ABDOU KOULDER BEN-NAOUM
YVES FÉLIX

Lefschetz number and degree of a self-map


<http://www.numdam.org/item?id=AFST_1997_6_6_2_229_0>
Lefschetz number and degree of a self-map(*)

ABDOU KOULDER BEN-NAOUM et YVES FÉLIX(1)

1. Introduction

Let $X$ be a connected topological space that has the homotopy type of a finite CW complex, and let $f$ be a self-map. We denote by $H_n(f)$ the linear map induced by $f$ in rational homology. The number $\lambda_f = \sum_i (-1)^i \text{tr} H_i(f)$ is called the Lefschetz number of $f$. It is well known that two homotopic self-maps $f$ and $g$ have the same Lefschetz number.

We here prove the following result.

**Theorem 1.1.** — *Suppose $H_\ast(\Omega f; \mathbb{Q}) = H_\ast(\Omega g; \mathbb{Q})$, then the Lefschetz numbers of $f$ and $g$ are equal, $\lambda_f = \lambda_g$. Moreover, if $X$ is a manifold then $\text{deg}(f) = \text{deg}(g)$.*

(*) Reçu le 27 avril 1995

(1) Institut de Mathématique, Université Catholique de Louvain, 2, chemin du Cyclotron, B-1348 Louvain-La-Neuve (Belgique)
This results suggests the following problem: Is it possible to deduce the number $\lambda_f$ from the sequence of linear maps $\pi_n(f) \otimes \mathbb{Q}$, when the space is 1-connected?

In the general case the answer is negative. For instance the spaces

$$X = P^2(\mathbb{C}) \times (S^3 \vee S^2) \quad \text{and} \quad Y = S^2 \times (S^5 \vee S^2 \vee S^4)$$

have the same rational homotopy groups but not the same Euler–Poincaré characteristic $\chi(X) = 3$, $\chi(Y) = 4$. Moreover the spaces $X$ and $Y$ appear as total spaces of homotopy fibrations with the same homotopy fibre $Z = \bigvee_{n \geq 4} S^n \vee (\bigvee_{n \geq 5} S^n)$ and with basis $X'$ and $Y'$ finite CW complexes with the same Euler–Poincaré characteristic:

$$X' = P^2(\mathbb{C}) \times S^3 \times S^2, \quad Y' = S^2 \times S^5 \times S^2.$$

Recall that a finite simply connected CW complex is elliptic if

$$\sum_n \dim \pi_n(X) \otimes \mathbb{Q} < \infty.$$

In the elliptic case a result of Halperin [8] shows that the Lefschetz number of a self-map $f$ can be deduced from the induced maps in homotopy.

**Theorem 1.2** [8]. Let $f$ be a self-map. Denote by $A_n$ the matrix representing the linear map $\pi_n(f) \otimes \mathbb{Q}$ in some basis. We then have

$$\lambda_f = \lim_{t \to 1} \prod_n \frac{\det(1 - t^{2n+2}A_{2n+1})}{\det(1 - t^{2n}A_{2n})}.$$

An elliptic space is a Poincaré duality space. Our next result is a formula giving the degree of $f$ in terms of the induced maps in rational homotopy.

**Theorem 1.3.** Denote by $\lambda_i$ the eigenvalues of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$ and by $\mu_j$ the eigenvalues of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$, then

$$\deg(f) = \frac{\prod_j \mu_j}{\prod_i \lambda_i}.$$
2. Bar construction

Let $A$ be an augmented graded differential $\mathbb{Q}$-algebra,

$$A = \bigoplus_{n \geq 0} A_n, \quad d : A_n \to A_{n-1}, \quad \varepsilon : A \to \mathbb{Q}. $$

The bar construction on $A$ is the coalgebra $(B(A), D) = (T(s\overline{A}), D)$ with $s\overline{A} = \ker \varepsilon$ the augmentation ideal, and $D := D_1 + D_2$. The differentials $D_i$ are given by:

$$D_1 [s a_1 | \ldots | s a_k] := -\sum_{i=1}^k (-1)^{\varepsilon_i} [s a_1 | \ldots | s a_i | \ldots | s a_k]$$

and

$$D_2 [s a_1 | \ldots | s a_k] := \sum_{i=2}^k (-1)^{\varepsilon_i} [s a_1 | \ldots | s a_{i-1} a_i | \ldots | s a_k]$$

with $\varepsilon_i = \sum_{j<i} |s a_j|$. By filtering $T(s\overline{A})$ by the coalgebras $T^{\leq k}(s\overline{A})$, we obtain a spectral sequence satisfying

$$(E^1, d_1) \cong (B(H_*(A), D_2)) , \quad E^2 \cong H_*(B(H_*(A), D_2))$$

and converging to $H_*(B(A))$ [3]. One interesting property of the bar construction is the following theorem of J.- C. Moore [11].

