ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 5, nº 2 (1996), p. 233-297

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Annales de la Faculté des Sciences de Toulouse Vol. V, n° 2, 1996

Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form(*)

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RÉSUMÉ. — L'article présente un système de coordonnées locales holomorphes pour l'espace des variétés hyperboliques de dimension 3 qui ont le groupe fondamental d'une surface. Ces coordonnées dépendent du choix d'une lamination géodésique sur la surface, et forment une complexification des coordonnées de décalage introduites par Thurston pour l'espace de Teichmüller. La partie imaginaire de ces coordonnées mesure la courbure d'une surface plissée réalisant la lamination géodésique. De plus, nous montrons comment ces coordonnées sont reliées, par l'intermédiaire de la forme symplectique de Thurston sur l'espace des laminations géodésiques mesurées, à la fonction longueur complexe et à sa différentielle.

ABSTRACT. — The article develops a system of local holomorphic coordinates for the space of hyperbolic 3-manifolds with the fundamental group of a surface. These coordinates depend on the choice of a geodesic lamination on the surface, and are a complexified version of Thurston's shear coordinates for Teichmüller space. The imaginary part of these coordinates measures the bending of a pleated surface realizing the geodesic lamination. We also show how these coordinates are related, via Thurston's symplectic form on the space of measured geodesic laminations, to the complex length function and to its differential.

AMS Classification : 57N02, 53C22 (1991).

KEY-WORDS : geodesic laminations, Teichmüller space, hyperbolic surface, pleated surface.

(*) Reçu le 16 septembre 1994

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This research was partially supported by N.S.F. grants DMS-8958665, DMS-9001895, DMS-9201466.

Introduction

Given two hyperbolic metrics m_1 and m_2 on a closed oriented surface S, W. P. Thurston described a way to pass from one to the other, called a *left earthquake* ([Th2], [Ke1], [EpM]). Such an earthquake has a fault locus λ which is a geodesic lamination, namely a closed family of disjoint simple m_1-geodesics in S. The earthquake process splits S along λ, then glues it back together so that any two pieces of $S - \lambda$ are shifted to the left of their original position with respect to each other. In this way, we obtain a new surface S' which is homeomorphic to S, and where the metric of $S - \lambda$ uniquely extends to a hyperbolic metric which is isotopic to m_2. If k is an arc transverse to λ, the amount by which the pieces of $S - \lambda$ meeting k are shifted to the left with respect to each other associates to k a number $\alpha(k) \geq 0$. It turns out that the map $k \mapsto \alpha(k)$ is countably additive, so that α defines a transverse measure for λ. The combination of λ and α forms what is known as a *measure lamination*. It is quite remarkable that this measured lamination completely determines the earthquake and is uniquely determined by the metrics m_1 and m_2.

In the first half of this paper, we consider a generalization of earthquakes, where we allow the pieces of $S - \lambda$ to be shifted to the right as well as to the left with respect to each other. Thurston calls this operation a *cataclysm*, although we will prefer the terminology of *shear map*. The amount of shifting to the left again associates a number $\alpha(k) \in \mathbb{R}$ to each arc transverse to λ, where a shift to the right is counted as negative. However, the map $k \mapsto \alpha(k)$ is not countably additive any more, but only finitely additive. This α defines what we call an $\mathbb{R}$-*valued transverse cocycle* for λ.

The $\mathbb{R}$-valued transverse cocycles for λ form a finite dimensional vector space $\mathcal{H}(\lambda; \mathbb{R})$ which is well understood, for instance in terms of weights on a train track carrying λ; see [Bo4].

We use these shear maps to parametrize the *Teichmüller space* $\mathcal{T}(S)$ of S, namely the space of isotopy classes of hyperbolic metrics on S. For this, we fix a maximal geodesic lamination λ; there are various ways to define this in a metric independent way. Then, we associate to each $m \in \mathcal{T}(S)$ a *shearing cocycle* $\sigma_m \in \mathcal{H}(\lambda; \mathbb{R})$ such that, if m_1 is transformed to m_2 by a shear map with fault locus λ, the transverse cocycle measuring the shifts to the left of this shear map is exactly $\sigma_{m_2} - \sigma_{m_1}$. We then prove the following results.

THEOREM A. — *The map $m \mapsto \sigma_m$ defines a real analytic homeomorphism from $\mathcal{T}(S)$ to an open convex cone $\mathcal{C}(\lambda)$ bounded by finitely many faces in $\mathcal{H}(\lambda;\mathbb{R})$.*

In particular, given two metrics m_1, $m_2 \in \mathcal{T}(S)$ and a maximal geodesic lamination λ, there is a unique shear map with fault locus λ that sends m_1 to m_2.

The vector space $\mathcal{H}(\lambda;\mathbb{R})$ carries a natural symplectic form τ, called the Thurston symplectic form. The convex cone in Theorem A can be explicitly described in terms of this form and of the space of transverse measures for λ. More precisely, we prove in section 6 the following theorem.

THEOREM B. — *The transverse cocycle $\mathcal{H}(\lambda;\mathbb{R})$ is the shearing cocycle of some hyperbolic metric if and only if $\tau(\alpha,\mu) > 0$ for every transverse measure μ for λ.*

Theorems A and B were essentially proved by Thurston in [Th3], although the connection to shearing cocycles and to the Thurston symplectic form is only outlined there (see also [Pa2]). The approach we use here is analytic, as opposed to Thurston's more topological point of view. One advantage of this analytic point of view is that it makes it easier to write down the details of a rigourous proof. But its main advantage is that the techniques developed also apply to another situation, where transverse cocycles can be used to measure the bending of pleated surfaces.

Indeed, there is another celebrated occurrence of measured laminations, as bending measures of locally convex pleated surfaces. A *pleated surface* with pleating locus the geodesic lamination λ is a map $f : S \to M$ from S to an oriented hyperbolic 3-dimensional manifold M such that f is a totally geodesic immersion on $S - \lambda$ and sends each geodesic of λ to a geodesic in M. Pleated surfaces have proved to be a very valuable tool to study the topology and geometry of hyperbolic 3-manifolds (see for instance [Th1], [CEG], [Mi], [Ca]). What prevents a pleated surface f from being totally geodesic is the fact that it may be bent along its pleating locus λ. It f is locally convex, namely if it always bends in the same direction, the amount of bending defines a transverse measure for λ ([Th1], [EpM]). In section 7, we show how to measure this amount of bending in the general case, expressed in terms of an $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle β_f which we call the *bending cocycle* of the pleated surface f.

The local geometry of a pleated surface $f : S \to M$ is not modified if we lift or project it through covering maps. It is therefore convenient to lift

the situation to universal coverings. We can then generalize the notion of pleated surface by defining an *(abstract) pleated surface* with pleating locus the geodesic lamination λ as a pair $f = (\widetilde{f}, \rho)$, where $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a homomorphism from the fundamental group of S into the group of orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$, where $f : \widetilde{S} \to \mathbb{H}^3$ is a pleated surface from the universal covering $\widetilde{S}$ of S into $\mathbb{H}^3$, with pleating locus the preimage $\widetilde{\lambda}$ of λ, and where $\widetilde{f}$ is equivariant with respect to ρ in the sense that $\widetilde{f}(\gamma x) = \rho(\gamma)\widetilde{f}(x)$ for every $x \in \widetilde{S}$, $\gamma \in \pi_1(S)$. When the image of ρ acts freely and properly discontinuously on $\mathbb{H}^3$, this is clearly equivalent to the previous definition. From now on, "pleated surface" will always mean "abstract pleated surface".

In this generalized sense, a pleated surface also has a bending cocycle $\beta_f \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$. A pleated surface also has a pull back hyperbolic metric m_f on S. We prove in sections 8 and 9 the following result.

THEOREM C.— *For every geodesic lamination λ on S, the map $f \mapsto (m_f, \beta_f)$ induces a homeomorphism from the space of all pleated surfaces with pleating locus λ to the space $\mathcal{T}(S) \times \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$. In addition, the space $\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ is homeomorphic to the union of 0 or 1 tori, whose number and dimension can be explicitly computed from λ.*

In the particular case where λ is maximal, a pleated surface $f = (\widetilde{f}, \rho)$ with pleating locus λ is completely determined by the homomorphism ρ. The space of such pleated surfaces is therefore identified to an open subset $\mathcal{R}(\lambda)$ of the complex algebraic set of homomorphisms $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$. By Theorems A and C, an element ρ of $\mathcal{R}(\lambda)$ is characterized by the bending cocycle $\beta_f \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ of the corresponding pleated surface f, and by the shearing cocycle $\sigma_{m_f} \in \mathcal{H}(\lambda; \mathbb{R})$ of the pull back metric m_f of f. We can combine these two cocycles in a complex cocycle $\Gamma_\rho = \sigma_m + i\beta_f \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$, called the *shear-bend cocycle*. Because λ is maximal, the space $\mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ is the disjoint union of two copies of $(\mathbb{C}/2\pi i\mathbb{Z})^{-3\chi(S)}$, where $\chi(S)$ is the Euler characteristic of S. In section 10, we prove:

THEOREM D.— *The map $\rho \mapsto \Gamma_\rho$ induces a biholomorphic homeomorphism from $\mathcal{R}(\lambda)$ to the open subset $\mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ of $\mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$, where $\mathcal{C}(\lambda) \subset \mathcal{H}(\lambda; \mathbb{R})$ is the open cone of Theorem A.*

It is not hard to see that the two components of $\mathcal{R}(\lambda)$ sit in different components of the space of all homomorphisms $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$, because the corresponding principal $\mathrm{Isom}^+(\mathbb{H}^3)$-bundles are non-isomorphic.

We have already encountered the Thurston symplectic form in Theorem B. One reason for its occurrence is that it is strongly related to a certain complex length 1-form on the manifold $\mathcal{R}(\lambda)$. More precisely, given a hyperbolic metric m on S, there is a unique continuous function $\ell_m : \mathcal{ML}(S) \to \mathbb{R}^+$ defined on the space of measured laminations $\mathcal{ML}(S)$ such that, if α consists of a simple closed m-geodesic endowed with the Dirac transverse measure of weight $a > 0$, $\ell_m(\alpha)$ is a times the length of this closed geodesic ([Th1], [Bo1], [Bo2]). This length function has a straightforward extension to geodesic laminations with transverse cocycles [Bo3], where it can be interpreted as the differential of the original function on $\mathcal{ML}(S)$. In section 3, we prove the following theorem.

THEOREM E.— *If α is a transverse cocycle for the maximal geodesic lamination λ, and if $\sigma_m \in \mathcal{H}(\lambda;\mathbb{R})$ is the shearing cocycle of the hyperbolic metric m,*

$$\ell_m(\alpha) = \tau(\alpha, \sigma_m)\,.$$

Similarly, if γ is a closed curve on S and if $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a homomorphism such that $\rho(\gamma)$ is a hyperbolic glide rotation, we can consider the translation length $\ell_\rho(\gamma)$ and the rotation angle $\mathrm{rot}_\rho(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$ of this glide rotation. The fact that the rotation angle is defined only modulo 2π turns out to be a problem, so it is better to consider a tangent vector $\dot\rho$ based at ρ for the space of all homomorphisms $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$. Then, we have a well defined variation $\mathrm{rot}_{\dot\rho}(\gamma) \in \mathbb{R}$ of $\mathrm{rot}_\rho(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$, as well as a variation $\ell_{\dot\rho}(\gamma) \in \mathbb{R}$ of $\ell_\rho(\gamma)$. If α is the measured lamination on S consisting of a closed geodesic λ_α with transverse Dirac measure of weight $a > 0$, we can then define $\ell_{\dot\rho}(\alpha) = a\ell_{\dot\rho}(\lambda_\alpha)$ and $\mathrm{rot}_{\dot\rho}(\alpha) = a\,\mathrm{rot}_{\dot\rho}(\lambda_\alpha)$. Note that the use of $\dot\rho$ is necessary for the rotation number to be defined since it is not possible to multiply an element of $\mathbb{R}/2\pi\mathbb{Z}$ by a real number.

In section 11, we extend this to the case where α is a transverse $\mathbb{R}$-valued cocycle for a geodesic lamination λ which can be realized by ρ, namely which is in the pleating locus of some pleated surface $(\widetilde{f}, \rho)$. We associate to α and to a tangent vector $\dot\rho$ at ρ a rotation number $\mathrm{rot}_{\dot\rho}(\alpha) \in \mathbb{R}$. We show that, for transverse measures, this extension is quite natural because it is continuous on the open subset of $\mathcal{ML}(S)$ consisting of those measured laminations which are realized by ρ.

THEOREM F.— *Given a homomorphism $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ and a tangent vector $\dot\rho$, the map $\alpha \mapsto \mathrm{rot}_{\dot\rho}(\alpha)$ is continuous and differentiable*

on the open subset $\mathcal{U}$ of $\mathcal{ML}(S)$ consisting of those measured laminations which transversely cross every geodesic lamination that is not realized by ρ. In addition, if we interpret a tangent vector $\dot\alpha$ of $\mathcal{U}$ as a geodesic lamination with a transverse $\mathbb{R}$-valued cocycle, the image of $\dot\alpha$ under the differential of this map is exactly $\mathrm{rot}_{\dot\rho}(\dot\alpha)$.

Theorem F is particular relevant when ρ is injective and has discrete image. In this case, there are only finitely many geodesic laminations which are not realized by ρ, and $\mathcal{U}$ is dense in $\mathcal{ML}(S)$ [Bo1].

Theorem F is the analog for rotation numbers of a similar result which we proved in [Bo3] for the length function ℓ_ρ (compare [Th1], [Bo1]).

We can also combine $\ell_{\dot\rho}(\alpha)$ and $\mathrm{rot}_{\dot\rho}(\alpha)$ into a *complex length*

$$L_{\dot\rho}(\alpha) = \ell_{\dot\rho}(\alpha) + i\,\mathrm{rot}_{\dot\rho}(\alpha)\,.$$

If we fix λ and α, this complex length can be interpreted as a closed holomorphic 1-form on the space of those ρ that realize λ.

As in the real case of Theorem E, this complex length is strongly related to the Thurston symplectic form and to the complex shear-bend cocycle $\Gamma_\rho = \sigma_{m_f} + i\beta_f$. Consider a transverse $\mathbb{R}$-valued cocycle α for the maximal geodesic lamination λ, and let $\rho \in \mathcal{R}(\lambda)$. If we differentiate the shear-bend cocycle $\Gamma_\rho \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ in the direction of the tangent vector $\dot\rho$, we get a $\mathbb{C}$-valued cocycle $\Gamma_{\dot\rho} \in \mathcal{H}(\lambda; \mathbb{C})$. Then we have the following result.

THEOREM G.— *For every $\alpha \in \mathcal{H}(\lambda; \mathbb{R})$, $L_{\dot\rho}(\alpha) = \tau(\alpha, \Gamma_{\dot\rho})$. In particular,*

$$\mathrm{rot}_{\dot\rho}(\alpha) = \tau(\alpha, \beta_f)\,.$$

All of these results have been stated for a compact connected orientable surface S without boundary, and most of the paper is written in this context. However, we can relax these hypotheses by allowing S to be non-orientable and to have non-empty boundary. For hyperbolic metrics, we have the option to require that each boundary component of S either is totally geodesic or corresponds to a cusp. Also, we can allow pleated surfaces to arrive in non-orientable hyperbolic 3-manifolds, or more generally to correspond to homomorphisms ρ from $\pi_1(S)$ to the group of all isometries of $\mathbb{H}^3$. In section 12, we briefly indicate how to extend our results to these various contexts. These extensions are fairly straightforward. They involve transverse cocycles valued in various coefficient bundles twisted by the appropriate local orientations, and satisfying certain boundary conditions.

1. Transverse cocycles for geodesic laminations

Consider a closed connected orientable surface S of negative Euler characteristic.

To define measured geodesic laminations on the surface S, one starts by endowing S with an auxiliary Riemannian metric m of negative curvature; such a metric exists because of our assumption that the Euler characteristic of S is negative. Then, an *m-geodesic lamination* of S is a partial foliation of S by m-geodesics, namely a closed subset $\lambda \subset S$ decomposed as a union of disjoint geodesics which are simple and do not transversely hit the boundary. Recall that a geodesic is simple if it does not cross itself; it may be closed or infinite. A geodesic lamination $\lambda \subset S$ covers only a small part of S, in the sense that it has Lebesgue measure 0, and even Hausdorff dimension 1 ([Th1, sect. 8], [BiS], [Th3, sect. 10]). In particular, the decomposition of the subset λ as a union of disjoint simple geodesics is unique; these geodesics are the *leaves* of λ.

It turns out that this definition can be made independent of the choice of the metric m. Indeed, consider another negatively curved metric m'. Every leaf g of λ is quasi-geodesic for the metric m', and consequently there is a unique m'-geodesic g' which can be homotoped to g by a homotopy moving points by a bounded amount. These m'-geodesics form an m'-geodesic lamination λ', and this establishes a natural correspondence between m-geodesic laminations and m'-geodesic laminations.

So, formally, we will define a *geodesic lamination* as an equivalence class of pairs (λ, m) where λ is an m-geodesic lamination for the negatively curved metric m on S, and where we identify two such (λ, m) and (λ', m') when λ' is the m'-geodesic lamination corresponding to λ. In practice, if there is a clear metric m under consideration, we will identify a geodesic lamination to its m-geodesic representative.

A geodesic lamination λ is *maximal* if it is contained in no larger geodesic lamination. This is easily seen to be equivalent to the property that the complement $S' - \lambda$ consists of finitely many triangles with vertices at infinity.

On the surface S, consider a geodesic lamination λ and let G be an abelian group. A *G-valued transverse cocycle* for λ is a map associating an element $\alpha(k) \in G$ to each unoriented arc k transverse to λ, which satisfies the following properties: α is *additive* in the sense that $\alpha(k) = \alpha(k_1) + \alpha(k_2)$

if we split k into two subarcs k_1 and k_2 with disjoint interiors; and α is λ-*invariant* in the sense that $\alpha(k) = \alpha(k')$ whenever the arc k can be deformed to the arc k' by a homotopy respecting λ. The reason for the use of the word "cocycle" is that α defines a 1-cocycle twisted by the local orientation of λ on a neighborhood of λ (see [Bo4] and compare sect. 3).

We will mostly be concerned with the case where the group G is either the real line $\mathbb{R}$ or the circle $\mathbb{R}/2\pi\mathbb{Z}$. When $G = \mathbb{R}$, a transverse cocycle for λ is just a finitely additive transverse signed measure for λ. If, in addition, the transverse cocycle is non-negative, then it is countably additive (see for instance [Bo4, Proposition 18]) and it defines a *transverse measure* for λ. We refer to [Th1], [CaB] and [PeH] for the theory of geodesic laminations with transverse measures.

A geodesic lamination has relatively few transverse measures, but many more transverse cocycles. More precisely, let $\mathcal{H}(\lambda; \mathbf{G})$ be the group of G-valued transverse cocycles for λ, and let $\chi(\lambda)$ be the Euler characteristic of λ, defined as the alternating sum of the ranks of its Čech cohomology groups (see [Bo4, sect. 4] for a more practical definition of $\chi(\lambda)$). A relatively elementary combinatorial argument shows:

PROPOSITION 1.— *If the geodesic lamination λ is connected, the group $\mathcal{H}(\lambda; G)$ is isomorphic to $G^{-\chi(\lambda)+1}$ if λ is orientable, and to $G^{-\chi(\lambda)} \oplus \{g \in G \mid 2g = 0\}$ if λ is non-orientable. In particular, if λ is maximal, then $\mathcal{H}(\lambda; G)$ is isomorphic to $G^{-3\chi(S)} \oplus \{g \in G \mid 2g = 0\}$.*

Proposition 1 is proved in detail in [Bo4, Theorem 15] and (essentially) in [PeH, § 2.1] when $G = \mathbb{R}$, and these proofs straightforwardly extend to the general case. By comparison, the dimension of the space of transverse measures for λ is at most $(3/2)\left|\chi(S)\right|$ (see [Ka], [Pa1]), and is equal to 1 for most geodesic laminations ([Ma], [Ve], [Re], [Ke2]).

We will frequently use another description of transverse cocycles by lifting the situation to the universal covering $\widetilde{S}$ of S, where λ has preimage $\widetilde{\lambda}$. Let a *plaque* of $\widetilde{S} - \widetilde{\lambda}$ be the closure in $\widetilde{S}$ of a component of $\widetilde{S} - \widetilde{\lambda}$.

Then, a G-valued transverse cocycle corresponds to a map associating an element $\alpha(P, Q) \in G$ to each pair of plaques P, Q of $\widetilde{S} - \widetilde{\lambda}$, and which satisfies the following three properties: α is symmetric, namely $\alpha(Q, P) = \alpha(P, Q)$; α is invariant under the action of $\pi_1(S)$ on $\widetilde{S}$; and α is additive, namely $\alpha(P, Q) = \alpha(P, R) + \alpha(R, Q)$ whenever the plaque R separates P from Q.

The correspondence is obtained by setting $\alpha(P,Q) = \alpha(k)$, where k is the projection to S of any arc $\widetilde{k}$ in $\widetilde{S}$ which joins P to Q, is transverse to $\widetilde{\lambda}$, and does not backtrack in the sense that it meets each leaf of $\widetilde{\lambda}$ at most once.

In [Bo4], it is shown that an $\mathbb{R}$-valued transverse cocycle for λ is also equivalent to the analytic notion of transverse Hölder distribution for λ. Incidentally, this explains our notation for $\mathcal{H}(\lambda; G)$. A *Hölder distribution* on a metric space is a (continuous) linear form on the space of Hölder continuous functions on this space. A *transverse Hölder distribution* for λ is the data of a Hölder distribution on each arc k transverse to λ, which is invariant under homotopy respecting λ in the sense that, if the arc k is sent ot the arc k' by a Hölder bicontinuous homotopy respecting λ, this homotopy sends the Hölder distribution defined on k to the Hölder distribution defined on k'.

THEOREM 2 [Bo4].— *There is a natural correspondence between $\mathbb{R}$-valued transverse cocycles and transverse Hölder distributions for a geodesic lamination λ, defined as follows. Given a transverse Hölder distribution α, the corresponding transverse cocycle associates to each transverse arc k the α-integral of the constant function 1 on k. Conversely, given an $\mathbb{R}$-valued transverse cocycle α, the corresponding transverse Hölder distribution is defined by the formula that, for every Hölder continuous function $\varphi : k \to \mathbb{R}$ defined on a transverse arc k,*

$$\alpha(\varphi) = \int \varphi \, \mathrm{d}\alpha = \alpha(k)\varphi(x_k^+) + \sum_d \alpha(k_d)\big(\varphi(x_d^-) - \varphi(x_d^+)\big)$$

where, having chosen an arbitrary orientation for k, x_k^+ is the positive end point of k, the sum is over all components d of $k - \lambda$, k_d is any subarc of k joining its negative end point x_k^- to any point in d, and x_d^+ and x_d^- are the positive and negative end points of d.

In this paper, the correspondence between $\mathbb{R}$-valued transverse cocycles and transverse Hölder distributions will not be used very much, except in sections 3 and 11. However, we will definitely use the spirit of this correspondence. In particular, sections 5 and 8 are based on non-commutative analogs of the formula of Theorem 2. In addition, what makes everything converge in this paper are the following relatively simple estimates, which were also among the key ingredients of [Bo3] and [Bo4].

LEMMA 3.— *In S endowed with a hyperbolic metric m, let k be a simple geodesic arc transverse to the geodesic lamination λ. Then there is a constant $A > 0$ and a number N such that every geodesic arc in λ which cuts k at least $n \geq N$ times has length at least $(n-1)A$.*

Proof.— This immediately follows from the fact that there is a positive lower bound to the length of any arc in λ going from k to itself. □

Note that, by adjusting the value of A, it is always possible to take $N = 2$ in the conclusion of Lemma 3. This also immediately follows from the proof of this lemma. However it is more convenient to state the lemma in this way, since we will later be interested in optimum values for A satisfying this precise statement.

Consider a geodesic arc k transverse to λ. Two arcs of $\lambda - k$ which are close enough are parallel with respect to k, namely the union of these two arcs and of two suitable arcs in k bounds a rectangle in S.

Now, let d be a component of $k - \lambda$ which does not contain an end point of k, and consider the two leaves g_d^+ and g_d^- of λ that pass through the end points of d. Orient these two leaves so that they determine the same transverse orientation for k, and identify the correspondingly oriented discrete sets $k \cap g_d^\pm$ to $\mathbb{Z}$ so that the end points of d correspond to 0. The *divergence radius* $r(d)$ of d with respect to k is the largest r such that, for every $n \in \mathbb{Z}$ with $-r < n \leq r$, the arc in $g_d^+ - k$ separating the $(n-1)$-point from the n-point is parallel with respect to k to the corresponding arc of $g_d^- - k$. By convention, $r(d) = 0$ for the two components of $k - \lambda$ containing the end points of k.

