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Uniformly Convex and Uniformly Smooth Convex Functions^(*)

DOMINIQUE AZÉ⁽¹⁾ and JEAN-PAUL PENOT⁽²⁾

RÉSUMÉ. — On présente plusieurs caractérisations de l'uniforme convexité et de l'uniforme lissité des fonctions convexes. La dualité entre ces deux classes de fonctions est utilisée de façon intense et les liens avec des propriétés de monotonie sont explorés.

ABSTRACT. — A number of characterizations of uniformly convex functions and uniformly smooth functions are presented. The duality between these two classes of functions is used extensively and the relationships with monotonicity properties are explored.

KEY-WORDS : Convexity, uniform convexity, smoothness, conjugate functions, duality mappings, monotone operators, modulus, gage.

MOTS-CLÉS : Convexité, convexité uniforme, conjugaison, application de dualité, gage, lissité, module, opérateur monotone.

AMS Classification : 49A55, 47H05.

1. Introduction

For some problems, such as fractional optimization problems for instance, a generalization of convexity is required. On the other hand it is sometimes useful to restrict one's attention to a special class of convex functions. The class of uniformly convex functions introduced by Poljak [31] and studied in [36], [38], [41] is such a class. Independently of its own interest this class of functions has been successfully used in a great variety of problems: classical

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algorithms (gradient methods [16], [25], proximal algorithm [34]), first and second order duality ([19], [20]), well-posedness of optimization problems ([24], [42]). Recently Attouch and Wets [4] introduced a related class of functions in view of devising a quantitative approach to the study of stability of optimization problems. Similarly, the notion of differentiability can be made uniform and again this variant is useful in a number of problems, especially for the geometry of Banach spaces and its applications.

The usefulness of differentiability and uniform differentiability does not require long comments. The purpose of this paper is to give characterizations of the two classes of uniformly convex functions and uniformly smooth (or moderately convex) functions. We also relate these two classes through convex conjugation and inversion of subdifferentials, completing the work pioneered by Šmul'yan [35] in connection with the geometry of Banach spaces and culminating in the works of Asplund [1], Asplund and Rockafellar [3], Vladimirov & al. ([39], [40]) and Zălinescu [41]. We show the complete equivalence of four properties characterizing uniform convexity (and the dual properties for uniform smoothness):

- (1) a convexity inequality,
- (2) a subdifferential inequality involving the Weierstrass excess function,
- (3) a monotonicity property of the subdifferential,
- (4) an expansivity property of the subdifferential.

Most of this program has already been carried out in [41]. But we get rid of restrictive hypothesis on the domain of the functions or the spaces as in ([38], [41]) and we prove the missing equivalences. A simple consequence of the Ekeland's variational principle akin to a result of Brøndsted and Rockafellar [9] plays a key role for this purpose. The dual characterization we give for (1) (Corollary 2.8) seems to be new and completes the results given for (2) and (4) in [41]. Our results include a simple duality relationship between strongly convex functions and weakly convex functions on Hilbert spaces ([36], [37]). We do not look for the best relationships between the different moduli occurring in the convexity, smoothness and monotonicity properties we consider, but this question would have some interest (see section 6 for some hints). In particular we deduce from our characterizations an estimate of the modulus of uniform continuity on balls of the duality map of a uniformly smooth Banach space in terms of the uniform convexity of its dual.

A table of the implications we prove is displayed at the end of the paper.

2. Uniform Convexity and Uniform Smoothness

In this paper X and Y denote two Banach spaces; their norms, closed unit balls, unit spheres are denoted by $|\cdot|$, B , S respectively or $|\cdot|_X$, B_X , S_X if there is any risk of confusion.

Throughout we suppose X and Y are in *metric duality*; this means that there exists a nondegenerate pairing $\langle \cdot, \cdot \rangle$ from $X \times Y$ into $\mathbb{R}$ such that

$$\begin{aligned} |x| &= \sup \{ \langle x, y \rangle : y \in B_Y \} \\ |y| &= \sup \{ \langle x, y \rangle : x \in B_X \}. \end{aligned}$$

Then Y (resp. X) can be identified with a closed subspace of the dual space X^* (resp. Y^*) of X (resp. Y). This framework is not only symmetric; it is also versatile since X can be chosen either as a Banach space with dual space Y or as a dual Banach space with predual Y . It also allows different situations such as products. We denote by $\Gamma_Y(X)$ the set of proper convex functions on X with values in $\mathbb{R}^\bullet = \mathbb{R} \cup \{+\infty\}$ which are l.s.c. for the weak topology $\sigma(X, Y)$ induced on X by the pairing; we adopt a similar definition for $\Gamma_X(Y)$. Equivalently $\Gamma_Y(X)$ is the set functions from X into $\mathbb{R}^\bullet$ which are suprema of nonempty families of mappings of the form $x \mapsto \langle x, y \rangle + c$ with $y \in Y$, $c \in \mathbb{R}$. It is well known that the conjugacy correspondence $f \rightarrow f^*$ with

$$f^*(y) = \sup_{x \in X} (\langle x, y \rangle - f(x))$$

defines a bijection of $\Gamma_Y(X)$ onto $\Gamma_X(Y)$. We set $\Gamma_0(X) = \Gamma_{X^*}(X)$. The domain of $f : X \rightarrow \mathbb{R}^\bullet$ is the set

$$\text{dom } f = \{x \in X \mid f(x) < +\infty\};$$

the indicator function of a subset C of X is the function ψ_C given by

$$\text{dom } i_C = C, \quad i_C(x) = 0 \text{ for } x \in C.$$

In the sequel we denote by A the set of $\alpha : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ such that $\alpha(0) = 0$ considered by Asplund in [1]; it can be identified with the set of nonnegative extended real-valued even functions on $\mathbb{R}$ with value 0 at 0. We denote by N the set of nondecreasing $\alpha \in A$. The (positive) conjugate $\varphi^*(s) = \sup \{rs - \varphi(r) \mid r \in \mathbb{R}_+\}$ of $\varphi \in A$ belongs to N since it is convex. If $f = \varphi \circ |\cdot|$ with $\varphi \in A$ then $f^* = \varphi^* \circ |\cdot|_*$ (see [1]).

An element $\varphi \in A$ is said to be *firm* if any sequence (t_n) such that $(\varphi(t_n)) \rightarrow 0$ converges to 0. When $\varphi \in N$, this is equivalent to $\varphi(t) > 0$ for $t > 0$. We denote by Φ the set of $\varphi \in A$ which are firm.

A *gage* is an element of N which is firm: the set G of gages is $G = \Phi \cap N$.

A *modulus* is an element of the set $M = \Lambda \cap N$ with

$$\Lambda = \left\{ \lambda \in A \mid \lim_{t \rightarrow 0_+} \lambda(t) = 0 \right\}.$$

An *hypermodulus* is an element of the set $H = \Omega \cap N$ where

$$\Omega = \left\{ \omega \in A \mid \lim_{t \rightarrow 0_+} t^{-1} \omega(t) = 0 \right\}$$

($\omega(t) = o(t)$ in Landau's notations).

An element α of A is said to be *subhomogeneous* of degree $d \in \mathbb{R}_+$ if $\alpha(ct) \leq c^d \alpha(t)$ for any $c \in [0, 1]$, $t \in \mathbb{R}_+$. For $d = 1$ this means that α is *starshaped* i.e. that its epigraph is starshaped at $(0, 0)$ or that $t \mapsto t^{-1} \alpha(t)$ belongs to N . Let us observe that if $\alpha \in A \setminus \{0\}$ is starshaped, then $\lim_{t \rightarrow +\infty} \alpha(t) = +\infty$ and $\alpha \in N$.

Let us summarize the preceding notations for the convenience of the reader.

Function	Definition	Notation
Asplund function	$\alpha : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ $\alpha(0) = 0$	A
	$\alpha \in A$ nondecreasing	N
	$\alpha \in A$ convex l.s.c.	Γ
firm function	$\alpha \in A, (\alpha(t_n)) \rightarrow 0$ implies $(t_n) \rightarrow 0$	Φ
gage	$\alpha \in A$ $\alpha(t) > 0$ for $t > 0$ nondecreasing	$G = \Phi \cap N$
	$\lim_{t \downarrow 0} \lambda(t) = 0$	Λ
modulus	$\mu \in A, \lim_{t \downarrow 0} \mu(t) = 0$ μ nondecreasing	$M = \Lambda \cap N$
	$\omega \in A, \lim_{t \downarrow 0} t^{-1} \omega(t) = 0$	Ω
hypermodulus	$\omega \in A, \lim_{t \downarrow 0} t^{-1} \omega(t) = 0$ ω nondecreasing	$H = \Omega \cap N$

The following lemma which provides a slight sharpening of Lemma 1 of [1] will be extremely useful.

