $\widetilde{T}^{(3)}$ and the dashed line delineates $\check{T}^{(3)}$.

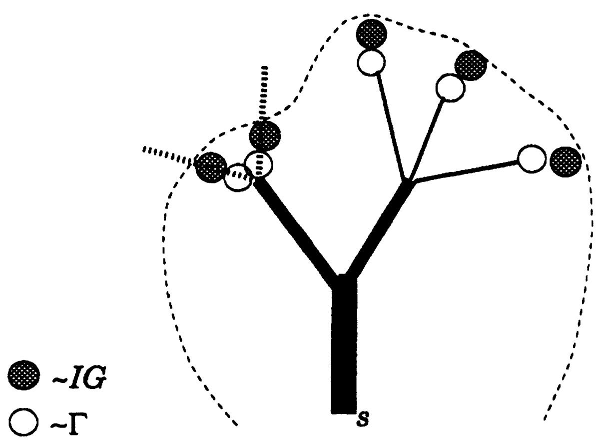

Fig. 3 $\widetilde{T}_c^{(3)}$ from Figure 1 delineated by the dashed line and the added random variables.
Notice that the left side has been cut in the edges.
The parts which have been cut are shown by dashed edges.

$\widetilde{T}_c^{(n)}$: Let

$$\varphi_n = \max\{\varphi(u) \mid u \in \partial T^{(n)}\}$$

and let $\widetilde{T}_c^{(n)}$ be the subtree with root s whose boundary is given by

$$\partial \widetilde{T}_c^{(n)} = \left\{ v \in \mathcal{V}_{|T^{(n)}} : \varphi(v) \geq \varphi_n \wedge \nexists\, w \in \mathcal{V}_{|T^{(n)}} : \varphi_n \leq \varphi(w) \leq \varphi(v) \right\} .$$

To each of these end vertices associate the gamma random variable $W_v = M^{-1}(\partial T_v)$ with distribution $\Gamma(1/2, (1/2)\gamma(\partial T_v)^2)$ and an inverse Gaussian random variable C_v with distribution $\mathrm{IG}(\varphi(v) - \varphi_n, \gamma(\partial T_v))$ (we interpret $\mathrm{IG}(0, \gamma(\partial T_v))$ as the degenerate distribution at 0). The random variables C_v are assumed to be independent and independent of M and of the edge resistances X_e of $\widetilde{\boldsymbol{T}}_c^{(n)}$. In view of the convolution property 2 of the inverse Gaussian distribution, if $v \in \partial\widetilde{T}_c^{(n)}$ the resistance X_e of any edge e with initial point v and endpoint u could from the start have been written as $C_v + Q_e$ where $Q_e \sim \mathrm{IG}(\varphi_n - \varphi(u), \gamma(\partial T_v))$ and C_v and Q_e are independent. So C_v may be viewed as corresponding to the part of the edge, starting from v, which have more potential than φ_n.

$\breve{\boldsymbol{T}}^{(n)}$: Take $\widetilde{\boldsymbol{T}}^{(n)}$ and to each end vertex $v \in \partial T^{(n)} \backslash \partial T$ associate an inverse Gaussian random variable Z_v with distribution $\mathrm{IG}(\varphi(v), \gamma(\partial T_v))$ and such that the Z_v's are independent and independent of M and $\{X_e \mid e \in \mathcal{E}\}$.

By this construction of $\widetilde{\boldsymbol{T}}^{(n)}$, $\widetilde{\boldsymbol{T}}_c^{(n)}$ and $\breve{\boldsymbol{T}}^{(n)}$, we have that

$$R(\widetilde{\boldsymbol{T}}_c^{(n)}) \le R(\widetilde{\boldsymbol{T}}^{(n)}) \le R(\breve{\boldsymbol{T}}^{(n)}) \tag{5}$$

Below we will show that:

(i) $R(\widetilde{\boldsymbol{T}}^{(n)})$ is increasing;

(ii) $R(\widetilde{\boldsymbol{T}}^{(n)}) \xrightarrow{\sim} \mathrm{RIG}(\varphi(s), \gamma(\partial T))$.

This will imply that $R(\widetilde{\boldsymbol{T}}^{(n)})$ converges a.s. to a random variable $R(\boldsymbol{T})$, with $\mathcal{L}(R(\boldsymbol{T})) = \mathrm{RIG}(\varphi(s), \gamma(\partial T))$, which we define as the total resistance of $\boldsymbol{T}$.

To establish (ii) we first note that

$$R(\breve{\boldsymbol{T}}^{(n)}) \sim \mathrm{RIG}(\varphi(s), \gamma(\partial T)) \tag{6}$$

independently of n. This follows from [2] since Theorem 3.1 therein applies directly as

$$\begin{cases} W_v + Z_v \sim \mathrm{RIG}(\varphi(v), \gamma(\partial T_v)) & v \in \partial\breve{T}^{(n)} \\ X_e \sim \mathrm{IG}(\psi(e), \gamma(\partial T_v)) & \text{otherwise} \end{cases}$$

and

$$\sum_{v \in \partial\breve{T}^{(n)}} \gamma(\partial T_v) = \gamma(\partial T).$$

Furthermore we have, again by [2, Theorem 3.1], that

$$R(\widetilde{\boldsymbol{T}}_c^{(n)}) \sim \mathrm{RIG}(\varphi(s) - \varphi_n, \gamma(\partial T))$$

since

$$\begin{cases} W_v + C_v \sim \mathrm{RIG}(\varphi(v) - \varphi_n, \gamma(\partial T_v)) & v \in \partial \widetilde{T}_c^{(n)} \\ X_e \sim \mathrm{IG}(\psi(e), \gamma(\partial T_v)) & \text{otherwise.} \end{cases}$$

By the assumption that $\varphi_n \to 0$ it follows that

$$R(\widetilde{\boldsymbol{T}}_c^{(n)}) \xrightarrow{\sim} \mathrm{RIG}(\varphi(s), \gamma(\partial T)) \,. \tag{7}$$

Together with (5) and (6), this implies (ii) as follows from standard properties of Laplace transforms.

