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## Spectral study of a self-adjoint operator on $L^2(\Omega)$ related with a Poincaré type constant<sup>(\*)</sup>

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**RÉSUMÉ.** — Soit  $\Omega$  un ouvert borné et connexe de  $\mathbb{R}^N$ ,  $N \geq 2$ , de frontière lipschitzienne. L'espace  $H_0^1(\Omega)$  est muni de la norme du gradient. L'inégalité suivante a lieu pour les éléments de  $L^2(\Omega)$ .

$$\left| u \right|_{L^2(\Omega)}^2 - \frac{1}{\text{mes}(\Omega)} \left| \int_{\Omega} u(x) \, dx \right|^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{H^{-1}(\Omega)}^2$$

où  $C(\Omega) > 0$  ne dépend que de  $\Omega$ . À l'aide d'un opérateur autoadjoint sur  $L^2(\Omega)$ , on caractérise la meilleure constante dans l'inégalité précédente. Lorsque  $\Omega$  est une boule de  $\mathbb{R}^N$ ,  $N \geq 2$ , on fait l'analyse spectrale de cet opérateur et on montre que la meilleure valeur de la constante est  $N$ .

**ABSTRACT.** — Let  $\Omega$  be a connected bounded open set in  $\mathbb{R}^N$ ,  $N \geq 2$ , with lipschitzian boundary.  $H_0^1(\Omega)$  is equipped with the gradient norm. The following inequality holds for the elements of  $L^2(\Omega)$ :

$$\left| u \right|_{L^2(\Omega)}^2 - \frac{1}{\text{mes}(\Omega)} \left| \int_{\Omega} u(x) \, dx \right|^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{H^{-1}(\Omega)}^2$$

where  $C(\Omega) > 0$  depends only on  $\Omega$ . This paper provides a characterization of the best constant in the previous inequality using a self-adjoint operator on  $L^2(\Omega)$ . When  $\Omega$  is a ball in  $\mathbb{R}^N$ ,  $N \geq 2$ , the spectral study of this operator is made and in this case, we obtain that the best constant is  $N$ .

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## 1. Introduction

Throughout this paper  $\Omega$  is a connected bounded open set in  $\mathbb{R}^N$ ,  $N \geq 2$ , and its boundary  $\Gamma$  is Lipschitz-continuous as [5]. The space  $H_0^1(\Omega)$  will always be equipped with the gradient norm. Derivates of functions on  $\Omega$  will be taken in the sense of distributions.

We denote by  $H^{-1}(\Omega)$  the dual space of  $H_0^1(\Omega)$  normed by:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \left( \frac{\langle f, v \rangle}{\|v\|_{H_0^1(\Omega)}} \right),$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ .

The following inequality

$$|u|_{L^2(\Omega)}^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{H^{-1}(\Omega)}^2, \quad \forall u \in L^2(\Omega), \quad \int_{\Omega} u(x) dx = 0, \quad (1)$$

or the equivalent inequality:  $\forall u \in L^2(\Omega)$ ,

$$|u|_{L^2(\Omega)}^2 - \frac{1}{\text{mes}(\Omega)} \left| \int_{\Omega} u(x) dx \right|^2 \leq C(\Omega) \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{H^{-1}(\Omega)}^2 \quad (2)$$

where  $C(\Omega) > 0$  depends only on  $\Omega$  occurs in very many problems in the mechanics of continuous media [4]. Constant  $C(\Omega)$  then occurs in the conditions for the uniqueness and sometimes for the existence of solutions. Knowledge of a value of  $C(\Omega)$  is also important for the Numerical Analysis of these problems.

This paper provides a characterization of the best constant  $P(\Omega)$  in inequalities (1)-(2) using a self-adjoint operator on  $L^2(\Omega)$ . Except if  $\Omega$  is a ball in  $\mathbb{R}^N$ , the explicit value of this best constant  $P(\Omega)$  is out of reach. In the particular case where  $\Omega$  is a rectangle in 2-D, we obtain an approximate value of this best constant  $P(\Omega)$ .

The reader will recall that if  $u \in H^1(\Omega)$ , we have the following inequality, called Poincaré's inequality:

$$\left| u \right|_{L^2(\Omega)}^2 - \frac{1}{\text{mes}(\Omega)} \left| \int_{\Omega} u(x) \, dx \right|^2 \leq K(\Omega) \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|_{L^2(\Omega)}^2,$$

where  $K(\Omega) > 0$  depends only on  $\Omega$ . It is well known ([3]) that the best constant in this inequality, called Poincaré's constant, is the inverse of the smallest (positive) eigenvalue of the operator  $-\Delta$  (in  $L^2(\Omega)$ ) for the Neumann problem. The explicit value of this constant is in general out of reach.

This paper is organized as follows.

Section 2 introduces some function spaces and we draw attention to an important inequality due to J. Nečas [6] for the elements of  $L^2(\Omega)$  which plays an essential role in the proof of inequalities (1)-(2).

In Section 3, we propose a proof of coercivity inequality (2) and, in particular, as a consequence, we obtain the following well known result:

$$\text{if } f \in D'(\Omega) \text{ with } \text{grad}(f) \in (H^{-1}(\Omega))^N, \text{ then } f \in L^2(\Omega).$$

In Section 4, we prove that the best constant in inequalities (1)-(2) is the inverse of the smallest spectral value of the operator  $T = -\text{div} \circ (-\Delta)^{-1} \circ \text{grad}$  on  $L^2(\Omega)/\mathbb{R}$ .