LEMMA 2.1

- (a) For any starshaped $g \in G$ one has $g^* \in H$.
- (b) For any $\omega \in \Omega$ one has $\omega^* \in \Phi$.
- (c) The correspondence $\gamma \rightarrow \gamma^*$ defines a bijection between $G \cap \Gamma$ onto $H \cap \Gamma$, where Γ is the set of convex and lower semicontinuous functions from $\mathbb{R}_+$ into $\overline{\mathbb{R}}_+$ with value 0 at 0.

Proof. — As $\gamma \mapsto \gamma^*$ maps N into N it suffices to prove that $g^* \in \Omega$ if $g \in G$ and $\omega^* \in \Phi$ if $\omega \in \Omega$.

(a) Let $g \in G$ be starshaped and let $\varepsilon > 0$. We claim that for $s \in]0, \delta]$ with $\delta = \varepsilon^{-1}g(\varepsilon) > 0$, we have $s^{-1}g^*(s) \leq \varepsilon$. Indeed, as $g(r) \geq r\varepsilon^{-1}g(\varepsilon)$ for $r \geq \varepsilon$, we have

$$\begin{aligned} g^*(s) &\leq \max \left(\sup_{0 \leq r \leq \varepsilon} (rs - g(r)), \sup_{r \geq \varepsilon} (rs - r\varepsilon^{-1}g(\varepsilon)) \right) \\ &\leq \max \left(\varepsilon s, \sup_{r \geq \varepsilon} r(s - \delta) \right) \\ &= \varepsilon s. \end{aligned}$$

(b) Let $\omega \in \Omega$ and let $s > 0$ be given. Taking $\varepsilon \in]0, s[$ we can find $\delta > 0$ such that $\omega \leq k$ where $k(r) = \varepsilon r$ for $r \in [0, \delta]$, $k(r) = +\infty$ for $r > \delta$. Then $\omega^*(s) \geq k^*(s) = (s - \varepsilon)\delta > 0$. Since h^* is convex and in A we have $h^* \in N$, hence $h^* \in G$.

(c) is a consequence of (a) and (b). $\square$

COROLLARY 2.2. — *For any starshaped function $\alpha \in A$ one has $\alpha \in G$ iff $\alpha^* \in H$ iff $\alpha^{**} \in G$.*

Proof. — Obviously, since any starshaped function $\alpha \in H$ is nondecreasing, $\alpha^{**} \in G$ implies $\alpha \in G$, and $\alpha \in G$ implies $\alpha^* \in H$ which in turn implies $\alpha^{**} \in G$ by Lemma 2.1. $\square$

We are now ready to introduce the notions we intend to study.

DEFINITION 1. — *Given $\sigma \in A$, a function $f : X \rightarrow \mathbb{R}^\bullet$ is said to be σ -smooth if*

$$(1_\sigma) \quad f((1-t)x_0 + tx_1) + t(1-t)\sigma(|x_0 - x_1|) \geq (1-t)f(x_0) + tf(x_1) \text{ for all } x_0, x_1 \in X, \text{ for all } t \in [0, 1].$$

Given $\rho \in A$, a function $g : Y \rightarrow \mathbb{R}^\bullet$ is said to be ρ -convex if

$$(1_\rho^*) \quad g((1-t)y_0 + ty_1) + t(1-t)\rho(|y_0 - y_1|) \leq (1-t)g(y_0) + tg(y_1) \text{ for all } y_0, y_1 \in Y, \text{ for all } t \in [0, 1].$$

If f is σ -smooth (or σ -flat) for some σ with $\lim_{t \rightarrow 0+} t^{-1}\sigma(t) = 0$ (i.e. $\sigma \in \Omega$) f is said to be *uniformly smooth* (or *moderately convex*).

If g is ρ -convex (or ρ -rotund) for some firm function ρ , g is said to be *uniformly convex*.

In the special case $\sigma(r) = (1/2)cr^2$ (resp. $\rho(r) = (1/2)cr^2$), f (resp. g) is said to be weakly convex (resp. strongly convex); a thorough study of these classes of functions is contained in [36], [37], [38] and [41].

LEMMA 2.3. — *Let us assume that $f : X \rightarrow \mathbb{R}^\bullet$ is σ -smooth for some $\sigma \in N$ with $\text{dom } \sigma \neq \{0\}$. Then $\text{dom } f = X$ if $\text{dom } f$ is nonempty.*

Proof. — Let $\bar{x} \in \text{dom } f$ and let $r > 0$ be such that $\sigma(r) < +\infty$. For $x_0 \in \bar{x} + \text{Int } rB_X$, we set $t = |x_0 - \bar{x}|/r \in [0, 1[$ and we define $x_1 \in X$ by $(1-t)x_0 + tx_1 = \bar{x}$. One has

$$f(\bar{x}) + t(1-t)\sigma(r) \geq (1-t)f(x_0) + tf(x_1)$$

hence $\text{dom } f + \text{Int } rB_X \subset \text{dom } f$; thus $\text{dom } f = X$. $\square$

DEFINITION 2. — *The modulus of uniform smoothness of a convex function $f : X \rightarrow \mathbb{R}^\bullet$ is the function $\sigma_f \in A$ given by*

$$\sigma_f(r) = \sup \left\{ t^{-1}(1-t)^{-1} ((1-t)f(x_0) + tf(x_1) - f((1-t)x_0 + tx_1)) : \right. \\ \left. 0 < t < 1, (1-t)x_0 + tx_1 \in \text{dom } f, |x_1 - x_0| = r \right\}.$$

The gage of uniform convexity (or uniform rotundity) of a convex function $g : Y \rightarrow \mathbb{R}^\bullet$ is the function $\rho_g \in A$ given by

$$\rho_g(r) = \inf \left\{ t^{-1}(1-t)^{-1} ((1-t)g(y_0) + tg(y_1) - g((1-t)y_0 + ty_1)) : \right. \\ \left. 0 < t < 1, y_0, y_1 \in \text{dom } g, |y_1 - y_0| = r \right\}.$$

Since it can be shown that σ_f and ρ_g are nondecreasing (Lemma 2.5 and Corollary 2.7 below) the following obvious interpretations of σ_f and ρ_g imply that f is uniformly smooth iff σ_f is an hypermodulus and g is uniformly convex iff ρ_g is a gage. This observation explains the abuse of language committed in Definition 2.

LEMMA 2.4. — *For any convex function f on X , σ_f is the infimum of the family $S(f)$ of $\sigma \in A$ such that f is σ -smooth. In fact $\sigma \in A$ belongs to $S(f)$ iff $\sigma \geq \sigma_f$.*

For any convex function g on Y , ρ_g is the supremum of the family $R(g)$ of $\rho \in A$ such that g is ρ -convex. In fact $\rho \in A$ belongs to $R(g)$ iff $\rho \leq \rho_g$.

When f (resp. g) is uniformly smooth (resp. uniformly convex) what follows shows that we also have $\sigma_f = \inf S(f) \cap H$, $\rho_g = \sup R(g) \cap G$. The following result is crucial.

LEMMA 2.5 ([38, Lemma 1]). — *For any convex function g on Y , for any $c \in [0, 1]$, $r \in \mathbb{R}_+$ one has $\rho_g(cr) \leq c^2 \rho_g(r)$. In particular ρ_g is starshaped and $\rho_g \in H$.*

The following duality result is also quite elementary but rich of consequences.