**Theorem [11].** — *If $X$ is connected, then there exists a natural quasi-isomorphism of differential graded coalgebras*

$$C_*(X) \congto B(C_*(\Omega X)).$$

When applied to $A = C_*(\Omega X)$, the bar construction gives thus a spectral sequence

$$H_*(B(H_*(\Omega X))) \Longrightarrow H_*(B(C_*(\Omega X))) \cong H_*(X).$$

This will be the main ingredient in the proof of Theorem 1.1.
THEOREM 2.1

a) Let X be a connected space, and f, g be endomorphisms of X inducing the same map on the rational homotopy groups, \( \pi_n(f) \otimes \mathbb{Q} = \pi_n(g) \otimes \mathbb{Q} \), for \( n \geq 2 \), and on the fundamental group, then for \( n \geq 0 \), we have

\[
\text{tr } H_n(f; \mathbb{Q}) = \text{tr } H_n(g; \mathbb{Q}).
\]

b) If \( \sum_i \dim H^i(X, \mathbb{R}) < \infty \), then \( \lambda_f = \lambda_g \).

c) If X is a manifold, then \( \deg(f) = \deg(g) \).

Proof. — Denote by \( \tilde{X} \) the universal cover of X. For any field k, the fibration

\[
\tilde{X} \rightarrow X \rightarrow K(\pi_1(X), 1)
\]

induces an exact sequence of Hopf algebras [4]:

\[
k \rightarrow H_*(\Omega \tilde{X}; k) \rightarrow H_*(\Omega X; k) \rightarrow k[\pi_1(X)] \rightarrow k.
\]

When \( k = \mathbb{Q} \), the Milnor–Moore theorem [12] implies that

\[
H_*(\Omega \tilde{X}; \mathbb{Q}) \cong U \pi_*(\Omega \tilde{X}) \otimes \mathbb{Q} \cong U \pi_{\geq 1}(\Omega X) \otimes \mathbb{Q},
\]

where \( U \) means the enveloping algebra functor. It clearly follows that \( f \) and \( g \) induce the same maps on \( H_*(\Omega X; \mathbb{Q}) \).

On the other hand \( f \) and \( g \) induce maps:

\[
C_*(\Omega X) \xrightarrow{C_*(f)} C_*(\Omega X), \quad C_*(\Omega X) \xrightarrow{C_*(g)} C_*(\Omega X)
\]

and therefore morphisms of spectral sequences

\[
E^i \xrightarrow{E^i(f)} E^i, \quad E^i \xrightarrow{E^i(g)} E^i.
\]

At the \( E^1 \) level, these are the morphisms induced by the bar construction:

\[
B(H_*(\Omega X; \mathbb{Q})) \xrightarrow{B(H_*(\Omega f))} B(H_*(\Omega X; \mathbb{Q}))
\]

\[
B(H_*(\Omega X; \mathbb{Q})) \xrightarrow{B(H_*(\Omega g))} B(H_*(\Omega X; \mathbb{Q})).
\]

This implies that \( E^i(f) = E^i(g) \) for all \( i \geq 1 \) and thus \( E^\infty(f) = E^\infty(g) \).
Recall now that $E^\infty(X)$ is the graded vector space associated to $H_*(X)$ for some filtration. It follows that for $n \geq 0$, we have

$$\text{tr} H_n(f; \mathbb{Q}) = \text{tr} H_n(g; \mathbb{Q}).$$

*Example.* Let $X = S^2 \vee (S^1 \times S^1)$ and $f, g$ be the maps

$$f : X \xrightarrow{f_1} S^2 \xrightarrow{i} X, \quad g : X \xrightarrow{g_1} S^2 \xrightarrow{i} X$$

where

$$f_1 := \text{id} \vee h \quad \text{et} \quad g_1 := \text{id} \vee q.$$

Here $h$ consists to collapse $S^1 \times S^1$ into a point and $q$ is the canonical projection $q : S^1 \times S^1 \to S^1 \wedge S^1 \cong S^2$. Clearly $\pi_*(f) = \pi_*(g)$. However, $f$ and $g$ are not homotopic because the maps $h$ and $q$ are not. Nevertheless, by Theorem 2.1, $\lambda_f = \lambda_g$

**Corollary 2.1.** A self-map of a simply connected compact manifold or of a simply connected finite CW complex that induces zero on the rational homotopy groups has a fixed point.