In Section 5, we prove that the inverse of this best constant  $P(\Omega)$  is the limit of a decreasing positive sequence which has for general term the smallest eigenvalue of the matrix corresponding to a positive definite quadratic form on a suitable finite dimensional euclidean space.

In Section 6, we consider two particular cases:

- $\Omega$  is a rectangle in 2-D. Using an appropriate basis of  $L^2(\Omega)$ , we obtain an approximate value of this constant  $P(\Omega)$ ;
- $\Omega$  is a ball in  $\mathbb{R}^N$  with  $N \geq 2$ . We make the complete spectral study of operator  $-\text{div} \circ (-\Delta)^{-1} \circ \text{grad}$  and we obtain that  $P(\Omega) = N$ .

## 2. Preliminaries

Throughout this paper we suppose for simplicity that all functions are real.

We use the usual product topology on the product spaces. We denote

$$\partial_i = \frac{\partial}{\partial x_i}.$$

For  $u = (u_1, \dots, u_N) \in (D'(\Omega))^N$ , we set  $\Delta u = (\Delta u_1, \dots, \Delta u_N)$ . For  $f = D'(\Omega)$ , we set  $\text{grad}(f) = (\partial_1 f, \dots, \partial_N f)$ .

In  $L^2(\Omega)$  the Hilbert norm and the scalar product are written  $|\cdot|_2$  and  $(\cdot, \cdot)_2$ .

Let  $M(\Omega)$  be the closed subspace of  $L^2(\Omega)$  of functions of zero mean (orthogonal to constants):

$$M(\Omega) = \left\{ u \in L^2(\Omega) \mid \int_{\Omega} u(x) \, dx = 0 \right\},$$

$M(\Omega)$  is equipped with the norm induced by Hilbert space  $L^2(\Omega)$ .

The quotient space  $L^2(\Omega)/\mathbb{R}$ , equipped with the usual quotient norm, is isometrically isomorphic to  $M(\Omega)$ . This isomorphism maps each equivalence class to its element of minimal norm, which is also the unique element of mean zero in the class. By convention we write  $L^2(\Omega)/\mathbb{R} \equiv M(\Omega)$ .

We recall that  $H_0^1(\Omega)$  is equipped with the gradient norm, denoted by  $\|\cdot\|$  and  $H^{-1}(\Omega)$  is equipped with the dual norm.  $(H_0^1(\Omega))^N$  is isomorphic to  $(H^{-1}(\Omega))^N$  and  $-\Delta$  is this isometric isomorphism.

For simplicity, in the remainder of this paper, we shall write indiscriminately  $\|\cdot\|$  for the norm on  $H_0^1(\Omega)$  or on  $(H_0^1(\Omega))^N$  and  $\|\cdot\|_{-1}$  (resp.  $((\cdot, \cdot))_{-1}$ ) for the norm (resp. scalar product) on  $H^{-1}(\Omega)$  or on  $(H^{-1}(\Omega))^N$ .

We introduce the following closed subspaces of  $(H_0^1(\Omega))^N$ :

$$V = \left\{ u \in (H_0^1(\Omega))^N \mid \text{div}(u) = 0 \right\},$$

$V^\perp$  : the subspace of  $(H_0^1(\Omega))^N$  orthogonal to  $V$ .

$V$  and  $V^\perp$  are equipped with the norm induced by  $(H_0^1(\Omega))^N$  (for properties of  $V$  see eg. [7]).

The important inequality which follows is proved in [6].

There exist a positive constant  $N(\Omega)$  which depends only on  $\Omega$  such that:

$$|u|_2 \leq N(\Omega) \left( \|u\|_{-1} + \|\text{grad}(u)\|_{-1} \right), \quad \text{for all } u \in L^2(\Omega). \quad (3)$$

### 3. Poincaré type inequality on $L^2(\Omega)$

PROPOSITION 1. — *There exist a constant  $C(\Omega) \geq 1$ , depending only on  $\Omega$ , such that:*

$$|u|_2^2 \leq C(\Omega) \|\text{grad}(u)\|_{-1}^2 + \frac{1}{\text{mes}(\Omega)} \left( \int_{\Omega} u(x) \, dx \right)^2, \quad \forall u \in L^2(\Omega).$$

*Proof.* —  $\text{grad} \in L(L^2(\Omega), (H^{-1}(\Omega))^N)$  and the kernel of  $\text{grad}$  is  $\mathbb{R}$  because  $\Omega$  is connected. Consequently,  $\text{grad}$  is a linear continuous injective mapping from  $L^2(\Omega)/\mathbb{R}$  into  $(H^{-1}(\Omega))^N$ .

Now we are going to show, by contradiction, that  $\text{grad}$  is bicontinuous.

We suppose that  $\text{grad}^{-1}$  is not bounded at 0. Then there exists a sequence  $\{\hat{u}_p\}$  of  $L^2(\Omega)/\mathbb{R}$  such that  $|\hat{u}_p|_{L^2(\Omega)/\mathbb{R}} = 1$  and

$$\lim_{p \rightarrow \infty} \|\text{grad}(\hat{u}_p)\|_{-1} = 0,$$

whence  $|u_p^0|_2 = 1$  and

$$\lim_{p \rightarrow \infty} \|\text{grad}(u_p^0)\|_{-1} = 0$$

where  $u_p^0$  is the unique element of  $\hat{u}_p$  of minimal norm.

Taking into account that the injection from  $L^2(\Omega)$  into  $H^{-1}(\Omega)$  is compact, there exist a subsequence  $\{u_{p_k}^0\}$  which converges in  $H^{-1}(\Omega)$ .

