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Perturbative computation of analytic invariants of resonant diffeomorphisms of $(C, 0)$[*]

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RÉSUMÉ. — On considère le problème de l'évaluation des invariants analytiques d'un difféomorphisme unidimensionnel, analytique, complexe et tangent à l'identité. Une approche perturbative relative à une sous-classe de difféomorphismes qui sont des petites perturbations du "*standard shift*" est présentée. On donne une procédure qui permet de calculer le développement perturbatif de l'invariant analytique à tous les ordres. On présente aussi la généralisation au cas d'un difféomorphisme résonnant générique, ainsi que quelques exemples.

ABSTRACT. — The problem of the evaluation of the analytic invariants of a one-dimensional complex analytic diffeomorphism tangent to the identity is considered; a perturbative approach relative to a subclass of diffeomorphisms which are small perturbations to the standard shift is outlined. A procedure which allows to calculate all the orders of the perturbative expansion of the analytic invariants is given. The generalization to generic resonant diffeomorphisms is presented; some examples are sketched.

1. Introduction

In this paper we consider mappings tangent to the identity, analytic in the neighbourhood of the origin; let $\mathcal{A}$ be the set of these mappings.

DEFINITION .— *Let f, $g \in \mathcal{A}$; then f and g are conjugated (or equivalent) if there exist a change of coordinates T such that $T(f) = g(T)$. According to the type of T we distinguish formal and analytic conjugation.*

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By formal classification we mean the partition of $\mathcal{A}$ in classes of elements which are formally conjugated. The formal invariant is the quantity which characterizes such class, i.e. f and g belong to the same class if and only if their formal invariant is the same. An analogous definition is given for the analytic classification and for the analytic invariants.

The formal classification of $\mathcal{A}$ can be worked out using elementary algebra and the explicit invariants can be constructed; on the contrary, the analytic classification requires more involved techniques: as it was first proved by Ecalle [1] and, independently, by Voronin [2], the analytic classification of $\mathcal{A}$ has a functional modulus, i.e. the analytic invariants are given by two functions Φ_+, Φ_-,

$$\Phi_\pm(w) = w + \sum_{k=0}^{+\infty} e^{2\pi i k w} c_k^\pm \tag{1.1}$$

which are analytic on the strips $\operatorname{Im} w > L$ and $\operatorname{Im} w < -L$ respectively. The exact computation of both the coefficients $c_k^\pm$ and the functions $\Phi_\pm$ in their dependence on the mapping f is a hard task, and has been explicitly carried out only for the case of the standard shift

$$F(z) = \frac{z}{1+z}, \tag{1.2}$$

which is the map which exhibits the simplest dynamics indise this class of diffeomorphisms; in fact $F(z)$ has trivial analytic invariants $c_k^\pm = 0$, $\Phi_\pm = w$.

Indeed, a perturbative approach is possible: if one considers small perturbations of the standard shift

$$f(z) = F(z) + \varepsilon h(z), \tag{1.3}$$

a theorem ([1]-[2]) states that the analytic invariants are convergent power series in ε; Elizarov [3] computed the first order of such series for perturbations $h(z) = O(z^4)$ which preserve the formal invariant, finding out that the first order shift in $c_k^\pm$ is proportional to the Borel transform of h evaluated at points $\pm 2\pi i k$.

Since the use of the coordinates $w = 1/z$ considerably simplifies the computation, we express the classe of mappings which are small perturbation to the standard shift preserving the formal invariant as follows:

$$\mathcal{A}_{2,0}^{\varepsilon} = \{ f \in \mathcal{A} \mid f(w) = w + 1 + \varepsilon h(w),\ h(w) = O(w^{-2}) \}, \tag{1.4}$$

where the first subscript denotes the lowest order of the nonlinearity and the second one the formal invariant. The main results of this paper is stated in the following Theorem.

THEOREM 1.1.— *The analytic invariants of $f \in \mathcal{A}^{\varepsilon}_{2,0}$ are given by*

$$\Phi_{\pm}(w) = \\ = \omega + \varepsilon \sum_{j=-\infty}^{+\infty} h\left(w + j + \varepsilon \sum_{i_1=-\infty}^{j-1} h\left(w + i_1 + \varepsilon \sum_{i_2=-\infty}^{i_1-1} h(w + i_2 + \cdots) \right)\right); \tag{1.5}$$

this expression allows to compute all the orders of the perturbative expansion. Moreover, the coefficients of the $c_k^{\pm}$ expansions in powers of ε are functionals of the Borel transform of the perturbation. Explicit formulas for the first three orders are given and some examples are sketched.

The generalization to generic perturbations which do not preserve the formal invariant

$$\mathcal{A}^{\varepsilon}_{2,\gamma} = \left\{ f \in \mathcal{A} \,\middle|\, f(w) = w + 1 + \varepsilon \frac{\gamma}{w} + \varepsilon g(w)\,,\ g(w) = \mathrm{O}(w^{-2}) \right\} \tag{1.6}$$

is given in Corollary 4.1, where it is proved that the same formal expression given in Theorem 1.1 holds, with

$$h(w) = g(w) + \frac{\gamma}{w} - \gamma \log \frac{f(w)}{w}\,. \tag{1.7}$$

We also considered the class $\mathcal{A}^{\varepsilon}_{r+1,\gamma}$, containing mappings of the form $z' = z + \mathrm{O}(z^{r+1})$, which exhibit many features of the resonant case. Here we have to apply a ramified transformation $z \longrightarrow z^{1/r}$ in order to recover the standard shift at order 0 in ε: this leads to series with rational powers $z^{j/r}$, defined on the Riemann surface $\mathbb{C}_r$. Nevertheless, it can be proved (Corollary 5.1) that the same formal expression (1.5) holds, the only difference being that now the analytic invariants are defined on $\mathbb{C}_r$, and therefore one has r sets of coefficients $c_k^{\pm(1)}, \ldots, c_k^{\pm(r)}$.

