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## Representations of the Whitehead manifold $Wh^3$ and Julia sets<sup>(\*)</sup>

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**RÉSUMÉ.** — Quand on étend à la variété de Whitehead le théorème de représentation pseudo-spine collapsible pour les sphères d'homotopie, les transversales tendues à l'ensemble des lignes doubles présentent un comportement chaotique bien connu. Ce comportement est engendré par une boucle de feedback dynamique qui engendre aussi des ensembles de Julia.

**ABSTRACT.** — When one extends to the Whitehead manifold the collapsible pseudo-spine representation theorem for homotopy 3-spheres, the tight transversals to the set of double lines present chaotic dynamical behaviour of a very well-known type. This behaviour is generated by a dynamical feedback loop which also generates Julia sets.

**KEY-WORDS :** Representations of open simply connected 3-manifolds, Whitehead manifolds, tight transversals to the set of double lines, Chaotic behaviour, Julia sets, Mandelbrot set.

**AMS Classification :** 57 M 30, 58 F 13.

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### Introduction

The present paper exhibits, possibly in the simplest form, something which many people expect to happen, namely a connection between wild differential topology in dimension 3 and chaotic behaviour of dynamical systems (see for instance section 6 from [FL]).

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The first author has given a certain kind of representations for homotopy 3-spheres the so called *collapsible pseudo-spine representations* [Po1]. Subsequently the same author has extended this kind of representations to arbitrary open simply connected 3-manifolds. Completely new features occur in this set up, namely the possibility of infinite accumulation of double points at the source of the representation; we will describe these things in detail later on in the present introduction, but right now let us say already that the point which this present paper makes, is the following. When you try to represent the celebrated Whitehead manifold by the procedure in question and if you take the most natural road for doing so, then the well known Julia sets from the iteration of quadratic polynomials appear very naturally. We will explain these things a bit more in detail. We will briefly review now how the collapsible pseudo-spine representations  $K^2 \xrightarrow{f} \Sigma^3$  from [Po1] are generalized. In the present paper we consider an open simply connected 3-manifold  $V^3$ , in lieu of  $\Sigma^3$  and an infinite but locally finite 2-dimensional simplicial complex  $X$  in lieu of  $K^2$ ; this  $X$  will be *almost arborescent* (this notion, which we will define precisely later on, is the substitute for almost-collapsibility in the context of infinite complexes).

With this we will be able to construct a non-degenerate map  $X \xrightarrow{f} V^3$  having properties:

- (i) The map  $f$  is a generic immersion except at a set  $\text{Sing}(f) \subset X$  of singularities; but *caution* we will need to leave the simplicial category,  $fX$  will not be a simplicial complex, not even a non-locally finite one, and accordingly  $f$  will not be simplicial. But  $fX$  will be a *generalized polyhedron*, in a sense which will be made explicit later on; it suffices to say here that the *open regular neighbourhood*  $\text{Nbd}(fX) \subset V^3$  continues to be well-defined.
- (ii) The double points of  $f$  are again commanded by the singularities (with the notation of [Po1] this means that  $\Psi(f) = \Phi(f)$ ). In plain english this also means that if you start zipping away from the singularities, then you catch all the double points.
- (iii) The open regular neighbourhood  $\text{Nbd}(fX) \subset V^3$  is isotopic to  $V^3 - \{\text{a closed totally discontinuous subset which is tame}\}$  (and this will also be made completely explicit later on).

So far we have for  $V^3$  a not completely trivial extension of the (almost-) collapsible pseudo-spine representation theorem for  $\Sigma^3$ , from [Po1].

But now we have a completely new phenomenon; the set of double points of  $f$ , i.e. of points  $x \in X$  such that  $\text{card}(f^{-1}fx) > 1$  which we will denote by  $M_2(f) \subset X$  is, generally speaking, *no longer closed* and the main concern of [Po2] and also of the present paper is the study of the *set of limit-points* of  $M_2(f)$ . In order to get our hands on this set, we will consider the various smooth lines  $\Lambda \subset X$  which are *transversal* to  $M_2(f)$  and look at the set of limit-points of  $\Lambda \cap M_2(f)$ , which we denote by  $\lim(\Lambda \cap M_2(f))$ ; here  $p \in \lim(\Lambda \cap M_2(f)) \subset \Lambda$  iff there exists an infinite sequence of distinct points  $p_1, p_2, \dots \in \Lambda \cap M_2(f)$  such that  $\lim p_n = p$ .

Before we go on, some more notations will have to be introduced.

By definition a generalized polyhedron of  $R^3$  is a subspace  $Y \subset R^3$  which up to a global diffeomorphism of  $R^3$ , can be described as follows. We are given to begin with, a countable (or possibly finite) collection of distinct affine planes  $R_1^2, R_2^2, \dots \subset R^3$  such that  $Y \subset \bigcup_i R_i^2$ . Moreover our collection of planes has fulfill the following conditions.

- (A1) No infinite subcollection of the  $R_i^2$ 's has a common point.
- (A2) There are positive numbers  $\varepsilon_1, \varepsilon_2, \dots$  such that the open subsets  $R_1^2 \times (-\varepsilon_1, \varepsilon_1), R_2^2 \times (-\varepsilon_2, \varepsilon_2), \dots$  of  $R^3$  meet *exactly* like the  $R_1^2, R_2^2, \dots$ . In particular this implies that if  $p \in R_\alpha^2 \cap R_\beta^2 - R_i^2$  then  $\text{dist}(p, R_i^2) > \varepsilon_i$ . Also the set of planes  $\{R_i^2\}$  is *discrete* in the sense that there are no "topological relations" (in the space planes of  $R^3$ ) of the type  $\lim_{m \rightarrow \infty} R_{i_m}^2 = R_{i_0}^2$ . Similarly, the set of lines  $\{R_i^2 \cap R_j^2\}$  and the set of points  $\{R_i^2 \cap R_j^2 \cap R_h^2\}$  are also discrete.
- (A3) For each  $R_i^2$  we give 2-dimensional piecewise linear submanifold with boundary  $L_i^2 \subset R_i^2 \subset R^3$ . We will assume these  $L_i^2$ 's to be of simplest type, like for instance  $R_i^2$  itself, a half plane, a quarter plane or rectangle (possibly with some edges at infinity, or simply deleted). But even in the case when each  $p \in Y$  has a neighbourhood in  $Y$  which is a locally finite simplicial complex (or, more generally, a neighbourhood which is a closed subset of  $R^3$ ),  $Y \subset R^3$  itself is *not* assumed to be a closed subset.

We do not have to go into more details here and now, since the generalized polyhedra considered in the present paper will be completely explicit.