PROPOSITION 2.6

(a) *If f is σ -smooth for some $\sigma \in A$ then f^* is σ^* -convex.*

(b) *If g is ρ -convex for some $\rho \in A$ then g^* is ρ^* -smooth.*

Proof

(a) Let y_0, y_1 in Y and let $t \in [0, 1]$. For any $x_0 \in X$, $v \in X$, setting $x_t = x_0 + tv$ and $y_t = (1 - t)y_0 + ty_1$, we have

$$\begin{aligned} (1 - t)f^*(y_0) + tf^*(y_1) &\geq \\ &\geq (1 - t)\langle x_0, y_0 \rangle + t\langle x_1, y_1 \rangle - (1 - t)f(x_0) - tf(x_1) \\ &\geq (1 - t)\langle x_0, y_0 \rangle + t\langle x_1, y_1 \rangle - f(x_t) - t(1 - t)\sigma(|x_0 - x_1|) \\ &\geq \langle x_t, y_t \rangle - f(x_t) + t(1 - t)(\langle x_0 - x_1, y_0 - y_1 \rangle - \sigma(|x_0 - x_1|)). \end{aligned}$$

As $x_0 \in X$ and v are arbitrary we get, taking suprema,

$$\begin{aligned} (1 - t)f^*(y_0) + tf^*(y_1) &\geq \sup_{v \in X} \sup_{x_0 \in X} \left\{ \langle x_0 + tv, y_t \rangle - f(x_0 + tv) \right. \\ &\quad \left. + t(1 - t)(\langle v, y_1 - y_0 \rangle - \sigma(|v|)) \right\} \end{aligned}$$

yielding

$$\begin{aligned} (1 - t)f^*(y_0) + tf^*(y_1) &\geq \sup_{v \in X} \left\{ f^*(y_t) + t(1 - t)(\langle v, y_1 - y_0 \rangle - \sigma(|v|)) \right\} \\ &\geq f^*(y_t) + t(1 - t)\sigma^*(|y_0 - y_1|). \end{aligned}$$

(b) Given x_0, x_1 in X and $t \in [0, 1]$ let r_0, r_1 in $\mathbb{R}$ be such that $r_0 < g^*(x_0)$, $r_1 < g^*(x_1)$. Then there exist y_0, y_1 in Y such that

$$r_0 \leq \langle x_0, y_0 \rangle - g(y_0), \quad r_1 \leq \langle x_1, y_1 \rangle - g(y_1).$$

Multiplying both sides of the first (resp. second) inequality by $1 - t$ (resp. t) and adding to both sides of the Young-Fenchel inequality

$$0 \leq g^*(x_t) + g(y_t) - \langle x_t, y_t \rangle,$$

where $x_t = (1 - t)x_0 + tx_1$, $y_t = (1 - t)y_0 + ty_1$, we get

$$\begin{aligned} (1 - t)r_0 + tr_1 &\leq \\ &\leq g^*(x_t) + g(y_t) - (1 - t)g(y_0) - tg(y_1) + t(1 - t)\langle x_0 - x_1, y_0 - y_1 \rangle \\ &\leq g^*(x_t) + t(1 - t)(|x_0 - x_1||y_0 - y_1| - \rho(|y_0 - y_1|)) \\ &\leq g^*(x_t) + t(1 - t)\rho^*(|x_0 - x_1|). \end{aligned}$$

As r_0 and r_1 are arbitrary in $]-\infty, g^*(x_0)[$ and $]-\infty, g^*(x_1)[$ respectively, we get that g^* is ρ^* -smooth. $\square$

COROLLARY 2.7

- (a) If $f \in \Gamma_Y(X)$, $\sigma \in A$ and if f is σ -smooth then f is σ^{**} -smooth.
- (b) For any $f \in \Gamma_Y(X)$, σ_f is convex and l.s.c.
- (c) For any $f \in \Gamma_Y(X)$ one has $\sigma_f = \rho_g^*$, where $g = f^*$.
- (d) For any $f \in \Gamma_Y(X)$, $c \in [0, 1]$, $r \in \mathbb{R}_+$ one has $\sigma_f(cr) \geq c^2\sigma_f(r)$.

Proof

(a) Taking $g = f^*$ in the preceding proposition we have $\sigma \in S(f)$ iff $\sigma^* \in R(g)$ iff $\sigma^{**} \in S(f^{**}) = S(f)$.

(b) As $\sigma_f^{**} \in S(f)$, Lemma 2.4 yields $\sigma_f^{**} \geq \sigma_f$ hence $\sigma_f^{**} = \sigma_f$.

(c) Since for $g = f^*$ we have $\rho_g \in R(g)$ we get $(\rho_g)^* \in S(f)$ and $\sigma_f \leq (\rho_g)^*$. On the other hand $(\sigma_f)^* \in R(g)$, hence $(\sigma_f)^* \leq \rho_g$ and $(\rho_g)^* \leq (\sigma_f)^{**} = \sigma_f$. Therefore $\sigma_f = (\rho_g)^*$.

(d) This follows from Lemma 2.5 and the inequality

$$c^2\sigma_f(r) = \sup_{s \geq 0} (c^2rs - c^2\rho_g(s)) \leq \sup_{s \geq 0} (crs - \rho_g(cs)) = \sigma_f(cr). \quad \square$$

COROLLARY 2.8

- (a) If $f \in \Gamma_Y(X)$ is uniformly smooth then $g = f^*$ is uniformly convex.
- (b) If $g \in \Gamma_Y(X)$ is uniformly convex then $f = g^*$ is uniformly smooth.

Proof

(a) Suppose there exists $\sigma \in S(f)$ with $\sigma \in \Omega$. Then, as $\sigma_f \leq \sigma$ one has $\sigma_f \in \Omega$, hence $\rho_g^* = \sigma_f \in H$, so that, by Corollary 2.2 and Lemma 2.5, $\rho_g \in G$.

(b) Suppose there exists $\rho \in R(g)$ with ρ firm. Then, as $\rho_g \in N$ by Lemma 2.5 and $\rho(g) \geq \rho$, $\rho_g \in G$. Therefore, by Lemma 2.2, $\sigma_f = \rho_g^* \in H$. $\square$

As a consequence, we derive the following result on the duality between strongly and weakly convex ([36], [37]) functions defined on a Hilbert space H .

Let $c \in \mathbb{R}^*$ and $f : H \rightarrow \mathbb{R}^*$. We say that f is c -convex if, for any $x_1, x_0 \in X$, $t \in [0, 1]$

$$f(tx_1 + (1+t)x_0) \leq tf(x_1) + (1-t)f(x_0) - t(1-t)\frac{c}{2}|x_1 - x_0|^2.$$

This is equivalent to $f - (c/2)|\cdot|^2$ is convex.

When $c > 0$ (resp. $c < 0$) f is said to be strongly (resp. weakly convex).

COROLLARY 2.9 (see also [40])

- (a) If f is c -convex with $c > 0$ then $-f^*$ is $-(1/c)$ -convex.
- (b) If f is c -convex with $c < 0$ and $\text{dom } f = H$ then $(-f)^*$ is $-(1/c)$ -convex.

Proof. — Obvious from Corollary 2.7. $\square$

3. Subdifferential characterizations

In this section we relate the properties of uniform smoothness and uniform convexity of f and g to properties of their subdifferentials:

$$\begin{aligned}\partial f(x) &= \{x^* \in X^* \mid f^*(x^*) + f(x) = \langle x, x^* \rangle\} \\ \partial_Y f(x) &= \{y \in Y \mid f^*(y) + f(x) = \langle x, y \rangle\}\end{aligned}$$

(and analogous definitions for g). Some care is needed since in general the inclusion $\partial_Y f(x) \subset \partial f(x)$ is strict. Let us begin with pointwise notions.

DEFINITION 3. — Given $f : X \rightarrow \mathbb{R}^\bullet$, $x_0 \in \text{dom } f$, $y_0 \in X^*$ we denote by $S(f, x_0, y_0)$ the set of $\eta \in A$ such that

- (2 $_\eta$) for all $x \in X$, $f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|)$
and we say that f is η -uniformly smooth at x_0 w.r.t. y_0 if

$$\eta \in S(f, x_0, y_0).$$

Given $g : Y \rightarrow \mathbb{R}^\bullet$, $y_0 \in \text{dom } g$, $x_0 \in Y^*$ we denote by $R(g, y_0, x_0)$ the set of $\gamma \in A$ such that

- (2 $_\gamma$) for all $y \in Y$, $g(y) \geq g(y_0) + \langle x_0, y - y_0 \rangle + \gamma(|y - y_0|)$
and we say that g is γ -uniformly convex at y_0 w.r.t. x_0 if

$$\gamma \in R(g, y_0, x_0).$$

Furthermore we set:

$$\begin{aligned} \eta_{f, x_0, y_0}(r) &= \inf \{ \eta(r) : \eta \in S(f, x_0, y_0) \} \\ &= \sup \{ f(x) - f(x_0) - \langle x - x_0, y_0 \rangle : |x - x_0| = r \}, \\ \gamma_{g, y_0, x_0}(r) &= \sup \{ \gamma(r) : \gamma \in R(g, y_0, x_0) \} \\ &= \inf \{ g(y) - g(y_0) - \langle x_0, y - y_0 \rangle : |y - y_0| = r \}. \end{aligned}$$

LEMMA 3.1

- (a) If f is convex then η_{f, x_0, y_0} is convex; if f is l.s.c. then η_{f, x_0, y_0} is l.s.c.
(b) If g is convex then γ_{g, y_0, x_0} is starshaped.

Proof. — The proof of both assertions is easy; see also [1, p. 36] and [41, p. 350] for the first one.