LEMMA 4.— *There is a uniform upper bound, independent of r, for the number of components d of $k - \lambda$ of such that $r(d) = r$.*

Proof.— Consider the hyperbolic surface with boundary $\widehat{S - \lambda}$ obtained from $S - \lambda$ by adding the (finitely many) leaves of λ which are adjacent to it. This is a surface of finite type, with finitely many spikes on its boundary (see for instance [CaB, sect. 4] or [CEG, sect. 4]). The components of $k - \lambda$ give arcs in $\widehat{S - \lambda}$ going from the boundary to the boundary (with the exception of the two components containing the end points of k). Since $\widehat{S - \lambda}$ has finite topological type and finitely many spikes, these arcs break into finitely many parallelism classes. By definition of $r(d)$, each of these parallelism classes contains at most one d with $r(d) = r$, for every $r \geq 0$. This proves the lemma. □

LEMMA 5.— *If A and N are constants satisfying the conclusions of Lemma 3, there exists a constant B such that the length of each component d of $k - \lambda$ is bounded by $B\, e^{-Ar(d)}$.*

Proof.— This immediately follows from a hyperbolic geometry estimate. The constant B depends on a positive lower bound for the angles made by k and λ at their intersection points, and on the lengths of the finitely many components d of $k - \lambda$ with $r(d) \leq N$. □

Fix a norm $\|\cdot\|$ on the (finite dimensional) vector space $\mathcal{H}(\lambda;\mathbb{R})$. Also, fix an arbitrary orientation for k. As in the statement of Theorem 2, for each component d of $k - \lambda$, let k_d be a subarc of k joining the negative end point of k to any point in d.

LEMMA 6.— *There is a constant C, depending only on the transverse geodesic arc k and on the norm $\|\cdot\|$, such that, for every transverse cocycle $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$ for λ and for every component d of $k - \lambda$,*

$$\alpha(k_d) \leq C\|\alpha\|\big(r(d) + 1\big)\,.$$

Proof.— We will have to refer to combinatorial arguments in [Bo4].

The components of $\lambda - k$ can be broken into finitely many parallelism classes with respect to k. This provides a train track T carrying λ, consisting of one switch located at k, and of one edge for each parallelism class. A transverse cocycle α associates a number $\alpha(e)$ to each edge e of T. Namely $\alpha(e) = \alpha(k_e)$ where k_e is an arc transverse to λ that cuts each arc of the corresponding parallelism class in one point and avoids λ everywhere else.

In [Bo4, Lemma 6], it is shown that, for any component d of $k - \lambda$, the number $\alpha(k_d)$ is a certain linear function of the edge weigths $\alpha(e)$. This linear function is determined by the pattern of intersection with k of the leaves g_d^+ and g_d^- passing through the end points of d, and its norm is bounded by a constant times $r(d) + 1$. The lemma immediately follows. □

2. The shearing cocycle of a hyperbolic metric

On the surface S, consider a hyperbolic metric m and a maximal geodesic lamination λ. Lift the situation to the universal covering $\widetilde{S}$, where λ has preimage $\widetilde{\lambda}$. Recall that a *plaque* of $\widetilde{S} - \widetilde{\lambda}$ is the closure in $\widetilde{S}$ of a component

of $\widetilde{S} - \widetilde{\lambda}$. Since λ is maximal, each plaque of $\widetilde{S} - \widetilde{\lambda}$ is an ideal triangle, namely a hyperbolic triangle with its vertices at infinity.

Given two leaves g and h of $\widetilde{\lambda}$, the geodesic lamination $\widetilde{\lambda}$ gives a preferred isometry $\theta^{gh} : g \to h$ defined as follows.

Indeed, consider the closure Σ of the component of $\widetilde{S} - g \cup h$ that is adjacent to both g and h. The leaves of $\widetilde{\lambda}$ that separate g from h provide a partial foliation of the strip Σ, which can be uniquely extended to a global foliation $\mathcal{G}$ of Σ by geodesics as follows: since λ is maximal, every component of the complement of these leaves of $\widetilde{\lambda}$ in Σ is a hyperbolic wedge, bounded by two asymptotic geodesics; and such a wedge admits a unique foliation by geodesics, all asymptotic on one side. An estimate in hyperbolic plane geometry shows that two disjoint geodesics which pass through two nearby points do so with directions differing by at most a constant times the distance between these two points (see for instance [CEG, § 5.2.6]). It follows that the normals to the leaves of $\mathcal{G}$ form a Lipschitz vector field on Σ. We can therefore integrate this vector field, to get a foliation $\mathcal{H}$ of Σ which is everywhere orthogonal to $\mathcal{G}$. Each leaf of $\mathcal{H}$ goes from g to h, and this defines a map $\theta^{gh} : g \to h$. Also, $\mathcal{H}$ respects distances on the leaves of $\mathcal{G}$ by the formula for the first variation, and it follows that θ^{gh} is an isometry. Note that $\theta^{gh} = \left(\theta^{gh}\right)^{-1}$.

Now, consider two plaques P and Q of $\widetilde{S} - \widetilde{\lambda}$. Let g be the leaf of $\widetilde{\lambda}$ in the boundary of P which is closest to Q, and let h be the leaf in the boundary of Q which is closest to P. Orient h as part of the boundary of Q with the orientation induced by the orientation of S. The plaque Q also determines a preferred base point on h, namely the projection to h of the third vertex of the ideal triangle Q. Similarly, the plaque P determines an orientation and a base point on the geodesic g. For the oriented isometric parametrization of h by $\mathbb{R}$ which sends 0 to the base point, let $\sigma(P, Q) \in \mathbb{R}$ be the coordinate of the image of the base point of g under $\theta^{gh} : g \to h$. In other words, for the isometric parametrization of g and h defined by the choices of orientation and base point, the isometry $\theta^{gh} : g \to h$ corresponds to the map $t \mapsto \sigma(P, Q) - t$.

Since $\theta^{hg} = \left(\theta^{gh}\right)^{-1}$, $\sigma(Q, P)$ is equal to $\sigma(P, Q)$.

Also, consider three plaques P, Q and R of $\widetilde{S} - \widetilde{\lambda}$ such that Q separates P from R. Let g be the leaf of $P \cap \widetilde{\lambda}$ that is closest to Q, h the leaf of $Q \cap \widetilde{\lambda}$ closest to P, k the leaf of $Q \cap \widetilde{\lambda}$ closest to R and ℓ the leaf of $R \cap \widetilde{\lambda}$ closest to Q. The map $\theta^{g\ell}$ decomposes as

$$\theta^{g\ell} = \theta^{k\ell} \circ \theta^{hk} \circ \theta^{gh} .$$

Since R admits an isometry exchanging h and k, the orientation-reversing map θ^{hk} sends the base point of h to the base point of k. It immediately follows that

$$\sigma(P,R) = \sigma(P,Q) + \sigma(Q,R)\,.$$

Therefore, the rule $(P,Q) \mapsto \sigma(P,Q)$ defines an $\mathbb{R}$-valued transverse cocycle σ for λ, in the sense of section 1. This transverse cocycle is the *shearing cocycle* of the hyperbolic metric m.

If we change the metric m to a metric m' by an isotopy $\varphi : S \to S$, then φ sends λ to the corresponding m'-geodesic lamination λ'. It immediatly follows that m and m' have the same shearing cocycles. Therefore, the shearing cocycle σ depends only on the class of m in $\mathcal{T}(S)$.

We can give another description of the number $\sigma(P,Q)$, which will be convenient later on.

Let $\widetilde{\lambda}_{PQ}$ be the set of those leaves of $\widetilde{\lambda}$ which separate P from Q, and orient these leaves to the left as seen from P. Let k be an oriented arc transverse to $\widetilde{\lambda}_{PQ}$ joining P to Q.

For each component d of $k - \widetilde{\lambda}$, let x_d^+ and x_d^- be its positive and negative end points, respectively. If d is not one of the components d^+ and d^- containing the positive and negative end points of k, respectively, then $x_d^\pm$ is contained in a leaf $g_d^\pm$ of $\widetilde{\lambda}_{PQ}$ which is adjacent to a component of $\widetilde{S} - \widetilde{\lambda}$. As before, the component of $\widetilde{S} - \widetilde{\lambda}$ containing d determines a base point on $g_d^\pm$, namely the projection of the third vertex. Let $f : g_d^\pm \to \mathbb{R}$ be the unique oriented isometry sending this base point to 0. This associates two numbers $f(x_d^+)$ and $f(x_d^-)$ to each component d of $k - \widetilde{\lambda}_{PQ}$ which is different from the end components d^+ and d^-. When $d = d^+$ or d^-, we can similarly define $f(x_{d^+}^-)$ and $f(x_{d^-}^+)$.

LEMMA 7.— *With the above data,*

$$\sigma(P,Q) = f(x_{d^-}^+) - f(x_{d^+}^-) + \sum_{d \neq d^+,\, d^-} \left(f(x_d^+) - f(x_d^-)\right)$$

where the sum is taken over all components d of $\widetilde{S} - \widetilde{\lambda}_{PQ}$ which are different from the end components d^+, d^-.

Proof.— We can parametrize the component Σ of $\widetilde{S} - P \cup Q$ that separates P from Q by a strip $\mathbb{R} \times [a\,,b]$ so that the leaves of $\mathcal{G}$ correspond to y = constant and the leaves of $\mathcal{H}$ correspond to x = constant. In

addition, since $\mathcal{H}$ respects the length along the leaves of $\mathcal{G}$, we can assume that this length along $\mathcal{G}$ is given by $|\mathrm{d}x|$. Finally, having oriented the leaves of λ_{PQ} from right to left as seen from P, we can require that this orientation corresponds to the orientation by increasing values of x on the lines $y =$ constant.

By definition of $\sigma(P,Q)$, it is immediate that

$$\sigma(P,Q) = \int_{x^+_{d-}}^{x^-_{d+}} \mathrm{d}x + f(x^+_{d-}) - f(x^-_{d+}) \,.$$

The subarc $[\,x^+_{d-}\,,\,x^-_{d+}\,]$ of k is the union of $k \cap \widetilde{\lambda}_{PQ}$ and of the subarcs $[\,x^-_d\,,\,x^+_d\,]$, with d ranging over all components of $k - \widetilde{\lambda}_{QP}$ different from the end components d^+, d^-. Since $k \cap \widetilde{\lambda}$ has measure 0 on k, the integral term can therefore be decomposed as

$$\int_{x^+_{d-}}^{x^-_{d+}} \mathrm{d}x = \sum_{d \neq d^+,\, d^-} \int_{x^-_d}^{x^+_d} \mathrm{d}x \,.$$

Consequently, it suffices to prove that

$$\int_{x^-_d}^{x^+_d} \mathrm{d}x = f(x^+_d) - f(x^-_d)$$

for every d.

Given a component $d \neq d^+, d^-$ of $k - \widetilde{\lambda}_{PQ}$, the component Σ_d of $\widetilde{S} - \widetilde{\lambda}_{PQ}$ that contains it is a wedge separated by the two geodesics g^+_d and g^-_d. This wedge admits an isometry exchanging g^+_d and g^-_d. This isometry respects $\mathcal{G} \cap \Sigma_d$, and therefore respects each leaf of $\mathcal{H} \cap \Sigma_d$. In particular, the base points of g^+_d and g^-_d are located on the same leaf of $\mathcal{H}$. It immediately follows that $\int_{x^-_d}^{x^+_d} \mathrm{d}x = f(x^+_d) - f(x^-_d)$. □

An immediate corollary of Lemma 7 is the following result.

LEMMA 8.— *With the data of Lemma 7,*

$$\left|\sigma(P,Q) - f(x^+_{d-}) + f(x^-_{d+})\right| \leq \ell_m(k - P \cup Q)$$

where ℓ_m denotes the length with respect to the metric m.

Proof. — By Lemma 7, it suffices to show that each term $|f(x_d^+)-f(x_d^-)|$ is bounded by the length of d. But we just saw that $f(x_d^+)-f(x_d^-)$ is equal to $\int_{x_d^-}^{x_d^+} \mathrm{d}x$ which, up to sign, is equal to the length of the projection of d to any leaf of $\mathcal{G}$ parallel to $\mathcal{H}$. Since this projection is distance non-increasing, the result follows. □

3. Lengths of transverse cocycles and the Thurston symplectic form

Given a hyperbolic metric m and a geodesic lamination λ, an $\mathbb{R}$-valued transverse cocycle α for λ has a well-defined *m-length* $\ell_m(\alpha)$. In this section, we show that this m-length can be described in terms of the shearing distribution σ_m of m and of Thurston's symplectic form on the space of transverse cocycles for λ.

The length function ℓ_m on the space $\mathcal{ML}(S)$ of measured laminations was introduced by Thurston in [Th1]. It is the unique continuous function such that, if $\alpha \in \mathcal{ML}(S)$ consists of a simple closed m-geodesic endowed with the transverse Dirac measure of weight $a > 0$, $\ell_m(\alpha)$ is equal to a times the length of this closed geodesic. Thurston's definition straightforwardly extends to geodesic laminations with transverse cocycles, and we showed in [Bo3] that this extension can be interpreted as the differential of Thurston's function $\ell_m : \mathcal{ML}(S) \to \mathbb{R}^+$.

The m-length $\ell_m(\alpha)$ of the transverse cocycle α for λ is defined as

$$\ell_m(\alpha) = \iint_\lambda \mathrm{d}\ell_m \, \mathrm{d}\alpha \,,$$

meaning that, locally, we first integrate the length measure $\mathrm{d}\ell_m$ along the leaves of λ, and then integrate the corresponding local function on the space of leaves of λ with respect to the transverse Hölder distribution associated to α. More precisely, cover λ by the interiors of finitely many flow boxes B_i, $i = 1, \ldots, n$, namely subsets for which there exists for each i a Hölder bicontinuous $\sigma_i : [0\,,1] \times [0\,,1] \to B_i \subset S$ such that $\sigma_i^{-1}(\lambda) = A_i \times [0\,,1]$ for some subset A_i of $[0\,,1]$. Choose a Hölder continuous partition of unity $\xi_i : S \to \mathbb{R}$, $i = 1, \ldots, n$, such that $\sum_{i=1}^n \xi_i = 1$ and such that the support of each ξ_i is contained in the interior of B_i. Identifying $[0\,,1]$ to any of the arcs $\sigma_i([0\,,1] \times t)$ transverse to λ, the transverse cocycle α defines a Hölder

distribution on $[0, 1]$, which is given by the formula of Theorem 2 and is independent of the choice of t. Then,

$$\ell_m(\alpha) = \sum_{i=1}^{m} \int_0^1 \int_0^1 \xi_i(\sigma_i(u,t))\, \mathrm{d}\ell_m(t)\, \mathrm{d}\alpha(u) = \sum_{i=1}^{n} \alpha(\varphi_i)$$

where $\varphi_i : [0, 1] \to \mathbb{R}$ is the Hölder continuous map defined by $\varphi_i(u) = \int_0^1 \xi_i(\sigma_i(u,t))\, \mathrm{d}\ell_m(t)$.

To connect the length $\ell_m(\alpha)$ to the Thurston symplectic form τ on the space $\mathcal{H}(\lambda;\mathbb{R})$ of $\mathbb{R}$-valued transverse cocycles for λ, we first define this form (compare [Pa1] and [PeH]). The general idea is that, given a small C^1 perturbation K' of a 1-dimensional object K on the oriented surface S, the sign of each intersection point of K' with K is independent of the choice of a local orientation for K. From this observation, it is possible to associate to two $\mathbb{R}$-valued transverse cocycles α and β for λ a homological intersection $\tau(\alpha,\beta) \in \mathbb{R}$. We can now be more precise.

An *orientation* for λ is a continuous choice or orientation for its leaves. The lamination λ admits an orientation covering $\widehat{\lambda} \to \lambda$. If U is a small neighborhood of λ, the covering $\widehat{\lambda} \to \lambda$ extends to a 2-fold covering $\widehat{U} \to U$ (the precise necessary condition on U is that it must avoid at least one point of each component of $S - \lambda$). Note that $\widehat{U}$ carries an orientation induced by the orientation of S, and that $\widehat{\lambda}$ is canonically oriented.

If $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$, it lifts to a transverse cocycle $\widehat{\alpha}$ for $\widehat{\lambda}$. The oriented lamination $\widehat{\lambda}$ together with this transverse cocycle $\widehat{\alpha}$ define an element $[\widehat{\alpha}] \in H_1(\widehat{U};\mathbb{R})$. A formal way to see this is to observe that $\widehat{\lambda}$ and the transverse Hölder distribution associated to $\widehat{\alpha}$ form a geometric current in the sense of [RuS], and therefore determine a closed de Rham current on $\widehat{U}$. This de Rham current associates to each differential form $\omega \in \Omega^1(\widehat{U})$ the number $\int \int_{\widehat{\lambda}} \omega\, \mathrm{d}\widehat{\alpha}$ obtained by locally integrating ω along the leaves of $\widehat{\lambda}$ and then integrating with respect to the distribution $\widehat{\alpha}$. Then, $[\widehat{\alpha}] \in H_1(\widehat{H};\mathbb{R})$ is the homology class defined by this de Rham current.

The homology class $[\widehat{\alpha}] \in H_1(\widehat{U};\mathbb{R})$ can be computed in a more practical way as follows. Select a family of disjoint transverse arcs $k_1, \ldots, k_n$ for $\widehat{\lambda}$ such that every leaf of $\widehat{\lambda}$ cuts at least one of the k_i. Then, the leaves of $\widehat{\lambda} - \bigcup_i k_i$ can be grouped into finitely many bunches of parallel arcs. Form an oriented graph Γ in $\widehat{U}$ by collapsing each k_i, to a point, and by collapsing each bunch of (oriented) parallel arcs of $\widehat{\lambda} - \bigcup_i k_i$ to an oriented edge joining the corresponding points. For each edge of Γ, the transverse cocycle $\widehat{\alpha}$

associates a number to the corresponding bunch of parallel arcs, namely the number associated to a transverse arc to $\widehat{\lambda}$ which crosses each of these arcs exactly once and does not meet $\widehat{\lambda}$ elsewhere. These weighted oriented edges define a real 1-chain in $\widehat{U}$, which is actually a cycle by additivity of $\widehat{\alpha}$. It immediately follows from definitions that the class of this chain in $H_1(\widehat{U};\mathbb{R})$ is equal to the class $[\widehat{\alpha}]$ defined by the Rham current defined above.

Given two transverse cocycles α and β for λ, we define $\tau(\alpha,\beta)$ to be $(1/2)[\widehat{\alpha}]\cdot[\widehat{\beta}]$, namely one half of the intersection number of the two classes $[\widehat{\alpha}], [\widehat{\beta}] \in H_1(\widehat{U};\mathbb{R})$. Clearly, τ defines an antisymmetric bilinear form on the vector space $\mathcal{H}(\lambda;\mathbb{R})$ of transverse Hölder distributions for λ. The bilinear form τ is the *Thurston symplectic form* on $\mathcal{H}(\lambda;\mathbb{R})$. The terminology is a little abusive since τ may be degenerate, which happens exactly when some end of $S-\lambda$ is adjacent to an even number of leaves of λ, as can be seen by adapting the arguments of [PeH, § 3.2]. But τ is non-degenerate in the generic case where λ is maximal, which is really the case of interest here.

This symplectic form has a nice expression when λ is carried by a train track T which is generic, in the sense that each switch is adjacent to exactly 3 edges. At each switch s of such a train track T, there is an incoming edge and two outgoing edges; let e_s^ℓ be the outgoing edge going to the left, and let e_s^r be the outgoing edge going to the right, as seen from the incoming edge and for the orientation of S. Then, if α, $\beta \in \mathcal{H}(\lambda;\mathbb{R})$, it easily follows from the above weighted graph description of the homology classes $[\widehat{\alpha}], [\widehat{\beta}] \in H_1(\widehat{U};\mathbb{R})$ that

$$\tau(\alpha,\beta) = \sum_s \big(\alpha(e_s^r)\beta(e_s^\ell) - \alpha(e_s^\ell)\beta(e_s^r)\big)$$

where the sum is taken over all switches of T, and where $\alpha(e)$, $\beta(e)$ are the weights associated by α and β to the edge e (compare [PeH, § 3.2]).

We can now state the main result of this section.

THEOREM 9.— *Given a maximal geodesic lamination λ, let σ_m be the shearing cocycle of the hyperbolic metric m. Then, for every transverse cocycle $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$ for the geodesic lamination λ, its length $\ell_m(\alpha)$ is equal to $\tau(\alpha,\sigma_m)$.*

Proof.— As before, let $\widehat{\lambda} \to \lambda$ be the orientation covering of λ. Choose a neighborhood U of λ such that each component of $U-\lambda$ is an open annulus; for instance, we could take U to consist of those points which are at distance less than ε from λ, for ε small enough. Extend $\widehat{\lambda} \to \lambda$ to a covering $\widehat{U} \to U$.

Along the leaves of the oriented lamination $\widehat{\lambda}$, the length measure induced by m defines a differential 1-form w_m. The direction of the leaf of $\widehat{\lambda}$ at a point $x \in \widehat{\lambda}$ is a Lipschitz function of x. Therefore, w_m can be extended to a closed Lipschitz differential 1-form $w_m \in \Omega^1_{\mathrm{Lip}}(\widehat{U})$. Since w_m is closed, it defines a cohomology class $[w_m] \in H^1(\widehat{U};\mathbb{R})$.

By definition of the length function,

$$\ell_m(\alpha) = \frac{1}{2}\,\ell_m(\widehat{\alpha}) = \frac{1}{2}\iint_{\widehat{\lambda}} w_m \,\mathrm{d}\widehat{\alpha} = \frac{1}{2}\,\langle [w_m]\,,\,[\widehat{\alpha}]\rangle$$

where the last term denotes the evaluation of the cohomology class $[\omega_m] \in H^1(\widehat{U};\mathbb{R})$ on the homology class $[\widehat{\alpha}] \in H_1(\widehat{U};\mathbb{R})$, and where the last equality comes from the realization of $[\widehat{\alpha}]$ by a geometric current supported by $\widehat{\lambda}$.

By definition of τ, the proof of Theorem 9 will therefore be completed by the following lemma.

LEMMA 10.— *For every homology class $c \in H_1(\widehat{U};\mathbb{R})$, the evaluation $\langle [w_m]\,,\,c\rangle$ is equal to the intersection number $c \cdot [\widehat{\sigma}_m]$.*

Proof.— Let W be a component of $U - \lambda$, By hypothesis on U, W is an open annulus bounded on one side by 3 leaves of λ. Consequently, its preimage $\widehat{W}$ in $\widehat{U}$ is an open annulus bounded on one side by 6 leaves of $\widehat{\lambda}$, with alternating orientations. Recall that each leaf of λ in the boundary of W has a preferred base point, coming from the projection of the third cusp of the component of $S - \lambda$ adjacent to that leaf. This determines a preferred base point O_g on each leaf g of $\widehat{\lambda}$ in the boundary of $\widehat{W}$.

Consider two consecutive leaves g, h of $\widehat{\lambda}$ in the boundary of $\widehat{W}$, and integrate w_m along an arc k joining O_g to O_h in $\widehat{W} \cup g \cup h$ which is made up of three pieces: first an arc in g joining O_g to a point x_g very close to the spike of $\widehat{W}$ separating g from h; then a small jump from x_g to its projection point x_h on h; and finally an arc in h joining x_h to O_h. By definition of w_m, the contribution of the first and last arcs to this integral are, in absolute value, respectively equal to the distances between O_g and x_g and between O_h and x_h. If x_g is far enough near the cusp, these two distances are approximately the same because O_g and O_h are at the same horocyclic distance from the spike (which comes from the fact that, as seen in section 1, the same property holds for their projections in S). Also, because of alternating orientations, the integral of w_m along the first and last arc have opposite signs; their sum is therefore very small. It follows that

the integral of w_m along k is arbitrarily small if we choose x_g close enough to the cusp. On the other hand, different choices for x_g give homotopic arcs, along which the integral of w_m is unchanged since w_m is closed. Therefore, the integral of w_m along k is actually 0.

Since $\widehat{W}$ is an annulus, it follows that the integral of w_m along every closed curve in $\widehat{W}$ is equal to 0. We can therefore define a function f_m on $\widehat{W}$ by the property that $f_m(x)$ is the integral of w_m along any arc in $\widehat{W}$ joining some base point O_g to x. This defines a function f_m on $\widehat{U} - \widehat{\lambda}$ such that $\mathrm{d}f_m = w_m$. There is of course no way to exend f_m to a global antiderivative of w_m over $\widehat{U}$. (The geodesic lamination λ carries at least one transverse measure, whose length has to be positive.)