We consider now in lieu of  $R^3$ , a smooth 3-manifold  $M^3$ , with  $\partial M^3 = \emptyset$ . A subset  $Z \subset M^3$  will be called a *generalized (sub-)polyhedron* of  $M^3$  if locally, i.e. in each coordinate chart  $R^3 \subset M^3$ , the object  $Z \cap R^3$  is a generalized polyhedron in the sense of (A1) to (A3).

We will define now the notion of a *singular generalized polyhedron* which is *locally finite at the source* (which we will just call a “singular” locally finite generalized polyhedron). This is a triple  $(X, f, M^3)$  where the following conditions are fulfilled.

- (B1)  $M^3$  is a smooth 3-manifold without boundary,  $X$  is a locally finite simplicial 2-dimensionnal complex and  $X \xrightarrow{f} M^3$  is a continous map which restricted to each simplex of  $X$  is a smooth embedding. The image  $fX \subset M^3$  is a generalized subpolyhedron of  $M^3$  (which also means that, generally speaking, the map  $f$  cannot be made simplicial). We will assume  $f$  to be generic.
- (B2) There is a closed discrete subset  $\text{Sing}(f) \subset X$  which is such that if  $x \in X - \text{Sing}(f)$ , then there exists an open neighbourhood  $x \in Y \subset X$  and a smooth coordinate neighbourhood  $f(x) \subset R^3 \subset M^3$  such that  $f|Y$  is an embedding of  $Y$  into  $R^3$ . So, outside the discrete and closed set  $\text{Sing}(f)$ ,  $f$  is an immersion.

*Caution* :  $Y$  is a locally finite simplicial complex while  $f(Y) \subset R^3$  is generally speaking (although isomorphic, abstractly speaking, to  $Y$ ) only a generalized subpolyhedron of  $R^3$ .

- (B3) For each singularity  $\sigma \in \text{Sing}(f)$ , we have the admissible local model from [Po1] (and [Po2]). Explicitely, this means the following (this is a reminder). For each *singularity*  $\sigma \in \text{Sing}(f)$ , there are two rectangles  $P_1, P_2$  of  $X$  such that  $P_1 \cup P_2 \subset X$  is a neighbourhood of  $\sigma$ , with the following properties: at the level of  $X$ ,  $P_1$  and  $P_2$  are *glued together along a common half-line*, the restrictions  $f|_{P_i}$  ( $i = 1, 2$ ) are embeddings, and at the target  $fP_1$  and  $fP_2$  *meet transversally* (fig. 1.1).

There is a line of double points of  $f$ , which starts at the singularity  $\sigma$ . Such points as we have just described, will be called *undrawable singularities* (or *admissible singularities*). It is assumed that  $f|_{\text{Sing}(f)}$  is injective, and also that  $f(\text{Sing}(f)) \subset M^3$  is a discrete subset which, generally speaking, need *not be closed*; it is only  $\text{Sing}(f) \subset X$  which is closed, at the source. On the other hand through each singularity  $\sigma \in X$  go now exactly two *smooth sheets* of  $X$ , namely  $P_1$  and  $P_2$ .

We will say that the *singular generalized polyhedron*  $X$  is *arborescent* if we can obtain it by a sequence of Whitehead dilatations starting with a point. Also “arborescent” does *not* imply “collapsible” ( $R^2$  for instance is arborescent). We will say that  $X$  is *almost arborescent* if it can be obtained

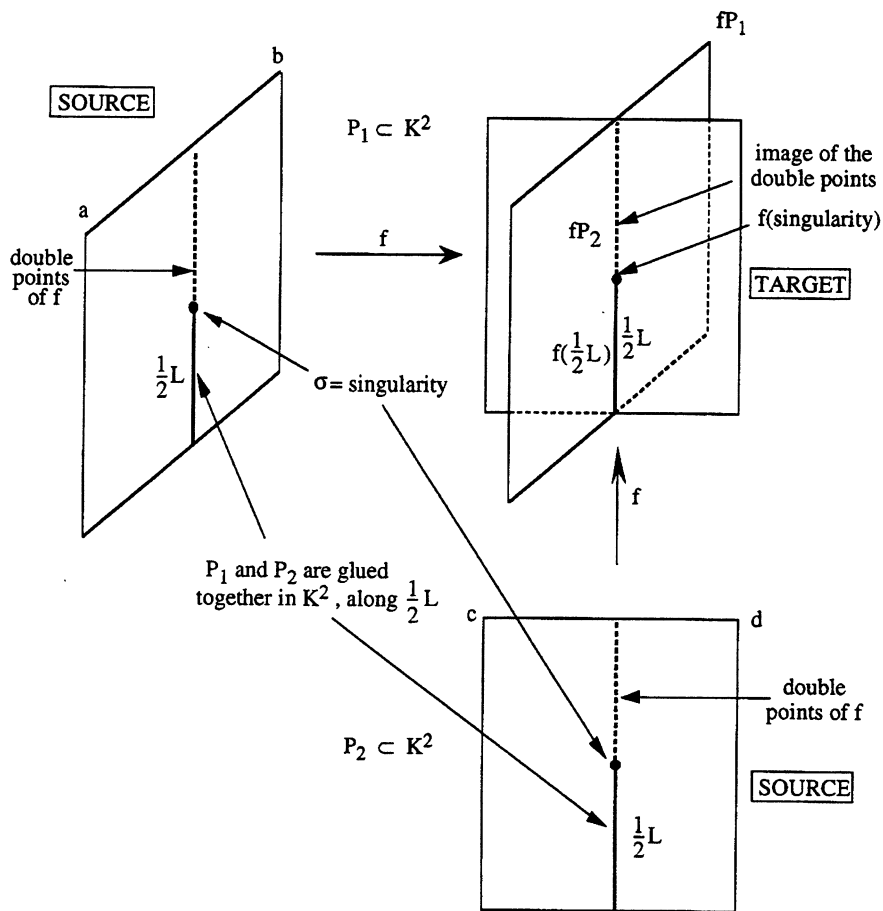


Fig. 1.1 Admissible or "undrawable" singularity.

by a sequence of Whitehead dilatations and of additions of 2-cells along their boundaries, starting with a point. In other words,  $X$  is *almost arborescent* if it admits an exhaustion by finite simplicial complexes

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X_\infty \stackrel{\text{def}}{=} X \quad (1.1)$$

such that  $X_0 = \{pt\}$  and that each inclusion  $X_i \subset X_{i+1}$  is a Whitehead dilatation or the addition of a 2-cell along its boundary.