The second one follows from the fact that the infimum of a family of starshaped functions is starshaped, since

$$\gamma_{g, y_0, x_0} = \inf_{|z| \leq 1} \gamma_{g, y_0, x_0, z}$$

with $\gamma_{g, y_0, x_0, z}(r) = g(y_0 + rz) - g(y_0) - \langle x_0, rz \rangle$. $\square$

PROPOSITION 3.2 ([41, Theorem 2.1, (iv) $\Leftrightarrow$ (v)])

- (a) If f is η -uniformly smooth at x_0 w.r.t. $y_0 \in \partial_Y f(x_0)$ then f^* is η^* -uniformly convex at y_0 w.r.t. x_0 .
(b) If g is γ -uniformly convex at y_0 w.r.t. $x_0 \in \partial_X f$ then g^* is γ^* -uniformly smooth at x_0 w.r.t. y_0 .

The proof follows by taking the conjugates of both sides of inequality (2_η) and (2_γ^*) respectively. Following the lines of the proofs of Corollaries 2.7 and 2.8, one gets:

COROLLARY 3.3. — *For any $f \in \Gamma_Y(X)$, $(x_0, y_0) \in \partial_Y f$, for $g = f^*$ one has*

$$\eta_{f, x_0, y_0} = (\gamma_{g, y_0, x_0})^*.$$

COROLLARY 3.4. — *For any $f \in \Gamma_Y(X)$, $g \in \Gamma_X(Y)$ with $g = f^*$ the following two statements are equivalent:*

- (2_Y) *there exists $\eta \in H$ such that $f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|)$ for all $(x_0, y_0) \in \partial_Y f$, $x \in X$,*
- (2_X^*) *there exists a starshaped $\gamma \in G$ such that $g(y) \geq g(y_0) + \langle x_0, y - y_0 \rangle + \gamma(|y - y_0|)$ for all $(y_0, x_0) \in \partial_X g$, $y \in Y$.*

Moreover one can take $\gamma = \eta^$ in the implication $(2_Y) \Rightarrow (2_X^*)$ and $\eta = \gamma^*$ in the implication $(2_X^*) \Rightarrow (2_Y)$.*

The following result is more subtle than Proposition 3.2. Its proof mimics the proof of [1, Proposition 1].

PROPOSITION 3.5. — *Let Z be a Banach space, let $h : Z \rightarrow \mathbb{R}^\bullet$ be l.s.c. and finite at $u \in Z$. Then the following three statements about h , its conjugate h^* and $v \in Z^*$ are equivalent and imply that*

$$h(u) + h^*(v) = \langle u, v \rangle :$$

(a) for some $\gamma \in G \cap \Gamma$, for any $z \in Z$

$$h(z) \geq h(u) + \langle z - u, v \rangle + \gamma(|z - u|),$$

(b) for some $\eta \in H \cap \Gamma$, for any $w \in Z^$*

$$h^*(w) \leq h^*(v) + \langle u, w - v \rangle + \eta(|w - v|),$$

(c) $v \in \text{Int dom } h^$ and any sequence (z_n) in Z such that*

$$\lim_{n \rightarrow \infty} (\langle z_n, v \rangle - h(z_n)) = h^*(v)$$

converges in norm to u .

COROLLARY 3.6.— *Let (X, Y) be a pair of Banach spaces in metric duality and let $f \in \Gamma_Y(X)$. Let $\omega \in \Omega$ and let $(x_0, x_0^*) \in \partial f \subset X \times X^*$ be such that f is ω -uniformly smooth at x_0 w.r.t. x_0^* . Then $x_0^* \in Y$ and f is Fréchet differentiable at x_0 with derivative x_0^* .*

Proof.— Let $Z = X^*$; we identify Y with its image in Z through the canonical isometric embedding. Let $g = f^* \mid Y$, and let $h : Z \rightarrow \mathbb{R}^\bullet$ be given by

$$h \mid Y \cup \{x_0^*\} = f^* \mid Y \cup \{x_0^*\},$$

$$h(x^*) = +\infty \text{ for } x^* \in Z \setminus Y, \quad x^* \neq x_0^*.$$

For all $x \in X$ we have

$$h^*(x) = \max \left(\sup_{y \in Y} (\langle x, y \rangle - f^*(y)), \langle x, x_0^* \rangle - f^*(x_0^*) \right).$$

As $\sup_{y \in Y} (\langle x, y \rangle - f^*(y)) = f(x)$ and $\langle x, x_0^* \rangle - f^*(x_0^*) \leq f(x)$, we get

$$h^*(x) = \sup_{y \in Y} (\langle x, y \rangle - f^*(y)) = f(x).$$

Using Lemma 2.1(b) and Proposition 3.2 we see that assertion (a) of Proposition 3.5 holds with $\gamma = \omega^*$. Taking a sequence (y_n) in Y such that $\lim_{n \rightarrow \infty} (\langle x_0, y_n \rangle - g(y_n)) = f(x_0)$ and observing that $g(y_n) = h(y_n)$, from Proposition 3.5(c) we get $(y_n) \rightarrow x_0^*$, hence $x_0^* \in Y$ as Y is complete. The fact that f is Fréchet differentiable at x_0 with $f'(x_0) = x_0^*$ is obvious. $\square$

PROPOSITION 3.7.— *Let X be a Banach space and let $f \in \Gamma_0(X)$. Suppose that for some $\eta \in A$ and for any $(x_0, y_0) \in \partial f$ the function f is η -uniformly smooth at x_0 with respect to y_0 . Then, for $\sigma = 2\eta$, f is σ -smooth:*

(1 $_\sigma$) for all $x_0, x_1 \in X$, for all $t \in [0, 1]$

$$f((1-t)x_0 + tx_1) + t(1-t)\sigma(|x_0 - x_1|) \geq (1-t)f(x_0) + tf(x_1).$$

Proof. — By assumption we have

$$\begin{aligned}\eta_f(r) &= \sup \{ f(x) - f(x_0) - \langle x - x_0, y_0 \rangle : (x_0, y_0) \in \partial f, |x - x_0| = r \} \\ &\leq \eta(r) ;\end{aligned}$$

and by Lemma 3.1, η_f is convex and l.s.c. Let $x_0, x_1 \in X$, $t \in [0, 1]$, and let $x_t = (1-t)x_0 + tx_1$. In order to prove (1_σ) we may suppose $x_t \in \text{dom } f$. Using the Brøndsted-Rockafellar Theorem ([9] and also [17]), we can find a sequence (w_n) with limit 0 in X and a sequence (y_n) with $y_n \in \partial f(x_t + w_n)$ and $f(x_t) = \lim_{n \rightarrow \infty} f(x_t + w_n)$. Then multiplying both sides of the two following inequalities by $1-t$ (resp. t)

$$\begin{aligned}f(x_0 + w_n) &\leq f(x_t + w_n) + \langle y_n, x_0 - x_t \rangle + \eta_f(|x_0 - x_t|), \\ f(x_1 + w_n) &\leq f(x_t + w_n) + \langle y_n, x_1 - x_t \rangle + \eta_f(|x_1 - x_t|)\end{aligned}$$

and adding, we get, as $\eta_f(0) = 0$ and as η_f is convex

$$\begin{aligned}(1-t)f(x_0 + w_n) + tf(x_1 + w_n) &\leq \\ &\leq f(x_t + w_n) + (1-t)\eta_f(|x_0 - x_t|) + t\eta_f(|x_1 - x_t|) \\ &\leq f(x_t + w_n) + 2t(1-t)\eta_f(|x_0 - x_1|) \\ &\leq f(x_t + w_n) + 2t(1-t)\eta(|x_0 - x_1|).\end{aligned}$$

Taking the limits as n goes to ∞ and using the lower semicontinuity of f we get (1_σ) with $\sigma = 2\eta$. $\square$

The following result fills a gap among the implications of [41, Theorem 2.2], even in the reflexive case.