### 3. Degree of a self-map of an elliptic space

In this section we prove Theorem 1.3.

**Theorem 3.1.** Let $X$ be an elliptic space and $f$ be a self-map. Denote by $\lambda_i$ the eigenvalues of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$ and by $\mu_j$ the eigenvalues of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$, then

$$\deg(f) = \frac{\prod_j \mu_j}{\prod_i \lambda_i}.$$

We prove the theorem by induction on the dimension of $\pi_*(X) \otimes \mathbb{Q}$ by using the theory of Sullivan minimal models [7]. When $\pi_*(X) \otimes \mathbb{Q} = 1$, a Sullivan minimal model of $X$ is given by $(\wedge u, 0)$, with $u$ of odd degree. Denote by $g$ a model of $f$. We have $g(u) = \mu \cdot u$ and $\deg(f) = \mu$.

We now suppose the theorem has been proved for self-maps of elliptic spaces $Z$ such that $\dim \pi_*(Z) \otimes \mathbb{Q} < n$ and we suppose that $f$ is a self-map...
of an elliptic space $X$ with $\dim \pi_*(X) \otimes \mathbb{Q} = n$. We denote by $(\bigwedge Z, d)$ the
Sullivan minimal model of $X$ and by $g : \bigwedge Z \to \bigwedge Z$ a minimal model for $f$. We denote also by $r$ the minimum degree $p$ with $\pi_p(X) \otimes \mathbb{Q} \neq 0$; we have $Z = Z^{\geq r}$ and $Z^r \neq 0$. We have to consider separately the case $r$ is odd and the case $r$ is even.

When $r$ is odd we form the KS extension (see [7] for the definition)

$$
(\bigwedge Z^r, 0) \to (\bigwedge Z, d) \to (\bigwedge Z^{> r}, d).
$$

The naturality of the Serre spectral sequence implies that

$$
\deg g = \deg g|_{\bigwedge Z^r} \cdot \deg \bar{g},
$$

where $\bar{g}$ denotes the projection of $g$ on $\bigwedge Z^{> r}$, $\bar{g} : \bigwedge Z^{> r} \to \bigwedge Z^{> r}$. We obtain the result by induction.

When $r$ is even, we work over the field of complex numbers and we choose an eigenvector $x$ of $Z^r : g(x) = \lambda x$. As $x$ is a cocycle of even degree, a power of $x$ has to be a coboundary: $[x^n] = 0$ for some $n$. We can take $n$ large enough so that $n \cdot |x| > 1 + |t|$ for all nonzero homogeneous elements $t$ in $Z$. We form the commutative differential graded algebra $(\bigwedge x \otimes \bigwedge y, d)$, $d(y) = x^m$. We extend $g$ to $(\bigwedge x \otimes \bigwedge y, d)$ by putting $g(y) = \lambda^m y$ and we form the KS extension of elliptic spaces

$$(\bigwedge x \otimes \bigwedge y, d) \to (\bigwedge Z \otimes \bigwedge y, d) \to (\bigwedge Z/\bigwedge x \cdot \bigwedge Z, d) = (\bigwedge Y, d).$$

By induction the formula for the degree is true for $(\bigwedge Y, d)$. The formula is trivially true for $(\bigwedge x \otimes \bigwedge y, d)$. By the Serre spectral sequence the formula is thus satisfied for $(\bigwedge Z \otimes \bigwedge y, d)$. A new application of the Serre spectral sequence applied to the KS extension

$$(\bigwedge Z, d) \to (\bigwedge Z \otimes \bigwedge y, d) \to (\bigwedge y, 0)$$

yields the result. □
4. Computation of the Lefschetz number

S. Halperin has shown [8] how the Lefschetz number of a self-map \( f \) of an elliptic space \( X \) can be deduced from the maps \( \pi_n(f) \otimes \mathbb{Q} \).

**Theorem 4.1 [8].** Denote by \( A_n \) the matrix representing the linear map \( \pi_n(f) \otimes \mathbb{Q} \) in some basis. We then have

\[
\lambda_f = \lim_{t \to 1} \prod_n \frac{\det(1 - t^{2n+2}A_{2n+1})}{\det(1 - t^{2n}A_{2n})}.
\]

This formula has been used by G. Lupton and J. Oprea [10] to prove that powers of self-maps of a Lie group always have a fixed point.

