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Sparsely totient numbers[*]

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RÉSUMÉ. — Soit $P_1(n) \geq P_2(n) \ldots$ la suite décroissante des diviseurs premiers de l'entier n. Nous montrons le résultat suivant : si $\big(m > n \Rightarrow \phi(m) > \phi(n)\big)$, alors $P_1(n) < C(\log n)^{37/20}$, où C est une constante absolue. Nous utilisons le crible de Harman et les estimations de Fouvry et Iwaniec pour les sommes trigonométriques.

ABSTRACT. — Let $P_j(n)$ be the j-th largest prime dividing n. We show the following result: if $\big(m > n \Rightarrow \phi(m) > \phi(n)\big)$, then $P_1(n) < C(\log n)^{37/20}$, where C is a an absolute constant. We use Harman's sieve and the estimates of Fouvry and Iwaniec for trigonometric sums.

1. Introduction

A positive integer n is said to be *sparsely totient* if

$$\phi(m) > \phi(n)$$

for all $m > n$. This definition was introduced by Masser and Shiu [9], who proved several interesting properties of sparsely totient numbers. Subsequently Harman [6] sharpened some results from [9]. In particular, he showed that $P_j(n)$, the j-th largest prime dividing n, satisfies

$$P_j(n) \leq \left(\frac{j}{j-1} + \varepsilon\right) \log n \tag{1.1}$$

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for a given $j \geq 2$ and $\varepsilon > 0$. Here we suppose $n \geq n_0(j, \varepsilon)$. For $j = 1$, the corresponding bound is of weaker order [6, theorem 1]:

$$P_1(n) < (\log n)^{2-8/65+\varepsilon}. \tag{1.2}$$

As regards $Q_j(n)$, the j-th smallest prime dividing n, Harman showed in [6] that

$$Q_j(n) > \left(\frac{j}{j+1} - \varepsilon\right) \log n$$

for $n > n_1(j, \varepsilon)$.

In the present paper we sharpen the bound (1.2).

THEOREM .— *Let n be a sparsely totient number. Then*

$$P_1(n) < C(\log n)^{37/20}$$

where C is an absolute constant.

The key to the improvement is work of Fouvry and Iwaniec [4] on exponential sums

$$\sum_m \sum_{m_1} \sum_{m_2} a_m b_{m_1 m_2} e(A m^\alpha m_1^{\alpha_1} m_2^{\alpha_2}) \tag{1.3}$$

vhere $e(\theta) = e^{2\pi i \theta}$. The sums we need here are of the particular form

$$\sum_h \sum_s \sum_t a_s b_t c_h e\left(\frac{hx}{st}\right). \tag{1.4}$$

It is well-known that there are devices for estimating this sum more efficiently than (1.3) (e.g. Iwaniec and Laborde [7], Baker [1], Fouvry and Iwaniec [4], Wu [10], Liu [8] and Baker and Harman [2]). These devices would make no difference to the final result if we employed them here; see the remark following the proof of Lemma 3.

We shall deduce the theorem from the following proposition.

PROPOSITION .— *For all x, v with v sufficiently large and*

$$v^{37/20} \le x \le v^2 ,$$

there are

$$\gg \frac{x}{v \log x}$$

solutions in primes p to

$$1 - \frac{x}{16\, v^2} < \left\{ \frac{x}{p} \right\} < 1 , \quad \textit{with } 2v < p < 3v . \tag{1.5}$$

The proposition is proved by the sieve method developed by Harman [5] and Baker, Harman and Rivat [3]. We are able to use the same numerical work as in [3]; this saves a great deal of space. Sums (1.4) arise, as one would expect, in bounding the remainder terms of the sieve.

The deduction of the Theorem from the Proposition follows [6] closely, but we give it here for completeness. Suppose that n is a sparsely totient number and

$$P_1(n) \ge C(\log n)^{37/20} ,$$

so that n is large. By (1.2), we know that

$$P_1(n) < (\log n)^2 .$$

Let $p_1 = P_1(n)$ and write $m = n/p_1$. We apply the Proposition with $x = p_1$, $v = \log n$. It follows that there are

$$\gg \frac{p_1}{v \log p_1}$$

solutions to (1.5). From (1.1), there are at most three primes between $2v$ and $3v$ which divide n. We deduce that (1.5) has a solution with $p \nmid n$. Let

$$r = \left[\frac{p_1}{p} \right] + 1 .$$

Evidently $mrp > n$. We now use (1.5) to show that $\phi(mrp) < \phi(n)$. We have

$$\phi(mrp) \le r\phi(m)p\left(1 - \frac{1}{p}\right) \le \frac{rp}{p_1}\, \frac{1 - 1/p}{1 - 1/p_1}\, \phi(n) . \tag{1.6}$$