Let us consider a generalized polyhedron  $R^3 \supset Y = \bigcup_i L_i^2$  with quantities  $\varepsilon_i > 0$  like in (A2) at the beginning of this section. Consider for every  $L_i^2$  an open tubular neighbourhood of width  $\eta_i < \varepsilon_i$ , let us call it  $L_i^2 \times (-\eta_i, \eta_i)$ . The open manifold:

$$\text{Nbd}(Y) = \bigcup_i (L_i^2 \times (-\eta_i, \eta_i)) \subset R^3,$$

which up to diffeomorphism only depends of  $(R^3, Y)$  is, by definition, the (*open*) *regular neighbourhood* of  $Y$ . With this we can define the regular neighbourhood of any :

$$X \xrightarrow{f} M^3 \quad (1.2)$$

as soon as  $\text{Sing}(f) = \emptyset$  (i.e. with an immersive  $f$ ) in particular for generalized polyhedra  $Z \subset M^3$ .

If  $V^3$  is any *open* 3-manifold, a singular locally finite generalized polyhedron:

$$X \xrightarrow{f} V^3$$

will be called an (*almost*) *arborescent representation* of  $V^3$  which is locally finite "at the source" if the following conditions are fulfilled:

(C1)  $X$  is (*almost*) arborescent.

(C2)  $\Psi(f) = \Phi(f)$ .

(C3) For the open submanifold  $\text{Nbd}(fX) \subset V^3$  there is an isotopy of smooth embeddings  $\text{Nbd}(fX) \xrightarrow{i_t} V^3$ , where  $t \in [0, 1]$ , such that  $i_0$  is the standard embedding and  $i_1$  is a *diffeomorphism*  $\text{Nbd}(fX) \xrightarrow[i_1]{\approx} V^3$ .

*Remark.* — Our  $i_t$  is not an ambient isotopy.

For a given open 3-manifold  $V^3$  we will define a type of submanifolds, which we call generically  $V_h^3$  which are defined by :

$$V_h^3 = V^3 - \bigcup_{n=1}^{\infty} C_n, \quad (1.3)$$

where the  $C_n$  are two-by-two disjoint compact discontinuous tame subsets, not accumulating at finite distance. Up to isotopy, the  $V_h^3$  is uniquely defined once the topological type of  $\bigcup_{n=1}^{\infty} C_n$  is determined.

Let  $L \subset X$  be a smooth sheet and  $\ell \subset L$  a smooth line meeting transversally the stratified set  $M_2(f) \subset X$ . We will say that the transversal  $\ell$  is *tight* if we cannot find any 2-cell  $D^2 \subset L$  resting with half of its boundary on  $\ell$  and with the other half on a smooth line in  $M_2(f)$ . In other words, for a *tight transversal* figure 1.2 is forbidden.

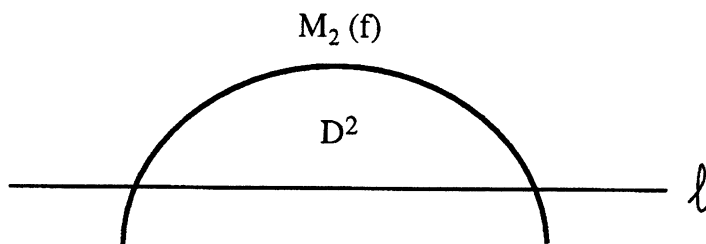


Fig. 1.2 This figure (supposedly living inside the smooth sheet  $L \subset X$ ) is forbidden for a tight transversal  $\ell$ .

We will only be interested in the following situation as far as singular  $X \xrightarrow{f} V^3$ 's are concerned. In any sufficiently small smooth sheet  $L \subset X$  the set  $M_2(f) \cap L$  is a finite union of subsets (actually only one or two), each of them contained in a family  $F \subset L$  of "parallel" lines with the following property. For any tight transversal  $\ell$ , the set  $F \cap \ell$  is discrete, i.e.  $F \cap \lim(F \cap \ell) = \emptyset$  (this behaviour is at the opposite pole with respect to laminations).

All this having been said we can state now the following theorem which is proved in [Po2] and which gives the general context for this paper. What we will call the *representation theorem* below is actually only part of the theorem proved in [Po2] but this part is the only relevant one for this context. So here is the theorem.

**REPRESENTATION THEOREM .—** *Let  $V^3$  be an open simply connected 3-manifold. We can always find a  $V_h^3$  (like in (1.3)) and a locally finite almost arborescent representation of  $V_h^3$*

$$X \xrightarrow{f} V_h^3 \subset V^3 \quad (1.4)$$

with the following property.

*For any tight precompact transversal  $\ell \subset X$  the limit set of  $M_2(f) \cap \ell$ , i.e.*

$$\lim(M_2(f) \cap \ell) \subset \ell$$

*is finite.*

This statement should be compared with the following.

(ALMOST) COLLAPSIBLE PSEUDO-SPINE REPRESENTATION THEOREM (see [Po1]) .— *Let  $\Sigma^3$  be a homotopy 3-sphere. We can always find a finite (non-void) set  $E \subset \Sigma^3$  and an (almost) collapsible representation of  $\Sigma^3 - E$ :*

$$K^2 \xrightarrow{g} \Sigma^3 - E.$$

(Here  $K^2$  is a bona fide finite simplicial complex and  $g$  a non-degenerate simplicial map.)

We will make now a few comments, first about the representation theorem itself. In some sense this theorem is the best possible, since one can prove the following fact. If  $X \xrightarrow{f} V_h^3 \subset V^3$  is like in the representation theorem and such that for any tight transversal  $\lim(M_2(f) \cap \ell) = \emptyset$  (i.e.  $\text{card}(M_2(f) \cap \ell) < \infty$ ) then  $\pi_1^\infty V^3 = 0$ . This fact is proved, among other things, in [Po2]. Only tight precompact transversals will be considered.

Now the representation (1.4) is, generally speaking, *highly redundant* in the sense that any a given double point

$$(x, y) \in M^2(f) \subset X \times X$$

of the map  $f$  can be reached in many distinct ways, via zippings starting from the singularities of  $f$ . Very much linked with this redundancy is the fact that in order to get the *finitenes* condition for our almost arborescent representation, we had to go from  $V^3$  to  $V_h^3$  thereby (highly) increasing the second homotopy group  $\pi_2$ . In the same spirit, when we realize  $f$  by elementary  $O(i)$  moves [Po1], many of these will be  $O(3)$ 's and each of these creates a new  $\pi_2$ -generator.