THEOREM 3.8. — *Let X, Y be a pair of Banach spaces in metric duality and let $f \in \Gamma_Y(X)$, $g = f^* \in \Gamma_X(Y)$. The following assertions are equivalent and are equivalent to the statements (2_Y) and (2_X^*) of Corollary 3.4.*

(1) *there exists $\sigma \in H$ such that for all $x_0, x_1 \in X$, $t \in [0, 1]$*

$$f((1-t)x_0 + tx_1) + t(1-t)\sigma(|x_0 - x_1|) \geq (1-t)f(x_0) + tf(x_1) ;$$

(1*) *there exists a starshaped $\rho \in G$ such that for all $y_0, y_1 \in Y$, $t \in [0, 1]$*

$$g((1-t)y_0 + ty_1) + t(1-t)\rho(|y_0 - y_1|) \leq (1-t)g(y_0) + tg(y_1) ;$$

(2) *there exists $\eta \in H$ such that for all $(x_0, y_0) \in \partial f$, for all $x \in X$*

$$f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|),$$

(2*) *there exists $\gamma \in G$ such that for all $(y_0, x_0) \in \partial g$, for all $y \in Y$*

$$g(y) \geq g(y_0) + \langle x_0, y - y_0 \rangle + \gamma(|y - y_0|).$$

Proof. — We already know that (1) $\Rightarrow$ (1*) with $\rho = \sigma^*$ (Proposition 2.6). Now (1*) $\Rightarrow$ (2*) with $\gamma = \rho$ by [38] or directly since for any $(y_0, x_0) \in \partial g$ and any $y_1 \in Y$ we have

$$\begin{aligned} \langle x_0, y_1 - y_0 \rangle &\leq g'(y_0; y_1 - y_0) \\ &\leq \lim_{t \downarrow 0} t^{-1} (tg(y_1) - tg(y_0) - t(1-t)\rho(|y_1 - y_0|)). \end{aligned}$$

Since (2*) $\Rightarrow$ (2*_X) $\Leftrightarrow$ (2_Y) with $\eta = \gamma^*$ (Corollary 3.4) and (2_Y) $\Leftrightarrow$ (2) (Corollary 3.6) and (2) $\Rightarrow$ (1) with $\sigma = 2\eta$ (Proposition 3.7) we get the complete equivalence. Moreover we see that (2*) $\Rightarrow$ (1*) with $\rho = \sigma^* = (2\eta)^* = 2\eta^*(\cdot/2) \leq \gamma^{**} \leq \gamma$, and equality holds when $\gamma \in \Gamma$. In particular when $\rho(t) = \gamma(t) = (1/2)t^2$, we have $\sigma(s) = (1/2)s^2$. $\square$

4. Links with Monotonicity and Uniform Continuity

In this section we relate the uniform smoothness of f with monotonicity properties and uniform continuity properties of ∂f and ∂f^* .

LEMMA 4.1. — *Let $f \in \Gamma_0(X)$ be such that for some $\eta \in A$ finite at some $r > 0$ and any $(x_0, y_0) \in \partial f$ one has*

$$\text{for all } x \in X, \quad f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|).$$

Then $\text{dom } f = X$.

Proof. — Thanks to the convexity of f we may assume η is nondecreasing. Given $x_1 \in \text{dom } f$, using the Brøndsted-Rockafellar Theorem guaranteeing the density of $\text{dom } \partial f$ in $\text{dom } f$ (see [9]) we can find $(x_0, y_0) \in \partial f$ such that $|x_0 - x_1| \leq r/2$. Then we get that f is bounded above on $x_0 + rB \supset x_1 + (r/2)B$ by $f(x_0) + r|y_0| + \eta(r)$. Thus $\text{dom } f + (r/2)B \subset \text{dom } f$; this implies $\text{dom } f = X$. $\square$

Remark 1. — It can be shown in a similar way that if $f \in \Gamma_0(X)$ and if, for some $\eta \in A$ finite at some $r > 0$ and for any $(x_0, y_0) \in \partial f$, one has $f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|)$ for all $x \in \text{dom } f$, then $\text{dom } f$ is closed.

PROPOSITION 4.2. — *For any $f \in \Gamma_0(X)$ the following assertions are equivalent :*

(2) *there exists $\eta \in H$ such that for all $(x_0, y_0) \in \partial f$, for all $x \in X$*

$$f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|)$$

(3) *$\text{dom } f = X$, there exist $\kappa \in H$ such that for all $(x_0, y_0), (x_1, y_1) \in \partial f$,*

$$\langle x_0 - x_1, y_0 - y_1 \rangle \leq \kappa(|x_0 - x_1|).$$

Moreover one can take $\kappa = 2\eta$ in (2) $\Rightarrow$ (3) and $\eta = \kappa$ in (3) $\Rightarrow$ (2).

Proof. — Taking the preceding lemma into account, the implication (2) $\Rightarrow$ (3) simply follows by addition. The implication (3) $\Rightarrow$ (2) is a direct consequence of the definition of ∂f . In fact, since $f \in \Gamma_0(X)$ with $\text{dom } f = X$, f is continuous and $\text{dom } \partial f = X$ so that, for each $x \in X$ and $y \in \partial f(x)$

$$\begin{aligned} f(x) &\leq f(x_0) + \langle x - x_0, y \rangle \\ &\leq f(x_0) + \langle x - x_0, y_0 \rangle + \langle x - x_0, y - y_0 \rangle \\ &\leq f(x_0) + \langle x - x_0, y_0 \rangle + \kappa(|x - x_0|). \quad \square \end{aligned}$$

In fact one can show the equivalence between the following two weaker assumptions.

PROPOSITION 4.3. — *The following assertions on $f \in \Gamma_0(X)$ are equivalent:*

($\tilde{2}$) *there exists $\eta \in H$ such that for all $(x_0, y_0) \in \partial f$, for all $x \in \text{dom } f$*

$$f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + \eta(|x - x_0|),$$

($\tilde{3}$) *there exists $\kappa \in H$ such that for all $(x_0, y_0), (x_1, y_1) \in \partial f$*

$$\langle x_0 - x_1, y_0 - y_1 \rangle \leq \kappa(|x_0 - x_1|).$$

Proof. — Again $(\tilde{2}) \Rightarrow (\tilde{3})$ by addition. Assuming $(\tilde{3})$, given $(x_0, y_0) \in \partial f$, $x \in \text{dom } f$ with $x \neq x_0$ and setting $w_n = (n+1)^{-1}(nx + x_0)$ we have $(w_n) \rightarrow x$, $|w_n - x_0| < |x - x_0|$ so that by the Brøndsted-Rokafellar Theorem [9] we can find $(x_n, y_n) \in \partial f$ with $|x_n - w_n| < |x - x_0| - |w_n - x_0|$, hence $|x_n - x_0| < |x - x_0|$. It follows that

$$\begin{aligned} f(x_n) &\leq f(x_0) + \langle x_n - x_0, y_n \rangle \\ &\leq f(x_0) + \langle x_n - x_0, y_0 \rangle + \kappa(|x_n - x_0|). \end{aligned}$$

Passing to the limit, using the facts that κ is nondecreasing and f is l.s.c. we get $(\tilde{2})$ with $\eta = \kappa$. $\square$

THEOREM 4.4. — *For any $f \in \Gamma_0(X)$ the following assertions on $g = f^*$ are equivalent to the assertions (2) and (3) of Proposition 4.2:*

(2*) *there exists a starshaped $\gamma \in G$ such that for all $(y_0, x_0) \in \partial g$, for all $y \in X^*$,*

$$g(y) \geq g(y_0) + \langle x_0, y - y_0 \rangle + \gamma(|y - y_0|);$$

(3*) *there exists a starshaped $\delta \in G$ such that for all $(y_0, x_0), (y_1, x_1) \in \partial g$,*

$$\langle x_0 - x_1, y_0 - y_1 \rangle \geq \delta(|y_0 - y_1|).$$

Proof. — Again $(2^*) \Rightarrow (3^*)$ follows by addition, with $\delta = 2\gamma$. The implication $(3^*) \Rightarrow (2)$ follows from the facts that $\partial f \subset (\partial g)^{-1}$ and $\delta^* \in H \cap \Gamma \subset H$ whenever $\delta \in G$ is starshaped and from the following inequalities in which $(y_0, x_0), (y_1, x_1) \in \partial g$, $t > 0$:

$$\begin{aligned} \langle x_0 - x_1, y_0 - y_1 \rangle &= (1+t)\langle x_0 - x_1, y_0 - y_1 \rangle - t\langle x_0 - x_1, y_0 - y_1 \rangle \\ &\leq t \left(t^{-1}(1+t)|x_0 - x_1||y_0 - y_1| - \delta(|y_0 - y_1|) \right) \\ &\leq t\delta^* \left(t^{-1}(1+t)|x_0 - x_1| \right); \end{aligned}$$

taking $t = 1$ we get (3) with $\kappa(r) = \delta^*(2r)$. Now, by Proposition 4.2 and Corollary 3.4, we have $(3) \Rightarrow (2)$ with $\eta = \kappa$ and $(2) \Rightarrow (2^*)$ with $\gamma = \eta^* = \kappa^* = \delta(\cdot/2)$. $\square$

The preceding proof shows that in (2^*) and (3^*) as in (2) and (3) we may replace G and H by $G \cap \Gamma$ and $H \cap \Gamma$ respectively.