It follows from (3) that there exist  $u^0 \in L^2(\Omega)$  such that:

$$\lim_{k \rightarrow \infty} u_{p_k}^0 = u^0 \text{ in } L^2(\Omega) \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\text{grad}(u_{p_k}^0)\|_{-1} = 0.$$

This result implies that  $\text{grad}(u^0) = 0$ , therefore

$$\lim_{k \rightarrow \infty} \widehat{u}_{p_k} = \widehat{0}$$

in contradiction with the definition of sequence  $\{\widehat{u}_p\}$ .

Consequently, there exist a positive constant  $C(\Omega)$ , depending only on  $\Omega$ , such that:

$$|u|_2^2 \leq C(\Omega) \|\text{grad}(u)\|_{-1}^2, \quad \text{for all } u \in M(\Omega) \equiv L^2(\Omega)/\mathbb{R}. \quad (4)$$

On the other hand, for each  $u \in L^2(\Omega)$ , there exist an unique element  $v \in M(\Omega)$  such that

$$u = v + \frac{1}{\text{mes}(\Omega)} \int_{\Omega} u(x) \, dx.$$

Hence

$$|u|_2^2 = |v|_2^2 + \frac{1}{\text{mes}(\Omega)} \left( \int_{\Omega} u(x) \, dx \right)^2.$$

Then, it follows from inequality (4),

$$|u|_2^2 \leq C(\Omega) \|\text{grad}(u)\|_{-1}^2 + \frac{1}{\text{mes}(\Omega)} \left( \int_{\Omega} u(x) \, dx \right)^2, \quad \forall u \in L^2(\Omega). \quad (5)$$

As  $|\text{div}(\phi)|_2 \leq \|\phi\|$  for all  $\phi \in (H_0^1(\Omega))^N$  [7, p. 140],

$$\|\text{grad}(u)\|_1 = \text{Sup}_{\|v\| \leq 1} \langle \text{grad}(u), v \rangle = \text{Sup}_{\|v\| \leq 1} (u, \text{div}(v))_2 \leq |u|_2 |\text{div}(v)|_2.$$

Therefore  $\|\text{grad}(u)\|_{-1} \leq |u|_2$ . Then, it follows from (4) that  $C(\Omega) \geq 1$ .

**COROLLARY 1.** —  $-\text{div} \circ (-\Delta)^{-1} \circ \text{grad}$  is an isomorphism from  $M(\Omega)$  onto  $M(\Omega)$ .

*Proof.* — We identify  $L^2(\Omega)$  with its dual. It follows from proposition 1 that  $\text{grad}$  is an isomorphism from  $M(\Omega)$  into  $(H^{-1}(\Omega))^N$ . Then its adjoint,  $-\text{div}$ , is an isomorphism from  $(H_0^1(\Omega))^N/V$  onto  $M(\Omega)$ . By transposition,  $\text{grad}$  is an isomorphism from  $M(\Omega)$  onto the dual space of  $(H_0^1(\Omega))^N/V$  that is onto  $V^0$  (the annihilator of  $V$ ). It is not difficult to see that  $V^0 = -\Delta(V^\perp)$ .

We deduce from corollary 1 the following well known result.

**COROLLARY 2.** — *If  $f \in D'(\Omega)$  and  $\text{grad}(f) \in (H^{-1}(\Omega))^N$ , then  $u \in L^2(\Omega)$ .*

*Proof.* — Let  $v \in (D(\Omega))^N$  such that  $\text{div}(v) = 0$ . Then we have:

$$\langle \text{grad}(f), v \rangle = -\langle f, \text{div}(v) \rangle = 0.$$

Therefore  $\text{grad}(f) \in V^0$ . Thus, there exist  $g \in L^2(\Omega)/\mathbb{R}$  such that  $\text{grad}(f) = \text{grad}(g)$ . It follows that  $f = g + C$  because  $\Omega$  is connected.

**NOTATION .** — *In the remainder of this paper, the best value of constant  $C(\Omega)$  in inequalities (4)-(5) is denoted by  $P(\Omega)$ :*

$$P(\Omega)^{-1} = \inf_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{\|\text{grad}(u)\|_{-1}^2}{|u|_2^2}.$$

#### 4. The operator related with the Poincaré type constant $P(\Omega)$

From corollary 1, the operator  $T = -\text{div} \circ (-\Delta)^{-1} \circ \text{grad}$  is an isomorphism from  $M(\Omega)$  onto  $M(\Omega)$ . Moreover, for all  $u \in M(\Omega)$ :

$$(Tu, u)_2 = \sum_{i=1}^N \langle \partial_i u, (-\Delta)^{-1} \circ \partial_i u \rangle = \|\text{grad}(u)\|_{-1}^2.$$

Consequently,

$$P(\Omega)^{-1} = \inf_{\substack{u \in M(\Omega) \\ |u|_2=1}} (Tu, u)_2.$$

Important properties of this operator  $T$  are as follows.

##### **THEOREM 1**

- 1)  $T$  is a self-adjoint and coercive operator.
- 2)  $P(\Omega)$  is the inverse of smallest spectral value of  $T$ .
- 3)  $T - I$  is a harmonic mapping in  $M(\Omega)$ .
- 4)  $\|T\| = 1$ , 1 is a eigenvalue of  $T$  and his eigenspace is infinite dimensional.



*Proof*

(1) For all  $(u, v) \in M(\Omega) \times M(\Omega)$  we have:

$$(Tu, v)_2 = \sum_{i=1}^N \left\langle \partial_i v, (-\Delta)^{-1} \circ \partial_i u \right\rangle = ((\text{grad}(u), \text{grad}(v)))_{-1} = (v, Tv)_2,$$

$$(Tu, u)_2 = \|\text{grad}(u)\|_{-1}^2 \geq P(\Omega)^{-1} |u|_2^2.$$

Then  $T$  is a self-adjoint and coercive operator and  $P(\Omega)^{-1}$  is the best value of the coercivity constant.