Let k be an oriented arc in $\widehat{U}$ that is transverse to $\widehat{\lambda}$. For each component d of $k - \widehat{\lambda}$, let x_d^+ and x_d^- be the positive and negative end points of d, respectively. From what precedes, and because $k \cap \lambda$ has Lebesgue measure 0,

$$\int_k w_m = \sum_d \int_d w_m = \sum_d (f_m(x_d^+) - f_m(x_d^-))$$

where the sum is over all components d of $k - \widehat{\lambda}$, and where $f_m(x_d^\pm)$ is defined by continuous extension of the restriction of f_m to d. (In particular, if d is adjacent to d' so that $x_d^+ = x_{d'}^-$ it may very well happen that $f_m(x_d^+) \neq f_m(x_{d'}^-)$.)

We can compare this formula to that of Lemma 7. Note that, if g is a leaf of $\widehat{\lambda}$ in the boundary of a component $\widehat{W}$ of $\widehat{U} - \widehat{\lambda}$, and if we continuously extend f_m to $\widehat{W} \cup g$, the extension of f_m to g is just the oriented isometry from g to $\mathbb{R}$ which sends the base point at 0. Consequently, if k is an arc transverse to $\widehat{\lambda}$ which is small enough so that the leaves of $\widehat{\lambda}$ cross k in the same direction,

$$\int_k w_m - f_m(x_k^+) + f_m(x_k^-) = \varepsilon \sigma_m(k') = \varepsilon \int_{k'} \sigma_m = \varepsilon \int_k \widehat{\sigma}_m$$

where k' is the projection of k to U, where $\varepsilon = +1$ if the leaves of $\widehat{\lambda}$ cross k from right to left, where $\varepsilon = -1$ if they cross from left to right, and where x_k^+ and x_k^- are the positive and negative end points of k, respectively. By interpreting $\widehat{\sigma}_m$ as a geometric current, we can incorporate the ε in an intersection number, and the above equality becomes

$$k \cdot \widehat{\sigma}_m = \int_k w_m - f_m(x_k^+) + f_m(x_k^-) \,.$$

By additivity, this equation actually holds for every arc k transverse to $\widehat{\lambda}$, without any restriction on the direction in which the leaves of $\widehat{\lambda}$ cross k.

Now, consider a class $c \in H_1(\widehat{U};\mathbb{R})$. This class can be represented by a cycle $\sum_{i=1}^n a_i k_i$, with $a_i \in \mathbb{R}$ and with the arcs k_i transverse to $\widehat{\lambda}$. Then,

$$\begin{aligned}\langle [w_m], c\rangle &= \sum_{i=1}^n a_i \int_{k_i} w_m = \sum_{i=1}^n a_i\big(k_i \cdot \widehat{\sigma}_m + f_m(x_{k_i}^+) - f_m(x_{k_i}^-)\big)\\ &= \sum_{i=1}^n a_i k_i \cdot \widehat{\sigma}_m = c \cdot [\widehat{\sigma}_m]\end{aligned}$$

where the f_m terms cancel out because $\sum_{i=1}^n a_i k_i$ has boundary 0. This concludes the proof of Lemma 10, and therefore of Theorem 9.

4. The shearing cocycle determines the metric

We want to show that, if two hyperbolic metrics have the same shearing transverse cocycle then they represent the same class in $\mathcal{T}(S)$.

Consider two hyperbolic metrics m_1 and m_2 and a maximal geodesic lamination λ. As indicated in section 1, λ can be represented by an m_1-geodesic lamination λ_1 and an m_2-geodesic lamination λ_2. Lift the situation to the universal covering $\widetilde{S}$, where λ, λ_1 and λ_2 have respective preimages $\widetilde{\lambda}$, $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$.

Since λ_i represents λ, there is a leaf of $\widetilde{\lambda}_i$, which is naturally associated to each leaf of $\widetilde{\lambda}$. Therefore, for each plaque P of $\widetilde{S} - \widetilde{\lambda}$, there is a plaque P_i of $\widetilde{S} - \widetilde{\lambda}_i$ which is naturally associated to P, as well as a homeomorphism $P \to P_i$ well defined up to isotopy. By composition, we get a preferred isotopy class of homeomorphisms $P_1 \to P_2$. Since any two ideal triangles are isometric, this isotopy class is represented by a unique isometry $\varphi_P : P_1 \to P_2$, called the *plaque map*.

Define the *shear map* $\varphi : \widetilde{S} - \widetilde{\lambda}_1 \to \widetilde{S} - \widetilde{\lambda}_2$ by the property that, on each component of $\widetilde{S} - \widetilde{\lambda}_1$, the map φ coincides with the corresponding plaque map $\varphi_P : P_1 \to P_2$. Note that φ is an isometry from the metric m_1 to the metric m_2.

LEMMA 11.— *If the two metrics m_1 and m_2 have the same shearing shearing cocycle, the shear map φ continuously extends to an isometry $(\widetilde{S}, m_1) \to (\widetilde{S}, m_2)$.*

Proof. — We first show that φ admits a continuous extension which is locally Lipschitz.

Consider two points $x_1, y_1 \in \widetilde{S} - \widetilde{\lambda}_1$, respectively contained in the plaques P_1 and Q_1 of $\widetilde{S} - \widetilde{\lambda}_1$. Let Σ_1 be the component of $\widetilde{S} - \mathrm{Int}(P_1 \cup Q_1)$ which separates P_1 from Q_1. As in section 2, there is a unique foliation $\mathcal{G}_1$ of Σ_1 by m_1-geodesics such that every leaf of $\widetilde{\lambda}_1$ separating x_1 from y_1 is a leaf of $\mathcal{G}_1$. Again as in section 2, let $\mathcal{H}_1$ be the foliation of Σ_1 orthogonal to $\mathcal{G}_1$.

Consider the m_1-geodesic arc α_1 joining x_1 to y_1, and let $u_1 = \alpha_1 \cap \Sigma_1 \cap P_1$ and $v_1 = \alpha_1 \cap \Sigma_1 \cap Q_1$ be the two end points of $\alpha_1 \cap \Sigma_1$. In Σ_1, u_1 and v_1 can be also be connected by the union of a leaf γ_1 of $\mathcal{H}_1$ and of an arc δ_1 contained in the leaf $\Sigma_1 \cap Q_1$ of $\mathcal{G}_1$. Let β_1 be the arc obtained from α_1 by replacing $\alpha_1 \cap \Sigma_1$ by $\gamma_1 \cup \delta_1$.

The projection of Σ_1 onto $\Sigma_1 \cap Q_1$ along the leaves of $\mathcal{H}_1$ is distance non-increasing. It follows that the length of δ_1 is bounded by the length of $\alpha_1 \cap \Sigma_1$, and therefore by the distance $d(x_1, y_1)$. By the Jacobi equation, the projection from Σ_1 to γ_1 along the leaves of $\mathcal{G}_1$ is locally Lipschitz, where the local Lipschitz constant can be taken to be the exponential of the projection distance. As in the case of the length of δ_1, this projection distance is bounded by $d(x_1, y_1)$. It follows that the length of γ_1 is bounded by $e^{d(x_1,y_1)}$ times the m_1-length of $\alpha_1 \cap \Sigma_1$, and therefore by $e^{d(x_1,y_1)} d(x_1, y_1)$. Altogether, we conclude that the length of β_1 is bounded by $(2 + e^{d(x_1,y_1)}) d(x_1, y_1)$.

Now, consider $x_2 = \varphi(x_1)$, $y_2 = \varphi(y_1)$. Let P_2 and Q_2 be the plaques of $\widetilde{S} - \widetilde{\lambda}_2$ respectively containing x_2 and y_2, and let Σ_2 be the closure of the component of $\widetilde{S} - P_2 \cup Q_2$ that separates P_2 from Q_2. As before, let $\mathcal{G}_2$ be the foliation of Σ_2 by m_2-geodesics such that every leaf of $\widetilde{\lambda}_2$ separating x_2 from y_2 is a leaf of $\mathcal{G}_2$, and let $\mathcal{H}_2$ be the orthogonal foliation. The point $u_2 = \varphi_P(u_1) \in P_2 \cap \Sigma_2$ can be joined to the point $v_2 = \varphi_Q(v_1) \in Q_2 \cap \Sigma_2$ by the union of a leaf γ_2 of $\mathcal{H}_2$ and of an arc δ_2 contained in the leaf $\Sigma_2 \cap Q_2$ of $\mathcal{G}_2$. Let β_2 be the arc connecting x_2 to y_2 which is the union of $\varphi(\alpha_1 \cap P_1)$, γ_2, δ_2 and $\varphi(\alpha_1 \cap Q_1)$.

Because m_1 and m_2 have the same shearing cocycle, the end point $\gamma_2 \cap Q_2$ of γ_2 is the image of $\gamma_1 \cap Q_1$ under the plaque map $Q_1 \to Q_2$. It follows that δ_2 is the image of δ_1 under the same plaque map; in particular, the m_1-length of δ_1 is equal to the m_2-length of δ_2.

In Σ_1, consider a wedge W_1 delimited by two asymptotic leaves of $\widetilde{\lambda}_1$ separating x_1 from y_1, such that the interior of W_1 does not meet $\widetilde{\lambda}_1$. Let $R_1 \subset W_1$ be the plaque of $\widetilde{S} - \widetilde{\lambda}_1$ that is adjacent to the same two leaves

of $\widetilde{\lambda}_1$, and let W_2 and R_2 be the wedge and plaque in Σ_2 respectively corresponding to W_1 and R_1. The fact that m_1 and m_2 have the same shearing cocycle implies that the plaque map $R_1 \to R_2$ sends the end point of $\gamma_1 \cap R_1$ that is closest to u_1 to the end point of $\gamma_2 \cap R_2$ that is closest to u_2. As a consequence, the isometric extension of this plaque map to $W_1 \to W_2$ sends $\gamma_1 \cap W_1$ to $\gamma_2 \cap W_2$. In particular, the m_1-length of $\gamma_1 \cap W_1$ is equal to the m_2-length of $\gamma_2 \cap W_2$.

Since $\gamma_1 \cap \widetilde{\lambda}_1$ has 1-dimensional Lebesgue measure 0, the length of γ_1 is equal to the (infinite) sum of the lengths of the $\gamma_1 \cap W_1$, where W_1 ranges over all wedges in Σ_1 as above. Since the same property holds for γ_2, it follows that the m_1-length of γ_1 is equal to the m_2-length of γ_2.

This proves that each of the four pieces forming β_1 has the same length as the corresponding piece of β_2. As a consequence, the m_1-length of β_1 is equal to the m_2-length of β_2.

Therefore,

$$d(x_2, y_2) \leq \ell_{m_2}(\beta_2) = \ell_{m_1}(\beta_1) \leq (2 + e^{d(x_1,y_1)} d(x_1, y_1)) .$$

Since this holds for every x_1, $y_1 \in \widetilde{S} - \widetilde{\lambda}_1$, it follows that φ admits a continuous extension $\varphi : (\widetilde{S}, m_1) \to (\widetilde{S}, m_2)$ which is locally Lipschitz.

We now prove that φ is distance non-increasing. For this, consider two points x_1 and $y_1 \in \widetilde{S}$ which are not on the same leaf of $\widetilde{\lambda}_1$, and let α_1 be the m_1-geodesic arc joining x_1 to y_1. Since $\alpha_1 \cap \widetilde{\lambda}_1$ has 1-dimensional Lebesgue measure 0 and since φ is locally Lipschitz, the image $\varphi(\alpha_1 \cap \widetilde{\lambda}_1)$ also has 1-dimensional Lebesgue measure 0 (for the metric m_2). Also, because φ is isometric on $\widetilde{S} - \widetilde{\lambda}_1$, the m_2-length of $\varphi(\alpha_1 - \widetilde{\lambda}_1)$ is equal to the m_1-length of $\alpha_1 - \widetilde{\lambda}_1$. Therefore, the m_2-length of $\varphi(\alpha_1)$ is equal to the m_1-length of α_1 and

$$d_{m_2}\big(\varphi(x_1)\,,\, \varphi(y_1)\big) \leq \ell_{m_2}\big(\varphi(\alpha_1)\big) = \ell_{m_1}(\alpha_1) = d_{m_1}(x_1, y_1) .$$

By density, this inequality $d_{m_2}\big(\varphi(x_1)\,,\, \varphi(y_1)\big) \leq d_{m_1}(x_1, y_1)$ holds for every x_1, $y_1 \in \widetilde{S}$, namely even if the two points are on the same leaf of $\widetilde{\lambda}_1$. In other words, $\varphi : (\widetilde{S}, m_1) \to (\widetilde{S}, m_2)$ is distance non-increasing.

By symmetry, the shear map $\varphi^{-1} : \widetilde{S} - \widetilde{\lambda}_2 \to \widetilde{S} - \widetilde{\lambda}_1$ extends to a distance non-increasing map $(\widetilde{S}, m_2) \to (\widetilde{S}, m_1)$. It follows that $\varphi : (\widetilde{S}, m_1) \to (\widetilde{S}, m_2)$ is an isometry. □

THEOREM 12.— *Two hyperbolic metrics m_1 and m_2 have the same shearing transverse distribution if and only if $m_1 = m_2$ in $\mathcal{T}(S)$.*

Proof.— If m_1 and m_2 have the same shearing transverse distribution, let $\varphi : (\widetilde{S}, m_1) \to (\widetilde{S}, m_2)$ be the isometry provided by Lemma 11. Since the shear map $\varphi : \widetilde{S} - \widetilde{\lambda}_1 \to \widetilde{S} - \widetilde{\lambda}_2$ commutes with the action of $\pi_1(S)$, φ induces an isometry $\psi : (S, m_1) \to (S, m_2)$ which is homotopic to the identity. In particular, m_1 and m_2 represent the same element of $\mathcal{T}(S)$. □

5. The local realization of shearing cocycles

In this section we show that, given a maximal geodesic lamination λ, the map $\mathcal{T}(S) \to \mathcal{H}(\lambda, \mathbb{R})$ which associates its shearing cocycle to a hyperbolic metric is open. By Theorem 12, this implies that this map is a homeomorphism onto its image. Its precise image will be determined in section 6.

PROPOSITION 13.— *Let m_0 be a hyperbolic metric with associated shearing cocycle σ_0 for the maximal geodesic lamination λ. Then, every $\sigma \in \mathcal{H}(\lambda; \mathbb{R})$ that is sufficiently close to σ_0 is the shearing cocycle of some hyperbolic metric m.*

Proof.— The proof will require several steps. Set

$$\alpha = \sigma - \sigma_0 \in \mathcal{H}(\lambda; \mathbb{R}).$$

Represent λ by an m_0-geodesic lamination which we will also denote by λ, and let $\widetilde{\lambda}$ be the preimage of λ in the universal covering $\widetilde{S}$. Consider two plaques P and Q of $\widetilde{S} - \widetilde{\lambda}$.

For every plaque R separating P from Q, let g_R^P and g_R^Q be the geodesics in the boundary of R which are closest to P and Q, respectively. Orient these geodesics to the left, as seen from P. Also, given an oriented geodesic g of $\widetilde{S}$ and a number $u \in \mathbb{R}$, let $T_g^u : \widetilde{S} \to \widetilde{S}$ denote the m_0-isometry which respects g and acts by translation of oriented amplitude u on g.

Let $\mathcal{P}_{PQ}$ be the set of all plaques of $\widetilde{S} - \widetilde{\lambda}$ that separate P from Q. Given a finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, index its elements as $P_1, P_2, \ldots, P_n$ so that the index i of P_i increases as one goes from P to Q, and consider

$$\varphi_{\mathcal{P}} = T_{g_1^P}^{\alpha(P,P_1)} T_{g_1^Q}^{-\alpha(P,P_1)} T_{g_2^P}^{\alpha(P,P_2)} T_{g_1^Q}^{-\alpha(P,P_2)} \cdots T_{g_n^Q}^{-\alpha(P,P_n)} T_{g_Q^P}^{\alpha(P,Q)}$$

where $g_i^P = g_{P_i}^P$, $g_i^Q = g_{P_i}^Q$ and g_Q^P is the geodesic in the boundary of Q that is closest to P. This formula is perhaps easier to read and understand if we notice that each P_i contributes a term $T_{g_i^P}^{\alpha(P,P_i)} T_{g_i^Q}^{-\alpha(P,P_i)}$.

Now, we let the finite subset $\mathcal{P}$ converge to $\mathcal{P}_{PQ}$ and we consider the limit

$$\varphi_{PQ} = \lim_{\mathcal{P} \to \mathcal{P}_{PQ}} \varphi_{\mathcal{P}} \,.$$

By convention, we decide that φ_{PP} is the identity. Of course, we first have to prove that the above limit exists.

LEMMA 14.— *Let k be the lift to $\widetilde{S}$ of a simple geodesic arc in S transverse to λ. Then, if $\alpha \in \mathcal{H}(\lambda)$ is sufficiently small (depending on k) and if the two plaques P and Q meet k, the map $\varphi_{\mathcal{P}}$ converges to an m_0-isometry φ_{PQ} as $\mathcal{P}$ tends to $\mathcal{P}_{PQ}$.*

Proof.— For notational convenience, set

$$\psi_{\mathcal{P}} = \varphi_{\mathcal{P}} T_{gQ^P}^{-\alpha(P,Q)} = T_{g_1^P}^{\alpha(P,P_1)} T_{g_1^Q}^{-\alpha(P,P_1)} T_{g_2^P}^{\alpha(P,P_2)} T_{g_1^Q}^{-\alpha(P,P_2)} \cdots T_{g_n^Q}^{-\alpha(P,P_n)} .$$

Identify the m_0-isometry group of $\widetilde{S}$ to the matrix group $\mathrm{SO}(2,1) \subset \mathrm{GL}_3(\mathbb{R})$, and endow it with the norm $\|A\| = \max_{x \in \mathbb{R}^3} \|Ax\|/\|x\|$. The main property we want is that this norm satisfies $\|AB\| \leq \|A\| \, \|B\|$.

We first show that the norm $\|\psi_{\mathcal{P}}\|$ is uniformly bounded, if α is small enough.

For every i, the distance between the geodesics g_i^P and g_i^Q is bounded by a constant times the length of $k \cap P_i$. By Lemma 5, this distance is therefore an $\mathrm{O}(e^{-Ar(k \cap P_i)})$ for some constant $A > 0$, where we identify k to its projection to S and $k \cap P_i$ to the corresponding component of $k - \lambda$. By an easy hyperbolic estimate, it follows that

$$T_{g_i^P}^{\alpha(P,P_i)} T_{g_i^Q}^{-\alpha(P,P_i)} = \mathrm{Id} + \mathrm{O}(e^{|\alpha(P,P_i)|} \, e^{-Ar(k \cap P_i)}) \,.$$

As a consequence,

$$\|\psi_{\mathcal{P}}\| \leq \prod_{i=1}^{n} (1 + \mathrm{O}(e^{|\alpha(P,P_i)|} \, e^{-Ar(k \cap P_i)})) \,.$$

By Lemmas 4 and 6, the series $\sum_{R\in\mathcal{P}_{PQ}} e^{|\alpha(P,P)|}\, e^{-Ar(k\cap R)}$ is bounded by the sum of finitely many geometric series $\sum_{r=0}^{\infty} e^{C\|\alpha\|(r+1)}\, e^{-Ar}$. It is therefore convergent if $\|\alpha\| < A/C$.

It follows that, if the transverse distribution α is small enough, the norm $\|\psi_{\mathcal{P}}\|$ is uniformly bounded.

Let $\mathcal{P}_n$, $n \in \mathbb{N}$ be an increasing sequence of finite subsets converging to $\mathcal{P}_{PQ}$, with the cardinal of $\mathcal{P}_n$ equal to n. The map $\psi_{\mathcal{P}_{n+1}}$, is obtained from $\psi_{\mathcal{P}_n}$ by inserting a term $T^{\alpha(P,R)}_{g^P_R} T^{-\alpha(P,R)}_{g^Q_R}$ in its expression. More precisely, there are subsets $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{P}_{PQ}$ such that

$$\psi_{\mathcal{P}_n} = \psi_{\mathcal{P}}\psi_{\mathcal{P}'} \quad \text{and} \quad \psi_{\mathcal{P}_{n+1}} = \psi_{\mathcal{P}} T^{\alpha(P,R)}_{g^P_R} T^{-\alpha(P,R)}_{g^Q_R} \psi_{\mathcal{P}'}\,.$$

Then

$$\begin{aligned}\|\psi_{\mathcal{P}_{n+1}} - \psi_{\mathcal{P}_n}\| &\le \|\psi_{\mathcal{P}}\|\, \|T^{\alpha(P,R)}_{g^P_R} T^{-\alpha(P,R)}_{g^Q_R} - \mathrm{Id}\|\, \|\psi_{\mathcal{P}'}\| \\ &= \mathrm{O}\big(e^{|\alpha(P,R)|}\, e^{-Ar(k\cap R)}\big) \\ &= \mathrm{O}\big(e^{C\|\alpha\|(r(k\cap R)+1)}\, e^{-Ar(k\cap R)}\big)\end{aligned}$$

by Lemmas 5 and 6, and because we just proved that $\|\psi_{\mathcal{P}}\|$ and $\|\psi_{\mathcal{P}'}\|$ are uniformly bounded.

By Lemma 4, it follows that the sequence $\psi_{\mathcal{P}_n}$ is Cauchy, and therefore convergent, if $\|\alpha\| < A/C$.

This proves that $\psi_{\mathcal{P}}$ has a limit ψ_{PQ} as $\mathcal{P}$ tends to $\mathcal{P}_{PQ}$, provided that $\alpha \in \mathcal{H}(\lambda)$ is small enough. The same clearly holds for $\varphi_{\mathcal{P}} = \psi_{\mathcal{P}} T^{\alpha(P,Q)}_{g^P_Q}$. □

For future reference, we note the following estimate.

LEMMA 15.— *Under the hypotheses of Lemma 14, there is a constant $B > 0$, depending on k and α, such that φ_{PQ} can be decomposed as $\varphi_{PQ} = \psi_{PQ} T^{\alpha(P,Q)}_{g^P_Q}$ with*

$$\|\psi_{PQ} - \mathrm{Id}\| = \mathrm{O}\Big(\sum_{R\in\mathcal{P}_{PQ}} e^{-Br(k\cap R)}\Big)\,.$$

Proof

$$\|\psi_{PQ} - \mathrm{Id}\| = \lim_{n\to\infty} \|\psi_{\mathcal{P}_n} - \psi_{\mathcal{P}_0}\| \leq \sum_{n=0}^{\infty} \|\psi_{\mathcal{P}_{n+1}} - \psi_{\mathcal{P}_n}\| \,. \square$$

Having proved the convergence in Lemma 14, we can now give another description of φ_{PQ} which is perhaps more intuitive. We are still assuming that P and Q meet the lift k of a simple geodesic arc transverse to λ in S.

Given an integer $r > 0$, let $\mathcal{P}^r_{PQ}$ consist of the finitely many $R \in \mathcal{P}_{PQ}$ such that $r(k \cap R) \leq r$. Index the elements of $\mathcal{P}^r_{PQ}$ as $P_1, P_2, \ldots, P_n$ so that the index i of P_i increases as one goes from P to Q. For notational convenience, set $P_0 = P$ and $P_{n+1} = Q$. For every i, choose a geodesic h_i, which separates the interior of P_i from the interior of P_{i+1}, and orient h_i to the left as seen from P. Then, set

$$\varphi^r_{PQ} = T_{h_0}^{\alpha(P_0,P_1)} T_{h_1}^{\alpha(P_1,P_2)} \ldots T_{h_n}^{\alpha(P_n,P_{n+1})} \,.$$

Compare [EpM, sect. 3].