For $v \in \partial T^{(n)}$, let

$$\widetilde{W}_v = X_{\{\zeta(v),v\}} + W_v \,. \tag{8}$$

Then the resistance $R(\widetilde{\boldsymbol{T}}^{(n)})$ can be written as a function $\widetilde{f}_n$,

$$R(\widetilde{\boldsymbol{T}}^{(n)}) = \widetilde{f}_n \left(\{X_e : e \in \mathcal{E}_{|T^{(n-1)}}\}, \{\widetilde{W}_v : v \in \partial T^{(n)}\} \right)$$

where $\mathcal{E}_{|T^{(n-1)}}$ denotes the set of edges in $T^{(n-1)}$.

It can be shown, using the same technique as in [2], that for any $u \in \partial T^{(n)}$ the function $\widetilde{f}_n$ can be expressed as a Möbius transformation

$$\widetilde{f}_n = \frac{A_u \widetilde{W}_u + B_u}{C_u \widetilde{W}_u + D_u}$$

where $A_u D_u - B_u C_u > 0$, and thereby we have that $\widetilde{f}_n$ is increasing in $\widetilde{W}_u$. Next, note that $R(\widetilde{\boldsymbol{T}}^{(n+1)})$ can be expressed in terms of $\widetilde{f}_n$, as

$$R(\widetilde{\boldsymbol{T}}^{(n+1)}) = \widetilde{f}_n \left(\{X_e : e \in \mathcal{E}_{|T^{(n-1)}}\}, \{\widetilde{W}'_v : v \in \partial T^{(n)}\} \right)$$

where

$$\widetilde{W}'_v = X_{\{\zeta(v),v\}} + \left(\sum_{i=1}^{n} (X_{\{v,u_i\}} + W_{u_i})^{-1} \right)^{-1} = X_{\{\zeta(v),v\}} + \left(\sum_{i=1}^{n} \widetilde{W}_{u_i}^{-1} \right)^{-1} \tag{9}$$

and

$$\zeta^{-1}(v) = \{u_1, u_2, \ldots, u_n\} \,.$$

See Figure 4 for an illustration.

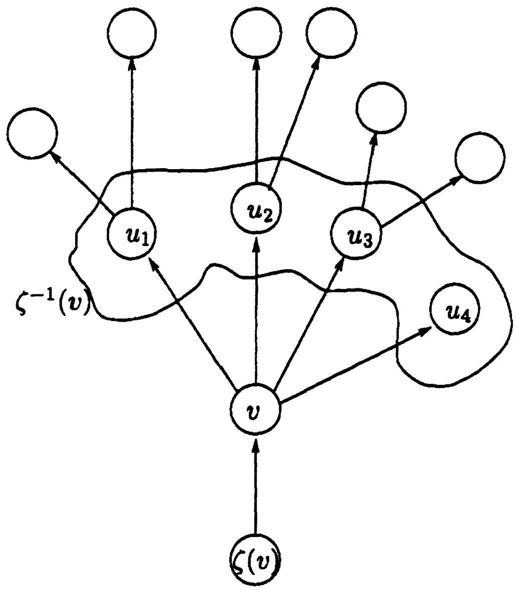

Fig. 4 Part of a tree showing the vertices v, $\zeta(v)$ and $\zeta^{-1}(v)$.

In order to establish (i) we need to show that $\widetilde{W}_v < \widetilde{W}'_v$, $\forall\, v \in \partial T^{(n)}$. By (8) and (9) we have

$$\widetilde{W}'_v - \widetilde{W}_v = \left(\sum_{i=1}^{n} \widetilde{W}_{u_i}^{-1}\right)^{-1} - W_v\,.$$

Further, noting that, by definition, $W_v^{-1} = M(\partial T_v)$ we find from (8) that $\widetilde{W}_v^{-1} < M(\partial T_v)$ and thereby

$$\sum_{i=1}^{n} \widetilde{W}_{u_i}^{-1} < \sum_{i=1}^{n} M(\partial T_{u_i}) = M(\partial T_v) = W_v^{-1}\,.$$

Hence (i) is established. □

Acknowledgement

We are greatful to Angelo Koudou for helpful comments.

References

[1] BARNDORFF-NIELSEN (O. E.) .— *A note on electrical networks and the inverse Gaussian distribution*, Adv. Appl. Probab. **26** (1994), pp. 63-67.

[2] BARNDORFF-NIELSEN (O. E.) and KOUDOU (A. E.) .— *Trees with random conductivities and the (reciprocal) inverse Gaussian distribution*, Adv. Appl. Probab. **30** (1998), pp. 409-424.

[3] JØRGENSEN (B.) .— *Statistical Properties of the Generalised Inverse Gaussian Distribution*, Lecture Notes in Statistics, Springer-Verlag, New York, **9** (1982).

[4] VALLOIS (P.) .— *La loi gausienne inverse généralisée, comme premier ou dernier temps de passage de diffusion*, Bull. Sc. Math., Série 2, **115** (1991), pp. 301-368.