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*Annales de la faculté des sciences de Toulouse 6e série*, tome 5, nº 4 (1996), p. 599-643

<http://www.numdam.org/item?id=AFST_1996_6_5_4_599_0>

Annales de la Faculté des Sciences de Toulouse Vol. V, n° 4, 1996

# Travelling wave analysis of an isothermal Euler–Poisson model[*]

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**RÉSUMÉ.** — Nous présentons une analyse en ondes progressives des équations d'Euler isothermes pour les électrons et les ions couplées par l'équation de Poisson. L'analyse est fondée sur une analyse par portrait de phase qui conduit à trois types de solutions génériques.

**ABSTRACT.** — We present a travelling wave analysis of the isothermal Euler equations for electrons and ions coupled by the Poisson equation. The analysis is based on a phase plane analysis which leads to three types of generic solutions.

## 1. Introduction

The Euler–Poisson system is used to describe the dynamics of a plasma consisting of electrons and ions in their self consistent electric field. In plasma physics, it is very often assumed that the plasma is quasineutral. The quasineutrality assumption can be viewed mathematically as a singular limit of the full Euler–Poisson model which leads to a different hyperbolic system. We shall refer to this limit system as the quasineutral Euler model. The aim of this paper is to perform a travelling wave analysis of the full Euler–Poisson system in order to determine the shock profiles of the quasineutral Euler system.

(*) Reçu le 27 mai 1994

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In this paper, we shall restrict to isothermal Euler models. The more complex case of full Euler models will be dealt with in a forthcoming paper. In section 2, we present the isothermal Euler–Poisson model in scaled form and we formally derive the quasineutral Euler model in section 3 when the scaled Debye length $\lambda$ tends to 0. In section 4, the travelling wave problem for the full Euler-Poisson system is stated. For the sake of simplicity, we shall assume that the mass of the electrons is zero. In reality the electron mass is small compared to the mass of the ions. We prove that for smooth solutions, the Euler-Poisson system can be reduced to a system of two ordinary differential equations. The phase plane analysis of such a dynamical systems has been initiated by Ascher, Markowich, Pietra, Schmeiser in [1] for semiconductor applications. In this paper, we shall apply similar techniques. We prove that there exists three different generic types of travelling wave solutions which we shall refer to as the solitary wave solutions, the periodic solutions and the shock solutions.

In section 5, we prove that the only smooth travelling wave solutions are solitary waves. These solutions can only be constructed for sufficiently small Mach numbers. They lead, when $\lambda \to 0$, to trivial solutions of the quasineutral Euler system.

In section 6, we show that shock solutions i.e. solutions which tend, when $\lambda \to 0$, to a shock wave for the quasineutral Euler model exist for sufficiently large shock strengths. These solutions connect a hydrodynamic shock on the ion hydrodynamic variables with the two states at infinity by a smooth curve.

In section 7, we construct periodic solutions. These solutions have already been described in basic books of plasma physics like [8]. Weak limits of such solution exist when $\lambda \to 0$, but these limits are not weak solutions of the quasineutral Euler–Poisson model in a classical sense.

Finally, in section 9, we give the guidelines to extend these results to the polytropic model, to the finite electrons mass model and to the multi-species plasma models.

## 2. The Euler–Poisson system

Let us consider a one-dimensional plasma consisting of electrons and ions. The electrons of mass $m_n$, charge $q_n = -e$ and of given constant temperature $T_n$ are described by their density $n_n$ and their velocity $u_n$;

the ions of mass $m_p$, charge $q_p = +e$ and of given constant temperature $T_p$ by $n_p$ and $u_p$ respectively. These scalar variables satisfy the isothermal mono-dimensional Euler system of conservation laws. The mass (or charge) conservation equation reads:

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial(n_\alpha u_{\bar{\alpha}})}{\partial x} = 0\,, \tag{2.1}$$

where $\alpha$ is the generic index for the species of particles with $\alpha = n$ (for the electrons) and $\alpha = p$ (for the ions). The momentum conservation equation can be written:

$$\frac{\partial(n_\alpha m_\alpha u_\alpha)}{\partial t} + \frac{\partial\big(n_\alpha m_\alpha u_\alpha^2 + P_\alpha(n_\alpha)\big)}{\partial x} = q_\alpha n_\alpha E = -q_\alpha n_\alpha \frac{\partial \phi}{\partial x}\,, \tag{2.2}$$

where $P_\alpha(n)$ denotes the pressure of the $\alpha$-th species. We need an equation of state to close the hydrodynamic system; in this paper, we consider the isothermal gas law:

$$P_\alpha(n) = nk_B T_\alpha\,, \tag{2.3}$$

with $k_B$ the Boltzmann constant and $T_\alpha$ the constant temperature. However, all the following analysis can be carried out with the polytropic gas law:

$$P_\alpha(n) = c_\alpha n^\gamma\,, \tag{2.4}$$

where $\gamma > 1$ and $c_\alpha$ is a constant. The self consistent electric field $E = -\partial\phi/\partial x$ is given by the Poisson equation:

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}(n_p - n_n)\,, \tag{2.5}$$

where $\varepsilon_0$ is the permitivity and $\phi$ the potential.

We now introduce the following scaling. We choose a characteristic length $L$, temperature $T_0$ and density $N_0$ from physical considerations. The masses are scaled by the ion mass and thus, $m_p = 1$ and $m_n = \eta \ll 1$; the charges are scaled to $+1$ for the ions and $-1$ for the electrons. Then, we define the velocity, time and electric field units by:

$$u_0 = \sqrt{\frac{k_B T_0}{m_p}}\,, \quad \tau = \frac{L}{u_0}\,, \quad E_0 = \frac{k_B T_0}{eL}\,, \tag{2.6}$$

respectively. The resulting scaled version of Euler-Poisson system reads

$$\partial_t n_p + \partial_x(n_p u_p) = 0\,, \tag{2.7}$$

$$\partial_t n_n + \partial_x(n_n u_n) = 0\,, \tag{2.8}$$

$$\partial_t(n_p u_p) + \partial_x(n_p u_p^2 + n_p T_p) = n_p E\,, \tag{2.9}$$

$$\eta\big(\partial_t(n_n u_n) + \partial_x(n_n u_n^2)\big) + \partial_x(n_n T_n) = -n_n E\,, \tag{2.10}$$

$$\lambda^2 \partial_x E = n_p - n_n\,, \tag{2.11}$$

where $n_n$, $u_n$, $T_n$ are now the scaled density, velocity and temperature of the electrons and $n_p$, $u_p$, $T_p$ are the corresponding quantities for the ions. The dimensionless paremeter $\lambda$ in the Poisson equation (2.11) is the scaled Debye length $\lambda = \lambda_D / L$ where

$$\lambda_D^2 = \frac{\varepsilon_0 k_B T_0}{N_0 e^2}\,.$$

The system (2.7)-(2.11) contains two small parameters $\lambda \ll 1$ and $\eta \ll 1$.

The quasineutral Euler model in obtained by formally letting $\lambda \to 0$. Usually, in typical plasmas, the parameter $\lambda$ is very small which physically explains this limit. The mathematical study of the Euler–Poisson model has been initiated by Degond and Markowich in [3] for semiconductors. The analysis of the singular quasineutral limit has been studied by Brenier for the Vlasov–Poisson equations [4] and in particular, the relations between the quasineutral limit of Vlasov-Poisson and the incompressible limit of the Euler equations. We also mention a recent work [5] about the defect measures of the limiting solutions when the Debye length tends to 0. Finally, the quasineutral limit has been studied by Schmeiser and Markowich [7] for a semiconductors model and Golse and Sentis [6] in the case of the Poisson equation with Maxwellian electrons and a fixed background on ions. See also [16] for related problems.

## 3. The Quasineutral Euler model

First, we let $\lambda = 0$ in the Poisson equation (2.11) and we get the so-called quasineutrality assumption:

$$n_n = n_p = n\,. \tag{3.1}$$

Physical justifications can be found in all introductions to plasma physics (for example) [9].

Then, the conservation equations (2.7) and (2.8) give:

$$(n_n u_n - n_p u_p) = J_0(t)\,. \tag{3.2}$$

This total current $J_0$ can be determined by the boundary conditions. We shall assume that $J_0(t) = 0$ and thus, we have a unique velocity $u_n = u_p = u$. The equation for $nu$ can be obtained by adding the two momentum equations (2.9) and (2.10) together:

$$\partial_t(nu) + \partial_x \left( nu^2 + n \frac{T_p + T_n}{m_p + m_n} \right) = 0\,. \tag{3.3}$$

The electric field does not appear explicitely in this limit system, but can be computed *a posteriori* from the electron momentum equation (2.10). Thus, the remaining variables $(n, nu)$ satisfy a gas dynamics system [11].

One aim of this paper is to provide some justification of this formal limit: we want to construct the shock wave solutions of the quasineutral Euler model as limits of travelling wave solutions of the full Euler–Poisson model.

However, only part of this program can be achieved. Indeed, we shall prove that there exist 3 distinct types of travelling wave solutions of the full Euler–Poisson model.

*First type.*— Solitary wave solutions are the only smooth travelling wave solutions of the full Euler–Poisson model (sect. 5). When $\lambda$ goes to 0, they lead to a trivial constant solution of the quasineutral Euler model. They are associated with velocities of the wave $\sigma$ which are close to the characteristic velocity of the quasineutral Euler model $\sigma_C$, more precisely $\sigma_C < \sigma < \sigma_C + \sigma_1$.

*Second type.*— Periodic solutions are bounded travelling wave solutions which involve one hydrodynamic shock for the ion state variables connected with a smooth periodic solution for increasing $x$ (sect. 7). They are associated with velocities $\sigma$ which are in between $\sigma_C + \sigma_1$ and $\sigma_C + \sigma_2$. When $\lambda \to 0$, they have a weak limit which is not a weak solution of the quasineutral Euler model.

*Third type.*— Shock solutions are travelling wave solutions consisting of two smooth curves connecting the two states at infinity with the two sides of a hydrodynamic shock for the ion state variables. When $\lambda \to 0$, these solutions actually tend to a shock wave solution of the quasineutral

Euler model with the correct speed $\sigma$ and the correct Rankine Hugoniot relations (sect. 6). However, such solutions only exist for $\sigma > \sigma_C + \sigma_2$. This indicates that shock wave solutions of the quasineutral Euler model are limits of travelling wave solutions of the full Euler–Poisson model only if the velocity of the shock wave is larger than some thershold velocity (or equivalently if the shock is strong enough).

This analysis will be extended to the non-isothermal Euler–Poisson model where the pressure law is replaced by an energy equation in a forthcoming paper. For such systems, the quasineutral limit leads to a non linear hyperbolic system in a non conservative form which brings some indetermination in the jump relations [14]. The relevant jump relations can be derived from a similar travelling wave analysis [2].

## 4. The travelling Wave Problem

Let $\lambda > 0$ be fixed. We construct travelling wave solutions of the Euler-Poisson system (2.7)-(2.11) of the form $U = (n_p, n_n, u_p, u_n, \phi)$:

$$U^\lambda(x,t) = U\left(\frac{x-\sigma t}{\lambda}\right) = U(\xi) \tag{4.1}$$

with $\xi = (x - \sigma t)/\lambda$, where $\sigma$ is the velocity of the travelling wave. The Euler–Poisson system can be written, for smooth solutions, as the following system of ordinary differential equations:

$$-\sigma n_p' + (n_p u_p)' = 0\,, \tag{4.2}$$

$$-\sigma n_n' + (n_n u_n)' = 0\,, \tag{4.3}$$

$$-\sigma(n_p u_p)' + (n_p u_p^2 + n_p T_p)' = -n_p \phi'\,, \tag{4.4}$$

$$\eta\left(-\sigma(n_n u_n)' + (n_n u_n^2)'\right) + (n_n T_n)' = -n_n \phi'\,, \tag{4.5}$$

$$-\phi'' = n_p - n_n\,, \tag{4.6}$$

where $'$ denotes the differentiation with respect to $\xi$. If the solution becomes discontinuous, this system has to be completed by the following jump relations:

$$\sigma[n_p] = [n_p u_p]\,, \tag{4.7}$$

$$\sigma[n_n] = [n_n u_n]\,, \tag{4.8}$$

$$\sigma[n_p u_p] = [n_p u_p^2 + n_p T_p]\,, \tag{4.9}$$

$$\eta(-\sigma[n_n u_n] + [n_n u_n^2]) + [n_n T_n] = 0\,, \tag{4.10}$$

$$[\phi] = [\partial_x \phi] = 0\,, \tag{4.11}$$

where $[\cdot]$ stands for the difference between the right and left limits of the corresponding function. Indeed, the electric field $E$ is continous and does not change the classical jump relations. We notice that the electron and ion jump relations are decoupled. The jump relations have to be supplemented by entropy conditions which will be made explicit in section 6.

We can now solve the mass (or charge) conservation equation by stating that the currents

$$J_n = n_n(u_n - \sigma)\,, \tag{4.12}$$

$$J_p = n_p(u_p - \sigma)\,, \tag{4.13}$$

are constant through the shocks.