(2) We denote by  $\sigma(T)$  the spectrum of  $T$ . It follows from (1) [1] that the residual spectrum of  $T$  is empty,  $\sigma(T)$  is closed and it lies in the closed interval  $[m, M]$  on the real axis, where

$$m = \inf_{\lambda \in \sigma(T)} \lambda = \inf_{\substack{u \in M(\Omega) \\ |u|_2=1}} (Tu, u)_2 = P(\Omega)^{-1},$$

$$M = \sup_{\lambda \in \sigma(T)} \lambda = \sup_{\substack{u \in M(\Omega) \\ |u|_2=1}} (Tu, u)_2 = \|T\|.$$

So,  $P(\Omega)^{-1} \in \sigma(T)$  and it is the smallest spectral value of  $T$ .

(3) For all  $u \in M(\Omega)$  we have, in the sens of distributions on  $\Omega$ :

$$\Delta \circ T(u) = - \sum_{i=1}^N \Delta \left( \partial_i \circ (-\Delta)^{-1} \circ \partial_i \right) (u) = \Delta u.$$

So,  $T(u) - u$  is a harmonic distribution on  $\Omega$ , thus it is a harmonic function on  $\Omega$ , for all  $u \in M(\Omega)$ .

(4) Let  $H(\Omega)$  be the closed subspace of  $M(\Omega)$  of harmonic functions:

$$H(\Omega) = \{u \in M(\Omega) \mid \Delta u = 0\}$$

and we denote by  $H(\Omega)^\perp$  the orthogonal complement of  $H(\Omega)$  in  $M(\Omega)$ .

Let  $v \in H(\Omega)^\perp$  be such that  $v \neq 0$  and  $w \in H(\Omega)$ . It follows from (3) that

$$(T(v) - v, w)_2 = (v, T(w) - w)_2 = 0.$$

Then  $T(v) - v \in H(\Omega)^\perp \cap H(\Omega)$ , consequently  $Tv = v$  and 1 is an eigenvalue of  $T$ .

Now let  $\phi \in D(\Omega)$  be with  $\Delta\phi \neq 0$ , we have that  $T(\Delta\phi) = \Delta\phi$ . Then, the eigenspace corresponding to the eigenvalue 1 is infinite dimensional.

On the other hand, for all  $v \in H(\Omega)^\perp$ ,  $v \neq 0$ , we have:

$$|v|_2^2 = (T(v), v)_2 \leq \|T\| |v|_2^2, \quad \text{from where } \|T\| \geq 1.$$

Moreover:

$$\|T\| = \sup_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{(T(u), u)_2}{|u|_2^2} = \sup_{\substack{u \in M(\Omega) \\ u \neq 0}} \frac{\|\text{grad}(u)\|_{-1}^2}{|u|_2^2} \leq 1.$$

Consequently,  $\|T\| = 1$ . Finally [1],

$$\|T\| = \sup_{\lambda \in \sigma(T)} \lambda.$$

So, the eigenvalue 1 is the largest spectral value of  $T$ .

**COROLLARY 3.** — *If  $u$  is an eigenvector of  $T$  corresponding to an eigenvalue  $\lambda \neq 1$ , then  $u$  is a harmonic function on  $\Omega$ .*

In the following section, we are going to give a method to approximate the constant  $P(\Omega)$ .

## 5. Approximation of the Poincaré type constant $P(\Omega)$

$M(\Omega)$  is separable. Let  $\{\varepsilon_j \mid 0 < j < \infty\}$  be an orthonormal basis of  $M(\Omega)$ , we consider the sequence  $\{M_K(\Omega)\}_{K>0}$  of finite dimensional subspaces of  $M(\Omega)$  defined as follows: for all integer  $K > 0$ ,  $M_K(\Omega)$  is spanned by the family of vectors  $\{\varepsilon_j \mid 0 < j \leq K\}$ .

Then  $M_K(\Omega) \subset M_{K+1}(\Omega)$ ,  $\forall K > 0$ , and for all  $u \in M(\Omega)$ , there exist a sequence  $\{u_K\}_{K>0}$ ,  $u_K \in M_K(\Omega)$ , such that  $\lim_{K \rightarrow \infty} |u - u_K|_2 = 0$ .

For all integer  $K > 0$ , let us put

$$\alpha_K = \inf_{\substack{w \in M_K(\Omega) \\ w \neq 0}} \frac{\|\text{grad}(w)\|_{-1}^2}{|w|_2^2} = \inf_{\substack{w \in M_K(\Omega) \\ w \neq 0}} \frac{(T(w), w)_2}{|w|_2^2}.$$

Then  $\alpha_K \geq P(\Omega)^{-1}$  and  $\alpha_K \geq \alpha_{K+1}$ ,  $\forall K > 0$ . Thus, the sequence  $\{\alpha_K\}_{K>0}$  is convergent and its limit  $\alpha$  is such that  $\alpha \geq P(\Omega)^{-1}$ . On the other hand, let  $u$  be some element of  $M(\Omega)$ . Then, there exist a sequence  $\{u_K\}_{K>0}$  with  $u_K \in M_K(\Omega)$ , such that if  $K \rightarrow \infty$ ,  $u_K \rightarrow u$  in  $L^2(\Omega)$  and therefore  $\text{grad}(u_K) \rightarrow \text{grad}(u)$  in  $(H^{-1}(\Omega))^N$ . Furthermore, we have  $\|\text{grad}(u_K)\|_{-1}^2 \geq \alpha_K |u_K|_2^2$ . Passing to limit when  $K \rightarrow \infty$ , we obtain  $\|\text{grad}(u)\|_{-1}^2 \geq \alpha |u|_2^2$ . This result implies that  $\alpha \leq P(\Omega)^{-1}$  and consequently  $P(\Omega)^{-1} = \alpha$ .