LEMMA 16.— *Under the hypotheses of Lemma 14, as r tends to infinity, φ^r_{PQ} tends to φ_{PQ} if $\alpha \in \mathcal{H}(\lambda)$ is small enough.*

Proof.— We will estimate the difference between φ^r_{PQ} and

$$\begin{aligned}\varphi_{\mathcal{P}^r_{PQ}} = {} & T_{g_1^P}^{\alpha(P_0,P_1)} T_{g_1^Q}^{-\alpha(P_0,P_1)} T_{g_2^P}^{\alpha(P_0,P_2)} T_{g_2^Q}^{-\alpha(P_0,P_2)} \ldots \\ & \ldots T_{g_n^Q}^{-\alpha(P_0,P_n)} T_{g_{n+1}^P}^{\alpha(P_0,P_{n+1})} \,.\end{aligned}$$

For this, it will be more convenient to rewrite φ^r_{PQ} as

$$\begin{aligned}\varphi^r_{PQ} = {} & T_{h_0}^{\alpha(P_0,P_1)} T_{h_1}^{-\alpha(P_0,P_1)} T_{h_1}^{\alpha(P_0,P_2)} T_{h_2}^{-\alpha(P_0,P_2)} \ldots \\ & \ldots T_{h_n}^{-\alpha(P_0,P_n)} T_{h_n}^{\alpha(P_0,P_{n+1})} \,,\end{aligned}$$

noting that $\alpha(P_i, P_{i+1}) = \alpha(P_0, P_{i+1}) - \alpha(P_0, P_i)$, and to consider

$$\begin{aligned}\psi^r_{PQ} &= \varphi^r_{PQ} T_{h_n}^{-\alpha(P_0,P_{n+1})} \\ &= T_{h_0}^{\alpha(P_0,P_1)} T_{h_1}^{-\alpha(P_0,P_1)} T_{h_1}^{\alpha(P_0,P_2)} T_{h_2}^{-\alpha(P_0,P_2)} \ldots T_{h_n}^{-\alpha(P_0,P_n)}\end{aligned}$$

and

$$\begin{aligned}\psi_{\mathcal{P}^r_{PQ}} &= \varphi_{\mathcal{P}^r_{PQ}} T^{-\alpha(P_0,P_{n+1})}_{g^P_{n+1}} \\ &= T^{\alpha(P_0,P_1)}_{g^P_1} T^{-\alpha(P_0,P_1)}_{g^Q_1} T^{\alpha(P_0,P_2)}_{g^P_2} T^{-\alpha(P_0,P_2)}_{g^Q_2} \cdots T^{-\alpha(P_0,P_n)}_{g^Q_n}.\end{aligned}$$

The isometry ψ^r_{PQ} is obtained from $\psi_{\mathcal{P}^r_{PQ}}$ by replacing each term

$$T^{\alpha(P_0,P_i)}_{g^P_i} T^{-\alpha(P_0,P_i)}_{g^Q_i} \quad \text{by } T^{\alpha(P_0,P_i)}_{h_{i-1}} T^{-\alpha(P_0,P_i)}_{h_i}.$$

For every i, the two geodesics g^Q_i and g^P_{i+1} follow the same edge path of length $2r$ in the train track associated to the arc k; otherwise, there would be another $R \in \mathcal{R}^r_{PQ}$ between P_i and P_{i+1}. Since h_i is between these two geodesics, it also follows the same edge path. Therefore, the distance between any two of these three geodesics is bounded by a constant times e^{-Ar}, for some constant $A > 0$ of Lemmas 3 and 5.

In particular, the distance from g^P_i to h_{i-1} and the distance from g^Q_i to h_i are both $\mathrm{O}(e^{-Ar})$. Also, the distance between g^P_i and g^Q_i is an $\mathrm{O}(e^{-Ar(k\cap P_i)})$ by Lemma 5. Since $r(k\cap P_i) \leq r$, it follows that the distance from h_{i-1} to h_i is also an $\mathrm{O}(e^{-Ar(k\cap P_i)})$.

From the second statement, it follows that

$$\begin{aligned}T^{\alpha(P_0,P_i)}_{h_{i-1}} T^{-\alpha(P_0,P_i)}_{h_i} &= \mathrm{Id} + \mathrm{O}\big(e^{|\alpha(P_0,P_i)|} e^{-Ar(k\cap P_i)}\big) \\ &= \mathrm{Id} + \mathrm{O}\big(e^{C\|\alpha\|(r(k\cap P_i)+1)} e^{-Ar(k\cap P_i)}\big)\end{aligned}$$

by Lemma 6. If ψ is any isometry obtained from $\psi_{\mathcal{P}^r_{PQ}}$ by replacing some of the n terms $T^{\alpha(P_0,P_i)}_{g^P_i} T^{-\alpha(P_0,P_i)}_{g^Q_i}$ by $T^{\alpha(P_0,P_i)}_{h_{i-1}} T^{-\alpha(P_0,P_i)}_{h_i}$ or by the identity, it follows as in the proof of Lemma 14 that

$$\begin{aligned}\log\|\psi\| &= \mathrm{O}\left(\sum_{i=1}^{n} e^{C\|\alpha\|(r(k\cap P_i)+1)} e^{-Ar(k\cap P_i)}\right) \\ &= \mathrm{O}\left(\sum_{R\in\mathcal{P}_{PQ}}^{n} e^{C\|\alpha\|(r(k\cap R)+1)} e^{-Ar(k\cap R)}\right)\end{aligned}$$

As a consequence, if $\|\alpha\| < A/C$, the norm of such a ψ is uniformly bounded.

Let ψ_i be obtained from $\psi_{\mathcal{P}^r_{PQ}}$ by replacing each

$$T^{\alpha(P_0,P_j)}_{g^P_j} T^{-\alpha(P_0,P_j)}_{g^Q_j} \quad \text{with } j \le i \text{ by } T^{\alpha(P_0,P_j)}_{h_{j-1}} T^{-\alpha(P_0,P_j)}_{h_j},$$

so that $\psi_0 = \psi_{\mathcal{P}^r_{PQ}}$ and $\psi_n = \psi^r_{PQ}$. To estimate the difference between ψ_{i-1} and ψ_i, note that we can write these as

$$\psi_{i-1} = \psi T^{\alpha(P_0,P_i)}_{g^P_i} T^{-\alpha(P_0,P_i)}_{g^Q_i} \psi' \quad \text{and} \quad \psi_i = \psi T^{\alpha(P_0,P_i)}_{h_{i-1}} T^{-\alpha(P_0,P_i)}_{h_i} \psi'$$

where ψ and ψ' are obtained from $\psi_{\mathcal{P}^r_{PQ}}$ by replacing some

$$T^{\alpha(P_0,P_j)}_{g^P_j} T^{-\alpha(P_0,P_j)}_{g^Q_j} \quad \text{by } T^{\alpha(P_0,P_j)}_{h_{j-1}} T^{-\alpha(P_0,P_j)}_{h_j}$$

or by the identity. By the above observation $\|\psi\|$ and $\|\psi'\|$ are uniformly bounded. Also, we noted that the distance from g^P_i to h_{i-1} and the distance from g^Q_i to h_i are both $\mathrm{O}(e^{-Ar})$. It follows that

$$\begin{aligned}\|\psi_{i-1} - \psi_i\| &\le \|\psi\|\,\|\psi'\| \|T^{\alpha(P_0,P_j)}_{g^P_j} T^{-\alpha(P_0,P_j)}_{g^Q_j} - T^{\alpha(P_0,P_j)}_{h_{j-1}} T^{-\alpha(P_0,P_j)}_{h_j}\| \\ &= \mathrm{O}\big(e^{2|\alpha(P_0,P_j)|} e^{-Ar}\big) = \mathrm{O}\big(e^{2C\|\alpha\|(r+1)} e^{-Ar}\big)\end{aligned}$$

and therefore that

$$\begin{aligned}\|\psi^r_{PQ} - \psi_{\mathcal{P}^r_{PQ}}\| &= \|\psi_n - \psi_0\| \\ &\le n\mathrm{O}\big(e^{2C\|\alpha\|(r+1)} e^{-Ar}\big) = \mathrm{O}\big(r e^{2C\|\alpha\|(r+1)} e^{-Ar}\big)\end{aligned}$$

since $n = \mathrm{O}(r)$ by Lemma 4.

It follows that ψ^r_{PQ} and $\psi_{\mathcal{P}^r_{PQ}}$ have the same limit as r tends to infinity, provided that $\|\alpha\| < A/2C$. On other hand, h_n converges to g^P_{n+1} as r tends to infinity. Therefore, the limit of $\varphi^r_{PQ} = \psi^r_{PQ} T^{\alpha(P_0,P_{n+1})}_{h_n}$ is the same as the limit of $\varphi_{\mathcal{P}^r_{PQ}} = \psi_{\mathcal{P}^r_{PQ}} T^{\alpha(P_0,P_{n+1})}_{g^P_{n+1}}$, namely is equal to φ_{PQ}, if α is small enough. □

There is natural generalization of Lemma 16, closer to the one used in [EpM] for the construction of earthquakes, which would lead to an even more intuitive definition of φ_{PQ}. As before, index the elements of a finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$ as $P_1, P_2, \ldots, P_n$, so that the index i of P_i increases as

one goes from P to Q. Then, we could expect φ_{PQ} to be the limit as P tends to $\mathcal{P}_{PQ}$ of $T_{h_0}^{\alpha(P_0,P_1)}T_{h_1}^{\alpha(P_1,P_2)}\cdots T_{h_n}^{\alpha(P_0,P_{n+1})}$, where the geodesic h_i separates P_i from P_{i+1} and is oriented to the left as seen from P. This approach would certainly lead to a more intuitive definition of φ_{PQ}, but is unfortunately too naive. Indeed, it is not hard to see that the above limit does not exist if the transverse distribution α associated to the transverse cocycle α is not a measure. So, only the restricted limit of Lemma 16 makes sense.

From Lemma 16, we get the following properties, which were not obvious from the definition of φ_{PQ}. (Note that even the second one is non-trivial if Q does not separate P from R.)

LEMMA 17.— *If $\alpha \in \mathcal{H}(\lambda)$ is small enough for the conclusions of Lemmas 14 and 16 to hold then, for every plaques P, Q, R of $\widetilde{S}-\widetilde{\lambda}$ meeting k, $\varphi_{PQ} = \varphi_{PQ}^{-1}$ and $\varphi_{PR} = \varphi_{PQ}\varphi_{QR}$.*

We can now get rid of the assumption that P and Q both meet a suitable arc k.

LEMMA 18.— *If $\alpha \in \mathcal{H}(\lambda)$ is sufficiently small then, for every plaques P, Q of $\widetilde{S}-\widetilde{\lambda}$, the map $\varphi_{\mathcal{P}}$ converges to an m_0-isometry as $\mathcal{P}$ tends to $\mathcal{P}_{PQ}$. In addition, $\varphi_{QP} = \varphi_{PQ}^{-1}$ and $\varphi_{PR} = \varphi_{PQ}\varphi_{QR}$ for every plaques P, Q, R.*

Proof.— Select in S finitely many simple geodesics arcs $k_1, \ldots, k_n$ transverse to λ, such that each component of $S-\lambda$ meets at least one of the k_i.

For each pair of plaques P, Q of $\widetilde{S}-\widetilde{\lambda}$, we can find a finite sequence of plaques $P = P_0, P_1, \ldots, P_n, P_{n+1} = Q$ such that each P_j separates P_{j-1} from P_{j+1} and such that P_j and P_{j+1} meet the same lift $\widetilde{k}_{i_j}$ of some k_{i_j}. Then, for $\|\alpha\|$ sufficiently small (depending on the k_i), Lemma 14 proves the existence of a limit $\varphi_{P_jP_{j+1}}$ for every j. This guarantees the existence of the limit

$$\varphi_{PQ} = \lim_{\mathcal{P}\to\mathcal{P}_{PQ}} \varphi_{\mathcal{P}} = \varphi_{P_0P_1}\,\varphi_{P_1P_2}\cdots\varphi_{P_nP_{n+1}}\,.$$

The second statement easily follows from Lemma 17. (Hint for the case where none of the three plaques P, Q, R separates the two other ones: consider the unique plaque N which separates any two of these plaques.) □

Now, consider the action of the fundamental group $\pi_1(S)$ on $\widetilde{S}$. By invariance of α under this action, we have that $\varphi_{(\gamma P)(\gamma Q)} = \gamma\varphi_{PQ}\gamma^{-1}$ for every $\gamma \in \pi_1(S)$ and every plaques P and Q.

Fix a base plaque P_0 of $\widetilde{S} - \widetilde{\lambda}$, and define $\rho(\gamma) = \varphi_{P_0(\gamma P_0)}\gamma$. Then, it immediately follows from Lemma 18 and the above property that ρ defines a group homomorphism from $\pi_1(S)$ to the group of m_0-isometries of $\widetilde{S}$.

By definition of φ_{PQ}, the interiors of $\varphi_{PQ}(Q)$ and P are always disjoint. In particular, for every $\gamma \in \pi_1(S)$ which is not the identity, $\rho(\gamma)P_0 = \varphi_{P_0(\gamma P_0)}(\gamma P_0)$ is disjoint from the interior of P_0. It follows that $\rho(\gamma)$ cannot be very close to the identity, and therefore that the representation ρ is a discrete homomorphism from $\pi_1(S)$ into the m_0-isometry group of $\widetilde{S}$.

Consider the surface $S' = \widetilde{S}/\rho$. The metric m_0 on $\widetilde{S}$ induces a hyperbolic metric m' on S'. Since ρ defines an isomorphism between $\pi_1(S)$ and $\pi_1(S')$, we have a preferred isotopy class of diffeomorphisms $\psi : S \to S'$. Let m be the hyperbolic metric on S obtained by pulling back the metric m' under ψ. Note that the class of m in $\mathcal{T}(S)$ does not depend on the choice of ψ.

The proof of Proposition 13 will then be completed by the following statement.

LEMMA 19.— *The shearing cocycle of the metric m is equal to $\sigma = \sigma_0 + \alpha$.*

Proof.— To understand the shearing cocycle σ_m of m, we first have to understand the m-geodesic lamination λ_m corresponding to λ.

Define a map $\widetilde{\varphi} : \widetilde{S} - \widetilde{\lambda} \to \widetilde{S}$ by the property that $\widetilde{\varphi}$ coincides with $\varphi_{P_0 P}$ on each plaque P of $\widetilde{S} - \widetilde{\lambda}$. This $\widetilde{\varphi}$ will more or less correspond to the shear map in the sense of section 4, modulo composition whith a suitable lift of ψ.

Note that $\widetilde{\varphi}\gamma = \rho(\gamma)\widetilde{\varphi}$ for every $\gamma \in \pi_1(S)$, and that $\widetilde{\varphi}$ is m_0-isometric. Therefore, $\widetilde{\varphi}$ induces an isometric map $\varphi : (S - \lambda, m_0) \to (S', m')$.

If P and Q are two distinct plaques of $\widetilde{S} - \widetilde{\lambda}$, it follows from the fact that the interior of $\varphi_{PQ}(Q)$ is disjoint from P that $\widetilde{\varphi}(P)$ and $\widetilde{\varphi}(Q)$ have disjoint interiors. As a consequence, φ is injective. Since $(S - \lambda, m_0)$ and (S', m') have the same area, we conclude that the image of φ is dense in S.

Therefore, the image of $\widetilde{\varphi}$ in dense in $\widetilde{S}$. In particular, every point in the complement $\widetilde{S} - \widetilde{\varphi}(\widetilde{S} - \widetilde{\lambda})$ is in the geodesic limit of a sequence $\widetilde{\varphi}(g_i)$, where each g_i is in the boundary of a plaque of $\widetilde{S} - \widetilde{\lambda}$. It follows that $\widetilde{S} - \widetilde{\varphi}(\widetilde{S} - \widetilde{\lambda})$ is a $\rho(\pi_1(S))$-invariant geodesic lamination $\widetilde{\lambda}'$ of $\widetilde{S}$, which projects to an m'-geodesic lamination λ' of S'.

The m-geodesic lamination $\lambda_m = \psi^{-1}(\lambda')$ of S is the m-geodesic lamination corresponding to λ. Indeed, a leaf of $\widetilde{\lambda}$ (resp. $\widetilde{\lambda}_m$) is completely determined by the way it separates the plaques of $\widetilde{S} - \widetilde{\lambda}$ (resp. $\widetilde{S} - \widetilde{\lambda}_m$), and $\widetilde{\psi}^{-1} \circ \widetilde{\varphi} : \widetilde{S} - \widetilde{\lambda} \to \widetilde{S} - \widetilde{\lambda}_m$ respects the combinatorics of these plaques while commuting with the action of $\pi_1(S)$.

We will prefer to work with λ' rather than λ_m, to avoid the interference of too many ψ with the arguments. Let $\sigma_{m'}$ be the shearing cocycle of the metric m' with respect to λ'. Then the isometry $\psi : (S, m) \to (S', m')$ sends σ_m to $\sigma_{m'}$. Namely, $\sigma_m(P, Q) = \sigma_{m'}(\widetilde{\varphi}(P), \widetilde{\varphi}(Q))$ if we note that $\widetilde{\varphi}(P)$ and $\widetilde{\varphi}(Q)$ are plaques of $\widetilde{S} - \widetilde{\lambda}'$ and are sent by the appropriate lift of ψ to the plaques of $\widetilde{S} - \widetilde{\lambda}_m$ corresponding to P and Q, respectively.

We want to show that $\sigma_m(P, Q) = \sigma_0(P, Q) + \alpha(P, Q)$ for every plaques P, Q of $\widetilde{S} - \widetilde{\lambda}$. By additivity of transverse cocycles, we can restrict attention to the case where P and Q both meet a transverse geodesic arc k whose projection to S is simple.

As usual, let $\mathcal{P}_{PQ}$ be the set of all plaques of $\widetilde{S} - \widetilde{\lambda}$ that separate P from Q. Choose a finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, index its elements as $P_1, P_2, \ldots, P_n$ so that the index i of P_i increases as one goes from P to Q, and set $P_0 = P$ and $P_{n+1} = Q$.

We want to compare $\sigma_m(P_i, P_{i+1}) = \sigma_{m'}(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}))$ to $\sigma_0(P_i, P_{i+1})$. Note that, to compute $\sigma_{m'}(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}))$, we do not need the whole lamination $\widetilde{\lambda}'$. We only need to know the metric m_0 on $\widetilde{S}$, the two triangles $\widetilde{\varphi}(P_i)$ and $\widetilde{\varphi}(P_{i+1})$, and the family $\widetilde{\lambda}'_{PQ}$ of those leaves of $\widetilde{\lambda}'$ that separate $\widetilde{\varphi}(P_i)$ and $\widetilde{\varphi}(P_{i+1})$ (the only requirement here being that two geodesics of $\widetilde{\lambda}'_{PQ}$ which are adjacent to the same component of $\widetilde{S} - \widetilde{\lambda}'_{PQ}$ are asymptotic). Let $s(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}) ; \widetilde{\lambda}'_{PQ})$ be the number defined by this procedure. Of course, in this case, $s(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}) ; \widetilde{\lambda}'_{PQ}) = \sigma_{m'}(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}))$ and, similarly, $s(P_i, P_{i+1} ; \widetilde{\lambda}_{PQ}) = \sigma_0(P_i, P_{i+1})$.

Now, remember that $\widetilde{\varphi}(P_i) = \varphi_{P_0 P_i}(P_i)$ and $\widetilde{\varphi}(P_{i+1}) = \varphi_{P_0 P_{i+1}}(P_{i+1})$. Since $\varphi_{P_0 P_i}$ is an isometry, we conclude that

$$s(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1}) ; \widetilde{\lambda}'_{PQ}) = s(P_i, \varphi_{P_i P_{i+1}}(P_{i+1}) ; \varphi^{-1}_{P_0 P_i}(\widetilde{\lambda}'_{PQ})) .$$

From Lemma 15, there is a constant $B > 0$ such that $\varphi_{P_iP_{i+1}}$ can be decomposed as $\varphi_{P_iP_{i+1}} = \psi_{P_iP_{i+1}} T_g^{\alpha(P_i,P_{i+1})}$ with

$$\|\psi_{P_iP_{i+1}} - \mathrm{Id}\| = \mathrm{O}\Big(\sum_{R \in \mathcal{P}_{P_iP_{i+1}}} e^{-Br(k \cap R)} \Big) ,$$

where g is the geodesic in the boundary of P_{i+1} that is closest to P_i, oriented to the left as seen from P_i.

By definition of s,

$$s\big(P_i, T_g^{\alpha(P_i,P_{i+1})}(P_{i+1})\,;\, \widetilde{\lambda}_{PQ}\big) = s(P_i, P_{i+1}; \widetilde{\lambda}_{PQ}) + \alpha(P_i, P_{i+1}) .$$

Let k_i be the subarc of k joining P_i to P_{i+1}. Note that k_i also joins P_i to $T_g^{\alpha(P_i,P_{i+1})}(P_{i+1})$. Within an error of $\ell_{m_0}(k_i)$, $s\big(P_i, T_g^{\alpha(P_i,P_{i+1})}(P_{i+1}); \widetilde{\lambda}_{PQ}\big)$ does not depend on the lamination $\widetilde{\lambda}_{PQ}$ by Lemma 8. An additional application of Lemma 8 gives that

$$\begin{aligned}
\sigma_m(P_i, P_{i+1}) &= s\big(\widetilde{\varphi}(P_i), \widetilde{\varphi}(P_{i+1})\,;\, \widetilde{\lambda}'_{PQ}\big) \\
&= s\big(P_i, \psi_{P_iP_{i+1}} T_g^{\alpha(P_i,P_{i+1})}(P_{i+1})\,;\, \varphi_{P_0P_i}^{-1}(\widetilde{\lambda}'_{PQ})\big) \\
&= s\big(P_i, T_g^{\alpha(P_i,P_{i+1})}(P_{i+1})\,;\, \widetilde{\lambda}_{PQ}\big) + \\
&\quad + \mathrm{O}\Big(\sum_{R \in \mathcal{P}_{P_iP_{i+1}}} e^{-Br(k \cap R)} \Big) + \mathrm{O}\big(\ell_{m_0}(k_i)\big) \\
&= s(P_i, P_{i+1}\,;\, \widetilde{\lambda}_{PQ}) + \alpha(P_i, P_{i+1}) + \\
&\quad + \mathrm{O}\Big(\sum_{R \in \mathcal{P}_{P_iP_{i+1}}} e^{-Br(k \cap R)} \Big) + \mathrm{O}\big(\ell_{m_0}(k_i)\big) \\
&= \sigma_0(P_i, P_{i+1}) + \alpha(P_i, P_{i+1}) + \\
&\quad + \mathrm{O}\Big(\sum_{R \in \mathcal{P}_{P_iP_{i+1}}} e^{-Br(k \cap R)} \Big) + \mathrm{O}\big(\ell_{m_0}(k_i)\big) .
\end{aligned}$$

Summing over all i, we obtain

$$\begin{aligned}\sigma_m(P,Q) &= \sigma_0(P,Q) + \alpha(P,Q) + \\ &\quad + \sum_{i=0}^{n} \mathrm{O}\Big(\sum_{R\in\mathcal{P}_{P_iP_{i+1}}} e^{-Br(k\cap R)} \Big) + \sum_{i=0}^{n} \mathrm{O}\big(\ell_{m_0}(k_i)\big) \\ &= \sigma_0(P,Q) + \alpha(P,Q) + \mathrm{O}\Big(\sum_{R\in\mathcal{P}_{PQ}-\mathcal{P}} e^{-Br(k\cap R)} \Big) + \\ &\quad + \mathrm{O}\Big(\ell_{m_0}\Big(k - P\cup Q\cup \bigcup_{R\in\mathcal{P}} R \Big)\Big).\end{aligned}$$

By Lemma 4, the series $\sum_{R\in\mathcal{P}_{PQ}} e^{-Br(k\cap R)}$ is convergent. Letting $\mathcal{P}$ tend to $\mathcal{P}_{PQ}$, this enables us to conclude that $\sigma_m(P,Q) = \sigma_0(P,Q) + \alpha(P,Q)$.

This completes the proof of Lemma 19, and therefore of Proposition 13. □

6. The global realization of shearing cocycles

In this section, we determine which transverse cocycles for λ can occur as shearing cocycles of hyperbolic metrics.

There is an obvious necessary condition for a given transverse cocycle α for λ to be the shearing cocycle σ_m of a hyperbolic metric m. Indeed, by Theorem 9, the m-length $\ell_m(\mu)$ of another $\mu \in \mathcal{H}(\lambda)$ is equal to $\tau(\mu,\sigma_m) = \tau(\mu,\alpha)$. If, in addition, this μ is a non-zero transverse measure for λ, then it follows from the definition of $\ell_m(\mu)$ that this length is positive. Consequently, for α to be the shearing distribution of some hyperbolic metric, it is necessary that $\tau(\mu,\alpha) > 0$ for every transverse non-zero measure μ for λ. Quite remarkably, this condition turns out to be sufficient.

THEOREM 20.— *A transverse Hölder distribution α for λ is the shearing distribution of some hyperbolic metric if and only if $\tau(\mu,\alpha) > 0$ for every non-zero transverse measure μ for λ.*

Proof.— As in section 5, let us endow $\mathcal{H}(\lambda;\mathbb{R})$ with a norm $\|\cdot\|$. For a hyperbolic metric m_0, Proposition 13 provides a ball $B(\sigma_{m_0},\varepsilon_0)\subset\mathcal{H}(\lambda;\mathbb{R})$ around σ_{m_0} such that every transverse cocycle in this ball is also the shearing distribution of some hyperbolic metric. Let us examine the proof of Proposition 13 in detail, to see what determines ε_0.