Since we are interested in the travelling wave solutions which have a chance to converge when $\lambda \to 0$ to a shock wave solution of the quasineutral model, we shall be mainly interested in those travelling waves which converge to a constant state for $\xi \to \pm\infty$. However, we shall see that this constraint is too strong and we shall relax the requirement on the behaviour at $\xi = +\infty$, and only require thus the solution to be bounded when $\xi \to +\infty$. In the last section of this paper, we shall complete the travelling wave analysis by giving all the bounded travelling wave solutions, but we shall not present the analysis.

Thus, we shall be looking at bounded travelling wave solutions such that the variable $U = (n_p, n_n, u_p, u_n, \phi)$ has a limit when $\xi \to -\infty$ which in compatible with the quasineutral hypothesis:

$$\lim_{\xi \to -\infty} (n_p, n_n, u_p, u_n, \phi) = (n_p^-, n_n^-, u_p^-, u_n^-, \phi^-) \tag{4.14}$$

with

$$n_n^- = n_p^- = n^-\,, \quad u_n^- = u_p^- = u^-\,. \tag{4.15}$$

If in addition $U$ has a limit when $\xi \to +\infty$:

$$\lim_{\xi \to +\infty} (n_p,\, n_n,\, u_p,\, u_n,\, \phi) = (n_p^+,\, n_n^+,\, u_p^+,\, u_n^+,\, \phi^+)\,; \tag{4.16}$$

then, we necessarily have:

$$n_n^+ = n_p^+ = n^+\,, \quad u_n^+ = u_p^+ = u^+\,. \tag{4.17}$$

Finally, we can impose $\phi^- = 0$ and we set $J_p = J_n = J = n_-(u_- - \sigma)$ the constant current.

In the remainder, we shall restrict ourselves to the case $J \geq 0$. Indeed, if the functions $n_n(\xi)$, $u_n(\xi)$, $n_p(\xi)$, $u_p(\xi)$, $\phi(\xi)$ are solution of problem (4.2)-(4.18) for given $J$ and $\sigma$, then $n_n(-\xi)$, $-u_n(-\xi)$, $n_p(-\xi)$, $-u_p(-\xi)$, $\phi(-\xi)$ is the solution for $-J$ and $-\sigma$.

The aim of this analysis is to determine the jump relations between the left and right limits when they exist. We summarize the problem as follows: for a given left state:

$$U_\ell = (n^-, u^-)\,, \tag{4.18}$$

and a velocity $\sigma$ being fixed, we want to construct a travelling wave solution satisfying:

- the differential equations (4.4)-(4.5)-(4.6) in the intervals of continuity of the solutions;
- the jump relations (4.9)-(4.10)-(4.11) together with entropy condition at a discontinuity
- the current conditions (4.12)-(4.13) which are equivalent to (4.2)-(4.3) in the smoothness intervals and to (4.7)-(4.8) at a shock;
- the asymptotic behaviour (4.14)-(4.15) for $\xi \to -\infty$ and (4.16)-(4.17) if the limit at $\xi \to \pm\infty$ exists.

The set of solutions of (4.1)-(4.18) can be parametrized by the velocity $\sigma$ of the wave.

## 5. Smooth solutions

In this section, we consider only smooth solutions of the travelling wave problem (4.1)-(4.18).

### 5.1 Reduction to a dynamical system

From (4.12) and (4.13), the momentum equations (4.4) and (4.5) can be written:

$$\left(Ju_p + n_pT_p\right)' = -n_p\phi' \,, \tag{5.1}$$

$$\eta(Ju_n + \left(n_nT_n\right)' = n_n\phi' \,, \tag{5.2}$$

moreover by use of (4.12) and (4.13), we shall express the velocities $u_n$ and $u_p$ in terms of the corresponding densities $n_n$ and $n_p$. Then, (5.1) and (5.2) lead to:

$$H_p(n_p)' = -\phi' \,, \tag{5.3}$$

$$H_n(n_n)' = \phi' \tag{5.4}$$

where we have set:

$$H_\alpha(n) = \frac{m_\alpha J^2}{2n^2} + T_\alpha \ln\left(\frac{n}{n_0}\right) , \tag{5.5}$$

the so called enthalpy of the $\alpha$-th species. We also define the derivative $h_\alpha$ of $H_\alpha$ with respect to $n$:

$$H_\alpha'(n) = h_\alpha(n) = \frac{T_\alpha}{n} - \frac{m_\alpha J^2}{n^3} \,. \tag{5.6}$$

The function $H_\alpha$ is increasing for $n > n_\alpha^s$ and decreasing for $n < n_\alpha^s$, where $n_\alpha^s$ is defined by $h_\alpha(n_\alpha^s) = 0$ i.e.:

$$n_p^s = \frac{J}{\sqrt{T_p}} , \quad n_n^s = \eta^{1/2}\frac{J}{\sqrt{T_n}} \,. \tag{5.7}$$

Since $\eta \ll 1$, we make the following assumption:

$$n_n > n_n^s \,. \tag{5.8}$$

We shall verify later on that this assumption does not restrict the analysis. In terms of the travelling wave velocity, the density $n_\alpha^s$ corresponds to:

$$n = n_\alpha^s \iff n\sqrt{T_\alpha} = nm_\alpha(u-\sigma)$$
$$\iff \sigma = u - \sqrt{\frac{T_\alpha}{m_\alpha}}.$$

Thus, the associated velocity $\sigma$ is the sonic velocity of the $\alpha$-th species and, therefore, $n_\alpha^s$ will be referred to as the sonic density of the $\alpha$-th species. We prove below that $n > n_p^s$ corresponds to subsonic states and $n < n_p^s$ to supersonic ones (sect. 6).

By adding the equations (5.3) and (5.4), we get:

$$H_n(n_n) + H_p(n_p) = d = \text{constant}. \tag{5.9}$$

This constant $d$ can be computed from the left boundary condition (4.18):

$$d = H_n(n^-) + H_p(n^-). \tag{5.10}$$

Then, (5.9) enables us to compute the electron density in terms of the ion density; indeed, from hypothesis (5.8), $n_n$ varies in a region where the function $H_n$ is monotonically increasing. We define the following function:

$$n_n(n_p) = H_n^{-1}(d - H_p(n_p)). \tag{5.11}$$

for $n_n > n_n^s$, or equivalently, since $H_n$ is increasing if $n_p$ is such that:

$$H_p(n_p) < d - H_n(n_n^s); \tag{5.12}$$

or,

$$n_p \in [p_{\min}, p_{\max}] \tag{5.13}$$

where $p_{\min}$ and $p_{\max}$ are defined by $H_p(p) = d - H_n(n_n^s)$ such that $0 < p_{\min} < n_- < p_{\max} < \infty$.

Hence, we have just proved that (4.4)-(4.5)-(4.6) can be equivalently written (for smooth solutions), as the following dynamical system:

$$\begin{cases} h_p(p)p' = E, \\ E' = p - n_n(p) \end{cases} \tag{5.14}$$

where, from now on, we shall denote by $p = n_p$ the ion density.

We have a first integral of system (5.14):

$$g(p) - \frac{1}{2}E^2 = g(n^-) = \text{constant}, \tag{5.15}$$

with

$$g(p) = pT_p + n_n(p)T_n + \frac{J^2}{p} + \frac{\eta J^2}{n_n(p)}. \tag{5.16}$$

We notice that this constant is preserved even through shocks from the jump relations (4.9)-(4.10)-(4.11) since the function $g(p)$ is the flux of the total momentum.

## 5.2 Phase plane analysis

Again, we recall that from now on, we shall write $p$ instead of $n_p$ to make notations simpler.

In order to give explicit calculations,wa shall consider the massless electrons approximation i.e. $\eta = 0$. We shall relax this hypothesis in section 9 provided (5.8) holds. Obviously, (5.8) is satisfied when $\eta = 0$. Then, the electron enthalpy $H_n$ reduces to $H_n(n) = T_n \ln(n/n_0)$ and the function $n_n$ given by (5.11) can be written in the form:

$$n_n(n_p) = n_0 \exp\left(\frac{d - H_p(n_p)}{T_n}\right) \tag{5.17}$$

and since $n_n^s = 0$, the condition (5.8) is always satisfied and there is no restriction, i.e. $p_{\min} = 0$ and $p_{\max} = +\infty$.

We recall that such dynamical systems have been studied in the semiconductors context in [1]. We follow the same lines being interested in the phase portrait of the system (5.14) in the $(p, E)$ plane.

At first we remark that the phase portrait is symmetric with respect to the $E = 0$ axis.

### *5.2.1 Stationary points*

The points of main are the *stationary points* of (5.14), because they will provide the possible asymptotic states when $\xi \longrightarrow \pm\infty$ of the travelling wave.

These points are defined by the equations:

$$\frac{E}{h_p(p)} = 0, \quad p = n_n(p).$$

The case $p = n_p^s$ will be studied threreafter. Assuming $p \neq n_p^s$ there exist exactly two stationary points of the form $(p, 0)$ with $p$ solution of:

$$p = n_n(p) \Longleftrightarrow H_n(p) + H_p(p) = d\,. \tag{5.18}$$

Because $p \mapsto H_n(p) + H_p(p)$ is monotonically decreasing for $p \in [\,0\,,\,n_c\,]$ and monotonically increasing for $p \in [\,n_c\,,\,+\infty\,]$ with the critical density $n_c$ defined by:

$$n_c = \sqrt{\frac{J^2}{T_n + T_p}}\,, \tag{5.19}$$

the equation (5.18) has two solutions $p = n^-$ (from the definition (5.10) of $d$) and $p^-$ such that the critical density $n_c$ lies between $n^-$ and $p^-$. We introduce the notation with $n_0 = \min\{n^-, p^-\}$ and $n_1 = \max\{n^-, p^-\}$ such that $n_0 \le n_c \le n_1$. In the following we shall prove that the existence of solutions of the travelling wave problem requires $n^- < n_c$, i.e. $n^- = n_0$.

We note that $n_c$ can be viewed as the sonic density for the global quasineutral fluid.

The critical line $n = n_c$ splits the phase plane $(p, E)$ into the subsonic domain $(n > n_c)$ and the supersonic domain $(n < n_c)$ for the quasineutral fluid.

Moreover, we shall refer to the region $p \in [\,n_0\,,\,n_1\,]$ of the $(p, E)$ plane where we have $p < n_n(p)$ and thus $E' < 0$ as the negatively charged region and to the region where $p \notin [\,n_0\,,\,n_1\,]$ as the positively charged region.

The sign of $p'$ depends on the sign of $E$ and on the position of $p$ with respect to the sonic line $p = n_p^s$. Thus, $p' > 0$ if $E > 0$ and $p > n_p^s$ or if $E < 0$ and $p < n_p^s$ and we have $p' < 0$ in the two other quadrants.

*5.2.2 Local analysis*

The local behaviour at the stationary points can be determined from a linearization of system (5.14); the jacobian of (5.14) is:

$$\begin{pmatrix} \dfrac{E\partial_p h_p(p)}{h_p(p)^2} & \dfrac{1}{h_p(p)} \\ 1 - \partial_p n_n(p) & 0 \end{pmatrix} \tag{5.20}$$

Since $\partial_p n_n(n_0) > 1$ and $\partial_p n_n(n_1) < 1$ (by construction), the nature of the stationary point depends on its position with respect to $n_p^s$. We have $n_0 < n_c$ and, by definitions (5.19) and (5.7), $n_c < n_p^s$; thus $h_p(n_0) < 0$ and $(n_0, 0)$ is always a saddle point.

On the other hand, if the density $n_1$, is such that $n_1 < n_p^s$, the point $(n_1, 0)$ is a center, whereas $n_1 > n_p^s$ implies that $(n_1, 0)$ is a saddle point.

Let us assume that $n_1 < n_p^s$. The trajectory starting at point $(n_0, 0)$ with $E < 0$ is given by the equation $E(p) = -\sqrt{2\big(g(p) - g(n_-)\big)}$. We have:

$$\frac{\partial E}{\partial p} = -\frac{\partial_p g(p)}{\sqrt{2\big(g(p) - g(n_-)\big)}} \tag{5.21}$$

and

$$\frac{\partial}{\partial p} g(p) = T_p - \frac{J^2}{p^2} + T_n \frac{\partial}{\partial p} n_n(p)\,. \tag{5.22}$$

But since we have from (5.6) and (5.17):

$$h_p(p) = \frac{T_p}{p} - \frac{J^2}{p^3}\,,$$

$$\frac{\partial}{\partial p} n_n(p) = -\frac{1}{T_n}\big(h_p(p)\big) n_n(p)\,,$$

we deduce that:

$$\frac{\partial}{\partial p} g(p) = p h_p(p) - n_n(p) h_p(p) = h_p(p)\big(p - n_n(p)\big)\,. \tag{5.23}$$

Since $h_p$ vanishes at $n_p^s$ and since $p = n_n(p) \Leftrightarrow p \in \{n_0, n_1\}$ the derivative of $g(p)$ with respect to $p$ vanishes at $n_0$, $n_1$ and $n_p^s$ with $n_0 < n_1 < n_p^s$. The graph of the function $g(p)$ is depicted on figure 1. It follows that $E$ decreases from $n_0$ to $n_1$ and increases for $p > n_1$.