In Theorem A below we consider a *non-redundant* representation of the classical Whitehead manifold and what we find is that if this representation "is the simplest possible" (that is what (c) in Theorem A is supposed to tell us), then there are tight transversals  $\ell$  such that  $\lim(M_2(f) \cap \ell)$  is uncountable (it actually exhibits chaotic behaviour). To put things slightly differently, a high redundancy is very likely *unavoidable* for a wild 3-manifold  $V^3$  (with  $\pi_1^\infty V^3 \neq 0$  like the Whitehead manifold), if one insists in getting an (almost) arborescent representation satisfying the *finiteness* property from the *representation theorem* above. It will turn out that the set  $M_2(f) \cap \Delta$ , where  $\Delta$  is a certain tight transversal, is generated by a dynamical feedback loop which is the same as the one which generates

the Julia set of a quadratic polynomial which has the property that under iteration the unique critical point escapes to infinity. In other words again, the *finiteness* property in our representation theorem is achieved only at the price of a lot of *redundancy*.

As far as the representation theorem itself is concerned, it is part of a general approach developed by the first author in connection with the problem of the simple connectivity at infinity for the universal covering space  $\widetilde{M}^3$  of a closed 3-manifold  $M^3$ , and our Theorem A is an addendum to the representation theorem in question; but it is *not* part of the approach to the proof of  $\pi_1^\infty \widetilde{M}^3 = 0$ .

Before we can state our result we have to open another parenthesis. We will consider a real quadratic polynomial  $f_\lambda$  which is such that iterates of the unique critical point go to infinity. We consider also a certain set of cardinality two  $E = \{x_0, y_1\} \subset R$  (fig. 2.1) and for this set we consider the set

$$E \cup f_\lambda^{-1}E \cup f_\lambda^{-2}E \cup \dots \subset [0, 1].$$

As we will see in the next section, the set of limit points of  $E \cup f_\lambda^{-1}E \cup f_\lambda^{-2}E \cup \dots$  is the Julia set (see for instance [B]) of the quadratic polynomial  $f_\lambda$ .

The goal of the next section of this paper is to exhibit the following fact which we believe to reflect a general kind of a pattern and which we will call.

**THEOREM A.** — *We can find an arborescent representation, which is locally finite at the source, of the Whitehead manifold  $\text{Wh}^3$ .*

$$K \xrightarrow{g} \text{Wh}^3,$$

with the following features:

(a) *We will have*

$$K = D_0 \cup D_1 \cup D_2 \cup \dots$$

*where each  $D_i$  is a 2-disk, for every  $i$  the disks  $D_{i-1}$  and  $D_i$  are glued together along a common arc and, in  $K$ , these arcs are two-by-two disjointed.*

(b) *The map  $g$  has no triple points, each connected component of  $M_2(g)$  is a finite tree starting at a singular point, and for each tight transversal  $\ell$  we have*

$$M_2(g) \cap (\lim (M_2(g) \cap \ell)) = \emptyset.$$

- (c) Each  $g(\partial D_i)$  is the middle curve  $S^1 \times pt \subset S^1 \times D^2 = T_i$  of the  $i$ 'th solid torus  $T_i$  of  $\text{Wh}^3$ ,  $g(D_i) \subset \text{Int } T_{i+1}$  and  $g(D_i)$  is a spine of  $T_{i+1}$ .
- (d) We can find a tight transversal  $\Lambda \subset D_0$  to  $M_2(g)$  with a disjointed partition

$$\Lambda \cap M_2(g) = \sum_{i=1}^{\infty} \underbrace{\{\text{double points involving } D_0 \text{ and } D_i\}}_{\text{call this } F_i} \subset [0, 1].$$

so that there is an order preserving bijective map

$$E \cup f_{\lambda}^{-1}(E) \cup f_{\lambda}^{-2}(E) \cup \dots \xrightarrow{\varphi} \Lambda \cap M_2(g)$$

which sends  $f_{\lambda}^{-n}E$  onto  $F_{n+1}$ .

- (e) The set of limit points of  $\Lambda \cap M_2(g)$  is a Cantor set, actually it is the Julia set the polynomial  $f_{\lambda}$ .

The following section owes a lot to the friendly help of John Hamal Hubbard who showed the first author how his algorithms concerning double points fitted into the context of the Julia sets.

## 2. Whitehead and Julia algorithms

The point we are trying to make in the present section is the following. As already said, the *finitess* in the representation theorem quoted in the introduction to this paper (and which is proved in [Po2]) was achieved at the cost of a lot *redundancy* (i.e. by going from  $V^3$  to  $V_h^3$ ). We will show here that for the simplest possible representation of the Whitehead manifold  $\text{Wh}^3$  (i.e. for a non-redundant representation of a wild simply-connected open 3-manifold), the set of limit points of double points, on a transversal to the double lines exhibits *chaotic behaviour*, in fact a chaotic dynamical behaviour of a very well-know type.

But before we can state exactly our result we need to recall a number of facts about holomorphic dynamics. Consider a (real) quadratic polynomial  $f_{\lambda}(x) = \lambda x(1-x) - \mu$  (with a fixed small  $\mu > 0$ ) with the parameter value  $\lambda > 0$  fixed in such a way that the sequence of iterates of the (unique) critical point of  $f_{\lambda}$  is *unbounded*. Figure 2.1 shows the graph of such a  $f_{\lambda}$  (with  $\lambda = 6.185$  and  $\mu = 0.5$ ); in this figure the critical point of  $f_{\lambda}$  is  $x = 1/2$  and as we can see, by iterating,





- ( $\beta$ ) If, in terms of figure 2.1, we think of  $f_\lambda$  as being a *not everywhere well-defined* map  $[0, 1] \rightarrow [0, 1]$ , then our  $J_\lambda \subset [0, 1]$  is exactly the subset where all the iterates of  $f_\lambda$  are well-defined.
- ( $\gamma$ ) The Julia set  $J_\lambda$  is the subset of the complex  $z$ -plane ( $z = x + iy$ ) which is such that if  $z_0 \in J_\lambda$  then there is *no* neighbourhood  $N$  of  $z_0$  such that, when restricted to  $z \in N$ , the set of iterates of  $f_\lambda(z)$  is a normal family. In other words this is the *chaotic set*, as far as  $f_\lambda$  is concerned.
- ( $\delta$ )  $J_\lambda$  is also the set of points the iterates of which do not escape to infinity (which, of course, is another way to say that the action of  $f_\lambda$  on this set is chaotic).