The following simple lemma seems to be new.

LEMMA 4.5. — *Let X be a Banach space and let $T : X \rightrightarrows X^*$ be a maximal monotone operator with nonempty domain D . Suppose D is closed and convex and T has bounded values. Then $D = X$ and T^{-1} is surjective.*

Proof. — It suffices to prove that the boundary δD of D is empty. Suppose the contrary. Then the Bishop-Phelps Theorem [8] on the density of support points in δD yields some $x_0 \in \delta D$ and some $w \in X^* \setminus \{0\}$ such that $\langle x_0, w \rangle \geq \langle x, w \rangle$ for each $x \in D$. Let $y_0 \in T(x_0)$. Then for each $(x, y) \in T$ and each $t \in \mathbb{R}_+$ we have

$$\langle x_0 - x, y_0 + tw - y \rangle = \langle x_0 - x, y_0 - y \rangle + t \langle x_0 - x, w \rangle \geq 0.$$

Since T is maximal monotone we get $y_0 + tw \in T(x_0)$, a contradiction with the boundedness of $T(x_0)$. $\square$

PROPOSITION 4.6. — *Let X be a Banach space and let $T : X \rightrightarrows X^*$ be a maximal monotone operator with a nonempty domain D . Suppose the closure C of D is convex and T is locally bounded on C . Then $D = X$ and T^{-1} is surjective. Here T is said to be locally bounded on C if for each $\bar{x} \in C$ there exist $r > 0$, $s > 0$ such that for any $x \in D \cap (\bar{x} + rB)$, $y \in T(x)$ one as $|y| \leq s$.*

Proof. — In view of the preceding lemma it suffices to show that $C = D$. Given $\bar{x} \in C$ let r, s be as above and let $(x_i)_{i \in I}$ be a net with limit $\bar{x}$ in $D \cap (\bar{x} + rB)$. Choosing $y_i \in T(x_i)$ for each $i \in I$ and observing that $(y_i)_{i \in I}$ is bounded, we can find a subnet $(y_j)_{j \in J}$ of $(y_i)_{i \in I}$ with weak* limit $\bar{y}$. Then for any $(x, y) \in T$ we have $\langle \bar{x} - x, \bar{y} - y \rangle = \lim \langle x_j - x, y_j - y \rangle \geq 0$. Since T is maximal monotone we get $(\bar{x}, \bar{y}) \in T$ and $\bar{x} \in D$. $\square$

Let us recall some notions about inversion of nondecreasing functions which will be needed. Given $f : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ nondecreasing with $f(0) = 0$, i.e. $f \in N$, we define as in [29, Definition 2.1] (see also [41, Proposition A2] and [28]) two canonical quasi-inverses of f by

$$\begin{aligned} f^e(s) &= \inf \{ r \in \mathbb{R}_+ \mid s \leq f(r) \}, \\ f^h(s) &= \sup \{ t \in \mathbb{R}_+ \mid f(t) \leq s \}. \end{aligned}$$

The proof of the following result is elementary.

LEMMA 4.7 ([28] and [29]). — *For each $g \in G$, g^e and g^h are in M . For each $f \in M$, $f^e \in G$.*

PROPOSITION 4.8. — *For any multifunction $T : X \rightrightarrows X^*$ the following assertions are equivalent:*

(a) *there exists $\mu \in M$ such that for all $(x_0, y_0) \in T$, for all $(x_1, y_1) \in T$,*

$$|y_1 - y_0| \leq \mu(|x_1 - x_0|),$$

(b) *there exists $\lambda \in G$ such that for all $(x_0, y_0) \in T$, for all $(x_1, y_1) \in T$,*

$$\lambda(|y_1 - y_0|) \leq |x_1 - x_0|.$$

Proof. — Taking $\lambda = \mu^e$ we have that (a) $\Rightarrow$ (b) by definition of μ^e and the fact that $\mu^e \in G$ when $\mu \in M$. Similarly, the implication (b) $\Rightarrow$ (a) follows from the definition of λ^h and the fact $\lambda^h \in M$ when $\lambda \in G$. $\square$

COROLLARY 4.9. — *Let X be a Banach space and let $f \in \Gamma_0(X)$, $g = f^*$. Then the following assertions are equivalent:*

(4) *there exists $\mu \in M$ such that for all $(x_0, y_0), (x_1, y_1) \in \partial f$*

$$|y_1 - y_0| \leq \mu(|x_1 - x_0|),$$

(4*) *$\text{dom } f = X$ and there exists $\lambda \in G$ such that for all $(x_0, y_0), (x_1, y_1) \in \partial f$*

$$\lambda(|y_1 - y_0|) \leq |x_1 - x_0|.$$

Proof. — This follows from Propositions 4.6 and 4.8 and from the fact that $T = \partial f$ is maximal monotone and locally bounded on the closure C of its domain D when (4) holds and that moreover $C = \text{cl dom } f$ is convex. In fact given $r > 0$ such that $\mu([0, r])$ is bounded above by s and given $\bar{x} \in C$ we can find $x_0 \in D$ with $|\bar{x} - x_0| < r/2$. Then for each $x \in D \cap (\bar{x} + (r/2)B)$ and for each $y \in T(x)$ we have $|y| \leq |y_0| + s$, with $y_0 \in \partial f(x_0)$ fixed. $\square$

THEOREM 4.10. — *Let X be a Banach space with dual space Y and let $f \in \Gamma_0(X)$, $g = f^*$. Then the assertions (1), (2), (3), (4), (1*), (2*), (3*), (4*) are all equivalent.*

Proof. — In view of Theorems 3.8 and 4.4 it suffices to prove that $(4) \Rightarrow (3)$ and $(3^*) \Rightarrow (4^*)$. By what precedes (4) ensures that $\text{dom } f = X$ and for any $(x_0, y_0), (x_1, y_1) \in \partial f$

$$\begin{aligned} \langle x_0 - x_1, y_0 - y_1 \rangle &\leq |x_0 - x_1| |y_0 - y_1| \\ &\leq |x_0 - x_1| \mu(|x_0 - x_1|). \end{aligned}$$

As $\kappa : r \mapsto r\mu(r)$ belongs to H when $\mu \in M$ we get that $(4) \Rightarrow (3)$. Finally the first part of the preceding inequalities and the inclusion $\partial f \subset (\partial g)^{-1}$ show that $(3^*) \Rightarrow (4^*)$ with $\lambda(r) = \gamma(r)/r$ and $\lambda \in G$ since we may suppose $\gamma \in G \cap \Gamma$ in (3^*) . $\square$

Remark 2. — Setting for $r \in \mathbb{R}_+$

$$\eta_f(r) = \sup \{ f(x) - f(x_0) - \langle x - x_0, y_0 \rangle \mid x_0 \in X, \\ y_0 \in \partial f(x_0), |x - x_0| = r \}$$

it is shown in [41, Theorem 2.1, (vii) $\Rightarrow$ (v)] that if f is such that (4) holds with some measurable $\mu \in M$, then $\eta_f \leq \eta$ where η is given by

$$\eta(r) = \int_0^r \mu(t) dt.$$

5. Application to the Geometry of Banach Spaces

As an example we give a simple application of our results to geometry of Banach spaces. Let us recall that a Banach space X is said to be uniformly convex if

$$\delta(r) := \inf \left\{ 1 - \left| \frac{x+y}{2} \right| \mid x \in S, y \in S, \frac{1}{2} |x-y| \geq r \right\}$$

is positive for each $r \in]0, 1]$. It is well known ([10], [14]) that X^* endowed with its dual norm $|\cdot|_*$ is uniformly convex iff the duality multifunction $J : X \rightrightarrows X^*$ given by

$$J(x) = \left\{ x^* \in X^* \mid |x^*|_* = |x|, \langle x, x^* \rangle = |x|^2 \right\}$$

is single-valued and uniformly continuous on balls. We intend to give a quantitative estimate of the modulus of continuity of J on balls. Let us begin with the following known remark.