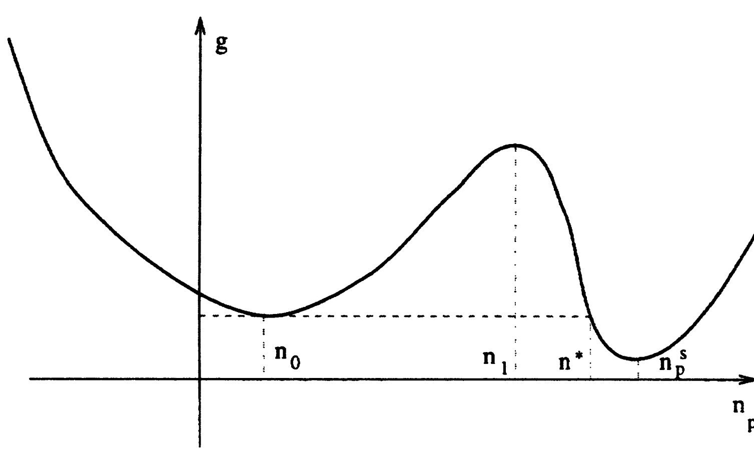


**Fig. 1** Graph of function $g$ for $n_1 < n_p^s$ and $g(n_0) > g(n_p^s) > 0$.

We want to characterize when the trajectory starting at $(n_0, 0)$ crosses the axis $E = 0$ at a point $(n^*, 0)$ with $n_p^s > n^* > n_1$ (case (i)) or when this trajectory leads to the sonic line (case (ii)). This depends on the sign of $g(n_0) - g(n_p^s)$. Indeed, the density $n^*$ is defined by $g(n^*) = g(n_0)$ and the equation $g(p) = g(n_0)$ has a solution if and only if $g(n_0) - g(n_p^s) > 0$. Thus case (i) corresponds to $g(n_0) - g(n_p^s) > 0$ and case (ii) to $g(n_0) < g(n_p^s)$.

The case $n_1 > n_p^s$ is called case (iii).

*5.2.3 Sonic line $p = n_p^s$*

The sonic line for the ion fluid $p = n_p^s$ splits the phase plane $(p, E)$ into the subsonic domain $p > n_p^s$ and the supersonic domain $p < n_p^s$.

This is a line of singularities of the system (5.14) since $h_p$ vanishes for $p = n_p^s$. Thus, it is only possible to cross this line at $E = 0$.

In fact, $(p, E) = (n_p^s, 0)$ is a point of non uniqueness of (5.14). Indeed the initial value problem for (5.14) with initial data $(p, E) = (n_p^s, 0)$ has two solutions in the case $n_1 < n_p^s$ (which has been previously defined as case (i) and (ii)) and none in the case $n_1 > n_p^s$ (refered as case (iii)).

In the first case, the trajectory leading to $(n_p^s, 0)$ (called sonic trajectory and denoted $T^s$) passes twice through $(n_p^s, 0)$, once on its way from subsonic to supersonic region and once on its way back. Then, we have to distinguish if the supersonic part of this sonic trajectory $T^s$ crosses the axis $E = 0$ at some point $p < n_p^s$, which corresponds to the case where the equation $g(n_p^s) = g(p)$ has a solution $n^*$, i.e. case (ii), or not in which case the two different branches of the supersonic part of $T^s$ ($E > 0$ and $E < 0$) never intersect; this is the case (i).

Because of the first equation of (5.14), we have $|p'| = \infty$ at all point $(n_p^s, E)$ with $E \neq 0$. At any point $(n_p^s, E)$ with $E > 0$ two trajectories start (one going into the supersonic region and one in the subsonic region) and at every point $(n_p^s, E)$ with $E < 0$ two trajectories end (one coming from the subsonic region and one from the supersonic region). All points at $p = n_p^s$ are reached for finite values of the independent variable $\xi$.

## 5.3 The phase portraits

Then, we have to distinguish between the three following generic cases.

### *5.3.1 The solitary wave case (or case (i))*

This is case (i): $g(n_0) - g(n_p^s) > 0$. There exists a homoclinic orbit starting from $(n_0, 0)$, the point $(n_1, 0)$ is a center and is such that $n_1 < n_p^s$. The phase portrait is depicted in figure 2.

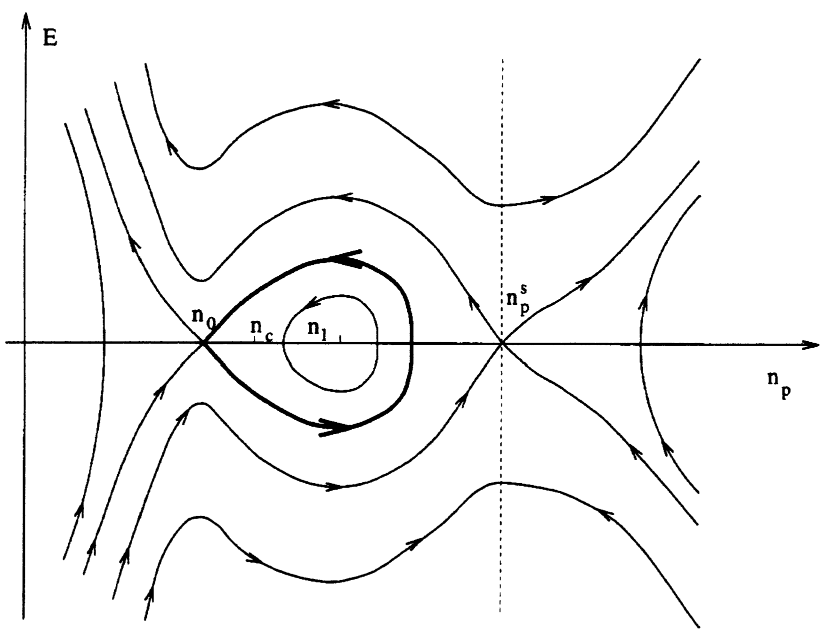


**Fig. 2** Soliton case.

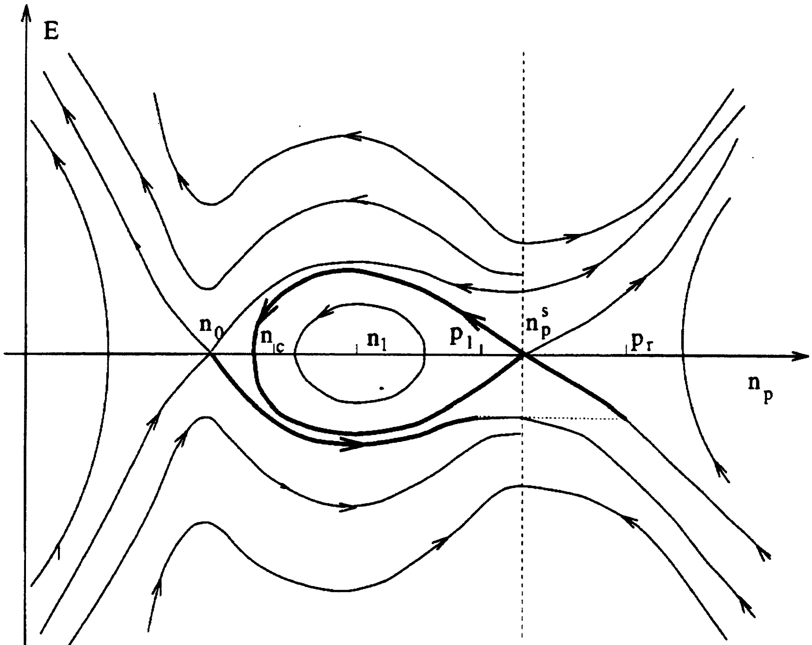


**Fig. 3** Periodic case.

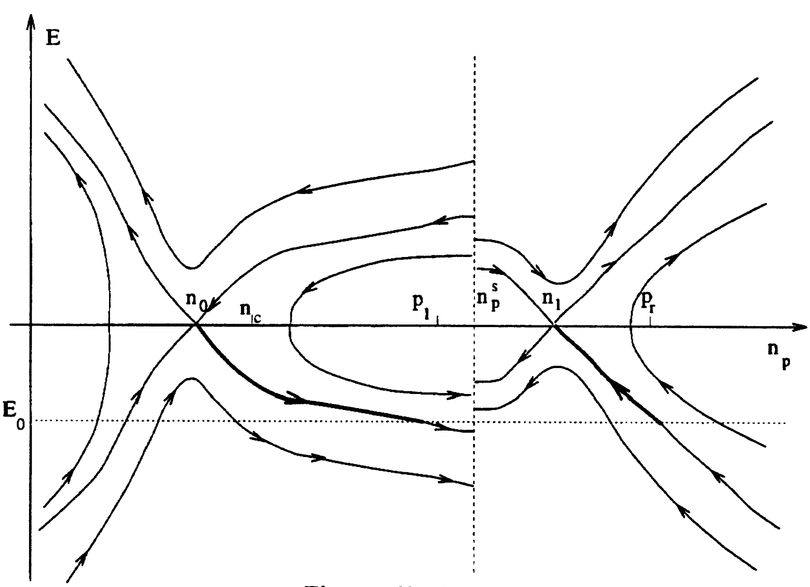


**Fig. 4** Shock case.

*5.3.2 The periodic case (or case (ii))*

This is case (ii): $g(n_0) - g(n_p^s) < 0$ and $n_1 < n_p^s$. There is no homoclinic orbity starting from $(n_0, 0)$, the point $(n_1, 0)$ is now a center. There exists one closed orbit passing through the sonic point $(n_p^s, 0)$ corresponding to a periodic solution. The corresponding phase portrait is given in figure 3.

Of course, there also exist periodic solutions in case (i). But for a reason to be explained later on, we shall refer to this case as the "periodic case".

*5.3.3 The shock case (or case (iii))*

This is case (iii): $n_1 > n_p^s$. The two stationary points are saddle points. This case is called the shock case because it will provide travelling wave solutions which tend, when $\lambda \to 0$, to a shock wave solution of the quasineutral Euler problem. Its phase portrait is shown on figure 4.

Moreover, the three following limiting case have to be considered.

*5.3.4 The trivial case*

This is the case where $n_0 = n_1 = n_c$. There is only one degenerate stationary point. Going back to the velocity of the wave, this case corresponds to the acoustic speed for the quasineutral Euler model

$$\sigma = u - \sqrt{T_p + T_n} \,. \tag{5.24}$$

The associated phase portrait is given on figure 5.

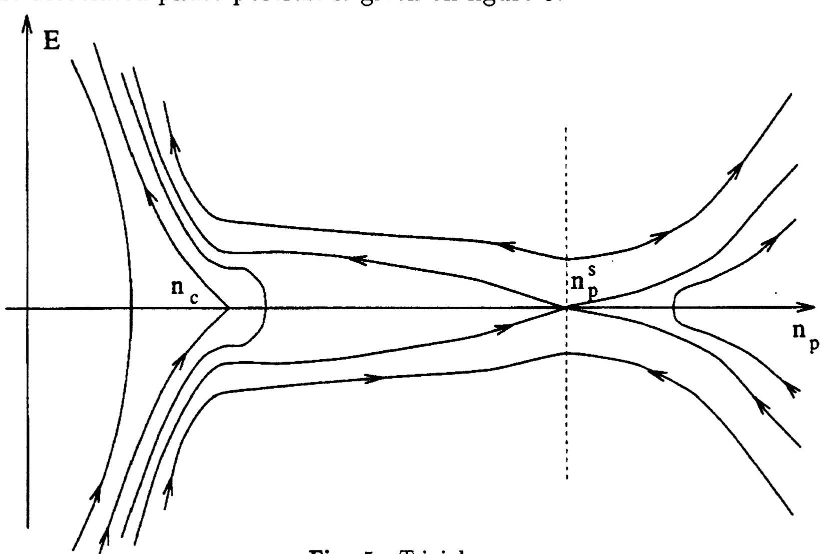


**Fig. 5** Trivial case.

*5.3.5 The critical case*

This is the case where $g(n_0) = g(n_p^s)$. The trajectory starting from $(n_0, 0)$ reaches in finite time the point $(n_p^s, 0)$. The associated phase portrait is given on figure 6.