Finally, we prove that in the resonant case

$$z' = f(z) = e^{2\pi i p/q} z + \sum_{i=2}^{\infty} f_i z^i \tag{1.8}$$

the computation of the analytic invariants can be done on the q-th iterate (Lemma 6.1), which is tangent to the identity and therefore can be analyzed using the previous results.

The plan of this paper is the following: in section 2 we recall some results on the analytic classification of diffeomorphisms tangent to the identity which are formally conjugated to the standard shift. In section 3 we prove the main result which allows to compute all the coefficients of the series. The generalization to diffeomorphisms which are not formally conjugated to the standard shift and whose nonlinearity do not start with a quadratic term are given respectively in section 4 and 5. The analysis of the resonant case ($q > 1$) is carried out in section 6.

2. Definition of analytic invariants

In this section we briefly recall some classical results on the analytic classification of diffeomorphisms tangent to the identity ([1], [2], [4], [5]).

We consider the set $\mathcal{A}$ of diffeomorphisms of $\mathbb{C}$ which have the origin as a fixed point, and which are tangent to the identity and analytic in the neighbourhood of the origin. Inside this class we analyze diffeomorphisms which have a quadratic nonlinearity; rescaling the second order coefficient to -1, such a diffeomorphism reads:

$$z' = f(z) = z - z^2 + \sum_{j=3}^{+\infty} f_j z^j\,, \quad f \in \mathcal{A}\,. \tag{2.1}$$

All the diffeomorphisms belonging to this class have the same topology [6], but inside it one can find elements with different both formal and analytic invariants ([1], [2]). We first consider the formal classification.

LEMMA 2.1.— *Let $g(z)$ and $h(z)$ belong to the class (2.1); then one can find a formal power series $\widehat{T}$ which conjugates g to h, namely*

$$g\big(\widehat{T}(z)\big) = \widehat{T}\big(h(z)\big) \tag{2.2}$$

if and only if $h_3 = g_3$.

A proof of this lemma can be found in [2]. Setting the third order coefficient to one we obtain a class of diffeomorphisms which are formally conjugated to the so called standard shift (1.2), which exhibits a particularly simple dynamics: it has a trivial iteration $F^{\circ n}(z) = z/(1 + nz)$ and is interpolated by the vector field $X(z) = -z^2(\mathrm{d}/\mathrm{d}z)$ (cf. [4]). Since most of the computation will be done using coordinates in the neighbourhood of the infinity, we express both F and f in terms of $w = 1/z$. The standard shift in the w coordinates becomes the translation $F(w) = w + 1$.

We consider the class of diffeomorphisms which are small perturbations of the standard shift, and which can be formally conjugated to it:

$$\mathcal{A}^{\varepsilon}_{2,0} = \left\{ f \in \mathcal{A} \;\middle|\; f(w) = w + 1 + \varepsilon h(w)\,,\ h(w) = \mathrm{O}\Big(\frac{1}{w^2}\Big) \right\}; \tag{2.3}$$

here the first subscript denotes the lowest order of the nonlinear term (in z coordinates), the second one is the formal invariant (i.e. the coefficient of the term of order $1/w$), and ε is a small complex parameter.

Taking $f \in \mathcal{A}^{\varepsilon}_{2,0}$, we conjugate it to the standard shift $F(\eta) = \eta + 1$ through a formal transformation $\widehat{T}$ tangent to the identity

$$\eta = \widehat{T}(w) = w + \widehat{t}(w)\,, \quad \widehat{t}(w) = \sum_{j=1}^{\infty} t_j w^{-j}\,; \tag{2.4}$$

following reference [5] we fixed the note terms of $\widehat{T}$ (which is a priori undetermined) to zero: this is not restrictive and simplifies the formulation of the analytic classification as given in ([1], [2]). The functional equation which defines $\widehat{T}$ is $F\big(\widehat{T}(w)\big) = \widehat{T}\big(f(w)\big)$ and substituting (2.4) one has

$$\widehat{t}\big(f(w)\big) - \widehat{t}(w) = -\varepsilon h(w)\,. \tag{2.5}$$

It is well known that the formal series $\widehat{t}$ is divergent in the generic case; nevertheless, using the Borel resummation technique or other methods one can prove the existence of solutions analytic on sectors.