We start with a topological data (independent of the metric m_0) consisting of simple arcs $\overline{k}_1,\dots,\overline{k}_n$, transverse to λ, such that every component of $S-\lambda$ meets some $\overline{k}_i$. We also require that, for any hyperbolic metric m_0 and after making λ m_0-geodesic by a first isotopy, each $\overline{k}_i$ can be isotoped respecting λ to a simple m_0-geodesic arc k_i. An easy way to achieve this is to choose each $\overline{k}_i$ contained in some non-backtracking simple closed curve transverse to λ, which we can always do.

Given a hyperbolic metric m_0 and geodesic arcs k_i isotopic to the $\overline{k}_i$ as above, Lemmas 3 and 6 associate constants A_i, N_i and C_i to each k_i. Note that A_i depends on k_i and on the metric m_0, but that C_i does not and depends only on the topology of $\overline{k}_i$ and λ.

Then, if we examine the proof of Proposition 13, and in particular the proof of Lemmas 14 and 16, we see that we can take $\varepsilon_0=\min_i A_i/2C_i$.

Now, let us change the perspective of the problem. Consider a hyperbolic metric m_0 whose shearing cocycle σ_0 is within ε of the complement of the image of the map $\Sigma:\mathcal{T}(S)\to\mathcal{H}(\lambda;\mathbb{R})$ which associates its shearing cocycle σ_m to each metric m.

This means that $\varepsilon_0\le\varepsilon$, and therefore that there is a k_i for which it is impossible to find constants A_i, N_i which satisfy the conclusion of Lemma 3 and such that $A_i/2C_i>\varepsilon$. In other words, there is a k_i such that, for every N, there is an arc b_N contained in a leaf of λ which cuts $n_N\ge N$ times the arc k_i and whose length $\ell_{m_0}(b_N)$ is such that $\ell_{m_0}(b_N)\le 2\varepsilon C_i(n_N-1)$.

Let n'_N denote the number of times the arc b_N crosses the union of the k_j. On each arc k transverse to λ, consider the Dirac measure μ_k^N of weight $1/n'_N$ based at the finite set $k\cap b_N$. Since every leaf of λ meets some k_j, the total mass of μ_k^N is uniformly bounded in N. It follows that we can extract a subsequence $(N_p)_{p\in\mathbb{N}}$ such that, for every transverse arc k, the measure $\mu_k^{N_p}$ weakly converges to some measure μ_k as p tends to ∞. Since $n'_{N_p}\ge n_{N_p}\ge N_p$ tends to ∞, these measures μ_k are invariant under homotopy of k respecting λ, and therefore define a transverse measure μ for λ.

By construction, the length $\ell_{m_0}(\mu)$ is equal to the limit of the

$$\frac{\ell_{m_0}(b_{N_p})}{n'_{N_p}} \leq 2\varepsilon C_i \frac{n_{N_p}-1}{n'_{N_p}} \leq 2\varepsilon C_i\,.$$

It follows that $\ell_{m_0}(\mu) \leq 2\varepsilon C_i$.

On the other hand, $\sum_j \mu(k_j)$ is the limit of the $\sum_j \mu_{k_j}^{N_p}(k_j) = 1$, and is therefore equal to 1. Since $\mu(k_j) \leq C_j||\mu||$ by definition of C_j in Lemma 6, we conclude that

$$||\mu|| \geq \Big(\sum_j C_j\Big)^{-1}$$

As a conclusion, if the shearing cocycle σ_0 of the metric m_0 is within ε of the complement of the image of the map $\mathcal{T}(S) \to \mathcal{H}(\lambda\,;\mathbb{R})$, there is a transverse measure μ for λ such that

$$\frac{\tau(\mu,\sigma_0)}{||\mu||} = \frac{\ell_{m_0}(\mu)}{||\mu||} \leq 2\varepsilon C_i\Big(\sum_j C_j\Big)^{-1} \leq 2\varepsilon\,.$$

By weak compactness of the space of transverse measures μ with $||\mu|| = 1$ and by continuity of τ it follows that, if $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$ is in the boundary of the image of $\mathcal{T}(S) \to \mathcal{H}(\lambda;\mathbb{R})$, then there exists a transverse measure μ with $\tau(\mu,\alpha) = 0$.

Therefore, the image of $\mathcal{T}(S) \to \mathcal{H}(\lambda;\mathbb{R})$ is closed in the set $\mathcal{C}(\lambda)$ of those $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$ such that $\tau(\mu,\alpha) > 0$ for every transverse measure μ. This image is also open in $\mathcal{C}(\lambda)$ by Proposition 13. Since $\mathcal{C}(\lambda)$ is defined by linear inequalities, it is connected. It follows that $\mathcal{C}(\lambda)$ is exactly equal to the image of $\mathcal{T}(S) \to \mathcal{H}(\lambda;\mathbb{R})$. □

COROLLARY 21.— *Consider the map $\Sigma : \mathcal{T}(S) \to \mathcal{H}(\lambda;\mathbb{R})$ which associates its shearing cocycle σ_m to each hyperbolic metric m. The image of Σ is an open convex cone in $\mathcal{H}(\lambda;\mathbb{R})$ bounded by finitely many faces.*

Proof.— By [Ka] (compare [Pa1], [PeH], [Bo4, sect. 4]), the geodesic lamination admits only finitely many ergodic transverse measures $\mu_1, \ldots, \mu_n$, and every transverse measure is a linear combination with non-negative coefficients of these μ_i. □

7. The bending cocycle of a pleated surface

A pleated surface with topological type S in a hyperbolic 3-dimensional manifold M is a map $f : S \to M$ such that:

(i) the path metric obtained by pulling back the hyperbolic metric of M by f is a hyperbolic metric m on S;

(ii) there is an m-geodesic lamination λ such that f sends each leaf of λ to a geodesic of M and is totally geodesic on $S - \lambda$.

In this case, we say that the pleated surface f admits the geodesic lamination λ as a pleating locus. Note that, with this definition, the pleating locus of a pleated surface is not unique. An extreme case is when f is totally geodesic, in which case every geodesic lamination is a pleating locus for f. See [Th1] and [CEG] for more details on pleated surfaces.

The local geometry of the pleated surface f is unchanged if we lift it to the covering of M with fundamental group $f_*(\pi_1(S))$. If we are only interested in this local geometry, it is therefore natural to extend the definition of pleated surfaces in the following way.

Let us define an *(abstract) pleated surface* with topological type S as a pair $(\widetilde{f}, \rho)$ where $\widetilde{f} : \widetilde{S} \to \mathbb{H}^3$ is a map from the universal covering $\widetilde{S}$ to the hyperbolic 3-space $\mathbb{H}^3$ and where $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ is a homomorphism from the fundamental group of S to the group of orientation-preserving isometries of $\mathcal{H}^3$, and such that:

(i) for every $\gamma \in \pi_1(S)$, $\widetilde{f}\gamma = \rho(\gamma)\widetilde{f}$;

(ii) the path metric obtained by pulling back the metric of $\mathbb{H}^3$ by $\widetilde{f}$ is a hyperbolic metric on $\widetilde{S}$ which, by (i), induces a hyperbolic metric m on S;

(iii) there is an m-geodesic lamination λ of S such that $\widetilde{f}$ sends each leaf of its preimage $\widetilde{\lambda}$ to a geodesic of $\mathbb{H}^3$ and is totally geodesic on $\widetilde{S} - \widetilde{\lambda}$.

Again, we then say that λ is a *pleating locus* for $(\widetilde{f}, \rho)$.

Note that, when the image of ρ is discrete and without torsion, the abstract pleated surface $(\widetilde{f}, \rho)$ induces a pleated surface in the hyperbolic 3-manifold $M = \mathbb{H}^3/\rho(\pi_1(S))$.

Consider an (abstract) pleated surface $f = (\widetilde{f}, \rho)$ with pleating locus λ. In general, what prevents $\widetilde{f} : \widetilde{S} \to \mathbb{H}^3$ from being totally geodesic is the fact

that it may be bent along the leaves of $\widetilde{\lambda}$. This section is devoted to giving a precise definition and measurement of this bending.

First consider the simple case of two plaques P and Q of $\widetilde{S} - \widetilde{\lambda}$ which meet along a leaf g of $\widetilde{\lambda}$. Orient g as part of the boundary of P. Then the way $\widetilde{f}$ is bent along g is clearly characterized by the angle $\theta(P,Q) \in \mathbb{R}/2\pi\mathbb{Z}$ from the totally geodesic triangle $\widetilde{f}(P)$ to the totally geodesic triangle $\widetilde{f}(Q)$, measured for the natural orientation of $\mathbb{H}^3$ with respect to the orientation of the geodesic $\widetilde{f}(g)$ defined by the orientation of g. Define in this case $\beta(P,Q) \in \mathbb{R}/2\pi\mathbb{Z}$ to be the external angle $\beta(P,Q) = \theta(P,Q) - \pi$.

If P and Q are separated by only finitely many leaves of $\widetilde{\lambda}$, we can similarly define a bending angle $\beta(P,Q) \in \mathbb{R}/2\pi\mathbb{Z}$ as the sum of the bending angles along these leaves. Defining a bending angle $\beta(P,Q)$ in general will be a little more difficult.

Consider two plaques P and Q of $\widetilde{S} - \widetilde{\lambda}$. Let Σ be the closure of the component of $\widetilde{S} - P \cup Q$ that separates P from Q, and let $\widetilde{\lambda}_{PQ}$ consist of those leaves of $\widetilde{\lambda}$ which separate P from Q, including the two leaves $P \cap \Sigma$ and $Q \cap \Sigma$. The leaves of $\widetilde{\lambda}_{PQ}$ decompose the strip Σ into hyperbolic strips. For each such strip W of $\Sigma - \widetilde{\lambda}_{PQ}$, the image $\widetilde{f}(W)$ is a 2-dimensional hyperbolic strip in $\mathbb{H}^3$, intersecting the sphere at infinity $\mathbb{H}^3_\infty$ in the disjoint union of two arcs, one of which may be reduced to a point.

As in section 2, the partial foliation $\widetilde{\lambda}_{PQ}$ of Σ can be extended to a foliation $\mathcal{G}$ of Σ by geodesics. Orient the leaves of $\mathcal{G}$ to the left as seen from P. As g ranges over the leaves of $\mathcal{G}$, the negative end points of the geodesics $\widetilde{f}(g)$ of $\mathbb{H}^3$ form a continuous arc γ in $\mathbb{H}^3_\infty$ going from the negative end point of $\widetilde{f}(P \cap \Sigma)$ to the negative end point of $\widetilde{f}(Q \cap \Sigma)$. Note that the natural projection map $\Sigma \to \gamma$ is Lipschitz. It follows that γ is a rectifiable curve in $\mathbb{H}^3_\infty$. Since λ has Hausdorff dimension 1 in S ([BiS], [Th3, sect. 10]), it also follows that, as g ranges over all leaves $\widetilde{\lambda}_{PQ}$, the negative end points of the $\widetilde{f}(g)$ form a subset of γ of Hausdorff dimension 0.

To define the amount of bending which occurs for $\widetilde{f}$ between P and Q, it is convenient to consider the upper half-space model for $\mathbb{H}^3$, for which the sphere at infinity $\mathbb{H}^3_\infty$ is identified to the euclidean plane $\mathbb{R}^2$ plus a point ∞. Without loss of generality, we can assume that the arc γ does not contain the point ∞. For every strip W of $\Sigma - \widetilde{\lambda}_{PQ}$, the projection of W to γ is a circle arc in $\mathbb{R}^2$, possibly straight and possibly reduced to a point. At the negative end point of $\widetilde{f}(P \cap \Sigma)$, let v_P be the unit vector in $\mathbb{R}^2$ tangent to $\widetilde{f}(P)$, oriented outwards; similarly, at the negative end point of $\widetilde{f}(Q \cap \Sigma)$, let v_Q be the unit vector tangent to $\widetilde{f}(Q)$, oriented inwards.

Intuitively, the angle $\theta(v_P, v_Q) \in \mathbb{R}/2\pi\mathbb{Z}$ from v_P to v_Q in $\mathbb{R}^2$ is the sum of the integral of the curvature of γ along the circle arcs corresponding to strips of $\Sigma - \widetilde{\lambda}_{PQ}$, plus the amount γ turns at the points of γ corresponding to the negative end points of all $\widetilde{f}(g)$ with g a leaf of $\widetilde{\lambda}_{PQ}$. We will define the bending angle $\beta(P, Q)$ to be this amount of turning at the leaves of $\widetilde{\lambda}_{PQ}$.

Orient the arc γ in $\mathbb{H}^3_\infty = \mathbb{R}^2$ so that it goes from the negative end point of $\widetilde{f}(P \cap \Sigma)$ to the negative end point of $\widetilde{f}(Q \cap \Sigma)$. For every strip W of $\Sigma - \widetilde{\lambda}_{PQ}$, consider the corresponding circle arc in γ, and let β_W be the integral of the signed curvature of this arc; in particular, $|\beta_W|$ is the quotient of the length of this circle arc by its radius, and is 0 if this arc is reduced to a point. We now define the *bending angle* $\beta(P, Q) \in \mathbb{R}/2\pi\mathbb{Z}$ to be

$$\beta(P, Q) = \theta(v_P, v_Q) - \sum_W \beta_W ,$$

where $\theta(v_P, v_Q)$ is the angle from v_P to v_Q and where W ranges over all the strips of $\Sigma - \widetilde{\lambda}_{PQ}$. Note that the $\widetilde{f}(W)$ meet a fixed compact subset of $\mathbb{H}^3$, which implies that the radii of the corresponding circle arcs in γ are bounded away from 0 and guarantees the convergence of the sum $\sum_W \beta_W$, since γ is rectifiable.

When the plaques P and Q meet along a geodesic g, the curve γ consists of a single point and $\beta(P, Q)$ is equal to the angle between the two vectors v_P and v_Q, which is also the external angle between the two geodesic triangles $\widetilde{f}(P)$ and $\widetilde{f}(Q)$. Therefore, $\beta(P, Q)$ is equal to the bending angle defined at the beginning of this section in this case.

The remainder of this section will be devoted to proving that the map $(P, Q) \mapsto \beta(P, Q)$ defines an $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle for λ. In particular, we will prove that $\beta(Q, P) = \beta(P, Q)$, which is far from being clear at this point. Indeed, the definitions of $\beta(Q, P)$ and $\beta(P, Q)$ each involve a different sum $\sum_W \beta_W$, where a W contributing to one sum may contribute 0 to the other one, and conversely.

For this, we will give a different definition of $\beta(P, Q)$.

As in section 5, let $\mathcal{P}_{PQ}$ be the set of all plaques of $\widetilde{S} - \widetilde{\lambda}$ that separate P from Q. Given a finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, index its elements as $P_1, P_2, \ldots, P_n$ so that the index i of P_i increases as one goes from P to Q, and set $P_0 = P$ and $P_{n+1} = Q$. Let $\widetilde{\lambda}_{\mathcal{P}}$ be the geodesic lamination of $\widetilde{S}$ which is obtained from $\widetilde{\lambda}$ by the following operations: for every $i = 0, \ldots, n$,

erase the leaves of $\widetilde{\lambda}$ which are contained in the interior of the strip Σ_i separating P_i from P_{i+1}, and replace them by a single "diagonal" geodesic joining the negative ends of the two geodesics delimiting Σ_i (endowing these geodesics with the boundary orientation). The choice of which diagonal we take is irrelevant here. Note that the diagonal will be equal to one of the leaves of $\widetilde{\lambda}$ in the degenerate cases where Σ_i is a wedge or a single geodesic, namely when the geodesics bounding Σ_i are asymptotic.

We now define a pleated surface $\widetilde{f}_{\mathcal{P}} : \widetilde{S} \to \mathbb{H}^3$ with pleating locus $\widetilde{\lambda}_{\mathcal{P}}$, without any assumption of equivariance with respect to a homomorphism $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$. The map $\widetilde{f}_{\mathcal{P}}$ coincides with $\widetilde{f}$ outside of the strips Σ_i. For each diagonal $d_i \in \widetilde{\lambda}_{\mathcal{P}}$ joining end points of the two geodesics g_i, h_i delimiting Σ_i, $\widetilde{f}_{\mathcal{P}}$ sends d_i to the geodesic of $\mathbb{H}^3$ that joins the corresponding end points of $\widetilde{f}(g_i)$ and $\widetilde{f}(h_i)$. In this construction of $\widetilde{f}_{\mathcal{P}}(k)$, we might worry that these end points of $\widetilde{f}(g_i)$ and $\widetilde{f}(h_i)$ might be equal; however, this problem is clearly not going to happen if g and h are close enough, and is therefore excluded if we assume that $\mathcal{P}$ is a large enough subset of $\mathcal{P}_{PQ}$. Now, each diagonal d_i separates the corresponding Σ_i into two wedges, possibly reduced to a geodesic in degenerate cases; there is basically a unique way to define $\widetilde{f}_{\mathcal{P}}$ on these wedges so that they are sent to totally geodesic wedges in $\mathbb{H}^3$.

For $\mathcal{P}$ a large enough finite subset of $\mathcal{P}_{PQ}$, we now have defined a geodesic lamination $\widetilde{\lambda}_{\mathcal{P}}$ and a pleated surface $\widetilde{f}_{\mathcal{P}} : \widetilde{S} \to \mathbb{H}^3$ with pleating locus $\widetilde{\lambda}_{\mathcal{P}}$. Of course, $\widetilde{\lambda}_{\mathcal{P}}$ is not invariant any more under the action of $\pi_1(S)$ on $\widetilde{S}$, and $\widetilde{f}_{\mathcal{P}}$ is not equivariant any more for some representation $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$. However, P and Q are still plaques of $\widetilde{S} - \widetilde{\lambda}_{\mathcal{P}}$, and we can measure the bending of $\widetilde{f}_{\mathcal{P}}$ between P and Q by a number $\beta_{\mathcal{P}}(P,Q) \in \mathbb{R}/2\pi\mathbb{Z}$. Since $\widetilde{\lambda}_{\mathcal{P}}$ has only finitely many leaves g_i between P and Q, this $\beta_{\mathcal{P}}(P,Q)$ is the sum over the g_i of the external angles between the two totally geodesic pieces of $\widetilde{f}_{\mathcal{P}}(\widetilde{S})$ meeting along $\widetilde{f}_{\mathcal{P}}(g_i)$.

LEMMA 22.— *For every two plaques P, Q of $\widetilde{S} - \widetilde{\lambda}$, the number $\beta_{\mathcal{P}}(P,Q) \in \mathbb{R}/2\pi\mathbb{Z}$ converges to $\beta(P,Q)$ as the finite subset $\mathcal{P}$ converges to the set $\mathcal{P}_{PQ}$ of all plaques separating P from Q.*

Proof.— Given a plaque $R \in \mathcal{P}_{PQ}$, exactly two of the geodesics in its boundary separate P from Q, and these two geodesics delimit a strip W_R in the strip Σ separating P from Q. Conversely, any strip delimited in Σ by two leaves of $\widetilde{\lambda}_{PQ}$ is associated in this way to a plaque of $\mathcal{P}_{PQ}$. We can then rephrase the definition of $\beta(P,Q)$ by saying that it is equal to the

angle from the vector v_P to the vector v_Q minus the sum $\sum_{R\in\mathcal{P}_{PQ}} \beta_{W_R}$, where β_W is the integral of the geodesic curvature of the circle arc γ_W in $\mathbb{H}^3_\infty$ obtained by projection of the strip W in the negative direction.

Similarly, $\beta_{\mathcal{P}}(P,Q)$ is equal to the same angle from v_P to v_Q minus a sum $\sum_{W\in\mathcal{W}} \beta_W$, where $\mathcal{W}$ is the finite set of strips delimited in Σ by the leaves of $\widetilde{\lambda}_{\mathcal{P}}$ separating P from Q. The set $\mathcal{W}$ contains all the W_R associated to the $R \in \mathcal{P}$. The other terms come from the two wedges obtained by splitting along the diagonal each strip Σ_i separating P_i from P_{i+1}, for $i = 0, \ldots, n$. Of these two wedges in Σ_i, at least one of them does not contribute to the sum since its projection to $\mathbb{H}^3_\infty$ in the negative direction consists of only one point; let W_i be the other wedge. Then,

$$\beta(P,Q) - \beta_{\mathcal{P}}(P,Q) = \sum_{R\in\mathcal{P}_{PQ}-\mathcal{P}} \beta_{W_R} - \sum_{i=0}^{n} \beta_{W_i}\,.$$

Since the radii of the circle arcs γ_W admits a positive lower bound, each term β_W is in absolute value bounded by a constant times the distance between the end points of γ_W. Also, each γ_{W_i} has the same end points as the arc projection of the strip Σ_i, where this projection is defined using the lamination $\widetilde{\lambda}_{PQ}$, and this projection of Σ_i consists of some arcs γ_{W_R} with $R \in \mathcal{P}_{PQ} - \mathcal{P}$ and of a set of Hausdorff dimension 0. It follows that

$$\beta(P,Q) - \beta_{\mathcal{P}}(P,Q) = \mathrm{O}\Big(\sum_{R\in\mathcal{P}_{PQ}-\mathcal{P}} \text{length } (\gamma_{W_R}) \Big)\,,$$

which tends to 0 as $\mathcal{P}$ tends to $\mathcal{P}_{PQ}$. □

Since there are only finitely many leaves $\widetilde{\lambda}_{\mathcal{P}}$ separating P from Q, we have that $\beta_{\mathcal{P}}(Q,P) = \beta_{\mathcal{P}}(P,Q)$. An immediate corollary of Lemma 22 is therefore.

LEMMA 23

$$\beta(Q,P) = \beta(P,Q)\,.$$

LEMMA 24.— *The bending angle $\beta(P,Q)$ is independent of the identification of $\mathbb{H}^3$ with the upper half-space model.*

Proof. — The bending angle $\beta_{\mathcal{P}}(P,Q)$ is equal to the intrinsic sum of the external angles between the finitely many totally geodesic pieces forming the pleated surface $\widetilde{f}_{\mathcal{P}}$ between P and Q, and is therefore independent of the choice of any model for $\mathbb{H}^3$. By Lemma 22, the same is therefore true for $\beta(P,Q)$. □

From Lemma 22, it immediately follows that $\beta(\gamma P,\gamma Q) = \beta(P,Q)$ for every $\gamma \in \pi_1(S)$. Also, if the plaques P, Q and R of $\widetilde{S} - \widetilde{\lambda}$ are such that Q separates P from R, it is immediate from the definitions that $\beta(P,R) = \beta(P,Q) + \beta(Q,R)$.

As a consequence, β defines an $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle for λ, called the *bending cocycle* of the pleated surface $f = (\widetilde{f},\rho)$.

We would like to conclude this section by a remark, indicating how to compute the bending angle $\beta(P,Q)$ if we use the Poincaré model, as opposed to the upper half space model, for the hyperbolic 3-space $\mathbb{H}^3$. In this model, the sphere at infinity $\mathbb{H}^3_\infty$ is identified to the unit sphere in $\mathbb{R}^3$. As in the definition of $\beta(P,Q)$, we associate to P and Q an arc γ in $\mathbb{H}^3_\infty$ going from the negative end point of $\widetilde{f}(P\cap\Sigma)$ to the negative end point of $\widetilde{f}(Q\cap\Sigma)$, where Σ is the closure of the component of $\widetilde{S} - P\cup Q$ separating P from Q. The leaves of $\widetilde{\lambda}$ separating P from Q decompose Σ into strips, and to each such strip is associated a circle arc in γ. We also have two vectors v_P and v_Q tangent to P and Q, respectively, at the end points of γ. To measure the angle from v_P to v_Q in $\mathbb{H}^3_\infty = S^2$, we now need to choose a differentiable arc δ going from the positive end point to the negative end point of γ. We can then measure an angle $\theta_\delta(v_P,v_Q) \in \mathbb{R}/2\pi\mathbb{Z}$ by using parallel transport along δ. As before, for each strip W of $\Sigma - \widetilde{\lambda}_{PQ}$, let β_W be the integral of the geodesic curvature of the corresponding circle arc in γ. Then,

$$\beta(P,Q) = -\theta_\delta(v_P,v_Q) + \sum_W \beta_W + A(\gamma\cup\delta) \in \mathbb{R}/2\pi\mathbb{Z}$$

where $A(\gamma\cup\delta)$ is the area of any cycle bounding $\gamma\cup\delta$, which is uniquely defined modulo 4π. This formula is an immediate consequence of the Gauss-Bonnet formula in the case when P and Q are separated by finitely many plaques, and follows from Lemma 22 in the general case. The fact that the sign is opposite to the one which could be expected comes from the fact that, when identifying the upper half-space model to the Poincaré model, the orientation of $\mathbb{R}^2\cup\infty$ is sent to the opposite of the orientation of S^2.