LEMMA 5.1 ([38, Theorem 6]). — *The Banach space $(X, |\cdot|)$ is uniformly convex iff for some (resp. any) $p \in]1, \infty[$ the mapping $x \mapsto |x|^p$ is uniformly convex on B .*

Proof. — It is known [7, p. 10] that $(X, |\cdot|)$ is uniformly convex iff for any $p \in]1, \infty[$ there exists a firm function $\delta_p : [0, 1] \rightarrow [0, 1]$ such that for any $r \in [0, 1]$ and any x, y in B with $(1/2)|x - y| \geq r$ one has

$$\left| \frac{1}{2}(x - y) \right|^p \leq (1 - \delta_p(r)) \left(\frac{1}{2}|x|^p + \frac{1}{2}|y|^p \right).$$

Then, as either $|x| \geq r$ or $|y| \geq r$ when $(1/2)|x - y| = r$, we get

$$\frac{1}{2}|x|^p + \frac{1}{2}|y|^p - \left| \frac{1}{2}(x + y) \right|^p \geq \frac{1}{2}r^p\delta_p(r) = 2^{-p-1}|x - y|^p\delta_p\left(\left|\frac{x - y}{2}\right|\right).$$

Using [41, Remark 2.1], we get that $x \mapsto |x|^p$ is uniformly convex on B with gage $r \mapsto 2^{-p}r^p\delta_p(r)$. Conversely, using the inequality $1 - a \geq p^{-1}(1 - a^p)$ for $a \in [0, 1]$ we get that if $x \mapsto |x|^p$ is uniformly convex on B with gage of convexity γ_p then for any $x, y \in S$ with $|x - y| = r$ we have

$$1 - \left| \frac{1}{2}(x + y) \right| \geq p^{-1} \left(1 - \left| \frac{1}{2}(x + y) \right|^p \right) \geq p^{-1} \frac{1}{4} \gamma_p(r)$$

so that $(X, |\cdot|)$ is uniformly convex. $\square$

PROPOSITION 5.2. — *Let $(X, |\cdot|)$ be a Banach space such that $(X, |\cdot|_*)$ is uniformly convex. If for $R > 0$ the gage of uniform convexity of the function $(1/2)|\cdot|_*^2$ on RB^* is denoted by ρ_R , then $\rho_R^{-1} \circ \rho_R^*$ is a modulus of uniform continuity of the duality mapping J of X on RB : for any x_0, x_1 in RB ,*

$$|J(x_0) - J(x_1)| \leq (\rho_R^{-1} \circ \rho_R^*)(|x_0 - x_1|).$$

Proof. — By homogeneity arguments one sees that there is no loss of generality in taking $R = 1$. Let $\sigma^* \in A$ be given by $\sigma^*(r) = (1/2)r^2$ for $r \in [0, 1]$, $\sigma^*(r) = +\infty$ for $r > 1$ so that σ^* is the conjugate function of σ given by $\sigma(r) = (1/2)r^2$ for $r \in [0, 1]$, $\sigma(r) = r - (1/2)$ for $r \geq 1$. Let f, g be given by $f(x) = \sigma(|x|)$, $g(y) = \sigma^*(|y|_*)$ for $x \in X$, $y \in X^*$, so that $f^* = g$. For any $x \in \text{Int } B$ we have $J(x) = \partial f(x)$. Since $(1^*) \Rightarrow (2^*)$ with

$\gamma = \rho = \rho_1$ (Theorem 3.8), $(2^*) \Rightarrow (3^*)$ with $\delta = 2\gamma$ and since $(3^*) \Rightarrow (3)$ with $\kappa(r) = \delta^*(2r) = 2\gamma^*(r)$ (Theorem 4.4) we get for any x_0, x_1 in $\text{Int } B$.

$$2\rho_1(|J(x_0) - J(x_1)|) \leq \langle x_0 - x_1, J(x_0) - J(x_1) \rangle \leq \rho_1^*(|x_0 - x_1|).$$

The result follows from the continuity of J and the fact that $\rho_1^{-1} \circ \rho_1^*$ is increasing. $\square$

When X is a Hilbert space, one has $\rho_R(t) = (1/2)t^2$ hence

$$(\rho_R^{-1} \circ \rho_R^*)(t) = t;$$

in this case the estimate is exact.

6. Recapitulative and Discussion

As mentioned in the introduction, it is of interest to delineate the links between the different modulus and gages appearing in the properties we studied. For the reader's convenience we summarize these relations below; we omit the quantifiers for the sake of brevity in the relations we recall.

In view of the precedind developments, especially in Lemma 2.5 and Remark 2, we reformulate the conditions we considered in terms of the slopes of the gages and modulus associated with f and g . We define the slope of a function $\varphi \in A$, i.e. $\varphi : \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$, $\varphi(0) = 0$ as the function $\widehat{\varphi} : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}}_+$ given by $\widehat{\varphi}(0) = 0$, $\widehat{\varphi}(t) = t^{-1}\varphi(t)$ for $t > 0$. Obviously the correspondence between φ and $\widehat{\varphi}$ is bijective and preserves the usual order. We denote by $(\widehat{k})$ the relation obtained by replacing in relation (k) the one variable function $\varphi = \sigma, \rho, \eta, \dots$ by its slope $\widehat{\varphi} = \widehat{\sigma}, \widehat{\rho}, \widehat{\eta}, \dots$. The correspondences we get which are summarized in our last table show that the use of slopes yields simpler links. Moreover some properties are more easily formulated with the slopes than with the functions themselves. For instance Lemma 2.5 allows us to take $\widehat{\rho}$ starshaped. It follows that we may assume that $\widehat{\sigma}$ is hypo-starshaped, i.e. that for $t \in \mathbb{R}_+$, $c \in [0, 1]$ one has $\widehat{\sigma}(ct) \geq c\widehat{\sigma}(t)$ or $-\widehat{\sigma}$ is starshaped; this follows from the relation $\sigma = \rho^*$ and the inequality

$$\sigma(ct) = \sup_{s \geq 0} \left(c \frac{s}{c} ct - \rho \left(c \frac{s}{c} \right) \right) \geq c^2 \sup_{s \geq 0} \left(\frac{s}{c} t - \rho \left(\frac{s}{c} \right) \right) = c^2 \sigma(t).$$

Tables for implications

Properties

- ($\hat{1}$) $\exists \hat{\sigma} \in M: f(x_t) + t(1-t)|x_1 - x_0|\hat{\sigma}(|x_1 - x_0|) \geq (1-t)f(x_0) + tf(x_1);$
- ($\hat{1}^*$) $\exists \hat{\rho} \in G: g(y_t) + t(1-t)|y_1 - y_0|\hat{\rho}(|y_1 - y_0|) \leq (1-t)g(y_0) + tg(y_1);$
- ($\hat{2}$) $\exists \hat{\eta} \in M: f(x) \leq f(x_0) + \langle x - x_0, y_0 \rangle + |x - x_0|\hat{\eta}(|x - x_0|);$
- ($\hat{2}^*$) $\exists \hat{\gamma} \in G: g(y) \geq g(y_0) + \langle x_0, y - y_0 \rangle + |y - y_0|\hat{\gamma}(|y - y_0|);$
- ($\hat{3}$) $\exists \hat{\kappa} \in M: \langle x_1 - x_0, y_1 - y_0 \rangle \leq |x_1 - x_0|\hat{\kappa}(|x_1 - x_0|);$
- ($\hat{3}^*$) $\exists \hat{\delta} \in G: \langle x_1 - x_0, y_1 - y_0 \rangle \geq |y_1 - y_0|\hat{\delta}(|y_1 - y_0|);$
- (4) $\exists \mu \in M: |y_1 - y_0| \leq \mu(|x_1 - x_0|);$
- (4*) $\exists \lambda \in M: \lambda(|y_1 - y_0|) \leq |x_1 - x_0|.$

Correspondences

- (4) $\Rightarrow$ (4*) $\lambda = \mu^e$ (Proposition 4.8)
- (4*) $\Rightarrow$ (4) $\mu = \lambda^h$ (Proposition 4.8)
- (4) $\Rightarrow$ ($\hat{3}$) $\hat{\kappa} = \mu$ (Cauchy-Schwarz)
- ($\hat{3}^*$) $\Rightarrow$ (4*) $\lambda = \hat{\delta}$ (Cauchy-Schwarz)
- (4) $\Rightarrow$ ($\hat{2}$) $\hat{\eta}(t) = \frac{1}{t} \int_0^t \mu(s) ds \leq \mu(t)$ (Remark 2)
- ($\hat{3}$) $\Rightarrow$ ($\hat{2}$) $\hat{\eta} = \hat{\kappa}$ (Proposition 4.2)
- ($\hat{3}^*$) $\Rightarrow$ ($\hat{2}^*$) $\hat{\gamma}(t) = \frac{1}{t} \int_0^t \hat{\delta}(s) ds$ (Remark 2)
- ($\hat{2}$) $\Rightarrow$ ($\hat{3}$) $\hat{\kappa} = 2\hat{\eta}$ (Proposition 4.2)
- ($\hat{2}^*$) $\Rightarrow$ ($\hat{3}^*$) $\hat{\delta} = 2\hat{\gamma}$ (Theorem 4.4)
- ($\hat{3}^*$) $\Rightarrow$ ($\hat{3}$) $\hat{\kappa} = \hat{\delta}^h$ (easy adaptation of Proposition 4.8)
- ($\hat{2}^*$) $\Rightarrow$ ($\hat{2}$) $\gamma = (\cdot)\hat{\gamma}, \eta = \gamma^* = (\cdot)\hat{\eta}$ (Corollary 3.4)
- ($\hat{2}^*$) $\Rightarrow$ ($\hat{1}$) $\hat{\sigma} = 2\hat{\eta}$ (analogous to Proposition 3.7)
- ($\hat{1}$) $\Rightarrow$ ($\hat{1}^*$) $\rho = \sigma^*, \sigma = (\cdot)\hat{\sigma}, \rho = (\cdot)\hat{\rho}$ (Proposition 2.6)