Now, we are going to specify the elements of sequence  $\{\alpha_K\}_{K>0}$ .

Let  $K$  be some positive integer. Each  $u \in M_K(\Omega)$  has a unique decomposition  $u = \sum_{j=1}^K \lambda_j \varepsilon_j$ . Put  $\chi = Gu = (\lambda_1, \lambda_2, \dots, \lambda_K) \in \mathbb{R}^K$ .

In what follows,  $M_K(\Omega)$  is equipped with the norm induced by  $L^2(\Omega)$  and  $\mathbb{R}^K$  is equipped with the usual euclidean norm  $\|\cdot\|_0$ . Then  $G$  is an isometric isomorphism from  $M_K(\Omega)$  onto  $\mathbb{R}^K$ .

On the other hand,

$$(Tu, u)_2 = (T \circ G^{-1}\chi, G^{-1}\chi)_2 \geq P(\Omega)^{-1} \|\chi\|_0^2.$$

Therefore

$$\chi \rightarrow Q_K(\chi) = (T \circ G^{-1}\chi, G^{-1}\chi)_2 = \sum_{i,j=1}^K \lambda_i \lambda_j (T\varepsilon_i, \varepsilon_j)_2 \quad (6)$$

is a positive definite quadratic form on  $\mathbb{R}^K$  and furthermore:

$$\alpha_K = \inf_{\substack{\chi \in \mathbb{R}^K \\ \chi \neq 0}} \frac{Q_K(\chi)}{\|\chi\|_0^2}.$$

So, we have proved the following result.

**PROPOSITION 2.** —  $\alpha_K$  is the smallest eigenvalue (minimum of a Rayleigh quotient) of the matrix  $A_K$  of the quadratic form defined on  $\mathbb{R}^K$  by formula (6) and  $\alpha_K \rightarrow P(\Omega)^{-1}$ , when  $K \rightarrow \infty$ .

## 6. Two particular cases

**6.1 The open is the rectangle**  $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L, 0 < y < \ell\}$

We use the previous results with  $N = 2$ .

We must now calculate the elements  $(T\varepsilon_i, \varepsilon_j)_2$  of the matrix  $A_K$ . For all  $(m, p) \in \mathbb{N}^2$ , we put

$$\omega_m = \frac{m\pi}{L}, \quad \delta_p = \frac{p\pi}{\ell}, \quad \alpha_{m,p} = \omega_m^2 + \delta_p^2$$

and introduce real positive numbers  $c_{m,p}$  defined by

$$c_{0,p}^2 = c_{m,0}^2 = \frac{1}{2} c_{m,p}^2 = \frac{2}{L\ell}, \quad \text{for } m \geq 1 \text{ and } p \geq 1.$$

We choose the following orthonormal basis of  $M(\Omega)$ :

$$\{e_{m,p}(x, y) = c_{m,p} \cos(\omega_m x) \cos(\delta_p y) \mid (x, y) \in \Omega, (m, p) \in \mathbb{N}^2, m + p \geq 1\}.$$

The elements of the matrix  $A_K$  are the following real numbers:

$$(Te_{m,p}, e_{j,q})_2 \quad \text{with } (m, p) \in \mathbb{N}^2, (j, q) \in \mathbb{N}^2, \\ 0 \leq m, p, j, q \leq K, m + p \geq 1 \text{ and } j + q \geq 1.$$

More precisely,  $(Te_{m,p}, e_{j,q})_2$  is the element of the  $(m(K+1) + p)$ -th row and  $(j(K+1) + q)$ -th column of the matrix  $A_K$ .

In order to give explicit values of  $(Te_{m,p}, e_{j,q})_2$ , it is convenient to introduce  $a(r, s)$  for  $(r, s) \in \mathbb{N}^2$  with  $r \geq 1$  and  $s \geq 1$ , given by

$$a(r, 0) = a(0, s) = 1, \quad a(r, s) = \sqrt{2}.$$

The calculation of the scalar products  $(Te_{m,p}, e_{j,q})_2$  is long but not very difficult. We obtain:

- (i)  $(Te_{m,p}, e_{j,q})_2 = 0$  if  $m \neq j$  and  $p \neq q$ ;
- (ii)  $(Te_{m,p}, e_{m,q})_2 = -\frac{\sqrt{2} a(p, q) \omega_m^3 (1 + (-1)^{p+q}) B_{m,p}}{\ell \alpha_{m,p} \alpha_{m,q}}$  if  $p \neq q$ ,

$$(iii) (Te_{m,p}, e_{j,p})_2 = -\frac{\sqrt{2} a(m, j) \delta_p^3 (1 + (-1)^{m+j}) \Lambda_{m,p}}{L \alpha_{m,p} \alpha_{j,p}} \text{ if } m \neq j,$$

$$(iv) (Te_{m,p}, e_{m,p})_2 = 1 - a(m, p)^2 \left( \frac{2\omega_m^3 B_{m,p}}{\ell \alpha_{m,p}^2} - \frac{2\delta_p^3 \Lambda_{m,p}}{L \alpha_{m,p}^2} \right),$$

where

$$B_{0,p} = 0, \quad B_{m,p} = \frac{\exp(\omega_m \ell) + \exp(-\omega_m \ell) - 2(-1)^p}{\exp(\omega_m \ell) - \exp(-\omega_m \ell)}, \quad \forall m \geq 1, \quad \forall p \geq 0$$

$$\Lambda_{m,0} = 0, \quad \Lambda_{m,p} = \frac{\exp(\delta_p L) + \exp(-\delta_p L) - 2(-1)^m}{\exp(\delta_p L) - \exp(-\delta_p L)}, \quad \forall m \geq 0, \quad \forall p \geq 1.$$

*Remark 1.* —  $A_K$  and consequently  $P(\Omega)^{-1}$ , depend only on the ratio of the dimensions of the rectangle.