THEOREM 2.1 (Kimura [7], Ecalle [1]).— *Let $f \in \mathcal{A}^{\varepsilon}_{2,0}$; we define the sectors*

$$\begin{aligned} \mathcal{D}_1 &= \Big\{ w \;\Big|\; \big|\arg(w - 1/\rho)\big| < \delta \Big\} \\ \mathcal{D}_2 &= \Big\{ w \;\Big|\; \big|\arg(w + 1/\rho) - \pi\big| < \delta \Big\} \end{aligned} \tag{2.6}$$

with $\delta \in \,]\,0\,,\pi\,[$ and $\rho = \rho(\delta)$ sufficiently large; then:

(i) exist T_i, analytic on $\mathcal{D}_i$ which conjugate f to F for $i = 1, 2$;

(ii) both T_1 and T_2 have the same asymptotic expansion $\widehat{T}$;

(iii) $\mathcal{D}_1$ *is the attractor sector: if* $w \in \mathcal{D}_1$, *then* $f(w) \in \mathcal{D}_1$ *and* $\lim_{n\to\infty} f^{\circ n}(w) = \infty$; $\mathcal{D}_2$ *is the repulsor sector: if* $w \in \mathcal{D}_2$, *then* $f^{\circ -1}(w) \in \mathcal{D}_2$ *and* $\lim_{n\to\infty} f^{\circ -n}(w) = \infty$.

Proof.— Proofs of this theorem can be found in ([1], [5], [7]). T_1 and T_2 are defined on a common domain $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{D}^+ \cup \mathcal{D}^-$, being $\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset$; the analytic classification is given in the following theorem.

THEOREM 2.2 (Ecalle [1], Voronin [2]).— *We define* $\Phi_\pm(w) = T_1 \circ T_2^{-1}|_{\mathcal{D}^\pm}$; *then on has:*

(i) $\Phi_\pm(w)$ *are analytic functions on* $\mathcal{D}^\pm$; $\Phi_\pm(w) - w$ *are periodic of period one and tend to zero when* $w \to \infty$;

(ii) $\Phi_\pm$ *can be analytically prolonged to* $\widetilde{\Phi}_\pm(w)$, *defined on*

$$\begin{aligned} \mathcal{D}_+ &= \{w \mid \operatorname{Im} w > L\}, \\ \mathcal{D}_- &= \{w \mid \operatorname{Im} w < -L\}, \end{aligned} \tag{2.7}$$

where $L > 0$ *depends on the parameters* ρ, δ *of* $\mathcal{D}^\pm$;

(iii) $\widetilde{\Phi}_\pm(w)$ *are analytic invariants of* $f(w)$, $f \in \mathcal{A}^\varepsilon_{2,0}$, *i.e. being* $\widetilde{\Psi}_\pm(w)$ *the analytic invariant of* $g \in \mathcal{A}^\varepsilon_{2,0}$, f *and* g *are analytically conjugated if and only if* $\widetilde{\Phi}_\pm \equiv \widetilde{\Psi}_\pm$.

Proof.— A simple proof of this theorem is given in [5]. For sake of simplicity, throughout the text the tilde will be suppressed and $\Phi_\pm$ will be defined on $\mathcal{D}_\pm$. A consequence of theorem 2.2 is that the analytic invariants are given by convergent Fourier series:

$$\Phi_\pm(w) = w + \sum_{k\geq 0} e^{2\pi i k w} c_k^\pm, \quad w \in \mathcal{D}_\pm, \tag{2.8}$$

whose coefficients $c_k^\pm$ are bounded by an exponential growth in k.

The dependence of the analytic invariants on a parameter is analyzed via the following theorem.

THEOREM 2.3 (Ecalle [1], Voronin [2]).— *If f belongs to a family of diffeomorphisms which analytically depend on a parameter ε, then $c_k^\pm$ and $\Phi_\pm(w)$ also analytically depend on ε, i.e. we can write*

$$\Phi_\pm(w) = w + \sum_{\ell=0}^{+\infty} \varepsilon^\ell \varphi_{\pm,\ell}(w) \tag{2.9}$$

$$c_k^\pm = \sum_{\ell=0}^{+\infty} \varepsilon^\ell c_{k,\ell}^\pm \,, \tag{2.10}$$

where both series are convergent for $|\varepsilon| < \varepsilon_0$.

3. Computation of the analytic invariants

The exact computation of the coefficients $c_k^\pm$ is trivial only for the case of the standard shift: here we have $h(w) = 0$, $T_1 = T_2 = w$ and $c_k^\pm = 0$; in this section we will show how to compute the perturbative expansion (2.9) and (2.10) of the analytic invariants, generalizing the first order results given by Elizarov [3].

LEMMA 3.1.— *If $f \in \mathcal{A}_{2,0}^\varepsilon$, then*

$$\begin{aligned} T_1(w) &= w + \varepsilon \sum_{j=0}^{+\infty} h\big(f^{\circ j}(w)\big)\,, \quad w \in \mathcal{D}_1\,, \\ T_2(w) &= w - \varepsilon \sum_{j=-\infty}^{-1} h\big(f^{\circ j}(w)\big)\,, \quad w \in \mathcal{D}_2\,. \end{aligned} \tag{3.1}$$

Proof.— Taking the functional equation (2.5) which defines $t(w)$, iterating n times and summing all the equations we have

$$t\big(f^{\circ n}(w)\big) - t(w) = -\varepsilon \sum_{j=0}^{n-1} h\big(f^{\circ j}(w)\big)\,. \tag{3.2}$$

If $w \in \mathcal{D}_1$, theorem 2.1 implies that $\lim_{n\to\infty} f^{\circ n}(w) = \infty$ and $t(\infty) = 0$: therefore taking the limit $n \to \infty$ one proves (3.1). Since $h(w) = \mathrm{O}(1/w^2)$ and $f^{\circ j}(w) = w + j + \mathrm{O}(1/w)$, the series absolutely converge and defines T_1 on $\mathcal{D}_1$; the analogous expression for T_2 can be found by iterating backward the functional equation and applying the same procedure.