It is a well-known fact that  $J_\lambda$  is a Cantor set for any polynomial  $f_\lambda$  which is like in figure 2.1 and as long as this is so, all those  $J_\lambda$ 's are the same object, up to a quasi-conformal homeomorphism of complex plane. By contrast with what we see in figure 2.1, the  $f_\lambda$  from the figure 2.2 has a connected "filled" Julia set and it corresponds to a value of the parameter living in the Mandelbrot set  $M$  (unlike our  $f_\lambda$ ).

For more details about all these things, one can consult [B] or any other standard reference on this subject.

*Remark.* — The dynamical behaviour displayed in figure 2.1 is very much related to the Smale horseshoe [S] (unlike the more subtle dynamics from figure 2.2, which is connected to the Feigenbaum cascade of period-doublings and transition to chaos [B], [F]). But for the discussion which follows the Julia set standpoint seems more natural than the horseshoe description; we will see this very soon.

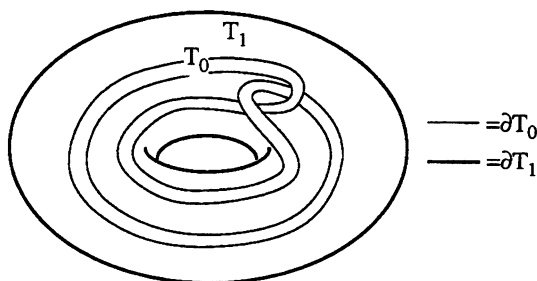


Fig. 2.3

In the figure 2.1 we can see two points  $x_0, y_1$  which are such that  $f_\lambda(x_0) = 0$ ,  $f_\lambda(y_1) = 1$ , the value  $x_0$  being chosen as small as possible while  $y_1$  is as large as possible. We will denote  $E = \{x_0, y_1\}$  and consider the set

$$\mathcal{E} \stackrel{\text{def}}{=} E \cup f_\lambda^{-1}E \cup f_\lambda^{-2}E \cup f_\lambda^{-3}E \cup \dots \subset [0, 1]. \quad (2.1)$$

As already said before the set of limit points of  $\mathcal{E}$  is the Julia set of our  $f_\lambda$ . Let us consider now some arc  $\Lambda$  and a countable subset  $\mathcal{F} \subset \Lambda$  endowed with a disjointed partition into finite subsets

$$\mathcal{F} = F_1 \cup F_2 \cup F_3 \cup \dots \subset \Lambda. \quad (2.2)$$

We will say that the partition (2.2) *obeys to the Julia algorithm* if there is an order preserving bijective map sending  $\mathcal{F}$  onto  $\mathcal{E}$  in such a way that for each  $n$  we have  $\varphi(F_{n+1}) = f_\lambda^{-n}E$ . All this having been said, we consider now the simplest Whitehead manifold  $\text{Wh}^3[\mathbb{R}]$ , which is obtained by iterating the embedding  $T_0 \subset T_1$ , or more explicitly

$$\{\text{solid torus of genus one}\} = T_0 \subset \text{Int } T_1 \subset T_1 = \{\text{solid torus of genus one}\}$$

from figure 2.3 (with null-framing). It is a well-know fact that for the open contractible 3-manifold  $\text{Wh}^3$  we have  $\pi_1^\infty \text{Wh}^3 \neq 0$ .

The goal of this section is to prove the theorem A from the introduction, which we restate below in a slightly modified form.

**THEOREM A.** — *We can find an arborescent representation of the Whitehead manifold  $\text{Wh}^3$*

$$K \xrightarrow{g} \text{Wh}^3,$$

*with the following features:*

- (1) (Reminder) *The space  $K$  is a locally finite simplicial complex which can be obtained starting with a point, by an infinite sequence of Whitehead dilatations (which is itself locally finite).*
- (2) *Actually we will have*

$$K = D_0 \cup D_1 \cup D_2 \cup \dots$$

*where each  $D_i$  is a 2-disk, for every  $i$  the disks  $D_{i-1}$  and  $D_i$  are glued together along a common arc and, in  $K$ , these arcs are two-by-two disjointed.*

- (3) The map  $g$  has no triple points and each connected component of  $M_2(g)$  is a finite tree, starting at a singular point. Moreover, for any tight transversal  $\ell$  to the  $M_2(g)$  we have  $M_2(g) \cap \lim(M_2(g) \cap \ell) = \emptyset$ .
- (4) (This point, which will be explained in much more detail, later on, is supposed to tell us that  $K \xrightarrow{g} \text{Wh}^3$  is the simplest possible representation of  $\text{Wh}^3$ ). Each  $g(\partial D_i)$  is the middle curve  $S^1 \times \text{pt} \subset S^1 \times D^2 = T_i$  of the  $i$ 'th solid torus  $T_i$  of  $\text{Wh}^3$ ,  $g(D_i) \subset \text{Int } T_{i+1}$  and  $g(D_i)$  is a spine of  $T_{i+1}$  (fig. 2.3.1).
- (5) We can find a tight transversal  $\Lambda \subset D_0$  to  $M_2(g)$  with a disjoint partition

$$\Lambda \cap M_2(g) = \sum_{i=1}^{\infty} \underbrace{\{\text{double points involving } D_0 \text{ and } D_i\}}_{\text{we call this } F_i} \subset [0, 1]. \quad (2.3)$$

so that there is an order preserving bijective map sending  $f_\lambda^{-n} E$  onto  $F_{n+1}$

$$E \cup f_\lambda^{-1}(E) \cup f_\lambda^{-2}(E) \cup \dots \xrightarrow{\varphi} \Lambda \cap M_2(g)$$

and hence (2.3) obeys the Julia algorithm.

- (6) The set of limit points  $\Lambda \cap M_2(g)$  is a Cantor set, it is actually the Julia set the polynomial  $f_\lambda$ .

All these things will be looked at in detail in the rest of this section

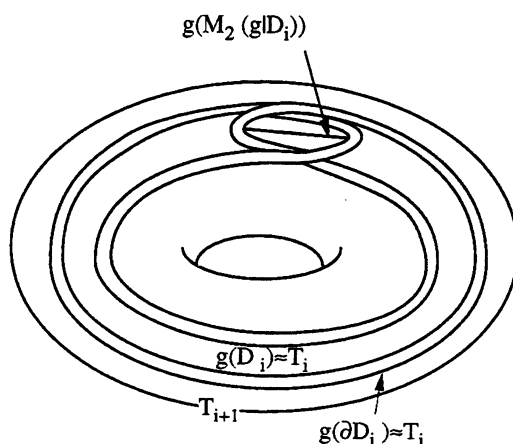


Fig. 2.3.1  $g(D_i) \not\subset T_{i+1}$ , i.e.  $g(D_i)$  is a spine of  $T_{i+1}$ .