Now

$$r - \frac{p_1}{p} = 1 - \left\{ \frac{p_1}{p} \right\} < \frac{p_1}{16\, v^2} < \frac{9\, p_1}{16\, p^2}$$

from (1.5). Hence

$$\frac{rp}{p_1} < 1 + \frac{9}{16\, p} . \tag{1.7}$$

Combining (1.6) and (1.7),

$$\begin{aligned}\phi(mrp) &\le \phi(n)\Big(1 - \frac{1}{p} + \mathrm{O}\Big(\frac{1}{p_1}\Big)\Big)\left(1 + \frac{9}{16\, p}\right) \\ &\le \phi(n)\Big(1 - \frac{7}{16\, p} + \mathrm{O}\Big(\frac{1}{p^{37/20}}\Big)\Big) .\end{aligned}$$

Since p is large, we have

$$\phi(mrp) < \phi(n) ,$$

which is absurd. The Theorem is proved. □

2. Exponential sums

Let ε be a sufficiently small positive number and let $\eta = \varepsilon^2$. Constants implied by " $\ll$ ", " $\gg$ " and " $\mathrm{O}_\varepsilon(\,\cdot\,)$ " will depend at most on ε. Constants implied by " $\mathrm{O}(\,\cdot\,)$ " will be absolute. We use the abbreviation " $m \sim M$ " for

$$M < m \le 2M .$$

We write $\alpha = 3/20$.

LEMMA 1.— *Let a_s $(s \sim M)$, b_t $(t \sim N)$ be complex numbers of modulus ≤ 1. Suppose that*

$$v^{2-\alpha} \le x \le v^2 , \tag{2.1}$$

$$v^{\alpha+\varepsilon} \ll N \ll v^{1-5\alpha-\varepsilon} . \tag{2.2}$$

Then

$$\sum_{h \le v^{\alpha+5\eta}} \Bigg| \sum_{\substack{s \sim M \\ 2v < st \le 3n}} \sum_{t \sim N} a_s b_t\, e\Big(\frac{hx}{st}\Big) \Bigg| \ll v^{1-6\eta} . \tag{2.3}$$

Proof.— This follows from Lemma 9 of [1], with $H \leq v^{\alpha+5\eta}$ and $Q = NH^{-1}v^{-\varepsilon}$, using the exponent pair $(1/2, 1/2)$.

LEMMA 2.— *The conclusion of Lemma 1 holds if the hypothesis (2.2) is replaced by*

$$v^{3\alpha+\varepsilon} \ll M \ll v^{1-3\alpha-\varepsilon}\,. \tag{2.4}$$

Proof.— Using the technique in [1, lemma 15], it suffices to show that

$$\sum_{n\sim H}\sum_{s\sim M}\sum_{t\sim N} a'_s b'_t c_h\, e\left(\frac{h_x}{st}\right) \ll v^{1-7\eta}$$

for $H \leq v^{\alpha+5\eta}$, where a'_s and b'_t have modulus ≤ 1. We obtain this bound by an appeal to Lemma 1 for [3] (a variant of a result in [4]), taking (M_1, M_2, M) to be either (H, M, N) or (H, N, M).

LEMMA 3.— *Let* $M \leq v^{3-3\alpha-\varepsilon}$. *We have*

$$\sum_{h\leq v^{\alpha+5\eta}}\left|\sum_{s\sim M} a_s \sum_{\substack{N<t\leq N_1\\ 2v<st<3v}} e\left(\frac{hx}{st}\right)\right| \ll v^{1-6\eta} \tag{2.5}$$

for any complex numbers a_s *of modulus* ≤ 1.

Proof.— Suppose first that

$$M \leq v^{1/2}\,.$$

Then (2.5) follows from Lemma 2 of [3], with X, K replaced by v, M, and with $H \leq v^{\alpha+5\eta}$. Now suppose that

$$v^{1/2} < M \leq v^{3-3\alpha-\varepsilon}\,.$$

Then Lemma 3 follows from Lemma 2.

We have not been able to improve the bound $v^{3-3\alpha-\varepsilon}$ in Lemma 3 by taking advantage of the fact that $b_t = 1$ in (2.5), together with the special features of sums (1.4) mentioned in the section 1. This is perhaps surprising.

3. Proof of the proposition

We write $\delta = x/(16\,v^2)$. Let $\mathcal{B}$ be the set of integers in $(2v, 3v)$, and let $\mathcal{A}$ be the set of k in $\mathcal{B}$ for which

$$1-\delta < \left\{\frac{x}{k}\right\} < 1\,.$$

For $\mathcal{E} = \mathcal{A}$ or $\mathcal{B}$, we write

$$\mathcal{E}_d = \{k \in \mathcal{E} : d \mid k\}\,,$$
$$S(\mathcal{E}, z) = \big|\{k \in \mathcal{E} : p \mid k \Rightarrow p \geq z\}\big|\,.$$

Thus the number of primes in $\mathcal{A}$ is $S\big(A, (3v)^{1/2}\big)$. We shall prove that

$$S\big(\mathcal{A}, (3v)^{1/2}\big) > \frac{\delta v}{4\log v} \tag{3.1}$$

which establishes the Proposition.