Now we are able to give an expression of the analytic invariants which allows to compute $c^{\pm}_{k,\ell}$ for all orders ℓ.

THEOREM 3.1.— *If $f \in \mathcal{A}^{\varepsilon}_{2,0}$, then, for small $|\varepsilon| < \varepsilon_0$ one has:*

(i) $\Phi_\pm(w) = w + \varepsilon \displaystyle\sum_{j=-\infty}^{+\infty} h\left(f^{\circ j}(T_2^{-1}(w))\right)\,, \quad w \in \mathcal{D}_\pm;$

(ii) $f^{\circ j}(T_2^{-1}(w))$ *satisfies the functional equation*

$$f^{\circ j}(T_2^{-1}(w)) = w + j + \varepsilon \sum_{i=-\infty}^{j-1} h\left(f^{\circ i}(T_2^{-1}(w))\right); \tag{3.3}$$

this expression allows to expand $\Phi_\pm$ in a power series in ε.

Proof of part (i).— We define $\tau(w)$ according to

$$T_2^{-1}(w) = w + \varepsilon\tau(w)\,; \tag{3.4}$$

since $T_2 \circ T_2^{-1}(w) = w$, using Lemma 3.1 we can prove that $\tau(w)$ satisfies the functional equation:

$$\tau(w) = \sum_{j=-\infty}^{-1} h\left(f^{\circ j}(w + \varepsilon\tau(w))\right). \tag{3.5}$$

The analytic invariants are given by $\Phi_\pm(w) = T_1 \circ T_2^{-1}(w)$, and substituting (3.5) and the definition of T_1 (3.1), one has

$$\Phi_\pm(w) = w + \varepsilon \sum_{j=-\infty}^{+\infty} h\left(f^{\circ j}(T_2^{-1}(w))\right)\,, \quad w \in \mathcal{D}_\pm\,. \tag{3.6}$$

Proof of part (ii).— Let us consider the case $j > 0$: since

$$f^{\circ j}(w) = w + j + \varepsilon \sum_{i=0}^{j-1} h(f^{\circ i}(w))\,, \quad j > 0\,, \tag{3.7}$$

we have

$$f^{\circ j}(T_2^{-1}(w)) = w + \varepsilon\tau(w) + j + \varepsilon \sum_{i=0}^{j-1} h\left(f^{\circ i}(T_2^{-1}(w))\right) \tag{3.8}$$

and substituting (3.5) we obtain the thesis. Similarly, one proves the case $j < 0$.

COROLLARY 3.1.— *If $f \in \mathcal{A}^{\epsilon}_{2,0}$, then*

$$\Phi_{\pm}(w) = w + \varepsilon \sum_{j=-\infty}^{+\infty} h\left(w + j + \varepsilon \sum_{i_1=-\infty}^{j-1} h\left(w + i_1 + \right.\right. \\ \left.\left. + \varepsilon \sum_{i_2=-\infty}^{i_1-1} h(w + i_2 + \cdots)\right)\right), \qquad w \in \mathcal{D}_{\pm}\,; \tag{3.9}$$

this formula is obtained by replacing (3.3) infinitely many times in (3.6).

Expanding this expression in power series of ε we compute the terms $\varphi_{\pm,\ell}(w)$: the first three orders read:

$$\begin{aligned} \varphi_{\pm,1}(\omega) &= \sum_{j=-\infty}^{+\infty} h(w+j) \\ \varphi_{\pm,2}(\omega) &= \sum_{j=-\infty}^{+\infty} h'(w+j) \sum_{i_1=-\infty}^{j-1} h(w+i_1) \\ \varphi_{\pm,3}(\omega) &= \frac{1}{2} \sum_{j=-\infty}^{+\infty} h''(w+j) \left(\sum_{i_1=-\infty}^{j-1} h(w+i_1)\right)^2 + \\ &\quad + \sum_{j=-\infty}^{+\infty} h'(w+j) \sum_{i_1=-\infty}^{j-1} h'(w+i_1) \sum_{i_2=-\infty}^{i_1-1} h(w+i_2)\,. \end{aligned} \tag{3.10}$$

The coefficients $c^{\pm}_{k,\ell}$ are functional of the Borel transform of the perturbation $h(w)$:

$$\begin{aligned} c^{\pm}_{k,1} &= \mp 2\pi i h_B(\mp 2\pi i k) \\ c^{\pm}_{k,2} &= \pm 2\pi i \left(t_1 h_B(t_1) \star h_B(t_2) \frac{e^{t_2}}{1-e^{t_2}} \right)\Bigg|_{\mp 2\pi i k} \\ c^{\pm}_{k,3} &= \mp 2\pi i \left(\frac{t_1^2}{2} h_B(t_1) \star h_B(t_2) \frac{e^{t_2}}{1-e^{t_2}} \star h_B(t_3) \frac{e^{t_3}}{1-e^{t_3}} + \right. \\ &\quad \left. + \; t_1 h_B(t_1) \star \sum_{i_1=1}^{\infty} t_2 h_B(t_2) e^{i_1 t_2} \star \sum_{i_2=i_1+1}^{\infty} h_B(t_3) e^{i_2 t_3} \right)\Bigg|_{\mp 2\pi i k} . \end{aligned} \tag{3.11}$$