LEMMA 2.1. — *There is an arborescent representation of the Whitehead manifold  $\text{Wh}^3$*

$$K \xrightarrow{g} \text{Wh}^3 \quad (2.4)$$

which is such that  $g$  has no triple points.

*Proof.* — We will explicitly construct such a  $K \xrightarrow{g} \text{Wh}^3$  which in some sense is the simplest possible one, before analyzing its further features. But it will be easier to start with another representation

$$K_1 \xrightarrow{g_1} \text{Wh}^3 \quad (2.5)$$

Which is *not* arborescent any longer, but which still has the following properties.

- (a) The space  $K_1$  is a locally finite contractible simplicial complex; but this  $K_1$  will not be *arborescent* (i.e. it is no longer obtainable from a point, via Whitehead dilatations). We also have  $M_3(g_1) = \emptyset$ . The next three points are just a reminder.
- (b) Except for admissible singularities like the ones in figure 1.1,  $g_1$  is a generic immersion without triple point.

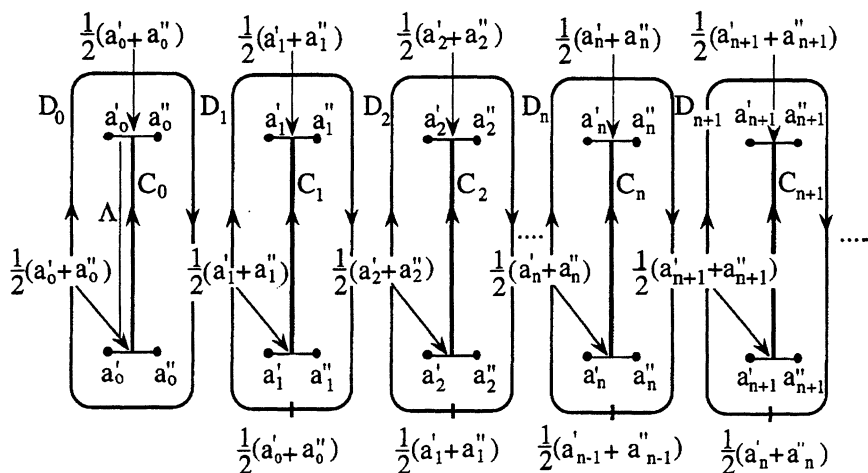


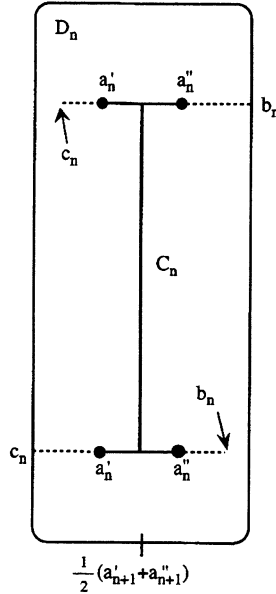
Fig. 2.4  $K_1$  is a quotient space of the disjoint union  $\sum_0^\infty D_n$  and points of  $\sum_0^\infty D_n$  with the same image in  $K_1$  carry the same letters in this figure.

- (c) We have  $\Psi(g_1) = \Phi(g_1)$ .
- (d) The subset  $g_1 K_1 \subset \text{Wh}^3$  is a *non-singular generalized polyhedron* and there is an (improper, i.e. non-ambient) isotopy of

$$\text{Nbd}(g_1 K_1) \xrightarrow{i_0} \text{Wh}^3$$

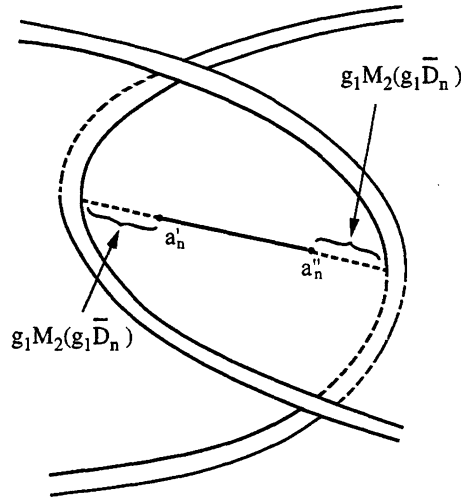
changing it into a diffeomorphism  $\text{Nbd}(g_1 K_1) \approx \text{Wh}^3$ .

We will eventually get our (2.4) by a very easy perturbation of the simpler object (2.5). From a purely abstract viewpoint (i.e. if we forget the map  $g_1$ ), the space  $K_1$  will be obtained by glueing together infinitely many copies of the 2-disk, namely  $D_0, D_1, D_2, \dots$ , (fig. 2.4). We actually will have to quotient the  $D_i$ 's before glueing them together.

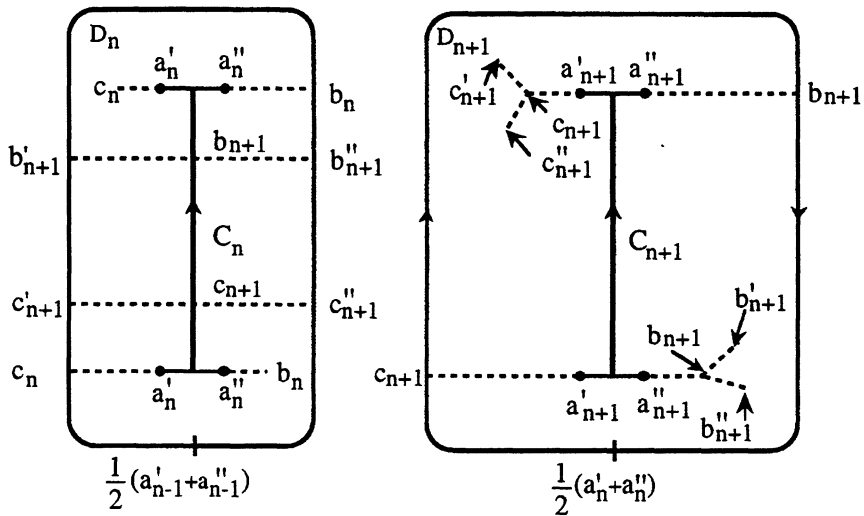


**Fig. 2.5**  $M_2(g_1 | \overline{D}_n)$ ; the double points are dotted; also  $\bullet$  = singularity; this figure is at the source of the map  $g_1$ .

The quotient and glueing pattern is the following (see again fig. 2.4).