8. The realization of bending cocycles

In this section we show that every $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle for the geodesic lamination λ is the bending cocycle associated to some pleated surface $f = (\widetilde{f}, \rho)$ with topological type S and with pleating locus λ. In addition, the pull back metric defined by f on S can be any hyperbolic metric m. The methods of proof will closely follow those used in section 5, but will be simpler.

Consider an $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle α for the maximal geodesic lamination λ, and let m be a hyperbolic metric on S. We want to construct a pleated surface $f = (\widetilde{f}, \rho)$, with topological type S and with pleating locus λ, whose bending transverse cocycle is α and whose pull back metric on S is m.

We start with any pleated surface $f_0 = (\widetilde{f}_0, \rho_0)$ with pleating locus λ and with pull back metric m. There clearly exists such a pleated surface with bending transverse cocycle 0, namely one which is totally geodesic, but it will be convenient to work in full generality. So, let β_0 be the bending transverse cocycle of f_0.

Set $\alpha = \beta - \beta_0$.

Given an oriented geodesic g of $\mathbb{H}^3$ and given a number $v \in \mathbb{R}/2\pi\mathbb{Z}$, let $R_g^v : \mathbb{H}^3 \to \mathbb{H}^3$ be the hyperbolic rotation of angle v around g.

Consider two plaques P and Q of $\widetilde{S} - \widetilde{\lambda}$. As in section 5, for every plaque R separating P from Q, let g_R^P and g_R^Q be the geodesics in the boundary of R which are closest to P and Q, respectively. Orient these geodesics to the left, as seen from P.

As usual, let $\mathcal{P}_{PQ}$ be the set of all plaques of $\widetilde{S} - \widetilde{\lambda}$ that separate P from Q. Given a finite subset $\mathcal{P}$ of $\mathcal{P}_{PQ}$, index its elements as $P_1, P_2, \ldots, P_n$ so that the index i of P_i increases as one goes from P to Q, and consider

$$\psi_{PQ} = \lim_{\mathcal{P} \to \mathcal{P}_{PQ}} R^{\alpha(P,P_1)}_{\widetilde{f}_0(g_1^P)} R^{-\alpha(P,P_1)}_{\widetilde{f}_0(g_1^Q)} R^{\alpha(P,P_2)}_{\widetilde{f}_0(g_2^P)} R^{-\alpha(P,P_2)}_{\widetilde{f}_0(g_2^Q)} \cdots R^{-\alpha(P,P_n)}_{\widetilde{f}_0(g_n^Q)} R^{\alpha(P,Q)}_{\widetilde{f}_0(g_Q^P)}$$

where $g_i^P = g_{P_i}^P$, $g_i^Q = g_{P_i}^Q$, and g_Q^P is the geodesic in the boundary of Q that is closest to P. The fact that the limit exists is proved by the same argument as Lemmas 14 and 16. The proof is actually much simpler because, since the $\mathbb{R}/2\pi\mathbb{Z}$-valued cocycle α is bounded, we do not have to worry about terms $e^{|\alpha(P,P_i)|} \leq e^{C\|\alpha\|(r(k \cap P_i)+1)}$ any more. In particular, the convergence holds without the assumption that α is small enough.

As in Lemma 18, we also have that, for every three plaques P, Q and R of $\widetilde{S}-\widetilde{\lambda}$, $\psi_{PQ}=\psi_{PR}\psi_{RQ}$ and $\psi_{QP}=\psi_{PQ}^{-1}$.

Fix a base plaque P_0 of $\widetilde{S}-\widetilde{\lambda}$. Define $\widetilde{f}:\widetilde{S}-\widetilde{\lambda}\to\mathbb{H}^3$ by the property that $\widetilde{f}$ coincides with $\psi_{P_0P}\widetilde{f}_0$ on each plaque P. We want to show that $\widetilde{f}$ extends to a pleated surface with pull back metric m_0 and with bending transverse cocycle $\beta=\beta_0+\alpha$.

For this, let $\mathcal{P}$ be a finite set of plaques of $\widetilde{S}-\widetilde{\lambda}$, and let $\widetilde{f}_{\mathcal{P}}:\widetilde{S}\to\mathbb{H}^3$ be defined as follows. For each plaque P of $\widetilde{S}-\widetilde{\lambda}$, let $P_1,\ldots,P_n$ be the elements of $\mathcal{P}$ which separate P_0 from P, where the index i of P_i increases as one goes from P_0 to P, and set $P_{n+1}=P$ if $P\in\mathcal{P}$. For each i, let g_i^- be the leaf of $\widetilde{\lambda}$ in the boundary of P_i that is closest to P_{i-1}, and let g_i^+ be the one that is closest to P_{i+1}. Then, $\widetilde{f}_{\mathcal{P}}$ coincides on P with

$$R^{\alpha(P_0,P_1)}_{\widetilde{f}_0(g_1^-)}R^{-\alpha(P_0,P_1)}_{\widetilde{f}_0(g_1^+)}R^{\alpha(P_0,P_2)}_{\widetilde{f}_0(g_2^-)}R^{-\alpha(P_0,P_2)}_{\widetilde{f}_0(g_2^+)}\cdots R^{-\alpha(P_0,P_n)}_{\widetilde{f}_0(g_n^+)}R^{\alpha(P_0,P_{n+1})}_{\widetilde{f}_0(g_{n+1}^-)}\widetilde{f}_0$$

if $P=P_{n+1}$ is in $\mathcal{P}$, and with

$$R^{\alpha(P_0,P_1)}_{\widetilde{f}_0(g_1^-)}R^{-\alpha(P_0,P_1)}_{\widetilde{f}_0(g_1^+)}R^{\alpha(P_0,P_2)}_{\widetilde{f}_0(g_2^-)}R^{-\alpha(P_0,P_2)}_{\widetilde{f}_0(g_2^+)}\cdots R^{-\alpha(P_0,P_n)}_{\widetilde{f}_0(g_n^+)}\widetilde{f}_0$$

if P is not in $\mathcal{P}$.

In particular, $\widetilde{f}_{\mathcal{P}}$ is obtained from $\widetilde{f}_0$ by bending it along those leaves of $\widetilde{\lambda}$ which are in the boundary of some plaque of $\mathcal{P}$. As a consequence, $\widetilde{f}_{\mathcal{P}}$ is a pleated surface with topological type $\widetilde{S}$ and with pleating locus $\widetilde{\lambda}$, although without any equivariance property with respect to a homomorphism $\pi_1(S)\to\mathrm{Isom}^+(\mathbb{H}^3)$.

We can measure a bending angle $\beta_{\mathcal{P}}(P,Q)$ of $\widetilde{f}_{\mathcal{P}}$ between the plaques P and Q. It is immediate from the definitions that $\beta_{\mathcal{P}}(P_0,P)=\beta_0(P_0,P)+\alpha(P_0,P)$ if $P\in\mathcal{P}$, and that $\beta_{\mathcal{P}}(P_0,P)=\beta_0(P_0,P)$ otherwise.

The general formula for $\beta_{\mathcal{P}}(P,Q)$ is a little more elaborate. Say that a plaque R is *between* P and Q if either it separates the interior of P from the interior of Q or it is equal to P or Q. Then, there is a unique plaque R_{P_0PQ} which is between P and Q, is between P and P_0 and is between Q and P_0. From the additivity property of transverse cocycles, it is immediate that $\gamma(P,Q)=\gamma(P_0,P)+\gamma(P_0,Q)-2\gamma(P_0,R_{P_0PQ})$ for every transverse cocycle γ for $\widetilde{\lambda}$. As a consequence:

LEMMA 25. — *If P, Q and R_{P_0PQ} are all in $\mathcal{P}$, then*

$$\beta_{\mathcal{P}}(P,Q)=\beta_0(P,Q)+\alpha(P,Q)\,.$$

We now let the finite set $\mathcal{P}$ tend to the set of all plaques of $\widetilde{S} - \widetilde{\lambda}$. Then, by definition of $\widetilde{f}_{\mathcal{P}}$, $\widetilde{f}$ and $\varphi_{P_0 P}$, $\widetilde{f}_{\mathcal{P}}$ converges to $\widetilde{f}$ on each plaque P. In addition, the estimates used to prove the convergence (namely arguments analogous to those of Lemmas 14 and 16) show that the convergence from $\widetilde{f}_{\mathcal{P}}$ to $\widetilde{f}$ is uniform on every compact set of $\widetilde{S}$. As a consequence, $\widetilde{f}$ has a continuous extension $\widetilde{f} : \widetilde{S} \to \mathbb{H}^3$.

Also, since $\widetilde{f}_{\mathcal{P}}$ is obtained from $\widetilde{f}_0$ by bending it along finitely many leaves of $\widetilde{\lambda}$, the pull back metric defined on $\widetilde{S}$ by $\widetilde{f}_{\mathcal{P}}$ is equal to m_0. It follows that the limit $\widetilde{f} : \widetilde{S} \to \mathbb{H}^3$ is a pleated surface, with pleating locus $\widetilde{\lambda}$, and that its pull back metric is still m_0 (see [CEG, § 5.2] for more details).

Define $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ by the property that $\rho(\gamma) = \psi_{P_0(\gamma P_0)}\rho_0(\gamma)$ for every $\gamma \in \pi_1(S)$. For every plaques P, Q, R, we saw that $\psi_{PQ} = \psi_{PR}\psi_{RQ}$, and it immediately follows from definitions that $\psi_{(\gamma P)(\gamma Q)} = \rho_0(\gamma)\psi_{PQ}\,\rho_0(\gamma)^{-1}$ for every $\gamma \in \pi_1(S)$. As a consequence, ρ is a group homomorphism, and $\widetilde{f}\gamma = \rho(\gamma)\widetilde{f}$ for every $\gamma \in \pi_1(S)$.

Therefore, $(\widetilde{f}, \rho)$ defines a pleated surface f with topological type S. By Lemma 22 and by the property that the convergence from $\widetilde{f}_{\mathcal{P}}$ to $\widetilde{f}$ is uniform on compact sets, the bending transverse cocycle β_f of $f = (\widetilde{f}, \rho)$ is equal to $\beta_0 + \alpha$. This achieves our goal to find a pleated surface with topological type S, with pull back metric m_0, and with bending transverse cocycle $\beta = \beta_0 + \alpha$. This proves:

THEOREM 26.— *For every hyperbolic metric $m \in \mathcal{T}(S)$ and every $\mathbb{R}/2\pi\mathbb{Z}$-valued cocycle β for the geodesic lamination λ on the surface S, there is a pleated surface $f = (\widetilde{f}, \rho)$ with topological type S with pull back metric m, with pleating locus λ, and with bending cocycle β.*

9. The pull back metric and the bending cocycle determine the pleated surface

In this section, we show that two pleated surfaces which have the same topological type, the same bending locus, the same pull back metric and the same bending transverse cocycle are equal. Of course, we first have to decide when we want to identify two pleated surfaces. The natural notion is to say that two pleated surfaces $f_1 = (\widetilde{f}_1, \rho_1)$ and $f_2 = (\widetilde{f}_2, \rho_2)$ with the same topological type S are *isomorphic* if there is a homeomorphism $\varphi : S \to S$ isotopic to the identity, a lift $\widetilde{\varphi} : \widetilde{S} \to \widetilde{S}$ of φ, and an isometry $\psi : \mathbb{H}^3 \to \mathbb{H}^3$ such that $\widetilde{f}_2 = \psi\widetilde{f}_1\widetilde{\varphi}$.

The main step in the proof is the following, which essentially proves this result when the pleated surfaces have bending transverse cocycle 0.

PROPOSITION 27.— *Let $f = (\widetilde{f}, \rho)$ be a pleated surface whose bending transverse cocycle is equal to 0. Then, $\widetilde{f}$ is a homeomorphism between $\widetilde{S}$ and a totally geodesic plane in $\mathbb{H}^3$.*

Proof.— Let P and Q be two plaques of $\widetilde{S} - \widetilde{\lambda}$. As usual, we consider the set $\widetilde{\lambda}_{PQ}$ of those leaves of $\widetilde{\lambda}$ which separate the interiors of P and Q, and the component Σ of $\widetilde{S} - \mathrm{Int}(P \cup Q)$ that separates the interiors of P and Q. As in section 7, there is a projection of Σ to a rectifiable arc γ in $\mathbb{H}^3_\infty = \mathbb{R}^2 \cup \infty$, and this curve is the union of a set of Hausdorff dimension 0 and of circle arcs, each corresponding to a component of $\Sigma - \widetilde{\lambda}_{PQ}$. For each such component W, let v_W be the unit tangent vector of the corresponding circle arc at its initial point, and let β_W be the integral of the curvature of this arc. Also, at the initial point of γ, let v_P be the outer unit tangent vector to P. Then, by definition of the bending transverse cocycle, the angle from v_P to v_W is equal to

$$\theta(v_P, v_W) = \sum_{P<W'<W} \beta_{W'} + \beta(P, P_W) = \sum_{P<W'<W} \beta_{W'},$$

where the sum is over all those components W' of $\Sigma - \widetilde{\lambda}_{PQ}$ which are between P and W, and where P_W is the plaque of $\widetilde{S} - \widetilde{\lambda}$ that is between P and Q and is contained in W.

From this formula, we conclude that γ admits a unit tangent vector v_x at each of its points x, and that this tangent vector depends continuously on x. Indeed, the angle $\theta(v_P, v_x)$ is equal to the integral of the curvature of γ (defined almost everywhere on γ) from the initial point of γ to x. Since the images under $\widetilde{f}$ of the components of $\Sigma - \widetilde{\lambda}_{PQ}$ meet a given compact subset of $\mathbb{H}^3$, the curvatures of the corresponding circle arcs is bounded. It follows that $\theta(v_P, v_x)$ is a Lipschitz function of x. In other words, the arc γ is of class $C^{1,1}$ when parametrized by arc length.

For every component W of $\Sigma - \widetilde{\lambda}_{PQ}$, $\widetilde{f}(W)$ is contained in a hyperbolic plane in $\mathbb{H}^3$, and therefore determines a euclidean circle in $\mathbb{H}^3_\infty = \mathbb{R}^2 \cup \infty$. If $y \in \Sigma - \widetilde{\lambda}_{PQ}$, there is a formula which gives the radius of the circle associated to the component containing y, in terms of $\widetilde{f}(y)$, of the projection x of $\widetilde{f}(y)$ to γ, and of the tangent vector v_x. In particular, this radius is a uniformly continuous function of y. Since this radius function is constant on each

wedge, we conclude that it is constant on Σ. If we arrange that one of these circles is a line passing through ∞, we conclude that each wedge is contained in a hyperbolic plane of $\mathbb{H}^3$ passing through ∞. If we go back to the formula for $\theta(v_P, v_W)$, we now see that this angle is constantly 0. As a consequence, all components W of $\Sigma - \widetilde{\lambda}_{PQ}$ have their images $\widetilde{f}(W)$ contained in the same hyperbolic plane $H \subset \mathbb{H}^3$.

Therefore, the image of $\widetilde{f}$ is contained in the hyperbolic plane H.

From the fact that the vectors v_W are constant, we conclude that $\widetilde{f} : \widetilde{S} \to H$ is "monotonic" on $\widetilde{\lambda}$ in the sense that, if the three leaves g, h, k of $\widetilde{\lambda}$ are such that h separates g from k, then $\widetilde{f}(h)$ separates $\widetilde{f}(g)$ from $\widetilde{f}(k)$. It follows that $\widetilde{f}$ is a homeomorphism onto its image. Since the pull back metric on $\widetilde{S}$ is complete, this image has to be all of H. □

THEOREM 28.— *Let $f_1 = (\widetilde{f}_1, \rho_1)$ and $f_2 = (\widetilde{f}_2, \rho_2)$ be two pleated surfaces which have the same topological type S, the same bending locus λ, the same pull back metric $m \in \mathcal{T}(S)$, and the same bending transverse cocycle $\beta \in \mathcal{H}(\lambda\,;\, \mathbb{R}/2\pi\mathbb{Z})$. Then, f_1 and f_2 are isomorphic.*

Proof.— Let us apply the process of section 8, and bend $\widetilde{f}_1$ and $\widetilde{f}_2$ along $\widetilde{\lambda}$ according to the transverse cocycle $-\beta$. This gives two new pleated surfaces $f_1' = (\widetilde{f}_1', \rho_1')$ and $f_2' = (\widetilde{f}_2', \rho_2')$ with pull back metric m, with pleating locus λ, and with bending transverse cocycle $\beta - \beta = 0$. Note that we can retrace our steps and that $\widetilde{f}_1$ and $\widetilde{f}_2$ are respectively obtained by bending $\widetilde{f}_1'$ and $\widetilde{f}_2'$ along λ according to the transverse cocycle β and following the same process of section 8. Therefore, it suffices to show that f_1' and f_2' are isomorphic.

By Proposition 27, each $\widetilde{f}_i'$ induces a homeomorphism between $\widetilde{S}$ and a totally geodesic plane H_i in $\mathbb{H}^3$. Composing $\widetilde{f}_i$ and $\widetilde{f}_i'$ with an isometry of $\mathbb{H}^3$ if necessary, we can assume that H_i is the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. Since the action of $\pi_1(S)$ on $\widetilde{S}$ is totally discontinuous, so is the action of $\rho_i'(\pi_1(S))$ on $\mathbb{H}^2$, and it follows that the homomorphism $\rho_i' : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2) \subset \mathrm{Isom}^+(\mathbb{H}^3)$ is discrete. Since $\widetilde{f}_1'$ and $\widetilde{f}_2'$ induce the same pull back metric m on $\widetilde{S}$, this immediately implies that f_1' and f_2' are isomorphic. □

10. The shear-bend complex cocycle

Consider a pleated surface $f = (\widetilde{f}, \rho)$ with pleating locus λ. Adding a few leaves if necessary, we can assume that λ is maximal.

We have associated to f a bending transverse cocycle $\beta_f \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$. But the pull back metric m of f also has a shearing transverse cocycle $\sigma_m \in \mathcal{H}(\lambda; \mathbb{R})$. We can combine these two transverse cocycles into the *shear-bend cocycle* $\Gamma_f = \sigma_m + i\beta_f \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$.

By Theorems 28 and 12, up to isomorphism, a pleated surface is characterized by its shear-bend cocycle Γ_f. Conversely, let $\mathcal{C}(\lambda)$ be the open convex cone appearing in Corollary 21, namely the cone consisting of those $\alpha \in \mathcal{H}(\lambda; \mathbb{R})$ such that $\tau(\alpha, \mu) > 0$ for every non-zero transverse measure μ for λ. Then Theorems 20 and 26 say that, in $\mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z}) = \mathcal{H}(\lambda; \mathbb{R}) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$, a $\mathbb{C}/2\pi i\mathbb{Z}$-valued cocycle is the shear-bend transverse cocycle of some pleated surface if and only if it is in $\mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$.

We will say that the geodesic lamination λ is *realized* by the homomorphism $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ if there exists a pleated surface $f = (\widetilde{f}, \rho)$ whose pleating locus contains λ. The following results are immediate extensions of classical properties of pleated surfaces in the case where ρ is discrete.

LEMMA 29.— *If the maximal geodesic lamination λ is realized by the homomorphism $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the pleated surface $f = (\widetilde{f}, \rho)$ is unique up to precomposition by the lift $\widetilde{\varphi} : \widetilde{S} \to \widetilde{S}$ of a homeomorphism of S that is isotopic to the identity.*

Proof.— Immediate extension of [Th1, § 8.10] or [CEG, § 5.3]. □

LEMMA 30.— *Given a maximal geodesic lamination λ, the set of those homomorphisms $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ which realize λ is open in the set of all representations $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$.*

Proof.— This is an immediate extension of arguments in [Th1, § 9.2]. If $f = (\widetilde{f}, \rho)$ realizes λ, it is possible to find a train track T carrying λ and to perturb $\widetilde{f}$ so that, if $\widetilde{T}$ is the preimage of T in $\widetilde{S}$, $\widetilde{f}(\widetilde{T})$ is a ρ-invariant train track graph of small curvature in $\mathbb{H}^3$. If ρ' is sufficiently close to ρ, $\widetilde{f}(\widetilde{T})$ can be deformed to a ρ'-invariant train track graph of small curvature. This provides a deformation of the restriction of $\widetilde{f}$ to $\widetilde{T}$ to a ρ'-equivariant map $\widetilde{g}$ sending each leaf of $\widetilde{\lambda}$ to a curve of small curvature. In particular, for every leaf h of $\widetilde{\lambda}$, there is a unique geodesic of $\mathbb{H}^3$ that stays at uniformly bounded distance from $\widetilde{g}(h)$. As in [Th1, § 8.10] or [CEG, § 5.3], this enables us to construct a pleated surface $(\widetilde{f}', r')$ realizing λ. □

By Lemma 29, there is a one-to-one correspondence between isomorphism classes of pleated surfaces with pleating locus λ and conjugacy classes of homomorphisms $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ realizing λ. (By definition, two such homomorphisms ρ and ρ' are in the same conjugacy class if there exists $\psi \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $\rho'(\gamma) = \psi\rho(\gamma)\psi^{-1}$ for every $\gamma \in \pi_1(S)$.)

Let $\mathcal{R}(\lambda)$ be the set of conjugacy classes of homomorphisms $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ realizing λ. By Lemma 30, $\mathcal{R}(\lambda)$ is open in the space of conjugacy classes of all homomorphisms $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$. Because $\mathrm{Isom}^+(\mathbb{H}^3)$ can be identified to the complex Lie group $\mathrm{PSL}_2(\mathbb{C})$, this space of conjugacy classes of homomorphisms has a natural structure of complex analytic manifold, except possibly near the reducible representations; see for instance [CuS]. If λ is a pleating locus for the pleated surface $(\widetilde{f}, \rho)$, the homomorphism ρ cannot be reducible. Indeed, if P is a plaque of $\widetilde{S} - \widetilde{\lambda}$, its image $\widetilde{f}(P)$ hits the sphere at infinity $\mathbb{H}^3_\infty$ in 3 distinct points, and it is possible to find $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(S)$ so that the $\gamma_i P$ are respectively close to each of these 3 points; then, the elements $\rho(\gamma_i\gamma_j^{-1})$ cannot have a common fixed point on $\mathbb{H}^3_\infty$, and therefore generate an irreducible subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. Therefore, the open subset $\mathcal{R}(\lambda)$ is in the manifold part of this representation space, and inherits a complex structure.

We want to show that the map $\rho \mapsto \Gamma_{(\widetilde{f},\rho)}$ is well behaved with respect to the complex structures of $\mathcal{R}(\lambda)$ and $\mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$.

THEOREM 31.— *The map $\mathcal{R}(\lambda) \to \mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$, which associates to each conjugacy class of homomorphism $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ realizing λ the shear-bend transverse cocycle $\Gamma_f \in \mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ of the corresponding pleated surface $f = (\widetilde{f}, \rho)$ with pleating locus λ, is a biholomorphic homeomorphism.*

Proof.— We will consider the inverse map $\mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z}) \to \mathcal{R}(\lambda)$. To show that this bijection is a biholomorphic homeomorphism, it suffices to show that it is a holomorphic map.

Let $\sigma_0 \in \mathcal{C}(\lambda)$. It is the shearing transverse cocycle of some hyperbolic metric $m_0 \in \mathcal{T}(S)$. If $\alpha \in \mathcal{H}(\lambda; \mathbb{R})$ is sufficiently small and if $\beta \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$, sections 5 and 8 provide an explicit pleated surface $(\widetilde{f}, \rho)$ whose shear-bend cocycle is $\sigma_0 + \alpha + i\beta$. We want to show that the homomorphism ρ depends holomorphically on $\Gamma = \alpha + i\beta$. For this, it will suffice to show that, for every $\gamma \in \pi_1(S)$, $\rho(\gamma)$ depends hololorphically on Γ.

Given an oriented geodesic g of $\mathbb{H}^3$ and given $z = u + iv \in \mathbb{C}/2\pi i\mathbb{Z}$, let $U_g^z = T_g^u R_g^v$ be the composition of the translation along g of amplitude $u \in \mathbb{R}$ and of the rotation around g of angle $v \in \mathbb{R}/2\pi\mathbb{Z}$. Note that the map $\mathbb{C}/2\pi i\mathbb{Z} \to \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$ defined by $z \mapsto U_g^z$ is holomorphic.