Among all simply connected spaces a very interesting class of spaces is given by the formal spaces. A 1-connected finite CW complex is called *formal* if \( X \) and its cohomology algebra have the same minimal model. For instance 1-connected compact Kähler manifolds are formal [2]. It follows from a general construction of Halperin-Stasheff [9] that a formal space \( X \) admits a special minimal model \((\Lambda Z, d)\) called its bigraded minimal model. This model is equipped with a bigradation \( Z = \bigoplus_{p,q \geq 0} Z_p^q \) satisfying:

1) \( d : Z_p^q \to (\Lambda \geq 2 Z)^{q+1}_{p-1} \)
2) \( H^*(\Lambda Z, d) = H_0^*(\Lambda Z, d) \).

Moreover if \( f \) is a continuous map from \( X \) into \( X \), then \( f \) admits a model \( f' : (\Lambda Z, d) \to (\Lambda Z, d) \) satisfying

\[
f'(Z_p^q) \subset (\Lambda Z)^q_{\leq p}.
\]

We denote by \( \phi_p^q \) the matrix representing the projection \( \theta_p^q \) of \( f' \) on \( Z_p^q \):

\[
\theta_p^q : Z_p^q \longrightarrow Z_p^q.
\]

We then have the following formula.

**Proposition 4.1.** With the previous notations

\[
\prod_{p,q} \left[ \det \left(1 - (-1)^{p+q} \phi_p^q \right) \right]^{(-1)^{q+1}} = \sum_n \text{tr} H^n(f; \mathbb{Q}) t^n.
\]
This formula enables the computation of the right hand side in terms of the characteristic polynomials of the matrices $\phi_p^q$ for $p + q \leq \dim X$.

Theorem 4.1 and Propositions 4.1 are in fact particular cases of a more general formula obtained for spaces equipped with a weight decomposition.

A space $X$ is said to have a weight decomposition if $X$ admits a minimal model $(\wedge Z, \partial)$ where $Z$ is given a bigradation $Z = \bigoplus_{p,q \geq 0} Z_p^q$ satisfying

$$d : Z_p^q \to \left( \wedge Z \right)_{p-1}^{q+1}.$$

**Proposition 4.2.** If $X$ is equipped with a weight decomposition, we have:

$$\prod_{p,q} \left[ \det \left( 1 - (-t)^{p+q} \phi_p^q \right) \right]^{-1} = \sum_n \left( \sum_{p+q=n} (-1)^p \text{tr} H_p^q(f) \right) t^n.$$

**Proof.** The Euler formula for the subcomplexes

$$0 \to \left( \wedge Z \right)_p^0 \to \left( \wedge Z \right)_{p-1}^1 \to \left( \wedge Z \right)_{p-2}^2 \to \cdots \to \left( \wedge Z \right)_0^p \to 0$$

gives the formula

$$\sum_i (-1)^{p-i} \text{tr} \left( \wedge f \right)_{p-i}^i = \sum_i (-1)^{p-i} \text{tr} H_{p-i}(f).$$

We then take the sum of these formulae and obtain:

$$\sum_n \left( \sum_i (-1)^{n-i} \text{tr} \left( \wedge f \right)_{n-i}^i \right) t^n = \sum_n \left( \sum_i (-1)^{n-i} \text{tr} H_{n-i}(f) \right) t^n.$$

This gives the result. \( \Box \)

A formal space admits a weight decomposition with cohomology concentrated in lower degree 0. Proposition 4.1 follows thus from Proposition 4.2.
Proof of Theorem 4.2. — Let $X$ be an elliptic space, $(\bigwedge Z, d)$ its minimal model and $g : (\bigwedge Z, d) \to (\bigwedge Z, d)$ the map induced by $f$. We denote by $d_\sigma$ the pure differential associated to $d$ [6]. This differential is defined by:

$$
\begin{cases}
d_\sigma(Z^{\text{even}}) = 0 \\
d_\sigma(Z^{\text{odd}}) \subset (\bigwedge Z^{\text{even}}) \\
(d - d_\sigma)(Z^{\text{odd}}) \subset (\bigwedge^+ Z^{\text{odd}}) \otimes (\bigwedge Z^{\text{even}}).
\end{cases}
$$