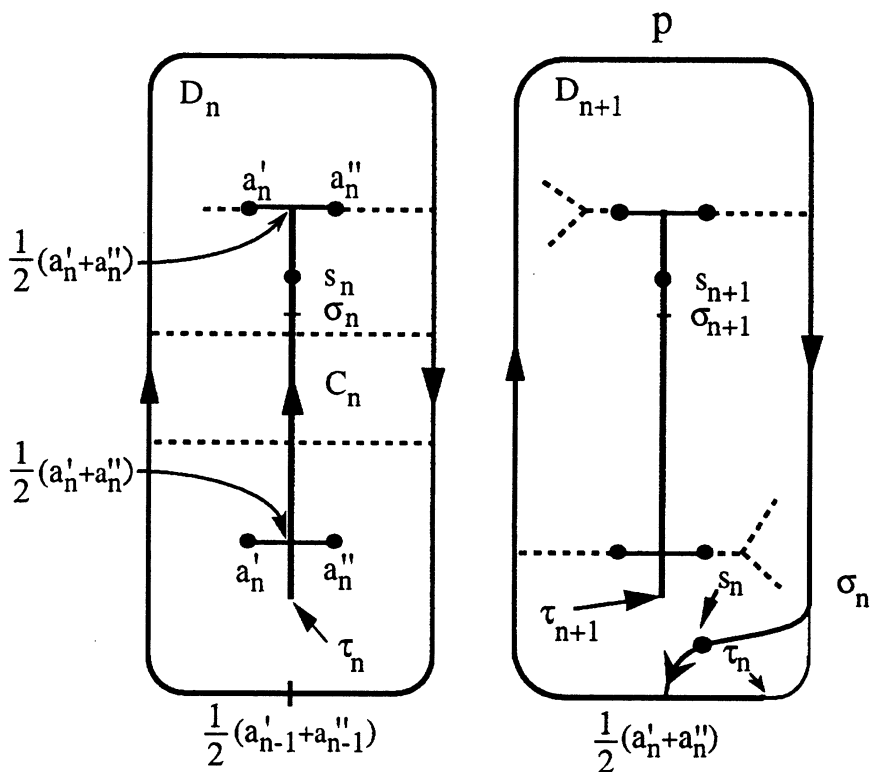


**Fig. 2.5.a** This figure which is at the target corresponds to figure 2.5;  
the fat arc  $[a'_n, a''_n]$  is not a double line;  
the  $a'_n, a''_n$  are undrawable singularities



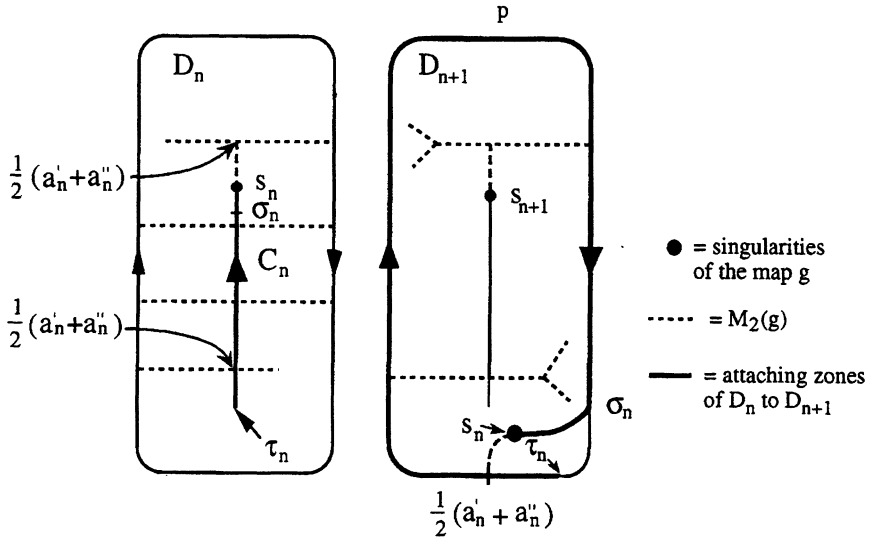
**Fig. 2.6**  $M_2(g_1 | (\bar{D}_n \cup \bar{D}_{n+1}))$ .

To begin with, at the level of each  $D_n$  (for  $n = 0, 1, 2, \dots$ ) we identify (in the orientable fashion) the two copies of the arc  $[a'_n, a''_n]$  which are contained in the interior of  $D_n$ .



**Fig. 2.7**  $\overline{D}_n \cup \overline{D}_{n+1} \subset K_2$  and  $M_2(g_2 | (\overline{D}_n \cup \overline{D}_{n+1})) = M_2(g_1 | (\overline{D}_n \cup \overline{D}_{n+1}))$ ;  
the points  $s_n, s_{n+1}$  are not yet singular;  
they will become so at the level of  $K$  (fig 2.8).

It is understood that the two middle points  $(1/2)(a'_n + a''_n)$  are glued together in this process. Let us denote by  $\overline{D}_0, \overline{D}_1, \overline{D}_2, \dots$  the quotient spaces of  $D_0, D_1, D_2, \dots$  obtained in this way. The image of the arc  $C_n \subset D_n$  (which joins the middle points of the two arcs  $[a'_n, a''_n]$ ) becomes a circle, which we denote by the same letter  $C_n \subset \overline{D}_n$ .



**Fig. 2.8**  $D_u \cup D_{n+1} \subset K$ ; all the dotted lines are double points of  $g$ ; they include  $M_2(g \mid (D_n \cup D_{n+1}))$ , but they are far from exhausting  $M_2(g) \cap (D_n \cup D_{n+1})$ , as we shall see explicitly later on.

Notice that  $\partial \overline{D_p} = \partial D_p$ . Next we glue for each  $n + 1$  ( $n = 0, 1, \dots$ ) the boundary  $\partial \overline{D_{n+1}} = \partial D_{n+1}$  to the circle  $C_n \subset \overline{D_n}$  via a homeomorphism. In terms of figure 2.4 it will be assumed (in order to make things precise) that the arrows of  $C_n$  and  $\partial D_{n+1}$  correspond to each other. The result of this operation is our  $K_1 = \overline{D_0} \cup \overline{D_1} \cup \dots$ , which will be the source of the map  $g_1$ .

Consider now the sequence of embedded solid tori of genus one, each one inside the interior of the next, which defines  $\text{Wh}^3$

$$T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_n \subset \cdots \subset \mathbf{Wh}^3. \quad (2.6)$$

The map  $K_1 \xrightarrow{g_1} \text{Wh}^3$  will be subjected to the following list of conditions.