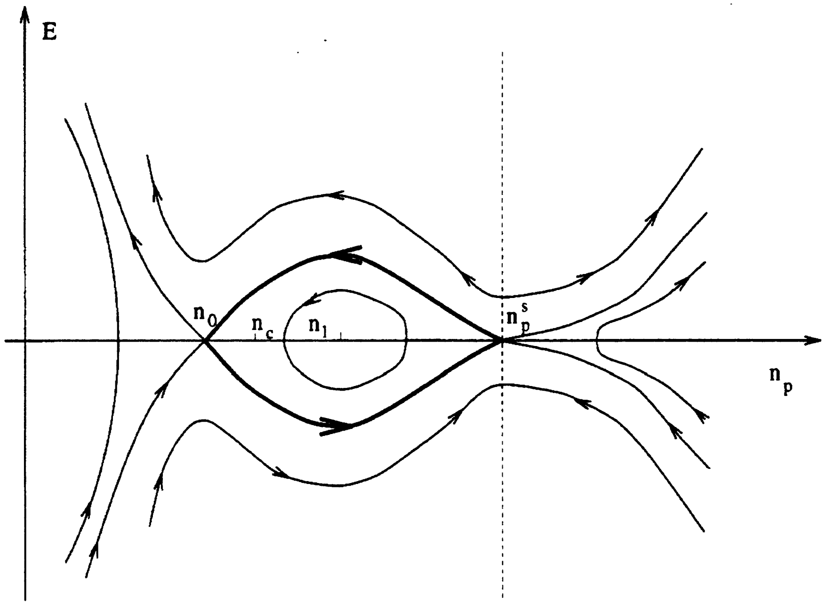


**Fig. 6** Critical case.

The determination of the associated velocity $\sigma$ cannot be done explicitely in the general case; but the equation $g(n_0) = g(n_p^s)$ can be written as an equation for the ion Mach number $\mathcal{M} = n_p^s/n_0$. We have from the definitions of $n_0$ and $n_1$ (see (5.18)): $d = H_n(n_0) + H_p(n_0)$, and thus:

$$g(n_0) = g(n_p^s) \Longleftrightarrow n_0(T_p + T_n) + \frac{J^2}{n_0} = n_p^s T_p + \exp\left(\frac{d - H_p(n_p^s)}{T_n}\right) T_n + \frac{J^2}{n_p^s}$$

and from (5.7),

$$(T_p + T_n) + \frac{J^2}{n_0^2} =$$

$$= 2\mathcal{M}T_p + \frac{T_n}{n_0} \exp\left(\frac{\frac{J^2}{2n_0^2} + (T_p + T_n)\ln(n_0) - \frac{J^2}{2(n_p^s)^2} - T_p \ln(n_p^s)}{T_n}\right)$$

or,

$$(T_p + T_n) + \frac{J^2}{n_0^2} = 2\mathcal{M}T_p + T_n \exp\left( \frac{\frac{J^2}{2n_0^2}(1 - \mathcal{M}^{-2}) - T_p \ln \mathcal{M}}{T_n} \right)$$

or again,

$$(T_p + T_n) + \frac{J^2}{n_0^2} = 2\mathcal{M}T_p + T_n \mathcal{M}^{-T_p/T_n} \exp\left( \frac{J^2}{2n_0^2 T_n}(1 - \mathcal{M}^{-2}) \right)$$

and again using (5.7)

$$\left(1 + \frac{T_n}{T_p}\right) + \mathcal{M}^2 = 2\mathcal{M} + \frac{T_n}{T_p}\mathcal{M}^{-T_p/T_n} \exp\left( \frac{T_p}{2T_n}(\mathcal{M}^2 - 1) \right)$$

since $\mathcal{M}^2 = J_2/(n_0^2 T_p)$. This equation has a unique solution $\mathcal{M} > 1$ that we denote by $\mathcal{M}_-$, which depends on $T_p$ and $T_n$. This equation can be simplified further if we choose a reference temperature equal to the electron one ($T_n = 1$),

$$(1 + T_p) + T_p\mathcal{M}^2 = 2T_p\mathcal{M}^2 + \mathcal{M}^{-T_p} \exp\left( \frac{T_p}{2}(\mathcal{M}^2 - 1) \right) . \tag{5.25}$$

In the limit of cold plasma i.e. $T_p \to 0$, the equation (5.25) is more conveniently written in terms of the electron Mach number

$$\mathcal{M}_e = \sqrt{\frac{J^2}{n_0^2 T_n}} = \sqrt{\frac{J^2}{n_0^2}}$$

which is also equal to $\mathcal{M}/\sqrt{T_p}$:

$$(1 + T_p) + \mathcal{M}_e^2 = 2\sqrt{T_p}\,\mathcal{M}_e + \left( \frac{\mathcal{M}_e}{\sqrt{T_p}} \right)^{-T_p} \exp\left( \frac{\mathcal{M}_e^2 - T_p}{2} \right) \tag{5.26}$$

and gives when $T_p \to 0$:

$$1 + \mathcal{M}_e^2 = \exp\left( \frac{\mathcal{M}_e^2}{2} \right) . \tag{5.27}$$

This is the Mach number equation for the existence of solitary waves via the Zagdeev potential theory [8] (see also [9] for its connection with the Bohm sheath criterion and [15] with an ion extraction model). The only positive solution is $\mathcal{M} \approx 1.5852$. In some sense (5.25) generalizes the criterion for the existence of solitary waves derived in [8] to finite temperature plasmas as we shall see later on.

#### *5.3.6 The sonic case*

This is the case where $n_1 = n_s$. In this case, the local behaviour at the stationary point cannot be determined from a linearization. However, we can prove that there exists exactly one trajectory starting from $(n_1, 0)$ with $E > 0$ and $p > n_1$ and one ending at $(n_1, 0)$ with $E < 0$ and $p > n_1$. Moreover we can prove that these trajectories have horizontal tangent at $(n_p^s, 0)$. This is left to the reader. The velocity of the wave corresponding to this case is the ion sonic velocity and thus, this case will be referred as the sonic case. Its phase portrait is given in figure 7.

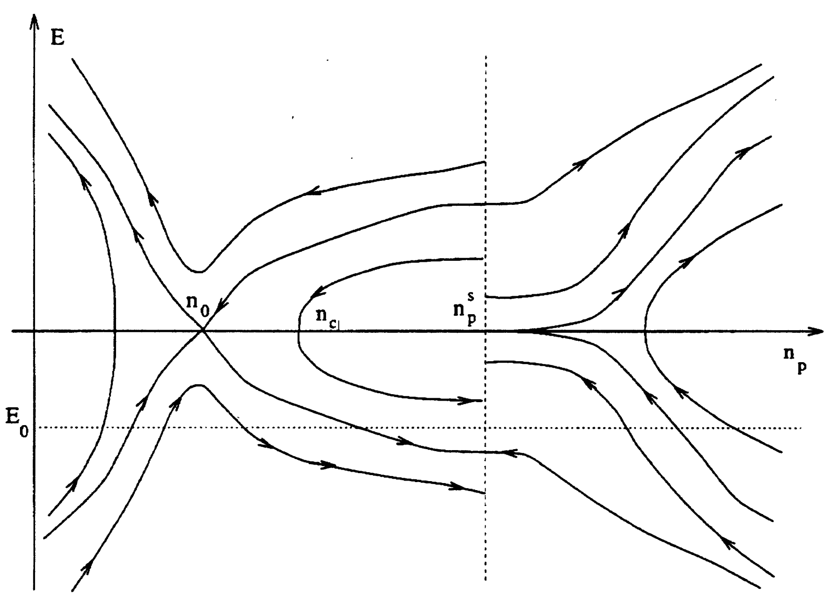


**Fig. 7** Sonic case.

### 5.4 Solitary wave solutions

We shall now describe the smooth travelling wave solutions of (4.1)-(4.18).

THEOREM 5.1.— *There exists a non constant smooth travelling wave solution of (4.1)-(4.18) if and only if the ion Mach number* $\mathcal{M} \stackrel{\text{def}}{=} n_p^s/n^-$ *is such that*

$$\sqrt{\frac{T_n + T_p}{T_p}} < \mathcal{M} \leq \mathcal{M}_- \tag{5.28}$$

*where* $\mathcal{M}_-$ *is the solution of (5.25) with* $\mathcal{M} > 1$. *Moreover, this solution satisfies:*

$$\lim_{\xi \to +\infty} n_n(\xi) = \lim_{\xi \to +\infty} n_p(\xi) = n^+ = n^- \,, \tag{5.29}$$

$$u^+ = u^- \,, \tag{5.30}$$

$$\int_{\mathbb{R}} E(\xi)\, \mathrm{d}\xi = 0 \,, \tag{5.31}$$

$$\int_{\mathbb{R}} \big(n_n(\xi) - n_p(\xi)\big)\, \mathrm{d}\xi = 0 \,. \tag{5.32}$$

The shape of such a solitary wave solution is given in figures 8 and 9.

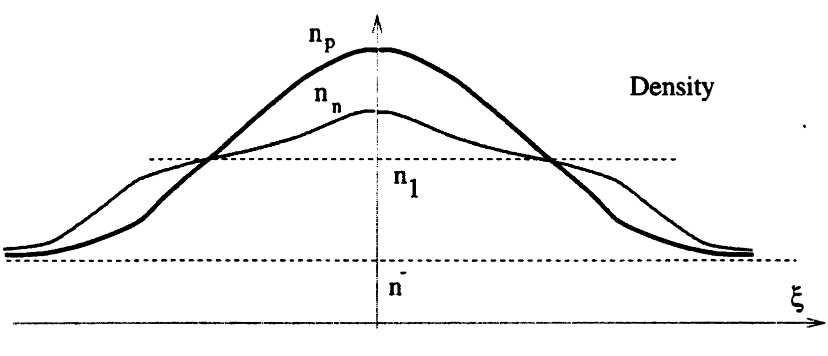


**Fig. 8** Solitary wave solution: densities.

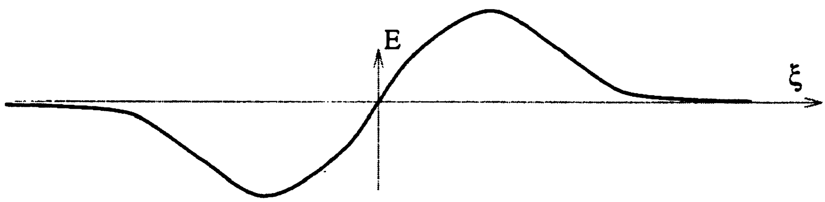


**Fig. 9** Solitary wave solutions: electric field.

*Proof.*— First of all, note that a non constant solution must start on the unstable manifold of a stationary saddle point. Thus, the point $(n_-, 0)$ can be either $(n_0, 0)$ in one of the three generic cases (i) to (iii) or $(n_1, 0)$ in the case (iii).

Moreover, the trajectory must also connect the stable manifold of another stationary point. Since there is no such heteroclinic orbit (which joins two stationary points) in any case, then the non constant solution of (4.1)-(4.18) must follow a homoclinic orbit. The only possible case for a homoclinic orbit is case (i). The orbit of such a solution is depicted on figure 2 in bold line.

The first condition $\sqrt{(T_n + T_p)/T_p} < \mathcal{M}$ arises directly from $n^- = n_0 < n_c$. The second condition comes from the characterization of case (i) (§ 5.3.5) which leads to equation (5.25).

The property (5.29) is obvious. By integrating the first equation of (5.14), we get:

$$\int_{\mathbb{R}} E(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} \big(H_p(n_p(\xi))\big)'\,\mathrm{d}\xi = H_p(n^+) - H_p(n^-) = 0$$

and the second equation leads to

$$\int_{\mathbb{R}} \big(n_n(\xi) - n_p(\xi)\big)\,\mathrm{d}\xi = \int_{\mathbb{R}} E(\xi)'\,\mathrm{d}\xi = 0\,. \ \square$$

The property (5.32) expresses the global neutrality of the plasma.

PROPOSITION 5.2.— *The necessary and sufficient condition (5.28) for the existence of non constant smooth solutions of problem (4.1)-(4.18) is equivalent to*

$$u^- - \sqrt{T_p + T_n} \geq \sigma > u^- - \mathcal{M}_- \sqrt{T_p}\,. \tag{5.33}$$

The left inequality $u^- - \sqrt{T_p + T_n} > \sigma$ is some sort of Lax entropy condition for the travelling wave solutions [13, p. 261], since $u^- - \sqrt{T_p + T_n}$ is the characteristic velocity of the waves going to the left for the quasineutral system.

We are now interested in the limit when $\lambda$ goes to 0 of such travelling waves profiles. Indeed, for the set $U^\lambda$ of solutions of (4.2)-(4.18), we have the following theorem.

THEOREM 5.3.— *Assume (5.28) holds. Let $n_p^\lambda$, $n_n^\lambda$, $u_p^\lambda$, $u_n^\lambda$, $\phi^\lambda$ be a non constant smooth solution of problem (4.1)-(4.18). Then, we have that*

$$\lim_{\lambda \to 0} (n_p^\lambda, n_n^\lambda, u_p^\lambda, u_n^\lambda, \phi^\lambda) = (n^-, n^-, u^-, u^-, 0) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{5.34}$$

Therefore, the solitary wave solutions lead to constant solutions in the limit $\lambda$ goes to 0. The proof is obvious since the functions $n_p^\lambda$, $n_n^\lambda$, $u_p^\lambda$, $u_n^\lambda$ and $\phi^\lambda$ are continuous and satisfy (5.29).