Implications

$$\begin{array}{ccccccc}
 \hat{(1)} & \Leftarrow & \hat{(2)} & \Longleftrightarrow & \hat{(3)} & \Leftarrow & \hat{(4)} \\
 \Downarrow & & \Uparrow & & \Uparrow & & \Downarrow \\
 \hat{(1^*)} & \Rightarrow & \hat{(2^*)} & \Longleftrightarrow & \hat{(3^*)} & \Rightarrow & \hat{(4)}
 \end{array}$$

References

- [1] ASPLUND (E.) . — *Fréchet Differentiability of convex functions*, Acta Mathematica **121** (1967), pp. 31-47.
- [2] ASPLUND (E.) . — *Positivity of duality mappings*, Bull. Am. Math. Soc. **73** (1967), pp. 200-203.
- [3] ASPLUND (E.) and ROCKAFELLAR (R. T.) . — *Gradients of convex functions*, Trans. Am. Math. Soc. **139** (1969), pp. 443-467.
- [4] ATTOUCH (H.) and WETS (R. J.-B.) . — *A quantitative approach via epigraphic distances to stability of strong local minimizers*, Technical report No 87-01, Univ. of Perpignan (1987).
- [5] AUBIN (J.-P.) and EKELAND (I.) . — *Nonlinear applied analysis*, J. Wiley, New York (1984).
- [6] AUSLENDER (A.) . — *Optimisation – Méthodes numériques*, Masson, Paris (1976).
- [7] BEAUZAMY (B.) . — *Introduction to Banach spaces and their geometry*, Mathematical Studies **68**, North-Holland, Amsterdam (1982).
- [8] BISHOP (E.) and PHELPS (R. R.) . — *The support functionals of a convex set*, In Convexity edited by P. Klee, Proc. Symp. Pure Math. **7**, Am. Math. Soc. Providence (1963), pp. 27-35.
- [9] BRÖNDSTED (A.) and ROCKAFELLAR (R. T.) . — *On the subdifferentiability of convex functions*, Proc. Am. Math. Soc. **16** (1965), pp. 605-611.
- [10] BROWDER (F. E.) . — *Problèmes non linéaires*, Presses de l'Université de Montréal **15** (1966).
- [11] BYNUM (J. C.) . — *Characterizations of uniform convexity*, Pac. J. Math. **38**, No 3 (1971), pp. 577-581.
- [12] CIARLET (P. G.) . — *Introduction à l'analyse numérique matricielle et à l'optimisation*, Masson, Paris (1982).
- [13] CIORANESCU (I.) . — *Duality mappings in the nonlinear functional analysis*, Ed. Acad. R.S.R., Bucharest (1974) (in Romanian).
- [14] DIESTEL (J.) . — *Geometry of Banach spaces, selected topics*, Lecture Notes in Mathematics **485**, Springer, Berlin (1975).
- [15] DONCHEV (A. L.) . — *Perturbations, approximations and sensitivity analysis of optimal control systems*, Springer Lecture Notes in Control **52** (1984).

- [16] DUNN (J. C.) .— *Convexity, monotonicity and gradient processes in Hilbert spaces*, J. Math. Anal. Appl. **53** (1976), pp. 145-158.
- [17] EKELAND (I.) .— *Two results in convex analysis*, in "Optimization and related fields", Ed. by R. Conti, E. de Giorgi and F. Giannessi, Lecture Notes in Mathematics **1190**, Springer Berlin (1986).
- [18] FIGIEL (T.) .— *On the moduli of convexity and smoothness*, Stud. Math. **56** (1976), pp. 121-155.
- [19] JEYAKUMAR (V.) .— *On subgradient duality with strong and weak convex functions*, J. Austr. Math. Soc. **40**, Ser. A (1986), 143-152.
- [20] JEYAKUMAR (V.) .— *ρ -convexity and second order duality*, Util. Math. **29** (1986), pp. 71-85.
- [21] KARMANOV (V.) .— *Programmation Mathématique*, French translation, Mir, Moscow (1975).
- [22] LEVITIN (E.) and POLJAK (B.) .— *Minimization methods in the presence of constraints*, Z. Vycisl. Mat. i. Fiz. **6**, No 5 (1966), pp. 787-823.
- [23] LEVITIN (E.) and POLJAK (B.) .— *Convergence of minimizing sequences in conditional extremum problems*, Sov. Math. Dokl. **7** (1967), pp. 764-767.
- [24] LUCCHETTI (R.) and PATRONE (F.) .— *Hadamard and Tyhonov well-posedness of a certain class of convex functions*, J. Math. Anal. Appl. **88** (1982), pp. 204-215.
- [25] LYUBICH (Y.) and MAISTROVSKI (G.) .— *The general theory of relaxation processes for convex functionals*, Russ. Math. Surv. **25** (1970), pp. 57-117.
- [26] MILMAN (V. D.) .— *A certain transformation of convex functions and a duality of the β and δ characteristics of a β space*, Dokl. Akad. Nauk. SSSR **187** (1969), pp. 33-45.
- [27] PSENITCHNY (B.) and DANILINE (Y.) .— *Méthodes numériques dans les problèmes d'extremum*, French translation, Mir, Moscow (1975).
- [28] PENOT (J.-P.) .— *Metric regularity, openness and Lipschitzian behaviour of multifunctions*, Nonlinear Anal. Theory Methods Appl. **13**, No 6 (1989), pp. 629-643.
- [29] PENOT (J.-P.) and VOLLE (V.) .— *Inversion of real valued functions and applications*, Z.O.R. Methods and Models of Operations Research **34** (1990), pp. 117-141.
- [30] PENOT (J.-P.) and VOLLE (M.) .— *On strongly convex and paraconvex dualities*, in "Generalized Convexity and Fractional Programming with Economics Applications", Proc. Pisa. Italy (1988), A. Cambini et al. Eds. Lecture Notes in Economics and Mathematical Systems **345**, Springer Verlag Berlin (1990), pp. 198-218.
- [31] POLJAK (B.) .— *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Sov. Math. Dokl. **7** (1967), pp. 72-75.
- [32] PRÜSS (J.) .— *A Characterization of uniform convexity and application to accretive operators*, Hiroshima Math. J. **11** (1981), pp. 229-234.
- [33] ROBERTS (A. W.) and VARBERG (D.) .— *Convex functions*, Academic Press, New York (1973).

- [34] ROCKAFELLAR (R. T.) . — *Monotone operators and the proximal point algorithm*, SIAM J. Control Opt. **14** (1976), pp. 877-898.
- [35] ŠMULYAN (V. L.) . — *Sur la dérivabilité de la norme dans l'espace de Banach*, Dokl. Acad. Naukl. SSSR, **27** (1940), pp. 643-648.
- [36] VIAL (J.-P.) . — *Strong convexity of sets and functions*, J. Math. Econ. **9** (1982) pp. 187-205.
- [37] VIAL (J.-P.) . — *Strong and weak convexity of sets and functions*, Math. Oper. Res. **8** (1983), pp. 231-259.
- [38] VLADIMIROV (A. A.), NESTEROV (YU. E.) and CHEKANOV (YU. N.) . — *On uniformly convex functionals*, Vest. Mosk. Univ. **3**, Ser. XV (1978), pp. 12-23.
- [39] VLADIMIROV (A. A.), NESTEROV (YU. E.) and CHEKANOV (YU. N.) . — *On uniformly quasi-convex functionals*, Vest. Mosk. Univ. **4**, Ser. XV (1978), pp. 18-27.
- [40] VOLLE (M.) Personal communication.
- [41] ZALINESCU (C.) . — *On uniformly convex functions*, J. Math. Anal. Appl. **95** (1983), pp. 344-374.
- [42] ZOLEZZI (T.) . — *On equiwellset minimum problems*, Appl. Math. Optimization **4** (1978), pp. 209-223.