*Numerical results.* — We have performed a few numerical tests. Let  $K$  be a positive integer. We have computed an approximate value of the smallest eigenvalue  $\alpha_K$  of the matrix  $A_K$  by means of the power of Mises [2, pp. 226-227]. We stopped this calculation when the relative error was less than  $10^{-9}$ . We have ascertained that sequence  $\{\alpha_K\}_{K>0}$  converges quickly.

The above mentioned values of the constant  $P(\Omega)^{-1}$  have been rounded up to the 3-th decimal place.

$$L = 1, \quad \ell = 1 : P(\Omega)^{-1} = 0.226$$

$$L = 2, \quad \ell = 1 : P(\Omega)^{-1} = 0.151$$

$$L = 4, \quad \ell = 1 : P(\Omega)^{-1} = 0.047.$$

**6.2 Then open is**  $\Omega = \{(x_1, x_2, \dots, x_{n+2}) \in \mathbb{R}^{n+2} \mid x_1^2 + \dots + x_{n+2}^2 < 1\}$   
**with**  $n \in \mathbb{N}$

In this case, we are able to give the spectrum of  $T$ . In that follows, the following proposition is essential.

**PROPOSITION 3.** — *Each harmonic homogeneous polynomial in  $\Omega$  of degree  $m \geq 1$  is an eigenvector of  $T$  corresponding to the eigenvalue  $m/(2m + n)$ .*

*Proof.* — Let  $u(x_1, x_2, \dots, x_{n+2})$  be a harmonic homogeneous polynomial in  $\Omega$  of degree  $m \geq 1$ . We have

$$\sum_{i=1}^{n+2} x_i \partial_i u = mu.$$

Hence, for all  $j = 1, 2, \dots, n+2$ ,

$$\partial_j u + \sum_{i=1}^{n+2} x_i \frac{\partial^2 u}{\partial x_j \partial x_i} = m \partial_j u. \quad (7)$$

On the other hand:

$$\begin{aligned} \Delta((x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \partial_j u) &= 2(n+2) \partial_j u + 4 \sum_{i=1}^{n+2} x_i \frac{\partial^2 u}{\partial x_i \partial x_j} + \\ &\quad + x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \partial_j (\Delta u), \end{aligned}$$

and from (7)

$$\begin{aligned} \Delta((x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \partial_j u) &= 2(n+2) \partial_j u + 4(m-1) \partial_j u \\ &= (2n+4m) \partial_j u. \end{aligned} \quad (8)$$

Since the function  $(x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \partial_j u \in D(\Omega)$ , it follows from (8) that

$$(-\Delta)^{-1}(\partial_j u) = -\frac{1}{4m+2n} (x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \partial_j u.$$

Therefore,

$$\begin{aligned} -\operatorname{div} \circ (-\Delta)^{-1} \circ \operatorname{grad}(u) &= \\ &= \frac{1}{4m+2n} \left( 2 \sum_{j=1}^{n+2} x_j \partial_j u + (x_1^2 + x_2^2 + \dots + x_{n+2}^2 - 1) \Delta u \right), \\ Tu &= \frac{1}{2m+n} \sum_{j=1}^{n+2} x_j \partial_j u = \frac{m}{2m+n} u. \end{aligned}$$

We denote by  $H_m(\Omega)$  the eigenspace of  $T$  corresponding to the eigenvalue  $m/(2m+n)$  with  $m \in \mathbb{N}^*$ .

We recall that  $H(\Omega)$  is the subspace of  $M(\Omega)$  of harmonic functions and  $H(\Omega)^\perp$  its orthogonal complement in  $M(\Omega)$ . We have the following result.

PROPOSITION 4 [3]. — *The family of harmonic homogeneous polynomials in  $\Omega$  of degree  $m \geq 1$  is free and total in  $H(\Omega)$ .*

PROPOSITION 5. — *The orthogonal complement  $H(\Omega)^\perp$  of  $H(\Omega)$  in  $M(\Omega)$  is the eigenspace of  $T$  corresponding to the eigenvalue 1.*

*Proof.* — We denote by  $H_{-n}(\Omega)$  the eigenspace of  $T$  corresponding to the eigenvalue 1. It is proved in theorem 1(4) that  $H(\Omega)^\perp \subset H_{-n}(\Omega)$ .

On the other hand, eigenvectors corresponding to different eigenvalues of  $T$  are orthogonal; from propositions 3 and 4 we have  $H_{-n}(\Omega) \subset H(\Omega)^\perp$ .

Then,  $H_{-n}(\Omega) = H(\Omega)^\perp$ , and consequently,

$$M(\Omega) = H_{-n}(\Omega) \oplus H(\Omega).$$

COROLLARY 4. — *The only eigenvalues of  $T$  are 1 and  $m/(2m+n)$  with  $m \in \mathbb{N}^*$ .*

$T$  is a bounded self-adjoint operator on  $M(\Omega)$ . Its spectrum  $\sigma(T)$  is partitioned into two disjoint sets: the point spectrum  $\sigma_p(T)$  and the continuous spectrum  $\sigma_c(T)$ .