## 6. Shock solutions

We have shown that there is no heteroclinic orbit which connects two different stationary points. If we want to construct travelling wave solutions with different limiting values at $\xi = \pm\infty$, then we have to consider solutions involving possible discontinuities. We first describe the admissible shocks from the jump relations and the entropy condition.

### 6.1 The admissible shocks

The (Rankine–Hugoniot) jump relations for a shock have been given in (4.7)-(4.11). We already notice that the electrons remain subsonic (see condition (5.8)) so that the electron state variables $n_n$ and $u_n$ have no jump. On the other hand, the electric quantities $E$ and $\phi$ are continous. Thus, the jump occurs only on the ion quantities $p = n_p$ and $u_p$.

A simple computation shows that the jump condition (4.9) is equivalent to the equation:

$$p_r p_\ell = \frac{J^2}{T_p} = \left(n_p^s\right)^2, \tag{6.1}$$

where if $\xi_0$ is a point of discontinuity of $p$, we have defined:

$$p_\ell = \lim_{\xi \to \xi_0^-} p(\xi), \quad p_r = \lim_{\xi \to \xi_0^+} p(\xi), \tag{6.2}$$

the left and right limits of $p$. To obtain a physically relevant solution, an additionnal entropy condition has to be imposed [13]. A way of stating it is to require that the shock occurs from a supersonic state to a subsonic state in the direction of the flow which is the increasing $x$ direction since $J > 0$. In the present situation, it can be written:

$$p_\ell < n_p^s < p_r \,. \tag{6.3}$$

In the sequel, values $p_\ell$ and $p_r$ of the ion density are called conjugate if they satisfy both (6.1) and (6.3). For weak shocks, the entropy condition (6.3) is equivalent to the following constraint on jumps:

$$\left[(pu_p^2 + 2T_p p \ln p)u_p\right] \leq \sigma\left[pu_p^2 + 2T_p p(\ln p - 1)\right] , \tag{6.4}$$

where $S(p, u_p) = pu_p^2 + 2T_p p(\ln p - 1)$ is the entropy and $F(p, u_p) = (pu_p^2 + 2T_p p \ln p)u_p$ the associated entropy flux.

### 6.2 The shock solutions

We shall now characterize the infinite limiting value when $\xi \to +\infty$ in terms of the one at $\xi \to -\infty$ by use of the first integral of (5.14) which is preserved even through the shocks. Since the infinite states are assumed to be quasineutral, the expression of the first integral $g$ can be simplified further:

$$g(n^+) = n^+ T_p + n_n(n^+) T_n + \frac{J^2}{n^+} = n^+(T_p + T_n) + \frac{J^2}{n^+} = g(n^-) \tag{6.5}$$

and the equation $g(n^+) = g(n^-)$ can be recast for $n^+ \neq n^-$ according to:

$$n^- n^+ = \frac{J^2}{T_p + T_n} = n_c^2 \,. \tag{6.6}$$

We now notice that the current $J$, the sonic density $n_p^s$ and critical densities $n_c$ are unchanged through the shock. Thus, equation (6.6) implies that $n_c$ lies between $n^-$ and $n^+$.

However, the phase diagram changes during the shock together with the constant $d$. Indeed, the function $H_p(p)$ is not preserved through a shock and so is $d$ (5.9).

We first consider the phase portrait of a smooth portion of the travelling wave solution before the occurence of a shock (the left phase portrait). The starting point $(n^-, 0)$ cannot be in the subsonic domain (case (iii)). Indeed the solution cannot leave this domain with a jump because the entropy condition prevents it and there is never two stationary points in the subsonic domain. Then, the starting point $(n^-, 0)$ has to be in the supersonic domain and more precisely, it must be a saddle point. Indeed if the solution would start from a center point, it would necessarily have a discontinuity which would lead to the point $(p_r, 0)$ in the subsonic domain conjugate to $(n^-, 0)$. But there is no trajectory either smooth or discontinuous connecting $(p_r, 0)$ to $(n^+, 0)$ by (6.6). Finally, $n^-$ must be the point $n_0$ of one of the three generic cases.

Then, we are interested in the phase portrait of the solution after the occurence of a shock (the rigth phase portrait). The entropy condition states that the point $(p_r, E_0)$ after the shock lies in the subsonic domain. This point must lie on the stable manifold of the stationary point $(n^+, 0)$. Thus, the point $(n^+, 0)$ is either a saddle point in the subsonic domain, i.e. $(n_1, 0)$ in case (iii), or a saddle point in the supersonic domain which is connecting the point $(p_r, E_0)$ in the subsonic domain, i.e. $(n_0, 0)$ in the critical case.

This last case is readily eliminated because we have both $n^- < n_c$ and $n^+ < n_c$ and this contradicts equation (6.6). In fact, such solutions correspond to the limiting case of solitary wave solutions when the homoclinic orbit reachs the sonic point $(n_p^s, 0)$ and return to the stationary point $(n_0, 0)$. Thus, this does not enter the set of shock solutions as characterized by (6.6).

Therefore, the point $(n^+, 0)$ is necessarily in case (iii) which in particular implies $n^+ \geq n_p^s$. We have the following theorem.

THEOREM 6.1.— *There exists a unique admissible travelling wave solutions of (4.1)-(4.18) with $n^- \neq n^+$ if and only if*

$$\mathcal{M} \stackrel{\text{def}}{=} \frac{n_p^s}{n^-} \geq \frac{T_n + T_p}{T_p}\,. \tag{6.7}$$

*In this case, the jump relations are given by (6.6) with $n^- < n_c < n^+$. The velocity $u^+$ is given by $J = n^-(u^- - \sigma) = n^+(u^+ - \sigma)$. Moreover, this solution satifies:*

$$E(\xi) \leq 0\,,\ \forall\, \xi \in \mathbb{R}\,, \quad \int_{\mathbb{R}} E(\xi)\,\mathrm{d}\xi = H_n(n^-) - H_n(n^+)\,, \tag{6.8}$$

$$\int_{\mathbb{R}} \big(n_n(\xi) - n_p(\xi)\big)\,\mathrm{d}\xi = 0\,. \tag{6.9}$$

The shape of such a shock solution is given in figures 10 and 11.

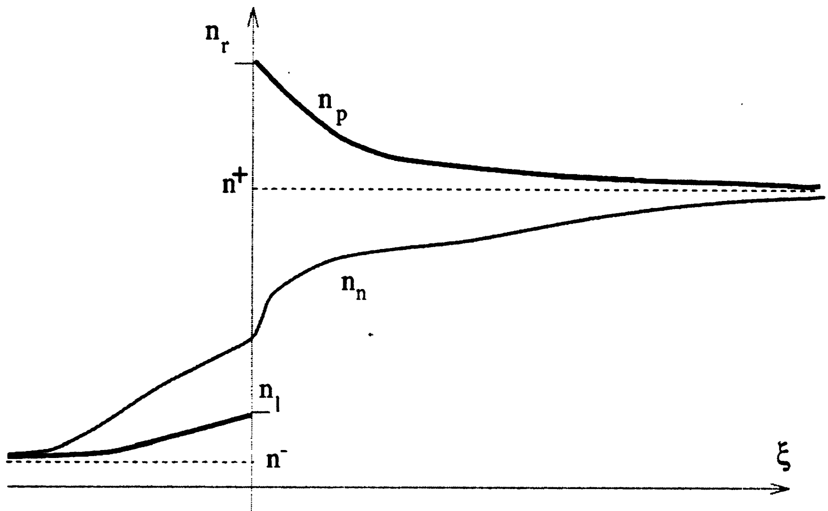


**Fig. 10** Shock solution: densities.

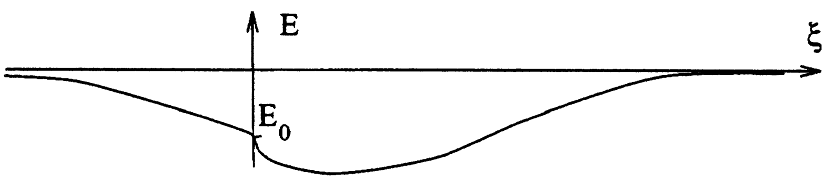


**Fig. 11** Shock solution: electric field.

*Proof.* — The condition (6.7) is equivalent to state that the point $(n^+, 0)$ is in the ion subsonic domain, i.e. $n^+ > n_p^s$. We recall that the latter condition is a necessary condition for the existence of shock solutions. Indeed, we have from the relation (6.6):

$$\begin{aligned} n^+ \geq n_p^s &\Longleftrightarrow \frac{n_c^2}{n^-} \geq n_p^s \\ &\Longleftrightarrow \mathcal{M} = \frac{n_p^s}{n^-} \geq \left(\frac{n_p^s}{n_c}\right)^2 = \frac{T_n + T_p}{T_p}, \end{aligned} \tag{6.10}$$

We now assume that (6.7) holds and we shall construct explicitly the travelling wave solution.

We have shown that the solution starts at the saddle point $(n^- = n_0, 0)$ in the supersonic domain, then it follows the trajectory either in the half plane $E > 0$ or in the half plane $E < 0$. This trajectory is given by:

$$E = E(p) = \pm\sqrt{2\big(g(p) - g(n^-)\big)}\,. \tag{6.11}$$

At one point $(p_\ell, E_0)$ of this trajectory, the solution has a shock which leads to the conjugate point $(p_r, E_0)$ in the subsonic domain. This point must lie on the unstable manifold of the saddle point $(n^+ = n_1, 0)$ of the subsonic domain (case (iii)).

Hence, the sign of $E$ is constant along the trajectory. We shall prove (6.8). Indeed, the electron density increases from $n^-$ to $n^+ > n^-$ and is given by (5.11) which is a decreasing function of $H_p(p)$. If $E > 0$, then $H_p(p)$ increases along the trajectory because of (5.6) and (5.14), and thus $n_n$ decreases. This is impossible; thus, $E$ has to be negative.

The second part of property (6.8) is obvious, by integrating the first equation of (5.4), we get:

$$\int_{\mathbb{R}} E(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} -\big(H_n(n_n(\xi))\big)'\,\mathrm{d}\xi = H_n(n^-) - H_n(n^+)$$

and the second equation of system (5.14) leads to (6.9):

$$\int_{\mathbb{R}} \big(n_n(\xi) - n_p(\xi)\big)\,\mathrm{d}\xi = \int_{\mathbb{R}} E(\xi)'\,\mathrm{d}\xi = 0\,.$$

Finally, we have to determine the intermediate shock i.e. the values of $p_\ell$, $p_r$ and $E_0 < 0$. Since $n^-$ and $n^+$ are known we can compute the constant $d$ for the left and right phase portraits:

$$d^\pm = H_p(n^\pm) + H_n(n^\pm)\,. \tag{6.12}$$

Then, the jump relation (6.1) allows us to express $p_r$ in terms of $p_\ell$ and the continuity of the electron density can be written as an implicit equation for the variable $p_\ell$:

$$\begin{aligned} n_n^-(p_\ell) = n_n^-(p_r) &\iff d^- - H_p(p_\ell) = d^+ - H_p(p_r) \\ &\iff \frac{T_p^2 p_\ell^2}{2J^2} + T_p \ln\Big(\frac{J^2}{T_p p_\ell}\Big) - \frac{J^2}{2p_\ell^2} - T_p \ln(p_\ell) = d^+ - d^- \\ &\iff \frac{T_p}{2}\left(2\ln\Big(\frac{J^2}{T_p p_\ell^2}\Big) + \frac{T_p p_\ell^2}{J^2} - \frac{J^2}{T_p p_\ell^2}\right) = d^+ - d^- \end{aligned} \tag{6.13}$$

where $n_n^{\pm}$ denotes the different functions $n_n(p)$ before and after the intermediate shock. We set $\mathcal{M}' = J/\sqrt{T_p}\, p_\ell = n_p^s/p_\ell$; this is the Mach number for the ions of the state before the shock and thus, $\mathcal{M}' > 1$ since $p_\ell$ is a supersonic state. Then, since $n_p^s$ is known, the implicit equation (6.13) in variable $p_\ell$ can be written in variable $\mathcal{M}'$ as follow:

$$\frac{T_p}{2}\,\Phi((\mathcal{M}')^2) = d^+ - d^- \quad \text{with } \Phi(X) = 2\ln X + \frac{1}{X} - X\,. \tag{6.14}$$

An obvious computation gives:

$$\begin{aligned} \Phi'(X) &= -\frac{(X-1)^2}{X^2} < 0\,, \\ \lim_{X\to+\infty} \Phi(X) &= -\infty\,, \\ \Phi(1) &= 0\,. \end{aligned}$$

Moreover, we shall prove that $d^+ < d^-$. Indeed,

$$\begin{aligned} d^+ - d^- &= H_p(n^+) + H_n(n^+) - H_p(n^-) - H_n(n^-) \\ &= \frac{J^2}{2(n^+)^2} + (T_p + T_n)\ln(n^+) - \frac{J^2}{2(n^-)^2} - (T_p + T_n)\ln(n^-) \\ &= (T_p + T_n)\left(\ln\left(\frac{n^+}{n^-}\right) + \frac{J^2}{2(T_p+T_n)}\left(\frac{1}{(n^+)^2} - \frac{1}{(n^-)^2}\right)\right). \end{aligned}$$

But, since $n_c^2 = n^+ n^-$ from (6.6):

$$\begin{aligned} d^+ - d^- &= \left(\frac{T_p + T_n}{2}\right)\left(2\ln\left(\frac{n^+}{n^-}\right) + \left(\frac{n^-}{n^+} - \frac{n^+}{n^-}\right)\right) \\ &= \left(\frac{T_p + T_n}{2}\right)\Phi\left(\frac{n^+}{n^-}\right). \end{aligned}$$

Thus, $n^+ > n^-$ and the properties of $\Phi$ imply $d^+ < d^-$.