By [6], $H^*(\bigwedge Z, d_\sigma)$ is finite dimensional. It is then easy to see that $g$ induces an endomorphism $g_\sigma$ of the differential graded algebra $(\bigwedge Z, d_\sigma)$. It follows that

$$
\lambda_{g_\sigma} = \lambda_g.
$$

As $(\bigwedge Z, d_\sigma)$ is equipped with a weight decomposition satisfying $Z^{\text{odd}} = Z_1$ and $Z^{\text{even}} = Z_0$, Theorem 4.2 follows from Proposition 4.2. □

Another case where $\lambda_f$ can be deduced from the characteristic polynomials associated to the maps $\pi_n(f) \otimes \mathbb{Q}$ is the case when the Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ has a finite dimensional cohomology:

$$\dim \text{Tor}^H_*(\Omega X; \mathbb{Q})(\mathbb{Q}, \mathbb{Q}) < \infty.$$

In this case, for a self-map $f$, we have the formula

$$
\left( \sum_{p, q} (-1)^{p+q} \frac{\text{tr} \text{Tor}^H_*(\Omega f; \mathbb{Q})(\mathbb{Q}, \mathbb{Q})}{(\sum_n \text{tr} H_n(\Omega f; \mathbb{Q})t^n)} \right) = 1.
$$

Therefore, denoting by $Q(t)$ the rational function $\sum_n \text{tr} H_n(\Omega f; \mathbb{Q})t^n$, we have

$$
\sum_{p, q} (-1)^{p+q} \frac{\text{tr} \text{Tor}^H_*(\Omega f; \mathbb{Q})(\mathbb{Q}, \mathbb{Q})}{Q(t)} = \lim_{t \to -1} \frac{1}{Q(t)}.
$$

On the other hand, there is a natural Milnor–Moore spectral sequence

$$
E^p,q_2 = \text{Tor}^H_*(\Omega f; \mathbb{Q})(\mathbb{Q}, \mathbb{Q}) \Rightarrow H_{p+q}(X; \mathbb{Q}).
$$

We deduce the following result.

**Proposition 4.3.** — If $\dim \text{Tor}^H_*(\Omega f; \mathbb{Q})(\mathbb{Q}, \mathbb{Q}) < \infty$, then

$$
\lambda_f = \lim_{t \to -1} \frac{1}{Q(t)},
$$

where $Q(t)$ denotes the Poincaré series $\sum_n \text{tr} H_n(\Omega f; \mathbb{Q})t^n$. 

- 237 -
Example. — Let $X$ be a formal space. For each $\mu \in \mathbb{Q} \setminus \{0\}$, the map

$$\overline{\mu} : H^*(X) \longrightarrow H^*(X), \quad x \longmapsto \overline{\mu}(x) := \mu^{|x|} \cdot x$$

can be realized by an automorphism $\overline{\mu}$ of $X$. Denote by $(\wedge Z, d)$ the bigraded model of $X$. The restriction of the map $\overline{\mu}$ at $Z^2_p$ consists in the multiplication by $\mu^{p+q}$. We remark that the formula given by Proposition 4.1 is a generalization of the formula given by Halperin and Stasheff [9] in the case $f = \text{id}$:

$$\prod_{p,q} \left[ 1 - (-t)^{p+q} \right] (-1)^{q+1} \dim Z^2_p = \sum_n \dim H^n(f; \mathbb{Q}) t^n.$$

The left hand side of the formula has a nonzero radius of convergence $R$. We now replace $t$ by $-\mu$ and we take $\mu < R$. The infinite product

$$\prod_{p,q} \left[ 1 - (\mu)^{p+q} \right] (-1)^{q+1} \dim Z^2_p$$

converges and we thus have this following corollary.

Corollary 4.1. — If $0 < \mu < R$, then

$$\lambda_{\overline{\mu}} = \prod_{p,q} \left[ 1 - (\mu)^{p+q} \right] (-1)^{q+1} \dim Z^2_p.$$
COROLLARY 5.1. — Let $X$ be a simply connected finite CW complex and $f$ be a self-map satisfying $\dim \text{Im} \pi_*(f) \otimes \mathbb{Q} < \infty$. Denoting by $r$ the multiplicity of $1$ as eigenvalue of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$ and by $s$ the multiplicity of $1$ as eigenvalue of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$, we have:

1. $s \geq r$,
2. if $s > r$, then $\lambda_f = 0$,
3. if $s = r$, then

$$\lambda_f = \frac{\prod_{\lambda_i \neq 0, 1}(1 - \lambda_i)}{\prod_{\mu_i \neq 0, 1}(1 - \mu_i)},$$

where the $\lambda_i$ are the eigenvalues of $\pi_{\text{odd}}(f) \otimes \mathbb{Q}$ and the $\mu_i$ the eigenvalues of $\pi_{\text{even}}(f) \otimes \mathbb{Q}$.