Now, we are going to specify two cases.

1)  $n = 0$ ,  $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$

In this case, the harmonic homogeneous polynomials of degree  $m \geq 1$  are eigenvectors of  $T$  corresponding to the eigenvalue  $1/2$ .

THEOREM 3. —  *$T$  has a pure point spectrum.  $1/2$  and 1 are the only eigenvalues of  $T$ .*

*Proof.* — It follows from corollary 4 that  $\sigma_p(T) = \{1/2, 1\}$ .

Now, let  $\alpha \in \mathbb{R}$  be different of  $1/2$  or 1, and  $v \in M(\Omega)$ . We have

$$v = v_{1/2} + v_0 \quad \text{with } v_{1/2} \in H_{1/2}(\Omega) \text{ and } v_0 \in H(\Omega)^\perp,$$

where  $H_{1/2}(\Omega)$  is the eigenspace of  $T$  corresponding to the eigenvalue  $1/2$ . We take

$$w = \frac{2}{1-2\alpha} v_{1/2} + \frac{2}{1-\alpha} v_0;$$

then  $w \in M(\Omega)$ , and we check that  $(T - \alpha I)w = v$ . Therefore,  $\alpha \notin \sigma(T)$ .

*Remark 2.*— The two eigenspaces of  $T$  are infinite dimensional.

$$2) \quad n \geq 1, \Omega = \{(x_1, \dots, x_{n+2}) \in \mathbb{R}^{n+2} \mid x_1^2 + \dots + x_{n+2}^2 < 1\}$$

**THEOREM 2.**—  $\sigma_p(T) = \{1, m/(2m+n) \mid m \in \mathbb{N}^*\}$  and  $\sigma_c(T) = \{1/2\}$ .

*Proof.*— From corollary 4,

$$\sigma_p(T) = \left\{ 1, \frac{m}{2m+n} \mid m \in \mathbb{N}^* \right\}.$$

Now, let  $\alpha \in \mathbb{R}$  be different of  $1/2$ ,  $1$  or  $m/(2m+n)$  with  $m \in \mathbb{N}^*$ , and  $v \in M(\Omega)$ . We have

$$v = \sum_{m=1}^{\infty} v_m + v_0 \quad \text{with } v_m \in H_m(\Omega) \text{ and } v_0 \in H(\Omega)^\perp.$$

We take

$$w = \sum_{m=1}^{\infty} \frac{2m+n}{m-\alpha(2m+n)} v_m + \frac{1}{1-\alpha} v_0;$$

then  $w \in M(\Omega)$ , and we check that  $(T - \alpha I)w = v$ . Therefore,  $\alpha \notin \sigma(T)$ .

Consequently, the limit  $1/2$  of the eigenvalues is the only element of the continuous spectrum of the operator  $T$ .

**COROLLARY 5.**— *The eigenspace of  $T$  corresponding to the eigenvalue  $m/(2m+n)$  is finite dimensional.*

*Proof.*— Let  $P_m$  be the vector subspace of  $M(\Omega)$  spanned by the harmonic homogeneous polynomials of degree  $m \geq 1$  ( $\dim P_m < +\infty$ ). It follows from propositions 3 and 4 that  $H(\Omega)$  is spanned by the family  $\{P_m \mid m \geq 1\}$ :

$$M(\Omega) = \bigotimes_{m \geq 1} P_m \oplus H(\Omega)^\perp.$$

On the other hand, suppose that there is an eigenvector  $u$  corresponding to the eigenvalue  $\lambda_m = m/(2m+n)$ , such that  $u \notin P_m$ . From corollary 3,  $u \in H(\Omega)$ . Since  $u \perp H(\Omega)^\perp$ ,  $u$  can be written

$$u = \sum_{\substack{j \geq 1 \\ j \neq m}} u_j, \quad \text{where } u_j \in P_j.$$



Then

$$\lambda_m u = Tu = \sum_{\substack{j \geq 1 \\ j \neq m}} \lambda_j u_j, \quad \text{with } \lambda_j = \frac{j}{2j+n}$$

and

$$\sum_{\substack{j \geq 1 \\ j \neq m}} (\lambda_j - \lambda_m) u_j = 0.$$

But this is impossible because the family  $\{u_j\}_{j \geq 1}$  is free.

Hence  $u \in P_m$ ; thus  $P_m$  is the eigenspace of  $T$  corresponding to the eigenvalue  $m/(2m+n)$  and the corollary is proved.

We recall that the eigenspace of  $T$  corresponding to 1 is infinite dimensional.

**COROLLARY 6.** — *If  $\Omega$  is the ball  $\Omega = \{(x_1, \dots, x_N) \mid x_1^2, \dots, x_N^2 < 1\}$  with  $N \geq 2$ , then  $P(\Omega) = N$ .*

*Remark 3.* — The spectrum of the operator  $T$  is independent of the radius of the ball in  $\mathbb{R}^N$ .

*Remark 4.* — Now we are going to give a family of eigenvectors of  $T$  which is total in  $M(\Omega)$ .

First we consider the orthogonal basis for  $M(\Omega)$  formed by the eigenvectors of  $-\Delta$  (in  $L^2(\Omega)$ ) for the Neumann problem.