Choose an isometric identification between $(\widetilde{S}, m_0)$ and $\mathbb{H}^2 \subset \mathbb{H}^3$. The action of $\pi_1(S)$ on $\widetilde{S} = \mathbb{H}^2$ embeds $\pi_1(S)$ into $\mathrm{Isom}^+(\mathbb{H}^2) \subset \mathrm{Isom}^+(\mathbb{H}^3)$. Also, we can assume λ to be m_0-geodesic.

Fix a base plaque P_0 of $\widetilde{S} - \widetilde{\lambda}$. For any other plaque P, let $\mathcal{P}_{P_0P}$ denote as usual the set of those plaques of $\widetilde{S} - \widetilde{\lambda}$ which separate P_0 from P. Given a finite subset $\mathcal{P}$ of $\mathcal{P}_{P_0P}$, write its elements as $P_1, \ldots, P_m$ so that the index i of P_i increases as one progresses from P_0 to P, and set $P_{m+1} = P$. For every i, let g_i^- and g_i^+ be the geodesics in the boundary of P_i that are closest to P_{i-1} and P_{i+1}, respectively. Similarly, if we need another finite subset $\mathcal{Q}$ of $\mathcal{P}_{P_0P}$, we will write its elements as $Q_1, \ldots, Q_n$ so that the index j of Q_j increases as one progresses from $P_0 = Q_0$ to $P = Q_{n+1}$, and h_j^- and h_j^+ will be the geodesics in the boundary of Q_j that are closest to Q_{j-1} and Q_{j+1}, respectively.

In section 5, to construct a hyperbolic metric m with shearing cocycle $\sigma_0 + \alpha$, we considered a shear map $\widetilde{\varphi} : \widetilde{S} - \widetilde{\lambda} \to \mathbb{H}^2$ whose restriction to each plaque P of $\widetilde{S} - \widetilde{\lambda} = \mathbb{H}^2 - \widetilde{\lambda}$ is defined by

$$\widetilde{\varphi}_{|P} = \lim_{\mathcal{P} \to \mathcal{P}_{P_0P}} T_{g_1^-}^{\alpha(P_0,P_1)} T_{g_1^+}^{-\alpha(P_0,P_1)} T_{g_2^-}^{\alpha(P_0,P_2)} T_{g_2^+}^{-\alpha(P_0,P_2)} \cdots$$
$$\cdots T_{g_m^+}^{-\alpha(P_0,P_m)} T_{g_{m+1}^-}^{\alpha(P_0,P)} .$$

We also had a homomorphism $\rho_m : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2)$ defined by the property that

$$\rho_m(\gamma) = \lim_{\mathcal{P} \to \mathcal{P}_{P_0(\gamma P_0)}} T_{g_1^-}^{\alpha(P_0,P_1)} T_{g_1^+}^{-\alpha(P_0,P_1)} T_{g_2^-}^{\alpha(P_0,P_2)} T_{g_2^+}^{-\alpha(P_0,P_2)} \cdots$$
$$\cdots T_{g_m^+}^{-\alpha(P_0,P_m)} T_{g_{m+1}^-}^{\alpha(P_0,\gamma P_0)} \gamma$$

for every $\gamma \in \pi_1(S)$. The map $\widetilde{\varphi}$ and the homomorphism ρ are connected by the property that $\widetilde{\varphi}\gamma = \rho_m(\gamma)\widetilde{\varphi}$ for every $\gamma \in \pi_1(S)$. The metric m is the pull back metric of an (arbitrary) homeomorphism $\psi : S \to \mathbb{H}^2/\rho_m(\pi_1(S))$ admitting a lift $\widetilde{\psi} : \widetilde{S} \to \mathbb{H}^2$ such that $\widetilde{\psi}\gamma = \rho_m(\gamma)\widetilde{\psi}$ for every $\gamma \in \pi_1(S)$. In addition, the m-geodesic λ_m of S corresponding to λ is such that, if $\widetilde{\lambda}_m$ is its preimage in $\widetilde{S}$, then $\widetilde{\psi}(\widetilde{\lambda}_m)$ is exactly the complement of $\widetilde{\varphi}(\widetilde{S} - \widetilde{\lambda})$ in $\mathbb{H}^2$.

Considering the map $\widetilde{\psi} : \widetilde{S} \to \mathbb{H}^2$ as arriving in $\mathbb{H}^3$ and the homomorphism $\rho_m : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2)$ as arriving in $\mathrm{Isom}^+(\mathbb{H}^3)$, the pair $(\widetilde{\psi}, \rho_m)$ can be interpreted as a pleated surface with pull back metric m and with bending cocycle 0. If we use the methods of section 8 to bend this pleated surface according to the transverse cocycle β, we obtain a pleated surface $(\widetilde{f}, \rho)$ with bending cocycle β where

$$\rho(\gamma) = \lim_{\mathcal{Q} \to \mathcal{P}_{P_0(\gamma P_0)}} R^{\beta(P_0,Q_1)}_{\widetilde{\psi}(h_1^-)} R^{-\beta(P_0,Q_1)}_{\widetilde{\psi}(h_1^+)} R^{\beta(P_0,Q_2)}_{\widetilde{\psi}(h_2^-)} R^{-\beta(P_0,Q_2)}_{\widetilde{\psi}(h_2^+)} \cdots$$
$$\cdots R^{-\beta(P_0,Q_n)}_{\widetilde{\psi}(h_n^+)} R^{\beta(P_0,\gamma P_0)}_{\widetilde{\psi}(h_{n+1}^-)} \rho_m(\gamma)$$

for every $\gamma \in \pi_1(S)$. In particular,

$$\rho(\gamma) = \lim_{\substack{\mathcal{P},\mathcal{Q} \to \mathcal{P}_{P_0(\gamma P_0)} \\ \mathcal{Q} \subset \mathcal{P}}} R^{\beta(P_0,Q_1)}_{\widetilde{\psi}(h_1^-)} R^{-\beta(P_0,Q_1)}_{\widetilde{\psi}(h_1^+)} R^{\beta(P_0,Q_2)}_{\widetilde{\psi}(h_2^-)} R^{-\beta(P_0,Q_2)}_{\widetilde{\psi}(h_2^+)} \cdots$$
$$\cdots R^{-\beta(P_0,Q_n)}_{\widetilde{\psi}(h_n^+)} R^{\beta(P_0,\gamma P_0)}_{\widetilde{\psi}(h_{n+1}^-)}$$
$$T^{\alpha(P_0,P_1)}_{g_1^-} T^{-\alpha(P_0,P_1)}_{g_1^+} T^{\alpha(P_0,P_2)}_{g_2^-} T^{-\alpha(P_0,P_2)}_{g_2^+} \cdots$$
$$\cdots T^{-\alpha(P_0,P_m)}_{g_m^+} T^{\alpha(P_0,\gamma P_0)}_{g_{m+1}^-} \gamma \,.$$

Note that, for any two geodesics g, h, and any two numbers $a \in \mathbb{R}$, $b \in \mathbb{R}/2\pi\mathbb{Z}$, we have that $R^b_h T^a_g = T^a_g R_{T_g^{-a}h}$. Also, since $\mathcal{Q} \subset \mathcal{P}$, each Q_j is equal to some P_i, in which case $h_j^\pm = g_i^\pm$. Therefore, we can rewrite the above limit as

$$\rho(\gamma) = \lim_{\substack{\mathcal{P},\mathcal{Q} \to \mathcal{P}_{P_0(\gamma P_0)} \\ \mathcal{Q} \subset \mathcal{P}}} V_{P_1} V_{P_2} \cdots V_{P_{m+1}} \gamma \,,$$

where $V_{P_i} = T^{\alpha(P_0,P_i)}_{g_i^-} T^{-\alpha(P_0,P_i)}_{g_i^+}$ if P_i is equal to no Q_j, where

$$V_{P_i} = V_{Q_j} = T^{\alpha(P_0,Q_j)}_{h_j^-} R^{\beta(P_0,Q_j)}_{\widetilde{\psi}_j(h_j^-)} R^{-\beta(P_0,Q_j)}_{\widetilde{\psi}_j(h_j^+)} T^{-\alpha(P_0,Q_j)}_{h_j^+}$$

with

$$\widetilde{\psi}_j = T_{g_i^-}^{-\alpha(P_0,P_i)} T_{g_{i-1}^+}^{\alpha(P_0,P_{i-1})} \cdots T_{g_2^+}^{\alpha(P_0,P_2)} T_{g_2^-}^{-\alpha(P_0,P_2)} T_{g_1^+}^{\alpha(P_0,P_1)} T_{g_1^-}^{-\alpha(P_0,P_1)} \widetilde{\psi}$$

if $P_i = Q_j$ with $i \leq m$, and where

$$V_{P_{m+1}} = V_{Q_{n+1}} = T_{h_{n+1}^-}^{\alpha(P_0,Q_{n+1})} R_{\widetilde{\psi}_{n+1}(h_{n+1}^-)}^{\beta(P_0,Q_{n+1})}$$

with $\widetilde{\psi}_{n+1}$ defined as above.

By the estimates of the proof of Lemma 14, the contribution to the product $V_{P_1} V_{P_2} \cdots V_{P_m}$ of the terms $V_{P_i} = T_{g_i^-}^{\alpha(P_0,P_i)} T_{g_i^+}^{-\alpha(P_0,P_i)}$ with $P_i \in \mathcal{P} - \mathcal{Q}$ is uniformly small if $\mathcal{Q}$ is large enough. Therefore,

$$\rho(\gamma) = \lim_{\substack{\mathcal{P},\mathcal{Q} \to \mathcal{P}_{P_0(\gamma P_0)} \\ \mathcal{Q} \subset \mathcal{P}}} V_{Q_1} V_{Q_2} \cdots V_{Q_{n+1}} \gamma .$$

Fix $\mathcal{Q}$ and let $\mathcal{P}$ tend to $\mathcal{P}_{P_0(\gamma P_0)}$. By definition of $\widetilde{\psi}$, $\widetilde{\psi}(h_j^{\pm})$ is equal to the image of $h_j^{\pm}$ under the restriction $\widetilde{\varphi}_{|Q_j}$, suitably extended to the boundary of Q_j. By definition of $\widetilde{\varphi}_{|Q_j}$, it follows that $\widetilde{\psi}_j(h_j^{\pm})$ tends to $h_j^{\pm}$ as $\mathcal{P}$ tends to $\mathcal{P}_{P_0(\gamma P_0)}$. Therefore, V_{Q_j} tends to

$$T_{h_j^-}^{\alpha(P_0,Q_j)} R_{h_j^-}^{\beta(P_0,Q_j)} R_{h_j^+}^{-\beta(P_0,Q_j)} T_{h_j^+}^{-\alpha(P_0,Q_j)} = U_{h_j^-}^{\Gamma(P_0,Q_j)} U_{h_j^+}^{-\Gamma(P_0,Q_j)}$$

as $\mathcal{P}$ tends to $\mathcal{P}_{P_0(\gamma P_0)}$ if $j \leq n$. Similarly, $V_{Q_{n+1}}$ tends to

$$T_{h_{n+1}^-}^{\alpha(P_0,Q_{n+1})} R_{h_{n+1}^-}^{\beta(P_0,Q_{n+1})} = U_{h_{n+1}^-}^{\Gamma(P_0,Q_{n+1})} .$$

It follows that

$$\rho(\gamma) = \lim_{\mathcal{Q} \to \mathcal{P}_{P_0(\gamma P_0)}} U_{h_1^-}^{\Gamma(P_0,Q_1)} U_{h_1^+}^{-\Gamma(P_0,Q_1)} U_{h_2^-}^{\Gamma(P_0,Q_2)} U_{h_2^+}^{-\Gamma(P_0,Q_2)} \cdots$$
$$\cdots U_{h_m^+}^{-\Gamma(P_0,Q_m)} U_{h_{n+1}^-}^{\Gamma(P_0,\gamma P_0)} \gamma .$$

For each j, the map $\Gamma \mapsto U_{h_j^{\pm}}^{\mp\Gamma(P_0,Q_j)}$ is holomorphic. It follows that the map $\Gamma \mapsto \rho(\gamma)$ is holomorphic for every $\gamma \in \pi_1(S)$.

This completes the proof of Theorem 31. □

The proof of Theorem 31 extends to show that the image of each plaque of $\widetilde{S} - \widetilde{\lambda}$ under the pleated surface $(\widetilde{f}, \rho)$ depends holomorphically on the shear-bend cocycle $\Gamma \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ of this pleated surface. More precisely, fix a base plaque P_0, and select a preferred vertex v_0 of P_0. Then, for every plaque P of $\widetilde{S} - \widetilde{\lambda}$ with a preferred vertex v, there is a unique element $\rho(P, v) \in \mathrm{Isom}^+(\mathbb{H}^3)$ which sends $\widetilde{f}(P_0)$ to $\widetilde{f}(P)$ and $\widetilde{f}(v_0)$ to $\widetilde{f}(v)$, respecting orientations. For instance, $\rho(\gamma P_0, \gamma v_0) = \rho(\gamma)$ when $\gamma \in \pi_1(S)$. This $\rho(P, v)$ is a well-defined function of the shear-bend cocycle $\Gamma \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ up to conjugation in $\mathrm{Isom}^+(\mathbb{H}^3)$, and is well-defined as an element of $\mathrm{Isom}^+(\mathbb{H}^3)$ if we normalize $\widetilde{f}$ so that it sends P_0 to a fixed ideal triangle in $\mathbb{H}^3$. An automatic extension of the proof of Theorem 31 shows that $\rho(P, v)$ is a holomorphic function of Γ. Applying Theorem 31, we obtain the following corollary.

COROLLARY 32.— *For every plaque P of $\widetilde{S} - \widetilde{\lambda}$ and every vertex v of P, the element $\rho(P, v) \in \mathrm{Isom}^+(\mathbb{H}^3)$ defined above depends holomorphically on the representation $\rho \in \mathcal{R}(\lambda)$.*

11. The rotation number of the realization of a transverse cocycle

In an oriented hyperbolic 3-manifold M, a closed geodesic α has a well defined length $\ell_M(\alpha)$ and a well defined rotation number $\mathrm{rot}_M(\alpha) \in \mathbb{R}/2\pi\mathbb{Z}$, corresponding to the rotation angle of the holonomy around γ. There is actually a natural way to lift $\mathrm{rot}_M(\alpha)$ to a number $\mathrm{rot}_{M,v}(\alpha) \in \mathbb{R}$ if we are given a vector field v along α which is nowhere tangent to α.

We want to generalize this to the situation where α is a measured lamination on a surface S which can be realized in M by a pleated surface $f : S \to M$. Since tangent vectors to the space of measured laminations can be interpreted as geodesic laminations with transverse cocycles (see [Bo3]), we even want to generalize this to the case where α is a transverse cocycle for the pleating locus λ of a pleated surface $f : S \to M$. Regarding the length $\ell_M(\alpha)$ of the realization of α in M, it is natural to define it as the length $\ell_m(\alpha)$ of α in S with respect to the pull back metric m of the pleated surface f; see [Th1] and [Bo2] for instance. In this section, we show how to generalize the rotation number $\mathrm{rot}_M(\alpha)$ to this setting, and prove an unexpected connection between this number, the bending transverse cocycle, and Thurston's symplectic form.

More generally, consider a pleated surface $f = (\widetilde{f}, \rho)$ in the sense of section 7, with pleating locus the geodesic lamination λ on S. And let α be a transverse cocycle for λ.

Choose a differentiable (say) vector field v defined on a neighborhood of $\widetilde{f}(\widetilde{\lambda})$ which is transverse to $\widetilde{f}(\widetilde{\lambda})$ and is invariant under ρ. Although this corresponds to the intuitive idea we should have of v, such a vector field may not necessarily exist if ρ is not discrete. So, a more formal (and mathematically correct) definition of v is the following: The group $\pi_1(S)$ acts freely properly discontinuously on the product $\widetilde{S} \times \mathbb{H}^3$ by the covering action on the $\widetilde{S}$ factor and by ρ on the $\mathbb{H}^3$ factor. Then v is a differentiable map defined on a neighborhood of the graph of $\widetilde{f}$ over $\widetilde{\lambda}$ and associates to each (x, y) in this neighborhood a vector $v(x, y) \in T_y\mathbb{H}^3$ such that the maps $x \mapsto v(x, y)$ are locally constant. In addition, $v(x, \widetilde{f}(x))$ is transverse to $\widetilde{f}(g)$ if x belongs to the leaf g of $\widetilde{\lambda}$, and v is invariant under the action of $\pi_1(S)$. Such a v can easily be constructed by considering the quotient manifold $\widetilde{S} \times \mathbb{H}^3/\pi_1(S)$. When there is no ambiguity about the x-neighborhood we are talking about, we will often write $v(y)$ for $v(x, y)$ since $x \mapsto v(x, y)$ is locally constant.

Let U be a neighborhood of λ, small enough so that v is defined on the graph of $\widetilde{f}$ over the preimage $\widetilde{U} \subset \widetilde{S}$ of U. As in section 3, let $\widehat{\lambda} \to \lambda$ be the orientation covering of λ, and extend it to a covering $\widehat{U} \to U$ (assuming that U avoids at least one point of each component of $S - \lambda$).

If w is a vector tangent to $\widehat{\lambda}$, let $\widehat{\omega}(w)$ be the rotation speed of v around $\widetilde{f}(\widetilde{\lambda})$ in the direction of w. Namely, lift w to a vector $\widetilde{w}$ tangent to $\widetilde{\lambda}$ in $\widetilde{S}$ and based at $\widetilde{x} \in \widetilde{S}$. Note that $\widehat{\lambda}$ is canonically oriented, so that the base point of w determines an orientation of the leaf g of $\widetilde{\lambda}$ containing $\widetilde{x}$. Choose a parallel vector field p along g which is orthogonal to g, orient the normal plane of g at $\widetilde{x}$ so that this orientation followed by the orientation of g gives the orientation of $\mathbb{H}^3$, and let θ be the angle from p to the projection of v to this normal plane. Then, $\widehat{\omega}(w)$ is the directional derivative of θ in the direction of w.

This $\widehat{\omega}$ defines a 1-form along the leaves of $\widehat{\lambda}$, which is locally the differential of the function θ. Therefore, we can extend it to a closed Lipschitz differential 1-form $\widehat{\omega} \in \Omega^1_{\mathrm{Lip}}(\widehat{U})$.

Now, if $\alpha \in \mathcal{H}(\lambda; \mathbb{R})$ is a transverse cocycle for λ, we define the *rotation number of α with respect to ρ and v* to be the real number

$$\mathrm{rot}_{\rho,v}(\alpha) = \frac{1}{2} \int_{\widehat{\alpha}} \omega \in \mathbb{R}\,,$$

where $\widehat{\alpha} \in H_1(\widehat{U};\mathbb{R})$ is the class associated to α in section 3. Note that this number is also obtained by locally integrating $\widehat{\omega}$ along the leaves of $\widehat{\lambda}$, and then integrating this with respect to the transverse Hölder distribution corresponding to α. Since the integral of a function with respect to a Hölder distribution depends only on the restriction of the function to support of the distribution [Bo4, Support Lemma 1], $\mathrm{rot}_{\rho,v}(\alpha)$ is independent of the extension of $\widehat{\omega}$ from $\widehat{\lambda}$ to $\widehat{U}$. On the other hand, $\mathrm{rot}_{\rho,v}(\alpha)$ does depend on v.

Note that, when α is the Dirac transverse measure of weight $a > 0$ associated to a closed leaf of λ and when ρ is discrete, $\mathrm{rot}_{\rho,v}(\alpha)$ is exactly a times the rotation number of the corresponding closed geodesic of $\mathbb{H}^3/\rho$ with respect to the transverse vector field v. Before going any further, let us show that $\mathrm{rot}_{\rho,v}(\alpha)$ also has a geometric significance in the general case.

PROPOSITION 33.— *Consider a family of measured laminations $\alpha_t \in \mathcal{ML}(S)$ defined for every t in some set of real numbers admitting 0 as an accumulation point, and assume that α_t converges to $\alpha_0 \in \mathcal{ML}(S)$ as t tends to 0. In addition assume that, as t tends to 0, the Hausdorff limit λ of the supports of the α_t exists and is a pleating locus for the pleated surface $f = (\widetilde{f}, \rho)$. Choose a ρ-invariant vector field v transverse to $\widetilde{f}(\widetilde{\lambda})$, as defined above. Then, $\mathrm{rot}_{\rho,v}(\alpha_0)$ is the limit of $\mathrm{rot}_{\rho,v}(\alpha_t)$ as t tends to 0. In addition, if $t \mapsto \alpha_t$ has a tangent vector $\dot{\alpha}_0$ at $t = 0$, interpreted as a transverse cocycle for λ, then the map $t \mapsto \mathrm{rot}_{\rho,v}(\alpha_t)$ has derivative $\mathrm{rot}_{\rho,v}(\dot{\alpha}_0)$ at $t = 0$.*

Proof.— In [Bo3, Theorem 29], we proved a very similar result (with slightly higher generality) for the length function $\alpha \mapsto \ell_\rho(\alpha)$. The crux of the argument was that $\ell_\rho(\alpha)$ is obtained by locally integrating a certain differential form along the leaves of λ, and then integrating this with respect to the transverse Hölder distribution corresponding to α. Since $\alpha \mapsto \mathrm{rot}_{\rho,v}(\alpha)$ is defined by a similar construction, the proof of [Bo3, Theorem 29] carries over to prove Proposition 33. □

In Proposition 33, the condition on the existence of a Hausdorff limit is relatively mild, and is for instance always satisfied if $t \mapsto \alpha_t$ is a piecewise linear path in $\mathcal{ML}(S)$; see [Bo3, sect. 3], for instance.

Approximating a measured lamination α by Dirac transverse measures for simple closed geodesics, Proposition 33 shows that our definition of $\mathrm{rot}_{\rho,v}(\alpha)$ is the only one that makes this function continuous and agrees with the geometric definition for closed geodesics. Similarly, the rotation

number function on the space of transverse cocycle is the differential of the same function on measured laminations.

A natural question is to ask how much the rotation angle $\mathrm{rot}_{\rho,v}(\alpha)$ depends on the transverse vector field v. Let us take another such ρ-invariant vector field v' transverse to $\widetilde{f}(\widetilde{\lambda})$, defined as above on a neighborhood of the graph of $\widetilde{f}$ in $\widetilde{S} \times \mathbb{H}^3$. There is an obstruction $o(v, v')$ to deform v to v' through a path of such vector fields, defined as follows.

Let U be a small neighborhood of λ in S. Pulling back the tangent bundle of $\mathbb{H}^3$ under $\widetilde{f}$, we get by invariance of $\widetilde{f}$ under ρ a 3-dimensional vector bundle $E \to U$. The ρ-invariant vector fields v and v' define sections of E. Extend the orientation covering $\widehat{\lambda} \to \lambda$ to a covering $\widehat{U} \to U$, and lift E to a bundle $\widehat{E} \to \widehat{U}$. Again, v and v' provide two sections of this bundle, which we will still denote by v and v'. For every $\widehat{x} \in \widehat{\lambda}$ and $\widetilde{x} \in \widetilde{S}$ lifting x, consider the unit tangent vector $\Lambda(\widetilde{x})$ of $\widetilde{f}(\widetilde{\lambda})$ at $\widetilde{f}(\widetilde{x})$, for the local orientation of $\widetilde{\lambda}$ specified by $\widehat{x}$; this defines a section Λ of $\widehat{E}$ above $\widehat{\lambda}$. Extend Λ to a section over all of $\widehat{U}$. If U is sufficiently small, Λ is everywhere transverse to the sections v and v'.

In each fiber of $\widehat{E}$, orient the subspace orthogonal to Λ so that this orientation followed by the orientation of Λ gives the orientation coming from the orientation of $\mathbb{H}^3$. Let $a_{v,v'} : \widehat{U} \to \mathbb{R}/2\pi\mathbb{Z}$ be the angle from the projection of v to the projection of v' in this orthogonal subspace. Then, consider the cohomology class $o(v, v') \in H^1(\widehat{U}; 2\pi\mathbb{Z})$ defined by the differential $\mathrm{d}a_{v,v'}$.

The following is immediate from definitions.

PROPOSITION 34.— *With the above data,*

$$\mathrm{rot}_{\rho,v'}(\alpha) - \mathrm{rot}_{\rho,v}(\alpha) = \frac{1}{2}\int_{\widehat{\alpha}} \mathrm{d}a_{v,v'} = \frac{1}{2}\left[o(v, v')\,,\, \widehat{\alpha}\right].$$

By taking neighborhoods U of λ which are smaller and smaller, there is a way to make this obstruction $o(v, v')$ independent of the choices made by interpreting it as an element of the Čech cohomology group $\check{H}^1(\widehat{\lambda}; 2\pi\mathbb{Z})$, but we will not need this.