Proof of the proposition 5.1. — Denote by $(\bigwedge Z, d)$ the minimal model of $X$, by $g$ a minimal model for $f$ and by $\varphi$ the linear part of $g$. We decompose $Z$ as the direct sum of the eigenspaces $V_\lambda$ associated to $\varphi$. We write $Z = V \oplus W$, with

$$V = \bigoplus_{\lambda \neq 0} V_\lambda \quad \text{and} \quad W = V_0.$$

We will show by induction on the degree that we can modify $V$ and $W$ in order to have $d(V) \subset \bigwedge V$, $g(V) \subset \bigwedge V$, $d(W) \subset \bigwedge^+ W \otimes \bigwedge V$ and $g(W) \subset \bigwedge^+ W \otimes \bigwedge V$. We take homogeneous basis $v_n$ of $V$ and $w_n$ of $W$ satisfying $|v_n| > |v_{n-1}|$ and $\varphi(v_n) - \lambda_n v_n \in \bigwedge(v_1, \ldots, v_{n-1})$, and a similar formula for the elements $w_n$.

Suppose the properties are satisfied for $v_1, \ldots, v_{n-1}$ and $w_1, \ldots, v_{m-1}$. We then have

$$\begin{cases}
  d(v_n) = \beta_n + \gamma_n \\
g(v_n) = \lambda_n v_n + u_n + \delta_n
\end{cases}$$

$$\begin{cases}
  \beta_n \in \bigwedge V \\
  \gamma_n \in \bigwedge V \otimes \bigwedge^+ W \\
  u_n \in \bigwedge^2 V \otimes (v_1, \ldots, v_{n-1}) \\
  \delta_n \in \bigwedge V \otimes \bigwedge^+ W.
\end{cases}$$

There exists integers $r$ and $s$ with $g^r(\gamma_n) = 0$ and $g^s(\delta_n) = 0$. We choose an integer $p$ greater than $r$ and $s$, and we replace $v_n$ by $g^p(v_n)$. We have

$$\begin{cases}
  d(g^p(v_n)) = g^p(\beta_n) \in \bigwedge V \\
g(g^p(v_n)) - \lambda_n v_n \in \bigwedge^2 V \oplus (v_1, \ldots, v_{n-1}).
\end{cases}$$
We now consider the element $w_n$. Suppose

$$d(w_n) = \alpha_m + \beta_m \quad \text{with} \quad \alpha_m \in \bigwedge^+ W \otimes \bigwedge V \quad \text{and} \quad \beta_m \in \bigwedge V.$$

Then $g^* (\beta_n)$ is a coboundary for some integer $s$. By the next lemma the element $\beta_n$ is therefore also a coboundary, $\beta_m = d(\gamma_m)$. Now by definition of $W$, $\gamma_m$ is a decomposable element. We can therefore replace $w_m$ by $\omega_m - \gamma_m$ in order to have $d(w_m) \in \bigwedge^+ W \otimes \bigwedge V$.

We now write

$$g(w_m) = \mu_m + \nu_m \quad \text{with} \quad \mu_m \in \bigwedge^{\geq 2} V \quad \text{and} \quad \nu_m \in \bigwedge^+ W \otimes \bigwedge V.$$

Of course by induction hypothesis, $g^* (\mu_m)$ is a cocycle, so that $\mu_m$ is also a cocycle. We replace $w_m$ by $w_m - g^{-1}(\mu_m)$, where $g^{-1}$ denotes the inverse of the function $g : (\bigwedge^V)^t \to (\bigwedge V)^t$, with $t = \deg(w_m)$.

**Lemma 5.1.** — Let $E$ be a finite type vector space over the complex numbers, let $f : E \to E$ be an isomorphism, and let $S$ be a graded sub vector space of $E$ invariant for $f$. Suppose $f^* x$ belongs to $S$ for some element $x$ and some integer $s$, then the element $x$ belongs to $S$.

**References**


[2] **Deligne (P.), Griffiths (P.), Morgan (J.) and Sullivan (D.)** . — Real homotopy theory of Kähler manifolds, Inventiones Math. 29 (1975), pp. 245-274.


Lefschetz number and degree of a self-map