Then, there exists a unique solution $\mathcal{M}_0 > 1$ of (6.14) and finally, the intermediate shock is uniquely determined. The corresponding electric field $E_0$ can be computed by (6.11). We notice that the Mach number of the intermediate shock $\mathcal{M}' = n_p^s/p_\ell$ for the ions is equal to the Mach number of the shock between the infinite values of the density $\mathcal{M}' = n_c/n^- = \sqrt{n^+/n^-}$ for the quasineutral fluid. □

The property (6.9) expresses the global neutrality of the plasma. The trajectory in phase space associated with the travelling wave solutions described in Theorem 6.1 is depicted on figure 4 in bold line.

*Remark* 6.2.— We point out that such a shock solution cannot have several jumps, since after the first shock, the solution is subsonic and cannot pass into the supersonic domain again. Moreover, we have uniquely determined the possible intermediate jump.

PROPOSITION 6.3.— *The necessary and sufficient condition (6.7) for the existence of solutions of problem (4.1)-(4.18) with $n^+ \neq n^-$ is equivalent to the following condition on the wave velocity:*

$$\sigma < u^- - \frac{T_p + T_n}{\sqrt{T_p}} \tag{6.15}$$

*or equivalently to the following constraint on the strength of the jump:*

$$\frac{n^+}{n^-} \geq \frac{T_p + T_n}{T_p}\,. \tag{6.16}$$

We are now interested in the limit when $\lambda$ goes to 0 of such travelling wave profiles. Indeed, we have constructed a set $U^\lambda$ of solutions of (4.2)-(4.18); we have the following theorem.

THEOREM 6.4.— *Assume (6.7) holds. Let $n_p^\lambda$, $n_n^\lambda$, $u_p^\lambda$, $u_n^\lambda$ and $\phi^\lambda$ be the solution of problem (4.1)-(4.18). Then, these functions converge in a distributional sense as follows:*

$$\lim_{\lambda\to 0} n_p^\lambda(x,t) = \lim_{\lambda\to 0} n_n^\lambda(x,t) = \begin{cases} n^- & \text{if } x < \sigma t \\ n^+ & \text{if } x > \sigma t \end{cases} \tag{6.17}$$

$$\lim_{\lambda\to 0} u_p^\lambda(x,t) = \lim_{\lambda\to 0} u_n^\lambda(x,t) = \begin{cases} u^- & \text{if } x < \sigma t \\ u^+ & \text{if } x > \sigma t \end{cases} \tag{6.18}$$

$$\lim_{\lambda\to 0} \phi^\lambda(x,t) = \begin{cases} \phi^- = 0 & \text{if } x < \sigma t \\ \phi^+ & \text{if } x > \sigma t \end{cases} \tag{6.19}$$

*where $n^+$ is given by (6.6), $u^+$ by $J = n^-(u^- - \sigma) = n^+(u^+ - \sigma)$ and $\phi^+ = \big(H_n(n^+) - H_n(n^-)\big)$. Moreover, the electric field $E^\lambda$ converges in a distributional sense:*

$$\lim_{\lambda\to 0} E^\lambda(x,t) = -\big(H_n(n^+) - H_n(n^-)\big)\,\delta_{x=\sigma t} \quad \text{in } \mathcal{D}'(\mathbb{R})\,. \tag{6.20}$$

*where $\delta_{x_0}$ represents the delta distribution located at $x_0$.*

The shock solutions lead to shock wave solutions of the quasineutral Euler model which propagate with velocity $\sigma$.

*Proof.*— The result follows from the fact that the travelling wave solution for $\lambda > 0$ is written:

$$U^{\lambda}(x,t) = U\left(\frac{x-\sigma t}{\lambda}\right)$$

where $U = U(\xi)$ is the above constructed solution. □

*Remark* 6.5.— The determination of the intermediate shock on the ion state variables and in particular $p_\ell$ and $p_r$ is not useful, if we are only interested in the limit of these solutions when $\lambda$ goes to 0. However, the monotony of $n_n$ along the trajectory will serve us to extend the analysis to the case $\eta \neq 0$. Finally, the shock relation for the quasineutral Euler model is described by the jump relation (6.6). This analysis also permits us to determine the strength of the delta function (6.20) which is the weak limit of the electric field and to justify the quasineutrality by (6.9).

## 7. Periodic solutions

We have shown that the existence of shock solutions requires sufficiently strong differences on the left and right densities $n^+$ and $n^-$ (Proposition 6.3). On the other hand smooth travelling wave solutions are solitary waves which satisfy $n^+ = n^-$. In order to fill the gap in between these two types of solutions in some way and find travelling wave solutions with ratio $n^+/n^-$ close to, but no equal to 1, we shall have to weaken the conditions on the asymptotic behaviour for $\xi \to +\infty$. We now look for solutions $U = (n_n,\, n_p,\, u_n,\, u_p,\, \phi)$ which remain bounded for $\xi \to +\infty$, without necessarily assuming a limit as $\xi \to +\infty$. We have following theorem.

THEOREM 7.1.— *We assume*

$$\mathcal{M}_- \leq \mathcal{M} \stackrel{\text{def}}{=} \frac{n_p^s}{n^-} < \frac{T_n + T_p}{T_p}, \tag{7.1}$$

*where $\mathcal{M}_-$ is the solution of (5.25) with $\mathcal{M} > 1$. Then, there exists a unique bounded travelling wave solutions of (4.1)-(4.18). Moreover, there exist $\xi_0 \in \mathbb{R}$, $\xi_1 > \xi_0$ and $T > 0$ such that:*

- *$n_n$, $n_p$, $u_n$, $u_p$ and $\phi$ are smooth solutions of (4.1)-(4.18) for $\xi < \xi_0$, with $n_n > n_n^s$ (subsonic electrons) and $n_p < n_p^s$ (supersonic ions);*
- *at the point $\xi_0$, the quantities $n_p$ and $u_p$ have an admissible discontinuity, but $n_n$, $\phi$ and $u_n$ are continuous;*
- *for $\xi \in \, ]\xi_0, \xi_1]$, the solution is smooth and follows the sonic trajectory $T^s$ in the subsonic domain. The solution passes through the sonic line at $\xi = \xi_1$;*
- *for all integer $n$, and for $\xi \in [\xi_1 + nT, \xi_1 + (n+1)T]$, the solution follows the supersonic loop of the sonic trajectory $T^s$ and has a periodic behaviour.*

The shape of such a periodic solution is given in figures 12 and 13. Its trajectory in phase space is depicted on figure 3 in bold lines.

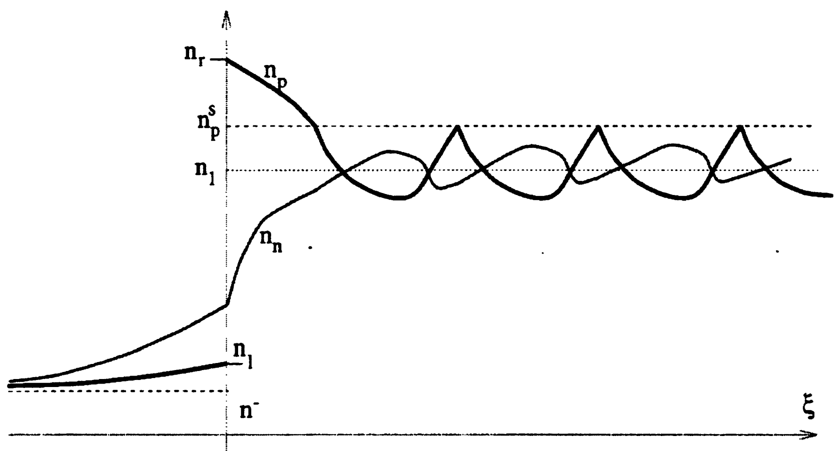


**Fig. 12** Periodic solution: densities.

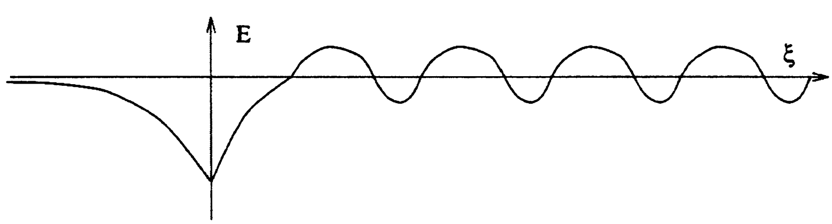


**Fig. 13** Periodic solution: electric field.

*Proof.*— Assume

$$\mathcal{M}_- \leq \mathcal{M} \overset{\text{def}}{=} \frac{n_p^s}{n^-} < \frac{T_n + T_p}{T_p},$$

i.e. case (ii). If a travelling wave solution with $n_p$ and $n_n$ tending to $n^+$ as $\xi \to +\infty$ would exist, then $n^+$ would be linked to $n^-$ by relation (6.6) with $n^- < n_c$. Then, $n^+$ would necessarily be the center point $(n_1, 0)$ in case (ii). But a center cannot be attained by a smooth solution.

Now, instead of looking for travelling wave solution such that $n_p$ and $n_n$ tend to $n^+$ as $\xi \to +\infty$, we look for travelling wave solutions which are bounded when $\xi \to +\infty$.

The only closed trajectory in case (ii) on which the solution can be bounded is the sonic one $T^s$ or any closed trajectory which in inside the sonic loop $T^s$. However, the solution starts at $\xi = -\infty$ from the hyperbolic stationary point $(n^-, 0)$. It must reach such a trajectory by a jump. However, a jump leads to the subsonic domain $n > n_p^s$. Thus, the only supersonic closed orbits which are reachable through a jump from $(n^-, 0)$ are those which are connected to the subsonic domain. Only one trajectory is in such case: the sonic trajectory $T^s$. Furthermore, the sonic point $(n_p^s, 0)$ is a point of non uniqueness so that the solution can follow the supersonic loop of $T^s$ infinitely many times.

The same ideas as in the shock case imply that the solution starts on the $E < 0$ branch of the unstable manifold of $(n_0 = n^-,\ 0)$. We have now to characterize the intermediate jump at the point $\xi_0$. This shock has to satisfy the conservation of momentum and global conservation of $g - (1/2)E^2$:

$$g(p_\ell) = g(p_r)\,, \quad g(p_\ell) - \frac{1}{2}E_0^2 = g(n^-)\,, \quad g(p_r) - \frac{1}{2}E_0^2 = g(n_p^s)\,.$$

Thus, we have to solve the same equation as in the critical case, i.e. $g(n^-) = g(n_p^s)$, but now as an equation for $d^+$ instead of $\sigma$:

$$g(n^-) = g(n_p^s)\,.$$

This leads to the following equation for $d^+$:

$$n^-(T_p + T_n) + \frac{J^2}{n^-} = n_p^s T_p + \exp\left(\frac{d^+ - H_p(n_p^s)}{T_n}\right) T_n + \frac{J^2}{n_p^s}\,, \qquad (7.2)$$

where $n^-$, $J$, $n_p^s$ are known. This can be written in the following form:

$$\exp\left(\frac{d^+ - H_p(n_p^s)}{T_n}\right) T_n = H_p(n^-) - H_p(n_p^s) + n^- T_n \,. \qquad (7.3)$$

The right hand side is non negative since $H_p$ takes its minimum at the point $n_p^s$. Then, like in the shock case, we can compute $p_\ell$ as the unique solution of the implicit equation (6.13). This concludes the proof. □

*Remark* 7.2.— We have the global neutrality over each loop on the sonic trajectory $T^s$:

$$\int_{\xi\in[\xi_1+nT\,,\,\xi_1+(n+1)T]} \big(n_n(\xi) - n_p(\xi)\big)\,\mathrm{d}\xi = 0\,. \qquad (7.4)$$

Moreover, we also have the global neutrality property on the first part of the periodic solutions:

$$\int_{\xi\in[-\infty\,,\,\xi_1]} \big(n_n(\xi) - n_p(\xi)\big)\,\mathrm{d}\xi = 0\,. \qquad (7.5)$$

The proof follows from the integration of the Poisson equation in (5.14) and using $E(\xi_1) = E(-\infty) = 0$ for (7.4) and the periodicity for (7.5).