We now relate the rotation number $\mathrm{rot}_{\rho,v}(\alpha)$ to the bending cocycle of a pleated surface $f = (\widetilde{f}, \rho)$ realizing λ. Increasing λ to the bending locus of f, and then increasing it more if necessary, we can assume that the geodesic lamination λ is maximal without loss of generality. We will require this condition to be satisfied for the rest of this section.

Consider ρ' close to ρ and connected to ρ by a small homotopy. By the proof of Lemma 30, the pleated surface $f' = (\widetilde{f}', \rho')$ realizing λ depends continuously on ρ' uniformly on compact subsets, provided it is suitably normalized; this can also be seen using section 10 and the explicit constructions of sections 5 and 8. In particular, using the equivariance property under the action of $\pi_1(S)$, v is still transverse to $\widetilde{f}'(\widetilde{\lambda})$ if ρ' is close enough to ρ. In this case, the rotation number $\mathrm{rot}_{\rho',v}(\alpha)$ is still defined.

Also, if v' is another vector field transverse to $\widetilde{f}(\widetilde{\lambda})$ and equivariant with respect to ρ, the obstruction $o(v, v') \in H^1(\widehat{U}; 2\pi\mathbb{Z})$ is a continuous function of ρ, and is therefore locally constant. By Proposition 34, it follows that the difference $\mathrm{rot}_{\rho',v}(\alpha) - \mathrm{rot}_{\rho,v}(\alpha)$ does not depend on the vector field v if ρ' is sufficiently close to ρ.

Let β, $\beta' \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ be the bending transverse cocycles of the pleated surfaces f, f', respectively. The short homotopy from ρ to ρ' gives a path from β to β', and provides a way to lift the difference $\beta' - \beta \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ to an $\mathbb{R}$-valued cocycle $\Delta\beta \in \mathcal{H}(\lambda; \mathbb{R})$.

We want to relate the change in rotation angle to this cocycle $\Delta\beta$.

THEOREM 35.— *If ρ' is sufficiently close to ρ, then*

$$\mathrm{rot}_{\rho',v}(\alpha) - \mathrm{rot}_{\rho,v}(\alpha) = \tau(\alpha, \Delta\beta)\,.$$

We will base the proof of Theorem 35 on the following formula for the bending cocycle β.

Let k be an oriented arc transverse to λ in U, which we will identify to one of its lifts in $\widetilde{S}$. For each component d of $k - \lambda$, let x_d^+ and x_d^- be the positive and negative end points of d, respectively. In $\mathbb{H}^3$, we now have two vectors at $\widetilde{f}(x_d^\pm)$. One is v, the other one is the normal n to the image under $\widetilde{f}$ of the plaque P of $\widetilde{S} - \widetilde{\lambda}$ containing d, oriented so that the orientation of $\widetilde{f}(P)$ (coming from the orientation of S) followed by the orientation of n gives the orientation of $\mathbb{H}^3$. If $x_d^\pm$ is not one of the end points of k, it belongs to a leaf g of $\widetilde{\lambda}$, which we orient from right to left with respect to k. Finally, we orient the normal plane of $\widetilde{f}(g)$ at $\widetilde{f}(x_d^\pm)$ so that its orientation followed by the orientation of $\widetilde{f}(g)$ gives the orientation of $\mathbb{H}^3$, and we let $a_{n,v}(x_d^\pm)$ be the angle from n to the projection of v to this normal plane. This angle $a_{n,v}(x_d^\pm) \in \mathbb{R}/2\pi\mathbb{Z}$ is clearly independent of the choice of the lift of k to $\widetilde{S}$, by ρ-equivariance of v and $\widetilde{f}$. However, there is a definite abuse of notation since $a_{n,v}(x_d^\pm)$ depends also on d; in particular, for two adjacent components d, d', we can have $a_{n,v}(x_d^+) \neq a_{n,v}(x_{d'}^-)$ even if $x_d^+ = x_{d'}^-$.

LEMMA 36. — *With the above data,*

$$\beta(k) = a_{n,v}(x_{d_-}^+) - a_{n,v}(x_{d_+}^-) + \sum_{d \neq d_+, d_-} \left(a_{n,v}(x_d^+) - a_{n,v}(x_d^-)\right) \in \mathbb{R}/2\pi\mathbb{Z}$$

where the sum is over all those components d of $k - \lambda$ which are different from the components d_+, d_- respectively containing the positive and negative end points of k.

Proof. — First, we have to make sense of this sum since the $a_{n,v}(x_d^\pm)$ are defined only modulo 2π. However, if d is small enough, the two angles $a_{n,v}(x_d^+)$ and $a_{n,v}(x_d^-)$ are close to each other, and their difference can be interpreted as a real number bounded by a constant times the length of d, since v is differentiable and $\widetilde{f}$ is isometric. Therefore, all but finitely many terms in the sum can be interpreted as real numbers, and their sum is convergent.

The formula is straightforward when $k \cap \lambda$ is finite. The general case follows from this one by locally approximating $\widetilde{f}$ by a pleated surface with finite pleating locus as in section 7, using Lemma 22 and the above convergence estimate. □

Proof of Theorem 35 in a special case. — We first restrict attention to the case where the pleated surfaces $f = (\widetilde{f}, \rho)$ and $f' = (\widetilde{f}', \rho')$ with pleating locus λ have the same pull back metric m. We could do the general case right away by the same methods, but it will be technically easier to do this case first, and then to deduce the general case from this one by an indirect argument.

The proof follows the line of the proof of Theorem 9. To alleviate the exposition we decide that, if a certain symbol represents a mathematical object associated to ρ, the same symbol with a prime $'$ represents the corresponding object associated to ρ'.

Let k be an oriented arc in $\widehat{U}$ which is transverse to $\widehat{\lambda}$ and such that the orientation of $\widehat{\lambda}$ always crosses k from right to left. Identifying k to its projection to U, Lemma 36 gives that

$$\beta(k) = a_{n,v}(x_{d_-}^+) - a_{n,v}(x_{d_+}^-) + \sum_{d \neq d_+, d_-} \left(a_{n,v}(x_d^+) - a_{n,v}(x_d^-)\right)$$

and

$$\beta'(k) = a'_{n,v}(x_{d_-}^+) - a'_{n,v}(x_{d_+}^-) + \sum_{d \neq d_+, d_-} \left(a'_{n,v}(x_d^+) - a'_{n,v}(x_d^-)\right) .$$

The short homotopy connecting ρ to ρ' provides a path from each $a_{n,v}(x_d^{\pm})$ to the corresponding $a'_{n,v}(x_d^{\pm})$. In particular, although $a_{n,v}(x_d^{\pm})$ and $a'_{n,v}(x_d^{\pm})$ are only defined modulo 2π, this homotopy enables us to define their difference as a real number $\Delta a_{n,v}(x_d^{\pm}) = a'_{n,v}(x_d^{\pm}) - a_{n,v}(x_d^{\pm}) \in \mathbb{R}$. Therefore,

$$\Delta\beta(k) = \Delta a_{n,v}(x_{d_-}^+) - \Delta a_{n,v}(x_{d_+}^-) + \sum_{d\neq d_+,\, d_-} \left(\Delta a_{n,v}(x_d^+) - \Delta a_{n,v}(x_d^-)\right).$$

Without loss of generality, we can assume that each component W of $U-\lambda$ is an annulus, bounded on one side by 3 leaves of λ. The preimage $\widehat{W}$ of W in $\widehat{U}$ then is an annulus bounded on one side by 6 leaves of $\widehat{\lambda}$ with alternating orientations. The periods of ω on W can be computed by integrating ω along the union of these 6 geodesics, defined using small jumps and a limiting process as in the proof of Lemma 10. To integrate ω along this cycle of 6 geodesics, we need a parallel vector field along the corresponding leaves of $\widetilde{f}(\widetilde{\lambda})$. A natural choice is to use the normal vector n of a plaque corresponding to W. This shows that ω coincides along this cycle with the differential $\mathrm{d}a_{n,v}$ where, if x belongs to one of the leaves g of $\widehat{\lambda}$ in the boundary of $\widehat{W}$, $a_{n,v}(x)$ is the angle from n to the normal part of v with respect to g, measured using the orientation of g as above.

For each such x, the angle $a_{n,v}(x)$ is only defined modulo 2π. However, the short homotopy connecting ρ to ρ' provides a path from $a_{n,v}(x)$ to $a'_{n,v}(x)$, and therefore defines a real number $\Delta a_{n,v}(x) = a'_{n,v}(x) - a_{n,v}(x) \in \mathbb{R}$. In particular, ω and ω' have the same periods, and there is a function f on $\widehat{W}$ such that $\Delta\omega = \omega' - \omega = \mathrm{d}f$. In addition, we can choose f so that its continuous extension coincides with $\Delta a_{n,v}$ on the boundary leaves of $\widehat{W}$.

Now,

$$\begin{aligned}
\Delta\beta(k) &= \Delta a_{n,v}(x_{d_-}^+) - \Delta a_{n,v}(x_{d_+}^-) + \sum_{d\neq d_+,\, d_-} \left(\Delta a_{n,v}(x_d^+) - \Delta a_{n,v}(x_d^-)\right) \\
&= f(x_{d_-}^+) - f(x_{d_+}^-) + \sum_{d\neq d_+,\, d_-} \left(f(x_d^+) - f(x_d^-)\right) \\
&= -f(x_k^+) + f(x_k^-) + \sum_d \left(f(x_d^+) - f(x_d^-)\right) \\
&= -f(x_k^+) + f(x_k^-) + \int_k \mathrm{d}f \\
&= -f(x_k^+) + f(x_k^-) + \int_k \Delta\omega\,.
\end{aligned}$$

where x_k^+ and x_k^- are the positive and negative end points of k, respectively.

Note that we would get the opposite sign if the orientation of $\widehat{\lambda}$ crossed k from left to right. We can combine the two cases by considering the current $\Delta\widehat{\beta} = \widehat{\beta}' - \widehat{\beta}$ in $\widehat{U}$. Then,

$$k \cdot \Delta\widehat{\beta} = -f(x_k^+) + f(x_k^-) + \int_k \Delta\omega\,,$$

and, by additivity, the formula holds for every arc k transverse to $\widehat{\lambda}$, without condition on the crossing orientation.

By cancellation of the boundary terms, we conclude that

$$c \cdot \Delta\widehat{\beta} = \int_c \Delta\omega$$

for every $c \in H_1(\widehat{U};\mathbb{R})$. In particular,

$$\mathrm{rot}_{\rho',v}(\alpha) - \mathrm{rot}_{\rho,v}(\alpha) = \frac{1}{2}\int_{\widehat{\alpha}} \Delta\omega = \frac{1}{2}\,\widehat{\alpha} \cdot \Delta\widehat{\beta} = \tau(\alpha, \Delta\beta)\,.$$

This concludes the proof of Theorem 35 in the case where the pleated surfaces $f = (\widetilde{f}, \rho)$ and $f' = (\widetilde{f}', \rho')$ have the same pull back metric. □

Proof of Theorem 35 in the general case.— By Proposition 1, there is a path $t \mapsto \beta_t \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ such that $\beta_0 = \beta$ and $\beta_1 \in \mathcal{H}(\lambda; \{0, \pi\})$. Let ρ_t correspond to the bending cocycle β_t and to the same pull back metric as ρ, and let ρ'_t correspond to the bending cocycle β_t and to the same pull back metric as ρ'. Note that ρ'_t is uniformly close to ρ_t if ρ' is close to ρ.

Cutting the interval $[0\,,\,1]$ into small pieces and using the special case already proved,

$$\begin{aligned}\mathrm{rot}_{\rho',v}(\alpha) - \mathrm{rot}_{\rho,v}(\alpha) &= \mathrm{rot}_{\rho',v}(\alpha) - \mathrm{rot}_{\rho'_0,v}(\alpha) + \mathrm{rot}_{\rho'_0,v}(\alpha) - \mathrm{rot}_{\rho_0,v}(\alpha)\\ &= \tau(\alpha, \Delta\beta) + \mathrm{rot}_{\rho'_1,w}(\alpha) - \mathrm{rot}_{\rho_1,w}(\alpha)\end{aligned}$$

for any vector field w for which the formula makes sense. Since β_1 takes only the values 0 and π, the pleated surfaces corresponding to ρ_1 and ρ'_1 have their images contained in a hyperbolic plane in $\mathbb{H}^3$. If we take w to be the normal vector field to this plane, we have $\mathrm{rot}_{\rho'_1,w}(\alpha) = \mathrm{rot}_{\rho_1,w}(\alpha) = 0$, which completes the proof. □

Theorem 35 provides a way to get rid of the choice of a vector field v, by considering the differential of the function $\rho \mapsto \mathrm{rot}_{\rho,v}(\alpha)$. Indeed, if $t \mapsto \rho_t$ is a path with tangent vector $\dot\rho_0$ at $t = 0$, Theorem 35 shows that the function $t \mapsto \mathrm{rot}_{\rho_t,v}(\alpha)$ has derivative $\tau(\alpha, \dot\beta_0)$, where $\dot\beta_0$ is the tangent vector to the corresponding path of bending cocycles. Note that his quantity does not depend on the choice of v.

This enables us to associate to each transverse cocycle α for λ a *rotation form* rot_α on the manifold $\mathcal{R}(\lambda)$ of those representations $\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$ realizing λ. This is the closed differential 1-form which associates to each tangent vector $\dot\rho$ to $\mathcal{R}(\lambda) \cong \mathcal{T}(S) \times \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ the number

$$\mathrm{rot}_\alpha(\dot\rho) = \tau(\alpha, \dot\beta)$$

where $\dot\beta$ is the tangent vector to the space of bending cocycles corresponding to $\dot\rho$.

We can combine this with the analysis of section 3, and complexify the situation. Define the *complex length of α with respect to ρ and v* as the complex number $L_{\rho,v}(\alpha) = \ell_\rho(\alpha) + i\,\mathrm{rot}_{\rho,v}(\alpha)$. By Proposition 34 and/or Theorem 35, the differential of this function is independent of v, and defines a closed differential 1-form L_α on $\mathcal{R}(\lambda) \cong \mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z}) \subset \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$. By Theorems 9 and 35, this differential is connected to the Thurston form by the following formula: If $\dot\rho$ is a tangent vector and if $\dot\Gamma \in \mathcal{H}(\lambda; \mathbb{C})$ is the corresponding tangent vector to the space of shear bend cocycles, then

$$L_\alpha(\dot\rho) = \tau(\alpha, \dot\Gamma)\,.$$

In particular, L_α is a closed holomorphic 1-form on $\mathcal{R}(\lambda)$, called the *complex length form* associated to α.

A particular case of interest is when the homomorphism ρ is quasi-Fuchsian or, more generally, when it is in the closure of the space of quasi-Fuchsian homomorphisms. Then, there is a preferred homotopy class of paths in $\mathcal{R}(\lambda)$ connecting ρ to Fuchsian representations. Integrating the 1-form L_α along such a path, we get a well-defined complex length $L_\rho(\alpha) \in \mathbb{C}$ associated to α. It is easy to check that this complex length function coincides with the one used in [McM] and [KeS], for instance.

12. The non-orientable and bounded cases

So far, we have assumed that the surface S was oriented and without boundary. Also, for pleated surfaces $(\widetilde{f}, \rho)$, we have restricted attention to the case where the homomorphism ρ arrives in the group of orientation-preserving isometries of $\mathbb{H}^3$. In this section, we briefly indicate how to lift these restrictions.

12.1 Non-orientable hyperbolic surfaces

First consider the case where S is a connected compact surface without boundary, possibly non-orientable. Assume the Euler characteristic of S negative, and let λ be a maximal geodesic lamination of S.

In the definition of the shearing cocycle σ_m of a hyperbolic metric m in the orientable case, we used the orientation of $\widetilde{S}$ on the plaque P of $\widetilde{S} - \widetilde{\lambda}$ to determine the sign of $\sigma_m(P, Q)$. It therefore makes sense to consider the orientation covering $\widehat{\widetilde{S}}$ of $\widetilde{S}$. Since $\widetilde{S}$ is orientable, $\widehat{\widetilde{S}}$ consists of two copies of $\widetilde{S}$ but its fundamental feature is that it carries a canonical orientation. The action of $\pi_1(S)$ on $\widetilde{S}$ canonically lifts to $\widehat{\widetilde{S}}$, in such a way that $\gamma \in \pi_1(S)$ exchanges the two components of $\widehat{\widetilde{S}}$ if an only if γ is orientation reversing.

Let $\varepsilon_S : \pi_1(S) \to \mathbb{Z}/2 = \{-1, +1\}$ be the orientation homomorphism. Let $\widehat{\widetilde{\lambda}}$ be the preimage of λ in $\widehat{\widetilde{S}}$. An $\mathbb{R}$-*valued* ε_S-*twisted transverse cocycle* for λ is a map α associating a number $\alpha(P, Q) \in \mathbb{R}$ to each pair of plaques P, Q of $\widehat{\widetilde{S}} - \widehat{\widetilde{\lambda}}$ which are contained in the same component of $\widehat{\widetilde{S}}$, and such that α satisfies the following properties:

(i) α is symmetric, namely $\alpha(Q, P) = \alpha(P, Q)$ for every P, Q;

(ii) α is additive, namely $\alpha(P, Q) = \alpha(P, R) + \alpha(R, Q)$ whenever the plaque R separates P from Q;

(iii) if ω is the canonical involution of $\widehat{\widetilde{S}}$ exchanging its two components and if $\gamma \in \pi_1(S)$, $\alpha(\omega P, \omega Q) = -\alpha(P, Q)$ and $\alpha(\gamma P, \gamma Q) = \varepsilon_S(\gamma)\alpha(P, Q)$ for every plaques P, Q.

This definition of ε_S-twisted transverse cocycles can also be translated in terms of arcs transverse to λ in k and in terms of the coefficient bundle $\widehat{\widetilde{S}} \times \mathbb{R}/(\pi_1(S) \times \mathbb{Z}/2)$, where $\pi_1(S)$ acts on $\widehat{\widetilde{S}}$ by the canonical action and on

$\mathbb{R}$ by ε_S and where $\mathbb{Z}/2$ acts on $\widehat{\widetilde{S}}$ by the identity and on $\mathbb{R}$ by multiplication by ± 1.

The definition is specially tailored so that section 2 associates to a hyperbolic metric m on S an $\mathbb{R}$-valued ε_S-twisted transverse cocycle σ_m for λ, called the *shearing cocycle* of m.

Let $\mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ be the space of $\mathbb{R}$-valued ε_S-twisted transverse cocycles for λ. By the methods of [Bo4, sect. 4] and [PeH, § 2.1], this is a vector space of dimension $3|\chi(S)|$. (For a general geodesic lamination λ, the dimension of $\mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ is equal to the sum of $|\chi(\lambda)|$ and of the number of those components of λ which are transversely orientable.) As in section 3, there is a pairing $\tau : \mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R}) \to \mathbb{R}$ such that, for every Hölder distribution $\alpha \in \mathcal{H}(\lambda;\mathbb{R})$ and every hyperbolic metric m, the length $\ell_m(\alpha)$ is equal to $\tau(\alpha, \sigma_m)$.

Then, the map $m \mapsto \sigma_m$ defines a homeomorphism from $\mathcal{T}(S)$ to an open subset of $\mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$. The image of this map is a cone bounded by finitely many faces, and consists of all those $\alpha \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ such that $\tau(\mu,\alpha) > 0$ for every non-trivial transverse measure μ for λ. The proofs are identical to those of sections 4-6.

For a pleated surface $f = (\widetilde{f}, \rho)$ with pleating locus λ, where ρ is a homomorphism $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the bending locus $\beta_f \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ is defined as in section 7 and the results of sections 7-11 immediately extend to this context.

21.2 Hyperbolic surfaces with totally geodesic boundary

If we want to allow the (compact connected) surface S to have non-empty boundary, there are at least two natural extensions of the space $\mathcal{T}(S)$. A first possibility is to consider on S hyperbolic metrics for which the boundary ∂S is totally geodesic (assuming the Euler characteristic $\chi(S)$ to be negative, to guarantee the existence of such metrics). Let $\mathcal{T}_g(S)$ be the space of isotopy classes of such metrics. This space is homeomorphic to an open ball of dimension $3|\chi(S)|$.

If S is endowed with a hyperbolic metric with totally geodesic boundary, a maximal geodesic lamination λ must contain all of the boundary ∂S. The results of sections 2-6 automatically extend to this situation: There is a well-defined shearing cocycle $\sigma_m \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ associated to each hyperbolic metric m with totally geodesic boundary on S. The map $m \mapsto \sigma_m$ defines a homeomorphism from $\mathcal{T}_g(S)$ to its image in $\mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$. The image of this

map can be expressed in terms of the pairing $\tau : \mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R}) \to \mathbb{R}$, and consists of all those $\alpha \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ such that $\tau(\mu,\alpha) > 0$ for every non-trivial transverse measure μ for λ. In particular, this image is an open cone bounded by finitely many faces.

For a pleated cocycle $f = (\widetilde{f},\rho)$ with pleating locus λ, where ρ is a homomorphism $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the bending locus $\beta_f \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ is defined as in section 7 and the results of sections 7-11 immediately extend to this context.

12.3 Hyperbolic surfaces with cusps

Another option for a surface S with boundary is to consider finite area complete hyperbolic metrics on the interior $\mathrm{Int}(S)$ of S. The ends of $\mathrm{Int}(S)$ then correspond to cusps. Let $\mathcal{T}_c(S)$ denote the space of isotopy classes of such metrics.

In this case, a geodesic lamination λ is contained in $\mathrm{Int}(S)$. It can be shown that it has only finitely many leaves going to the cusps. If, in addition, λ is maximal, we associate to each metric $m \in \mathcal{T}_c(S)$ a shearing cocycle $\sigma_m \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ as in the previous cases. However, the completeness of the metric imposes a new condition. Namely, for each cusp, the integral of σ_m over a curve transverse to λ and going once around the cusp is equal to 0. Let $\mathcal{H}^0(\lambda;\mathbb{R})$ (resp. $\mathcal{H}^0_{\varepsilon_S}(\lambda;\mathbb{R})$) denote the set of those transverse cocycles (resp. ε_S-twisted transverse cocycles) which satisfy this cusp condition.

A transverse cocycle $\alpha \in \mathcal{H}^0(\lambda;\mathbb{R})$ has a well defined m-length $\ell_m(\alpha) \in \mathbb{R}$, where the cusp condition is necessary for this length to be finite. This length can be expressed in terms of the Thurston pairing $\tau : \mathcal{H}^0(\lambda;\mathbb{R}) \times \mathcal{H}^0_{\varepsilon_S}(\lambda;\mathbb{R}) \to \mathbb{R}$ by the property that $\ell_m(\alpha) = \tau(\alpha,\sigma_m)$ for every $\alpha \in \mathcal{H}^0(\lambda;\mathbb{R})$. This provides a parametrization of $\mathcal{T}_c(S)$ by the convex cone in $\mathcal{H}^0_{\varepsilon_S}(\lambda;\mathbb{R})$ consisting of those α such that $\tau(\mu,\alpha) > 0$ for every non-trivial compactly supported transverse measure μ for λ.

For a pleated cocycle $f = (\widetilde{f},\rho)$ with pleating locus λ, where ρ is a homomorphism $\pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^3)$, the bending locus $\beta_f \in \mathcal{H}_{\varepsilon_S}(\lambda;\mathbb{R})$ is defined as in section 7. It actually turns out that β_f is in $\mathcal{H}^0_{\varepsilon_S}(\lambda;\mathbb{R})$. The results of sections 7-11 immediately extend to this context, provided we restrict attention to $\alpha \in \mathcal{H}^0(\lambda;\mathbb{R})$ to define rotation angles in section 11.

12.4 Non-orientable hyperbolic 3-manifolds

We can also consider pleated surfaces in non-orientable hyperbolic 3-manifolds or, more generally, pleated surfaces $f = (\widetilde{f}, \rho)$ where the homomorphism can arrive in the group $\mathrm{Isom}(\mathbb{H}^3)$ of all isometries of $\mathbb{H}^3$ (orientation-preserving or not). Let $\varepsilon_\rho : \pi_1(S) \to \mathbb{Z}/2$ be the composition of ρ with the orientation homomorphism $\mathrm{Isom}(\mathbb{H}^3) \to \mathbb{Z}/2$. Then, the bending cocycle β_f is twisted by the product $\varepsilon_S \varepsilon_\rho$. Otherwise, the above results immediately extend to this case (except that the rotation angles of section 11 have to be defined in a twisted coefficient bundle).

Acknowledgments

We would like to thank Steve Kerckhoff for a particularly illuminating conversation which led to this work.

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