We prove this last statement for the first order coefficients; the generalization to the higher orders is straightforward. $c^+_{k,1}$ are the Fourier coefficients of $\varphi_{+,1}(w)$:

$$c^+_{k,1} = \int_{i\beta}^{i\beta+1} \varphi_{+,1}(w)\, e^{-2\pi i k w}\, \mathrm{d}w\,, \quad \beta > L \tag{3.12}$$

(where L is the lowest imaginary part of $w \in \mathcal{D}_\pm$, see (2.7)). Substituting (3.10) we have

$$c^+_{k,1} = \int_{i\beta-\infty}^{i\beta+\infty} h(w)\, e^{-2\pi i k w}\, \mathrm{d}w\,. \tag{3.13}$$

We recall [9] that the Borel transform of a function analytic in a neighbourhood of infinity is given by

$$h_B(t) = \frac{1}{2\pi i}\int_\Gamma e^{tw} h(w)\, \mathrm{d}w\,, \tag{3.14}$$

where Γ is a circle of radius R sufficiently large so that it lies in the analyticity domain of $h(w)$. We divide Γ in two arcs: Γ_1, above the line $\operatorname{Im} w = \beta$, and Γ_2, below it. Since $h(w)$ is analytic on a neighbourhood of infinity, Cauchy theorem implies

$$\lim_{R\to\infty}\int_{\Gamma_1} e^{-2\pi i k w} h(w)\, \mathrm{d}w = -\int_{i\beta-\infty}^{i\beta+\infty} e^{-2\pi i k w} h(w)\, \mathrm{d}w = -c^+_{k,1}\,. \tag{3.15}$$

On the other hand, the integration on Γ_2 vanishes in the limit $R \to \infty$, since the exponential is bounded by $e^{2\pi k\beta}$ and h satisfies the estimate $\left|h(R\, e^{i\theta})\right| < AR^{-2}$. In a similar way one proves the analogous expression for $c^-_{k,1}$.

Example.— We consider $f(w) = w + 1 + \varepsilon\sum_{i=2}^{+\infty} w^{-i}$; then we have $h_B(t) = e^t - 1$ and therefore

$$c^\pm_{k,1} = 0\,, \quad c^\pm_{k,2} = -4\pi^2 k \pm 4\pi^3 i k^2\,. \tag{3.17}$$

4. Generalization to generic diffeomorphisms tangent to the identity

We examine the case of an analytic diffeomorphism which is not formally conjugated to the standard shift, i.e. the class

$$\mathcal{A}^{\varepsilon}_{2,\gamma} = \left\{ f \in \mathcal{A} \;\middle|\; f(w) = w + 1 + \varepsilon \frac{\gamma}{w} + \varepsilon g(w) \,,\; g(w) = \mathrm{O}(w^{-2}) \right\}; \quad (4.1)$$

a generic f expressed in z coordinates and starting with a quadratic nonlinearity is transformed to this form by the coordinate change $w = 1/z$, where $\varepsilon\gamma$ is the formal invariant; theorems 2.1-2.3 can be generalized to this case ([1], [8], [10]). The computation of the analytic invariants is given by the following

COROLLARY 4.1.— *If $f \in \mathcal{A}^{\varepsilon}_{2,\gamma}$, then the analytic invariants of f are given by formula (3.9), where $h(w) = \mathrm{O}(w^{-2})$ is an analytic function of w in a neighbourhood of infinity and is defined according to:*

$$h(w) = g(w) + \frac{\gamma}{w} - \gamma \log \frac{f(w)}{w} \,. \quad (4.2)$$

Proof.— We recall ([4], [8], [10]) that in order to conjugate $f \in \mathcal{A}^{\varepsilon}_{2,\gamma}$ to the standard shift we have to insert in T a logarithmic singularity:

$$\eta = T(w) = w + t(w) - \varepsilon\gamma \log w \,. \quad (4.3)$$

Using this Ansatz, one can find a formal power series for t and resum it to two functions t_1, t_2, analytic on $\mathcal{D}_1$, $\mathcal{D}_2$. In fact, the functional equation reads:

$$t\big(f(w)\big) - t(w) = -\varepsilon \left(g(w) + \frac{\gamma}{w} - \gamma \log \frac{f(w)}{w} \right) = \mathrm{O}(w^{-2}) \,; \quad (4.4)$$

therefore one can carry out the computation such as in the case $\gamma = 0$, defining $-\varepsilon h(w)$ as the r.h.s of (4.4).

Example.— Let $f \in \mathcal{A}^{\varepsilon}_{2,\gamma}$; then the first order analytic invariants read

$$c^{\pm}_{1,k} = \mp 2\pi i \big(g_B(\mp 2\pi i) + \gamma \big) \,. \quad (4.5)$$

5. Generalization to higher order nonlinearities

We consider the family of diffeomorphisms

$$\mathcal{A}_{r+1,\gamma} = \{d \in \mathcal{A} \mid d(z) = z + \mathrm{O}(z^{r+1})\} \tag{5.1}$$

here $r \geq 2$, $r \in \mathbb{N}$ and γ is the formal invariant.

The analysis of this class exhausts all the possible diffeomorphisms tangent to the identity, and exhibits some features of the resonant case. We first rewrite the class $\mathcal{A}_{r+1,\gamma}$ in the coordinates w which explicitly exhibits the formal invariant.