PROPOSITION 7.3.— *The necessary and sufficient condition (7.1) is equivalent to the following condition on the wave velocity:*

$$u^- - \frac{T_n + T_p}{\sqrt{T_p}} < \sigma < u_- - \mathcal{M}_- \sqrt{T_p}\,. \qquad (7.6)$$

We now consider the limit when $\lambda \to 0$ of such periodic solutions. We have the following result.

THEOREM 7.4.— *Assume (7.1) holds. Let $n_p^\lambda$, $n_n^\lambda$, $u_p^\lambda$, $n_n^\lambda$ and $\phi^\lambda$ be the unique bounded travelling wave solution of problem (4.1)-(4.18). Then, these functions converge in the distributional sense as follows:*

$$\lim_{\lambda\to 0} n_p^\lambda(x,t) = \lim_{\lambda\to 0} n_n^\lambda(x,t) = \begin{cases} n^- & \textit{if } x < \sigma t \\ \overline{n} & \textit{if } x > \sigma t \end{cases} \qquad (7.7)$$

$$\lim_{\lambda\to 0} u_p^\lambda(x,t) = \lim_{\lambda\to 0} u_n^\lambda(x,t) = \begin{cases} u^- & \textit{if } x < \sigma t \\ \overline{u} & \textit{if } x > \sigma t \end{cases} \qquad (7.8)$$

$$\lim_{\lambda\to 0} \phi^\lambda(x,t) = \begin{cases} \phi^- = 0 & \textit{if } x < \sigma t \\ \phi^+ = \overline{\phi} & \\ \quad = \overline{H_n(n_n)} - H_n(n^-) & \textit{if } x > \sigma t \end{cases} \qquad (7.9)$$

*where $\overline{n}$, $\overline{u}$ and $\overline{\phi}$ are the average values of the density $n_p$, velocity $u_p$ and potential $\phi$ over one loop of the supersonic part of the sonic trajectory $T^s$. More precisely, we define the length of travel $T/2$ over the half loop (from $(n_p^s, 0)$ to $(n^*, 0)$ defined by $g(n^*) = g(n_p^s)$ in case (ii) along $T^s$) by*

$$\frac{T}{2} \stackrel{\text{def}}{=} \int_{n_p^s}^{n^*} \frac{h_p(n)}{\sqrt{2\big(g(n) - g(n_-)\big)}}\, \mathrm{d}n \tag{7.10}$$

*and the average value of a function $f(\xi)$ over a loop is*

$$\overline{f} \stackrel{\text{def}}{=} \frac{2}{T} \int_{\xi_1}^{\xi_1 + T/2} f(\xi)\, \mathrm{d}\xi \tag{7.11}$$

*for symmetric functions of $\xi$ i.e. $f(\xi_1+T/2+h) = f(\xi_1+T/2-h)$. Moreover, the electric field $E^\lambda$ converges in a distributional sense:*

$$\lim_{\lambda \to \infty} E^\lambda(x,t) = -\overline{\phi}\delta_{x=\sigma t} \quad \text{in } \mathcal{D}'(\mathbb{R})\,. \tag{7.12}$$

*Proof.* — The computation of the length of travel $T/2$ over the half loop comes from the first equation of (5.14):

$$\frac{\mathrm{d}p(\xi)}{\mathrm{d}\xi} = \frac{E}{h_p\big(p(\xi)\big)}\,, \tag{7.13}$$

then, we obtain, from the first integral of (5.14):

$$E = -\sqrt{2\big(g(n) - g(n_-)\big)}\,, \tag{7.14}$$

since $E < 0$. Thus, we have:

$$\mathrm{d}(\xi) = \frac{-h_p\big(p(\xi)\big)}{\sqrt{2\big(g(n) - g(n_-)\big)}}\, \mathrm{d}p \tag{7.15}$$

and by integrating from $(n_p^s, 0)$ to $(n^*, 0)$, we obtain (7.10).

Then, the weak convergences (7.7) comes from the weak convergences of periodic functions $f^\lambda$ to their average value when the period i.e. $\lambda T$ tends to 0. (7.8) is consequence of the periodicity through the relation $p(u_p - \sigma) =$ constant. The electric field tends to a delta function; indeed, because of its symmetry over each period: $E(\xi_1 + T/2 + h) = -E(\xi_1 + T/2 - h)$. The mean values of $E$ over its period are 0 and thus, the electric potential $\phi^\lambda$ is a periodic function for $\xi > \xi_1$. Thus, (7.9) comes from integrating the electric field $E$ from $-\infty$ to $\xi_1$:

$$\phi(\xi_1) = -\int_{-\infty}^{\xi_1} E(\xi)\,\mathrm{d}\xi = \int_{n^-}^{n_p^s} (H_n(n))'\,\mathrm{d}n = H_n(n_p^s) - H_n(n^-) \quad (7.16)$$

and we have

$$\overline{\phi} = \phi(\xi_1) + \frac{1}{T}\int_{\xi_1}^{\xi_1+T}\int_0^t -E(\xi)\,\mathrm{d}\xi\,\mathrm{d}t$$

and

$$\begin{aligned}\frac{1}{T}\int_{\xi_1}^{\xi_1+T}\int_0^t -E(\xi)\,\mathrm{d}\xi\,\mathrm{d}t &= \frac{1}{T}\int_{\xi_1}^{\xi_1+T} \big(H_n(n(t)) - H_n(n_p^s)\big)\,\mathrm{d}t \\ &= \overline{H_n(n_n)} - H_n(n_p^s)\,.\end{aligned}$$

Then, we have

$$\overline{\phi} = \phi(\xi_1) + \overline{H_n(n_n)} - H_n(n_p^s) = \overline{H_n(n_n)} - H_n(n^-)\,.$$

Therefore, $\phi^\lambda$ tends to a Heaviside function from 0 to $\overline{\phi}$ given by (7.9) and $E^\lambda = -\partial_x\phi^\lambda$ tends to a delta function in the distributional sense when $\lambda$ goes to 0.

The property $n_n^- = n_p^-$ arises from the global neutrality (Remark (7.4)). □

*Remark* 7.5.— If we identify the "right state" as the average value $\overline{p}$ and $\overline{u}_p$, we obtain "jump relations" for the limits as $\lambda \to 0$ of the solutions (7.7)-(7.9). However the momentum is no more preserved. Indeed the flux of momentum is given by $g(n)$ and we have from the first integral:

$$g\big(p(\xi)\big) - \frac{1}{2}E^2(\xi) = g(n^-)$$

and by averaging:

$$\overline{(g(p))} = g(n^-) + \frac{1}{2}\overline{(E^2)}\,.$$

Thus, a part of the momentum flux has been converted into electric field energy since $\overline{(E^2)}$ is not zero in general.

Now, we consider the charge conservation equation. The average velocity is computed from the current conservation:

$$u_p = \frac{J}{p(\xi)} + \sigma$$

and by averaging, we get

$$\overline{u}_p = \overline{\left(\frac{J}{p(\xi)}\right)} + \sigma \neq \frac{J}{\overline{p}} + \sigma .$$

Thus, the mass (or charge) conservation is also not preserved for the weak limit of the periodic solution, and we need to compute separatly the average values of $p$ and $u_p$. Thus, weak limits when $\lambda \to 0$ of periodic travelling wave solutions are not weak solutions of the quasineutral Euler model.

When $\mathcal{M} \to \mathcal{M}_-$, the phase portraits before and after the shock both "tend" to the critical case portrait. Then, the constant $d^+$ solution of (7.2) tends to $d^-$ and the intermediate shock becomes weaker and weaker.

Moreover, the solution passes infinitely many times near the stationary point $(n_0, 0)$ in case (ii) and since "it spends a lot of time near $n_0$" we have:

$$\overline{n} \longrightarrow n^- , \quad \overline{u} \longrightarrow u^- .$$

But, since the velocity of these waves $\sigma \approx u^- - \mathcal{M}_- \sqrt{T_p}$ is not the velocity for small shocks of the quasineutral Euler model $\sigma \approx u^- - \sqrt{T_p + T_n}$, then, the momentum flux has still a jump accross the shock of order 1:

$$\overline{(g(p))} = g(n^-) + \frac{1}{2}\overline{(E^2)} ,$$

with $\overline{(E^2)} > 0$ and $\overline{(E^2)} \not\to 0$ as $\mathcal{M} \to \mathcal{M}_-$.

From these considerations, it is clear that the discontinuous solution from the left state $(n^-, u^-)$ to the right state $(\overline{n}, \overline{u})$ is not a weak solution of the quasineutral system. Such solutions can be associated to a special choice of the Lipschitz continuous path in the definition of the nonconservative products developed in [14].

When $\mathcal{M} \to 1 + T_n/T_p$, the loop becomes very small, i.e. $n^* \to n_p^s$ and we have:

$$\overline{n} \longrightarrow n_p^s , \quad \overline{u} \longrightarrow u_- - \sqrt{T_p} , \quad \overline{E^2} \longrightarrow 0 .$$

Then, the jump relations obtained with the periodic solutions extend continuously to the jump relations for shock solutions. Therefore, for

$\mathcal{M} \to 1 + T_n/T_p$, there is in some sense a continuous matching of the set of solutions obtained in Theorem 7.4 with the shock curve of the quasineutral Euler model.

The three generic solutions we constructed have been already described by plasma physicists. We presented the problem arising from the limit of periodic solutions. The plasma physicists usually deal with this problem by stating that the energy of plasma oscillations is dissipated. The mathematical analysis of these dispersive phenomenom has to be investigated further.

## 8. Stationary solutions

In this section, we are interested in the stationary solutions of the isothermal Euler–Poisson model, i.e. in the travelling waves of (4.1)-(4.18) with $\sigma = 0$. It can be naturally extended to polytropic model as in section 9.2.

We set $J = n^-(u^- - \sigma)$ the total current. The existence of a solution which has a limit when $\xi \to -\infty$ imposes $n^- = n_0 < n_c$. Then, we have the following theorem.

THEOREM 8.1.— *Let $n^- > 0$ and $u^- > 0$ be given. Define $\mathcal{M} = n_p^s/n^-$ the ion Mach number of the left state.*

- *If $\sqrt{(T_n + T_p)/T_p} < \mathcal{M} < \sqrt{\mathcal{M}_-}$ where $\mathcal{M}_-$ is defined by (5.25), then there exists a unique non constant smooth travelling wave solution of (4.1)-(4.18) with $\sigma = 0$ as constructed in Theorem 5.1.*
- *If $\mathcal{M} \geq (T_n + T_p)/T_p$ then there exists a unique admissible travelling wave solutions of (4.1)-(4.18) with $\sigma = 0$; this is a shock solution as described in section 6.*

*In these two cases, the solution verifies:*

$$\int_{\mathbb{R}} (n_n(\xi) - n_p(\xi)) = 0\,.$$

*In between, i.e. for $\mathcal{M}_- \leq \mathcal{M} < (T_n + T_p)/T_p$, there is no non-constant stationary solution with a limit $U^+$ when $\xi \to +\infty$, but there exists a bounded stationary solution of the Euler–Poisson model with a periodic behaviour for $\xi > \xi_1$ as described in Theorem 7.1.*

## 9. Extensions

We first complete the set of bounded travelling waves by solutions which satisfy weaker condition at $-\infty$. Then, we extend the analysis to polytropic models and to small but finite electron mass models.

### 9.1 Completing the set of travelling wave solutions

In this section, we shall complete the set of non constant bounded travelling wave solutions by relaxing the boundary condition when $\xi \to -\infty$.

A solution of (4.1)-(4.18) is bounded as $\xi \to -\infty$ if and only if it has a limit $U^-$ (case previously analyzed) or it has a periodic behaviour, i.e. it stays on a closed orbit. The only closed orbits are in the supersonic domain in case (i) and (ii).

All the trajectories which are inside the homoclinic orbit issued from $n_0$ in case (i) or inside the sonic trajectory $T^s$ in case (ii) are smooth periodic solutions of (4.1)-(4.18). These solutions can be characterized by the values $g > 0$ of $g(p) - (1/2)E^2$, $d > 0$ of $H_n(p) + H_p(p)$ and $J > 0$ of the current which are constant. These invariants play the role of the initial values $n^-$, $u^-$ and the velocity $\sigma$ in the preceeding cases.

For given $g$, $d$ and $J$, we define the functions $H_p$, $H_n$, $g$, $n_n$ of $p$ and the sonic and critical densities $n_p^s$ and $n_c$ as usual. We also define the densities $n_0$ and $n_1$ by the equation $(H_n + H_p)(p) = d$ such that $n_0 < n_c < n_1$. Then, these constants correspond to a closed orbit in case (i) and (ii) if and only if $n_1 < n_p^s$ (or equivalently $(H_p + H_n)(n_p^s) > d$ since $(H_p + H_n)(p)$ is increasing for $p > n_c$) and $g > \max(g(n_0), g(n_p^s))$. Then, we have the following result.