LEMMA 4.— *Let a_s, $s \leq 2M$, and b_t, $t \sim N$, by complex numbers with $|a_s|$, $|b_t| \ll v^{\eta}$.*

For $M \leq v^{1-3\alpha-\varepsilon}$, we have

$$\sum_{s\leq M} a_s|\mathcal{A}_s| = \delta v \sum_{s\leq M} \frac{a_s}{s} + \mathrm{O}(\delta v^{1-3\eta})\,. \tag{3.2}$$

For M in any of the intervals

$$\left[v^{\alpha+\varepsilon}, v^{1-5\alpha-\varepsilon}\right], \quad \left[v^{3\alpha+\varepsilon}, v^{1-3\alpha-\varepsilon}\right], \quad \left[v^{5\alpha+\varepsilon}, v^{1-\alpha-\varepsilon}\right], \tag{3.3}$$

we have

$$\sum_{\substack{st\in\mathcal{A}\\ s\sim M\\ t\sim N}} a_s b_t = \delta v \sum_{\substack{st\in\mathcal{B}\\ s\sim M\\ t\sim N}} \frac{a_s b_t}{st} + \mathrm{O}(\delta v^{1-3\eta})\,. \tag{3.4}$$

Proof.— We prove (3.1) by combining the argument of Lemma 2 of [5] with Lemma 3. The proof of (3.4) is similar, using Lemma 1 or Lemma 2 in place of Lemma 3.

We may now follow the analysis of [3] very closely indeed. In place of Lemmas 6 and 7 of [3] we have Lemma 4, with v, α playing the roles of X and $1-\gamma$. We summarize the results obtained, leaving the details of proof to the reader. Let

$$P(y)=\prod_{p<y} p\,,\quad g(x)=\exp\left(1-\frac{1}{x}\log\frac{1}{x}\right).$$

LEMMA 5.— *For $M \le x^{11/20-\varepsilon}$, we have*

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, v^{\varepsilon}) =$$

$$=\delta\sum_{m\sim M} a_m S(\mathcal{B}_m, x^{\varepsilon})\Big(1+\mathrm{O}\big(g(\lambda)\big)+\mathrm{O}_{\varepsilon}(L^{-1})\Big)+\mathrm{O}_{\varepsilon}(\delta v^{1-2\eta}).$$

Here $\lambda = 30\,\varepsilon$, $|a_m| \ll v^{\eta}$ and $a_m = 0$ unless $\big(m, P(x^{\varepsilon})\big)=1$.

LEMMA 6.— *Let $u \ge 1$ be given and suppose that $\mathcal{D} \subset \{1, \ldots, u\}$ and M lies in one of the intervals (3.3). Then*

$$\sum_{p_1}\cdots\sum_{p_u}{}^{*} S(\mathcal{A}_{p_1\ldots p_u}, p_1)=$$

$$=\delta\sum_{p_1}\cdots\sum_{p_u}{}^{*} S(\mathcal{B}_{p_1\ldots p_u}, p_1)\big(1+\mathrm{O}_{\varepsilon}(L^{-1})\big)+\mathrm{O}(\delta v^{1-2\eta}).$$

*Here * indicates that $p_1, \ldots, p_u$ satisfy*

$$v^{\varepsilon}\le p_1<\cdots<p_u\,,\quad \prod_{j\in\mathcal{D}} p_j \sim M\,,$$

together with no more than ε^{-1} further conditions which take the form

$$R\le\prod_{j\in\mathcal{F}} p_j \le S\,.$$

LEMMA 7.— *Let $M \le v^{11/20-\varepsilon}$. We have*

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, v^{1/10-2\varepsilon}) =$$

$$=\delta\sum_{m\sim M} a_m S(\mathcal{B}_m, v^{1/10-2\varepsilon})\Big(1+\mathrm{O}\big(g(\nu)\big)+\mathrm{O}_{\varepsilon}(L^{-1})\Big)+\mathrm{O}(\delta v^{1-2\eta})$$

where $\nu = 100\,\varepsilon$, $0 \le a_m \ll v^{\eta}$ and $a_m = 0$ unless $\big(m, P(v^{1/10-2\varepsilon})\big)=1$.

We may now carry out the decomposition of $\mathcal{S}(\mathcal{A},(3v)^{1/2})$ in exactly the same fashion as [3, sect. 5] with v in the role of X. A few changes by a factor v^{ε} in the endpoints of intervals will occur. These will in turn alter the coefficients in the inequalities defining the regions of integration by $O(\varepsilon)$. this clearly does not alter the final result, which is the lower bound (3.1). This completes the proof of the Proposition.

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