LEMMA 5.1.— *Let $d \in \mathcal{A}_{r+1,\gamma}$; then d is analytically equivalent to a diffeomorphism which is analytic on $\mathbb{C}_r$ and reads*

$$f(w) = w + 1 + \frac{\gamma}{w} + \sum_{i=2r}^{+\infty} f_i w^{-i/r} \equiv w + 1 + \frac{\gamma}{w} + g(w)\,. \tag{5.2}$$

Proof.— We conjugate $d(z)$ to $e(y)$

$$e(y) = y - \frac{1}{r}\,y^{r+1} + e_{2r+1}y^{2r+1} + \mathrm{O}(y^{3r+1}) \tag{5.3}$$

using a transformation which is a polynomial of finite order, and therefore analytic (see [2]). Then, we apply the ramified transformation $w = y^{-r}$ which transforms $e(y)$ to $f(w)$:

$$f(w) = w + 1 + \frac{\gamma}{w} + \sum_{i=2r}^{+\infty} f_i w^{-i/r}\,, \quad w \in \mathbb{C}_r\,, \tag{5.4}$$

where $\gamma = 1 - re_{2r+1} + (1-r)/2r$.

When $\gamma = 0$, $f(w)$ is formally conjugated to the standard shift of order r which in the coordinates y reads $E(y) = y/(1+y^r)^{1/r}$, and which has trivial iteration $E^{\circ j}(y) = y/(1+jy^r)^{1/r}$ and interpolating vector field $Y(y) = -y^{r+1}\,\mathrm{d}/\mathrm{d}y$ (see [4]).

The transformation $T(w)$ which conjugates $f(w)$ to the standard shift is formally equal to the case $r=1$ (see (4.3)), but here $\widehat{t}(w)$ is a formal power series in $w^{-1/r}$. A generalization of theorem 2.1 can be proved ([1], [8]).

THEOREM 5.1 (Ecalle [1], Martinet-Ramis [4]).— *Let* $f \in \mathcal{A}_{r+1,\gamma}$ *and let*

$$\begin{aligned}\mathcal{D}_1^{(s)} &= \left\{ w \in \mathbb{C}_r \;\Big|\; \left|\arg\Big(w - \frac{1}{\rho}\Big) - 2\pi(s-1)\right| < \delta \right\} \\ \mathcal{D}_2^{(s)} &= \left\{ w \in \mathbb{C}_r \;\Big|\; \left|\arg\Big(w + \frac{1}{\rho}\Big) - 2\pi\Big(s - \frac{1}{2}\Big)\right| < \delta \right\}\end{aligned} \qquad s = 1, \ldots, r \quad (5.5)$$

(throughout this section the index s *will always assume the* r *integer values* $s = 1, 2, \ldots, r$*) be* $2r$ *sectors of aperture* $2(\pi - \delta)$ *of* $\mathbb{C}_r$*; then:*

(i) *there are* $t_1^{(s)}$, $t_2^{(s)}$, *analytic on* $\mathcal{D}_1^{(s)}$, $\mathcal{D}_2^{(s)}$ *respectively, which conjugate* f *to the standard shift* F*;*

(ii) *all the* $t_1^{(s)}$, $t_2^{(s)}$ *have the same asymptotic expansion* $\widehat{t}$ *around the infinity;*

(iii) $\mathcal{D}_1^{(s)}$ *are attractor sectors of* f, *and* $\mathcal{D}_2^{(s)}$ *are repulsor sectors of* f.

$T_1^{(s)}$ *and* $T_2^{(s)}$ *have common domains: we define*

$$\mathcal{D}^{+(s)} = \mathcal{D}_1^{(s)} \cap \mathcal{D}_2^{(s)} \tag{5.6}$$

and similarly $T_2^{(s)}$ *and* $T_1^{(s+1)}$ *are defined on a common domain*

$$\mathcal{D}^{-(s)} = \mathcal{D}_1^{(s+1)} \cap \mathcal{D}_2^{(s)} \quad \textit{where } \mathcal{D}_1^{(r+1)} \equiv \mathcal{D}_1^{(1)}\,. \tag{5.7}$$

The analytic classification of $\mathcal{A}_{r+1,\gamma}$, $r \geq 2$, *is given by the composition of the transformations on the common domains.*

THEOREM 5.2 (Ecalle [1], Martinet-Ramis [4]).— *We define*

$$\begin{aligned}\Phi_+^{(s)} &= T_1^{(s)} \circ \big(T_2^{(s)}\big)^{-1}, \quad w \in \mathcal{D}^{(s)} \\ \Phi_-^{(s)} &= T_1^{(s+1)} \circ \big(T_2^{(s)}\big)^{-1}, \quad w \in \mathcal{D}^{(s)}\end{aligned} \tag{5.8}$$

then:

(i) $\Phi_{\pm}^{(s)}$ *are analytic on* $\mathcal{D}^{\pm(s)}$; $\Phi_{\pm}^{(s)}(w) - w$ *are periodic of period one and tend to zero when* $w \to \infty$;