THEOREM 9.1.— *Let $J > 0$, $d > 0$ and $g > 0$ be given such that*

$$(H_p + H_n)(n_p^s) > d\,, \quad g > \max(g(n_0, g(n_p^s))) \tag{9.1}$$

*where $H_\alpha$ for $\alpha = n$, $p$, $g$, $n_0$, $n_p^s$ and $n_c$ are defined by (5.5), (5.16), (5.9), (5.7) and (5.19). Then, there exists a unique bounded non constant periodic travelling wave solution of (4.1)-(4.18). At one point $(p_1, E_0)$ on this closed orbit with $E_0 < 0$, the solution can have a jump such that:*

*If $n_p^s(2T_p + T_n) < g$, the solution has a limit when*

$$\lim_{\xi \to +\infty} n(\xi) = n^+ \,, \tag{9.2}$$

*such that $(T_p + T_n)n^+ + J^2/n^+ = g$.*

*Otherwise, the solution returns in the supersonic domain and has a periodic behaviour.*

The latter condition indicates the conservation of the total momentum.

## 9.2 Polytropic models

We shall extend the travelling wave analysis to polytropic models where the isothermal law is replaced by the following pressure law:

$$P_\alpha(n) = c_\alpha n^\gamma \,, \tag{9.3}$$

where $\gamma > 1$ and $c_\alpha$ are constant for each species. We shall present this analysis with the same polytropic constant $\gamma$ for electrons and ions for the sake of simplicity. Indeed, this analysis will serve us in the study of the complete Euler model, i.e. with energy equations. The travelling wave analysis can be carried out for smooth solutions exactly like in section 5. We obtain the dynamical system (5.14) with:

$$H_\alpha(p) = \frac{\gamma c_\alpha}{\gamma - 1} p^{\gamma-1} + \frac{J_\alpha^2}{2p^2} \,. \tag{9.4}$$

We have the following theorem for smooth solutions.

THEOREM 9.2.— *There exists a non constant smooth travelling wave solutions of (4.1)-(4.18) with $\gamma > 1$ if and only if the ion Mach number*

$$\mathcal{M} \overset{\text{def}}{=} \left(\frac{n_p^s}{n^-}\right)^{\gamma+1} \quad \textit{with } n_p^s = \left(\frac{J^2}{\gamma c_p}\right)^{1/(\gamma+1)}$$

*or equivalently,*

$$\mathcal{M} = \frac{(u-\sigma)^2}{\gamma T_p} = \left(\frac{J^2}{\gamma c_p n_0^{\gamma+1}}\right)$$

*is such that*

$$\frac{T_p + T_n}{\gamma T_p} < \mathcal{M} \leq \mathcal{M}_- \tag{9.5}$$

*where $\mathcal{M}_-$ is the solution of*

$$1 + c_p + \gamma c_p \mathcal{M} = (\gamma+1) c_p \mathcal{M}^{\gamma/(\gamma+1)} + \\ + \left(1 + c_p + \frac{\gamma-1}{2} c_p \mathcal{M} - \frac{\gamma+1}{2} c_p \mathcal{M}^{(\gamma-1)/(\gamma+1)}\right)^{\frac{\gamma}{\gamma-1}} \tag{9.6}$$

*with $\mathcal{M} > 1$. Moreover, this solution satisfies (5.29)-(5.32).*

The shape of such a solitary wave solution is the same as in figure 8. The lower bound in (9.5) comes from the condition:

$$n^- = n_0 < n_c \stackrel{\text{def}}{=} \left(\frac{J^2}{c_p + c_n}\right)^{1/(\gamma+1)} .$$

The upper bound given by (9.6) generalizes the condition of existence of solitary wave solutions given in [8] for polytropic plasma models.

We are now interested in shock solutions. Then, we have the following theorem.

THEOREM 9.3.— *There exists a unique admissible travelling wave solutions of (4.1)-(4.18) with $n^- \neq n^+$ and $\gamma > 1$ if and only if the Mach number $\mathcal{M}$ is such that*

$$\mathcal{M} \stackrel{\text{def}}{=} \left(\frac{n_p^s}{n^-}\right)^{\gamma+1} = \frac{(u-\sigma)^2}{\gamma T_p} = \frac{J^2}{\gamma c_p n_0^{\gamma+1}} \geq \mathcal{M}_* \tag{9.7}$$

*with $\mathcal{M}_* = X^{-\gamma-1}$ where $X$ is the solution of*

$$\frac{X - X^{\gamma+1}}{1 - X} = \frac{3(T_p + T_n)}{5 T_p} .$$

*In this case, the jump relations are given by the usual Rankine–Hugoniot relations for the quasineutral Euler model which can be written:*

$$n^+ n^- \frac{(n^+)^\gamma - (n^-)^\gamma}{n^+ - n^-} = n_c^{\gamma+1} \tag{9.8}$$

*with*

$$n^- < n_c \stackrel{\text{def}}{=} \left(\frac{J^2}{c_p + c_n}\right)^{1/\gamma+1} < n^+ .$$

*The velocity* $u^+$ *is given by* $J = n^-(u^- - \sigma) = n^+(u^+ - \sigma)$. *Moreover, this solution satisfies:*

$$E(\xi) \leq 0\,,\ \forall\, \xi \in \mathbb{R}\,, \quad \int_{\mathbb{R}} E(\xi)\,\mathrm{d}\xi = H_n(n^-) - H_n(n^+)\,, \tag{9.9}$$

$$\int_{\mathbb{R}} (n_n(\xi) - n_p(\xi))\,\mathrm{d}\xi = 0\,. \tag{9.10}$$

The condition (9.7) comes from $n^+ > n_p^s$ and using the relation (9.8) which arises from the conservation of the total momentum, i.e. the first integral $g(p) = J^2/p + c_p p^\gamma + c_n n_n(p)^\gamma$. The shape of such solutions is the same as in figure 10. In between, we shall again construct periodic solutions.

THEOREM 9.4.— *We assume:*

$$\mathcal{M}_- \leq \mathcal{M} \stackrel{\mathrm{def}}{=} \left(\frac{n_p^s}{n^-}\right)^{\gamma+1} < \mathcal{M}_* \tag{9.11}$$

*where* $\mathcal{M}_-$ *is the solution of (9.6) with* $\mathcal{M} > 1$ *and* $\mathcal{M}_*$ *is defined in (9.7). Then, there exists a unique bounded travelling wave solutions of (4.1)-(4.18). Moreover, there exists* $\xi_0 \in \mathbb{R}$, $\xi_1 > \xi_0$ *and* $T > 0$ *such that:*

- $n_n$, $n_p$, $u_n$, $u_p$ *and* $\phi$ *are smooth solutions of (4.1)-(4.18) or* $\xi < \xi_0$, *with* $n_n > n_n^s$ *(subsonic electrons) and* $n_p < n_p^s$ *(supersonic ions);*
- *at the point* $\xi_0$ *the quantities* $n_p$ *and* $u_p$ *have an admissible discontinuity, but* $n_n$, $\phi$ *and* $u_n$ *are continuous;*
- *for* $\xi \in \,]\xi_0\,,\xi_1\,]$, *the solution is smooth and follows the sonic trajectory* $T^s$ *in the subsonic domain. The solution passes through the sonic line at* $\xi = \xi_1$*;*
- *for all integer* $n$ *and for* $\xi \in \left[\xi_1 + nT\,,\, \xi_1 + (n+1)T\right]$, *the solution follows the supersonic loop of the sonic trajectory* $T^s$ *and has a periodic behaviour.*

The shape of such a periodic solution is given in figure 12. Its trajectory in phase space is depicted on figure 3 in bold lines.

### 9.3 Small electron mass

In this paragraph, we shall extend the travelling wave analysis to small but finite electron mass models. We present the ideas for the isothermal case. The key point is to express the electron density in terms of the ions one using (5.11). For this, we need to ensure the monotony of the electron enthalpy $H_n$ all along the range of the solution $n_n(\xi)$ or equivalently that hypothesis (5.8) is satisfied by $n_n(\xi)$ for all $\xi \in \mathbb{R}$. Indeed, we prove that if the condition (5.8) is satisfied for $n^-$, i.e. if

$$n^- > n_n^s = \eta^{1/2} \frac{J}{\sqrt{T_n}},$$

or equivalently if

$$\sigma > u^- - \sqrt{\frac{T_n}{\eta}}, \tag{9.12}$$

then, condition (5.8) is satisfied along the trajectory of the solitary solutions, of the shock solutions and of the periodic solutions. Indeed, the electron density increases when $E < 0$ since $n_n$ increases with decreasing $H_p(p)$ and $\big(H_p(p)\big)' = E$. Thus, the electron density increases above the value $n^-$ on the unstable branch of $(n^-, 0)$ since $E$ becomes negative. This argument allows thus to state the three existence theorems.

*Solitary wave solutions*

THEOREM 9.5.— *Assume (9.12) holds. There exists a non constant smooth travelling wave solution of (4.1)-(4.18) if and only if the ion Mach number $\mathcal{M} \stackrel{\text{def}}{=} n_p^s/n^-$ is such that*

$$\sqrt{\frac{T_n + T_p}{T_p(1+\eta)}} < \mathcal{M} \leq \mathcal{M}_- \tag{9.13}$$

*where $\mathcal{M}_-$ is the solution of:*

$$\left(1 + \frac{T_n}{T_p}\right) + (1+\eta)\mathcal{M}^2 = (2+\eta)\mathcal{M} + \frac{T_n}{T_p}\mathcal{M}^{-T_p/T_n} \exp\left(\frac{T_p(1+\eta)}{2T_n}(\mathcal{M}^2 - 1)\right) \tag{9.14}$$

*with $\mathcal{M} > 1$. Moreover, this solution satisfies (5.29)-(5.32).*

For the solitary wave solution, the electron density reaches its maximum value when $p = p^*$ (i.e. when $E = 0$) such that $p^* > n^- > n_n^s$ and afterwards, it decreases from $p^*$ to $n_0$.

*Shock wave solutions*

THEOREM 9.6.— *Assume (9.12) holds. There exists a unique admissible travelling wave solution of (4.1)-(4.18) with* $\eta \neq 0$ *if and only if*

$$\mathcal{M} \stackrel{\text{def}}{=} \frac{n_p^s}{n^-} \geq \frac{T_n + T_p}{T_p(1+\eta)} . \tag{9.15}$$

*In this case, the relation between the states at infinity is:*

$$n^- n^+ = \frac{J^2(1+\eta)}{T_p + T_n} = n_c^2$$

*with*

$$n^- < n_c \stackrel{\text{def}}{=} \sqrt{\frac{J^2(1+\eta)}{T_p + T_n}} < n^+ .$$

*The velocity* $u^+$ *is given by* $J = n^-(u^- - \sigma) = n^+(u^+ - \sigma)$. *Moreover, this solution satisfies:*

$$E(\xi) \leq 0 \, , \ \forall \, \xi \in \mathbb{R} \, , \quad \int_{\mathbb{R}} E(\xi) \, \mathrm{d}\xi = H_n(n^-) - H_n(n^+) \tag{9.16}$$

$$\int_{\mathbb{R}} \big(n_n(\xi) - n_p(\xi)\big) \, \mathrm{d}\xi = 0 . \tag{9.17}$$

For shock solutions, the electric field is always negative and thus the electron density increases monotonically along the trajectory from $n^-$ to $n^+$.

*Periodic solutions*

THEOREM 9.7.— *We assume (9.12) holds and*

$$\mathcal{M}_- \leq \mathcal{M} \stackrel{\text{def}}{=} \frac{n_p^s}{n^-} < \frac{T_n + T_p}{T_p(1+\eta)} \tag{9.18}$$

*where* $\mathcal{M}_-$ *is the solution of (9.14) with* $\mathcal{M} > 1$. *Then, there exists a unique bounded travelling wave solutions of (4.1)-(4.18) as described in Theorem 7.1.*

For periodic solutions, the electric field is negative for $\xi \in \, ]-\infty \, , \xi_1]$ (where $\xi_1$ is defined in Theorem 7.1) and thus the electron density increases.

Then, on the first half period $T/2$, the electron density decreases ($E > 0$) from $n_n(n_p^s)$ to $n_n(n^*)$. Moreover, $(n^*, 0)$ is in the negatively charged region and thus $n_n(n^*) > n^* > n_0$. Finally, $(n_n(p))$ remains larger than $n^-$ for the periodic solutions.

We mention that (9.14) generalizes the condition of existence of solitary wave solutions given in [8] for isothermal plasma models with small but non zero electon mass. On the other hand, (9.12) can be interpreted as a condition for the Mach numbers

$$\mathcal{M} < \left(\frac{T_n}{T_p}\eta\right)^{-1/2} .$$

Thus, the shock solutions cannot be constructed for arbitrary large shock.

### 9.4 Several ions

When several species of ions are taken into account, it is proved in [10] that the obtained quasineutral system is not always hyperbolic. However, it seems possible to extend the travelling wave analysis. This will be done in future work.

This analysis will be extended to the full Euler equations, i.e. with energy equations instead of pressure laws [2]. In this case, the quasineutral limiting system is in a non-conservative form which brings some indetermination in the jump relations. We construct the three generic solutions and we get jump relations for the shock solutions. Using these relations we can solve the Riemann problem for the quasineutral system [12].

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