To write these functions, the appropriate polar co-ordinates are

$$\begin{aligned} x_1 &= \rho \cos \theta_1, \\ x_2 &= \rho \sin \theta_1 \cos \theta_2, \\ x_3 &= \rho \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ &\dots \\ x_{n+1} &= \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_n \cos \varphi, \\ x_{n+2} &= \rho \sin \theta_1 \sin \theta_2 \cdots \sin \theta_n \sin \varphi; \end{aligned}$$

with  $0 \leq \varphi \leq 2\pi$ ,  $0 \leq \theta_1, \theta_2, \dots, \theta_n \leq \pi$  and  $0 \leq \rho < 1$ .

For simplicity, we put:

$$L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(t) = P_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(t), \quad t \in [0, 1],$$

if  $n - i$  is a positive even integer and

$$L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(t) = Q_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(t), \quad t \in [0, 1],$$

if  $n - i$  is a positive odd integer, where  $P_\mu^\eta$  is the associate Legendre function of the first kind of order  $\mu$  and degree  $\eta$ , and  $Q_\mu^\eta$  is the associate Legendre function of the second kind of order  $\mu$  and degree  $\eta$  [8].

In these polar co-ordinates this orthogonal basis for  $M(\Omega)$  is the family of functions  $\{\Phi_{\nu,k}^q \mid k \in \mathbb{N}\} \cup \{\Psi_{\nu,k}^q \mid k \in \mathbb{N}^*\}$  defined by

$$\begin{aligned} \Phi_{\nu,k}^q &= \rho^{-n/2} J_{\nu_1+n/2}(\lambda_{\nu_1+n/2,q}\rho) \times \\ &\times \left( \prod_{i=1}^n \left( L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(\cos \theta_i) (\operatorname{cosec} \theta_i)^{(n-i)/2} \right) \right) \cos(k\varphi), \end{aligned}$$

$$\begin{aligned} \Psi_{\nu,k}^q &= \rho^{-n/2} J_{\nu_1+n/2}(\lambda_{\nu_1+n/2,q}\rho) \times \\ &\times \left( \prod_{i=1}^n \left( L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(\cos \theta_i) (\operatorname{cosec} \theta_i)^{(n-i)/2} \right) \right) \sin(k\varphi). \end{aligned}$$

with  $q \in \mathbb{N}^*$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{N}^n$ ,  $k = \nu_{n+1} \leq \nu_n \leq \dots \leq \nu_1$ ; where  $J_{\nu_1+n/2}$  is the Bessel function of the first kind of order  $\nu_1 + n/2$  [9], and  $\lambda_{\nu_1+n/2,q}$  are the positive roots of equation

$$\left. \frac{d}{d\rho} (\rho^{-n/2} J_{\nu_1+n/2}(\lambda\rho)) \right|_{\rho=1} = 0.$$

If we write the harmonic homogeneous polynomials of degree  $\nu_1 \geq 1$  in these polar co-ordinates, we have

$$\begin{aligned} u_{\nu,k}^{(1)}(\rho, \theta_1, \dots, \theta_n, \varphi) &= \\ &= \rho^{\nu_1} \left( \prod_{i=1}^n \left( L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(\cos \theta_i) (\operatorname{cosec} \theta_i)^{(n-i)/2} \right) \right) \cos(k\varphi), \quad k \in \mathbb{N}, \end{aligned}$$

$$\begin{aligned}
 u_{\nu,k}^{(2)}(\rho, \theta_1, \dots, \theta_n, \varphi) &= \\
 &= \rho^{\nu_1} \left( \prod_{i=1}^n \left( L_{\nu_i+(n-i)/2}^{\nu_{i+1}+(n-i)/2}(\cos \theta_i) (\operatorname{cosec} \theta_i)^{(n-i)/2} \right) \right) \sin(k\varphi), \quad k \in \mathbb{N}^*
 \end{aligned}$$

where  $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{N}^n$ ,  $k = \nu_{n+1} \leq \nu_n \leq \dots \leq \nu_1$ . We say that this family of harmonic homogeneous polynomials is an orthogonal basis for  $H(\Omega)$  and, as proved in proposition 3, they are eigenvectors of  $T$  corresponding to the eigenvalue  $\nu_1/(2\nu_1 + n)$ .

Now we consider the functions  $\Phi_{0,0}^q$ ,  $\Phi_{\nu,k}^q - \gamma_{\nu,k}^q u_{\nu,k}^{(1)}$  and  $\Psi_{\nu,k}^q - \delta_{\nu,k}^q u_{\nu,k}^{(2)}$ , with  $q \in \mathbb{N}^*$ ,  $k \in \mathbb{N}^*$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{N}^n$ ,  $k = \nu_{n+1} \leq \nu_n \leq \dots \leq \nu_1$ ; where

$$\gamma_{\nu,k}^q = \frac{(\Phi_{\nu,k}^q, u_{\nu,k}^{(1)})_2}{|u_{\nu,k}^{(1)}|_2^2} \quad \text{and} \quad \delta_{\nu,k}^q = \frac{(\Psi_{\nu,k}^q, u_{\nu,k}^{(2)})_2}{|u_{\nu,k}^{(2)}|_2^2}.$$

We obtain that these functions are eigenvectors of  $T$  corresponding to the eigenvalue 1.

Finally, we prove that the family of eigenvectors of  $T$

$$\{u_{\nu,k}^{(1)}, u_{\nu,k}^{(2)}, \Phi_{0,0}^q, \Phi_{\nu,k}^q - \gamma_{\nu,k}^q u_{\nu,k}^{(1)}, \Psi_{\nu,k}^q - \delta_{\nu,k}^q u_{\nu,k}^{(2)}\}$$

with  $q \in \mathbb{N}^*$ ,  $k \in \mathbb{N}^*$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{N}^n$ ,  $k = \nu_{n+1} \leq \nu_n \leq \dots \leq \nu_1$  is total in  $M(\Omega)$ .

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