(ii) $\Phi_{\pm}^{(s)}(w)$ *can be analytically prolonged to functions* $\widetilde{\Phi}_{\pm}^{(s)}(w)$ *defined on* $\mathcal{D}_{\pm}^{(s)}$, *where* $\mathcal{D}^{\pm(s)} \subset \mathcal{D}_{\pm}^{(s)}$:

$$\begin{aligned} \mathcal{D}_{+}^{(s)} &= \{w \mid |\arg w - (2s-1)\pi| < \pi\,,\ \operatorname{Im} w > L\} \\ \mathcal{D}_{-}^{(s)} &= \{w \mid |\arg w - (2s-1)\pi| < \pi\,,\ \operatorname{Im} w < -L\}\ . \end{aligned} \tag{5.9}$$

Therefore, $\widetilde{\Phi}_{\pm}^{(s)}(w)$ *can be expressed by Fourier series*

$$\widetilde{\Phi}_{\pm}^{(s)}(w) = w + \sum_{k=0}^{+\infty} e^{\pm 2\pi i k w} c_k^{\pm(s)}\,, \quad w \in \mathcal{D}_{\pm}^{(s)}\,; \tag{5.10}$$

the coefficients $c_k^{\pm(s)}$ *are bounded by an exponential growth in* k;

(iii) $\widetilde{\Phi}_{\pm}^{(s)}$ *are the analytic invariants of* $f \in \mathcal{A}_{r+1,\gamma}$.

Such as in section 2, we suppress the tilde for sake of simplicity. One can also generalize Lemma 3.1.

LEMMA 5.2.— *Let* $f \in \mathcal{A}_{r+1,\gamma}$; *then*

$$\begin{aligned} T_1^{(s)}(w) &= w + \sum_{j=0}^{+\infty} h\big(f^{\circ j}(w)\big)\,, \quad w \in \mathcal{D}_1^{(s)}\,, \\ T_2^{(s)}(w) &= w - \sum_{j=-\infty}^{-1} h\big(f^{\circ j}(w)\big)\,, \quad w \in \mathcal{D}_2^{(s)}\,, \end{aligned} \tag{5.11}$$

where

$$h(w) = g(w) + \frac{\gamma}{w} - \gamma \log \frac{f(w)}{w} = \sum_{i=2r}^{+\infty} h_i w^{-i/r}\,. \tag{5.12}$$

Proof.— The proof follows the same scheme of Lemma 3.1 and Corollary 4.1. Therefore, one can generalize Theorem 3.1.

COROLLARY 5.1.— *Let $f \in \mathcal{A}^{\varepsilon}_{r+1,\gamma}$, i.e.*

$$f(w) = w + 1 + \frac{\varepsilon\gamma}{w} + \varepsilon \sum_{i=2r}^{+\infty} g_i w^{-i/r} ,$$

analytic on $\mathbb{C}_r$ in a neighbourhood of the infinity; then:

(i) the analytic invariant of f read

$$\begin{aligned}\Phi^{(s)}_{\pm}(w) = w + \varepsilon \sum_{j=-\infty}^{+\infty} h\Bigg(w + j + \varepsilon \sum_{i_1=-\infty}^{j-1} h\Bigg(w + i_1 + \\ + \varepsilon \sum_{i_2=-\infty}^{i_1-1} h(w + i_2 + \cdots)\Bigg)\Bigg) , \quad w \in \mathcal{D}^{(s)}_{\pm} ;\end{aligned} \tag{5.13}$$

(ii) the coefficients $c^{\pm(s)}_{k,\ell}$ are functionals of the Borel transform of $h(w)$; the same formal expressions (3.11) hold. For instance, one has

$$c^{\pm(s)}_{k,1} = \mp 2\pi i h^{(s)}_B(\mp 2\pi i k) , \tag{5.14}$$

where the determination of the Borel transform is taken on the same sheet of $\mathbb{C}_r$ where $\Phi^{(s)}_{\pm}(w)$ is defined.

Proof.— Part (i) can be proved following the same scheme of the case $r = 1$; in order to prove part (ii) we first recall the definition of the Borel transform of a function analytic on $\mathbb{C}_r$ [9].

DEFINITION .— *Let $h(w) = \sum_{i=r}^{+\infty} h_i w^{-i/r}$, $w \in \mathbb{C}_r$ be analytic in a neighbourhood of infinity; then*

$$h^{(s)}_B(t) = \frac{1}{2\pi i} \int_{\Gamma^{(s)}} e^{tw} h(w)\, \mathrm{d}w \tag{5.15}$$

is the Borel transform, entire on $\mathbb{C}_r$; $\Gamma^{(s)}$ is a path on the s-th sheet of $\mathbb{C}_r$; $\Gamma^{(s)} = \Gamma^{(s)}_1 - \Gamma^{(s)}_2 + \Gamma^{(s)}_3$:

$$\begin{aligned}&\Gamma^{(s)}_1 : R\, e^{i\theta} , \quad \theta \in [\theta_0 + \delta\, , \theta_0 - \delta + 2\pi] \\ &\Gamma^{(s)}_2 : \rho\, e^{i\theta_0 + \delta} , \quad \rho \in [R , +\infty] \\ &\Gamma^{(s)}_3 : \rho\, e^{i\theta_0 - \delta + 2\pi} , \quad \rho \in [R , +\infty] ;\end{aligned} \tag{5.16}$$

here R is arbitrarily large so that the path lies in the analyticity domain of h; $\theta_0 \in [0\, , 2\pi r]$ defines the cut and the sheet of both h and h_B.

A proof of this property can be found in [9].

We now prove formula (5.14); the Fourier coefficients of the analytic invariants are given by

$$c_{k,1}^{\pm(s)} = \int_{i\beta-\infty}^{i\beta+\infty} h(w)\, e^{-2\pi ikw}\, \mathrm{d}w\,, \quad w \in \mathcal{D}_{+}^{(s)}\,,\ \beta > L\,. \tag{5.17}$$

On the other hand, we can exploit expression (5.15); we put the cut close to $\mathbb{R}^+$ at $\theta_0 \in \,]\,2\pi s - (1/2)\pi\,,\, 2\pi s\,[$ so that $\Gamma_1^{(s)}$ is divided by the line $\operatorname{Im} w = \beta$ into two parts $\Gamma_{10}^{(s)}$ ans $\Gamma_{11}^{(s)}$ respectively above and below it; the integral over $\Gamma_{11}^{(s)}$ vanishes in the limit $r \to \infty$ such as in the case $r = 1$; moreover

$$\begin{aligned} &\lim_{R\to+\infty} \left| \int_{\Gamma_2^{(s)}} e^{-2\pi kw} h(w)\, \mathrm{d}w \right| = \\ &= \lim_{R\to+\infty} \left| \int_R^\infty e^{-2\pi ik\rho\, e^{i(\theta+\delta)}} \sum_{j=2r}^{+\infty} h_j \rho^{-j/r}\, e^{i(\theta_0+\delta)j/r}\, e^{i(\theta_0+\delta)}\, \mathrm{d}\rho \right| \le \\ &\le \lim_{R\to+\infty} \int_R^\infty e^{2\pi k \sin(\theta_0+\delta)\rho} \sum_{j=2r}^{+\infty} |h_j|\, \rho^{-j/r}\, \mathrm{d}\rho = 0\,, \end{aligned} \tag{5.18}$$

since $\sin\theta_0 < 0$. Similarly one can prove that the integration over $\Gamma_3^{(s)}$ gives no contribution, and therefore $\int_{i\beta-\infty}^{i\beta+\infty} = -\int_{\Gamma_{10}^{(s)}}$ and formula (5.14) holds; the formulae for the higher orders can be proved following the same scheme.

6. Generalization to the resonant case

We define $\mathcal{A}_{r+1,\gamma,q}$ as the set of diffeomorphisms whose linear part is $\lambda_q z$, where $\lambda_q = \exp(2\pi ip/q)$ and whose q-th iterate belongs to $\mathcal{A}_{rq+1,\gamma,1} \equiv \mathcal{A}_{rq+1,\gamma}$. The computation of the analytic invariants of this class can be done directly on the q-th iterate.

LEMMA 6.1.— *Let f_1, $f_2 \in \mathcal{A}_{r+1,\gamma,q}$ and let e_1, e_1 be the q-th iterates, belonging to $\mathcal{A}_{qr+1,\gamma,1}$; then f_1 and f_2 are analytically conjugated if and only if e_1 and e_1 are analytically conjugated.*

Proof.— If $f_1 = h \circ f_2 \circ h^{-1}$, h analytic diffeomorphism in a neighbourhood of the fixed point, then, iterating q times, one has $e_1 \equiv f_1^{\circ q} = h \circ f_2^{\circ q} \circ h^{-1} = h \circ e_2 \circ h^{-1}$. In order to prove the other sense of the implication, we first show that given $e_1 \in \mathcal{A}_{rq+1,\gamma,1}$, $e_1 = f_1^{\circ q}$, $f_1 \in \mathcal{A}_{r+1,\gamma,q}$, then there exist only one $f \in \mathcal{A}_{r+1,\gamma,q}$ which satisfies

$$e_1 = f^{\circ q} \tag{6.1}$$

and therefore $f = f_1$. A solution of equation (6.1) can be built using the functional equation which conjugates e_1 to its normal form:

$$e_1(z) = \widehat{T}^{-1}(\zeta) \circ \left(\frac{\zeta}{\left(1 + \zeta^{kq}\right)^{1/kq}} \right) \circ \widehat{T}(z) \tag{6.2}$$

where $\widehat{T}(z)$ is a formal power series which can be resumed to $2kq$ functions analytic on sectors of aperture $2\pi/kq - \delta$, as outlined in section 5. The unique formal q-th iterative root can be constructed using the above equation [1]:

$$\widehat{f}(z) \equiv e_1^{\circ 1/q} = \widehat{T}^{-1}(\zeta) \circ \left(\frac{\lambda_q \zeta}{\left(1 + \zeta^{kq}/q\right)^{1/kq}} \right) \circ \widehat{T}(z) \; ; \tag{6.3}$$

since we know a priori the existence of a solution $f(z)$ analytic in a neighbourhood of zero, the formal series $\widehat{f}(z)$ is convergent, coincides with $f(z)$, and therefore $f(z)$ is unique.

This result allows to prove the second part of the Lemma: in fact if $f_1^{\circ q} = h \circ f_2^{\circ q} \circ h^{-1}$, then $f_1^{\circ q} = \left(h \circ f_2 \circ h^{-1}\right)^{\circ q}$, and the uniqueness of the q-th iterative root implies $f_1 = h \circ f_2 \circ h^{-1}$.

The above Lemma allows the computation of the analytic invariants of a resonant analytic diffeomorphism $f \in \mathcal{A}_{r+1,\gamma,q}$ by computing its q-th iterate ad applying the procedure outlined in the previous sections; this exhausts the aims of this paper.

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