### Conditions on $g_1$

- (0) The image of  $\partial \overline{D}_n \subset K_1$  is contained in  $\text{Int } T_n$  and  $g_1 \partial \overline{D}_n \subset T_n$  is a spine of  $T_n$ .
- (1) The image of  $\overline{D}_n \subset K_1$  is contained in  $\text{Int } T_{n+1}$  and  $g_1 \overline{D}_n \subset T_{n+1}$  is a spine of  $T_{n+1}$ .

- (2) The double points of the map  $g_1 | \bar{D}_n$  are like in figure 2.5 and the singularities of  $g_1$  are exactly

$$\text{Sing}(g_1) = \{a'_0, a''_0, a'_1, a''_1, \dots, a'_n, a''_n, \dots\} \subset K_1.$$

In figure 2.5 (and also in figure 2.6) we use the convention that points on the dotted lines denoted with the same letter (like  $b_n, c_n, \dots$ ) are identified to each other via the map  $K_1 \xrightarrow{g_1} \text{Wh}^3$ .

- (3) The set  $g_1 \bar{D}_n$  is very thin with respect to  $g_1 \bar{D}_{n+1}$  for each  $n$ ; in more precise terms  $g_1 \bar{D}_n$  is concentrated inside a thin tubular neighbourhood of  $g_1 \partial \bar{D}_{n+1}$  and the double points of  $g_1 | (\bar{D}_n \cup \bar{D}_{n+1})$  are like in figure 2.6.

These conditions still leave a lot of margin of maneuver but they completely determine the subset  $M_2(g_1) \subset K_1$  modulo a *not necessarily ambient isotopy*. (Our “margin of maneuver” allows for an arbitrary reparametrization of the source, and also for an arbitrary non-ambient isotopy of the image  $g_1 K_1 \subset \text{Wh}^3$  inside the target.) We will make this set  $M_2(g)$  completely explicit later on. It should be obvious that  $M_2(g_1)$  is acyclic and with  $M_3(g_1) = \emptyset$ . Moreover  $\text{Nbd}(g_1 K_1) \subset \text{Wh}^3$  is completely determined modulo an isotopy of embeddings which is not ambient; proprieties (a), (b), (c), (d) are satisfied. It will be convenient to change  $K_1 \xrightarrow{g_1} \text{Wh}^3$  into a variant  $K_2 \xrightarrow{g_2} \text{Wh}^3$  before our real  $K \xrightarrow{g} \text{Wh}^3$  can be defined. In going from  $K_1$  to  $K_2$  we keep the same principle  $K_2 = \bar{D}_0 \cup \bar{D}_1 \cup \bar{D}_2 \dots$  but the glueing pattern of  $\bar{D}_n$  to  $\bar{D}_{n+1}$  will be slightly different. In terms of figure 2.7,  $C_n \subset \bar{D}_n$  will be identified now with the curve

$$\left[ \frac{1}{2} (a'_n + a''_n), p, \sigma_n, \frac{1}{2} (a'_n + a''_n) \right]$$

contained in  $\bar{D}_{n+1}$  and which coincides with  $\partial \bar{D}_{n+1} = \partial D_{n+1}$  except for the small arc

$$\left[ \sigma_n, s_n, \frac{1}{2} (a'_n + a''_n) \right]$$

which is close to the arc

$$\left[ \sigma_n, \tau_n, \frac{1}{2} (a'_n + a''_n) \right] \subset \partial \bar{D}_{n+1}$$

but sits in the interior. Also the arc  $[\text{the lower } (1/2)(a'_n + a''_n), \tau_n] \subset \bar{D}_n$  is glued now to  $[(1/2)(a'_n + a''_n), \tau_n] \subset \partial \bar{D}_{n+1}$ . It is understood that the arc

$$\left[ \sigma_n, s_n, \text{the upper } \frac{1}{2} (a'_n + a''_n) \right] \subset C_n \subset \bar{D}_n$$

is very short so that the passage  $K_1 \Rightarrow K_2$  is far from the double points of the map  $g_1$ . The double points of  $g_2$  are rigorously the same as those of  $g_1$ . One finally gets  $(K, g)$  from  $(K_2, g_2)$  by keeping fixed the image  $g_2 K_2 = gK \subset \text{Wh}^3$  but by unglueing  $K_2$  along each tripod

$$\left[ s_n; \text{upper } \frac{1}{2} (a'_n + a''_n); a'_n, a''_n \right] \quad \text{for } n = 0, 1, 2, \dots$$

So, in passing from  $(K_2, g_2)$  to  $(K, g)$ , figure 2.7 changes into 2.8. The arc  $[\tau_n, \sigma_n, s_n] \subset D_n$  is glued now isomorphically (at the level of  $K$ ) to the fat curve  $[\tau_n, p, \sigma_n, s_n] \subset D_{n+1}$  for  $n = 0, 1, 2, \dots$  and this defines the space  $K$  which is arborescent. This  $K$  is again a quotient of  $\bigcup_0^\infty D_n$  but now each  $D_n$  injects into  $K$ . The pair of singularities  $a'_n, a''_n$  is replaced by the unique singularity  $s_n$  and we continue to have  $M_3(g) = \emptyset$  (the lower  $(1/2)(a'_n + a''_n) \in D_n$  continues to be glued to  $(1/2)(a'_n + a''_n) \in D_{n+1}$  at the source of  $g$ ). This finishes the description of  $K \xrightarrow{g} \text{Wh}^3$ .

#### Remarks

(A) The double point structures of  $g$  and  $g_1(g_2)$  differ only in the neighbourhood of  $s_n$  ( $n = 0, 1, 2, \dots$ ). In particular, in the neighbourhood of  $[\text{lower } (1/2)(a'_n + a''_n), \sigma_n] \subset C_n$ , the double points of  $g$  and of  $g_1$  are exactly the same.

(B) The construction of  $(K \xrightarrow{g} \text{Wh}^3)$  leaves a lot of margin of maneuver; actually exactly the same as for  $(K_1 \xrightarrow{g_1} \text{Wh}^3)$ . We will give now.

The algorithm which generates the transverse structure  $M_2(g) \subset K$

We will consider in  $D_0$  a line  $\Lambda$  parallel to the  $[\text{lower } (1/2)(a'_0 + a''_0), \sigma_0]$  (see the left side of figure 2.7, with  $n = 0$ ) and which is slightly shorter than our  $[(1/2)(a'_0 + a''_0), \sigma_0]$ . This line which is contained inside the smooth part of  $K$  is a transversal to  $M_2(g)$  which is tight with respect to  $M_2(g)$  and it is disjoint from  $M_2(g|D_0)$ . Moreover our tight transversal  $\Lambda$  (which we can also see in figure 2.4) is supposed to be long enough so that it meets (exactly once) every arc in

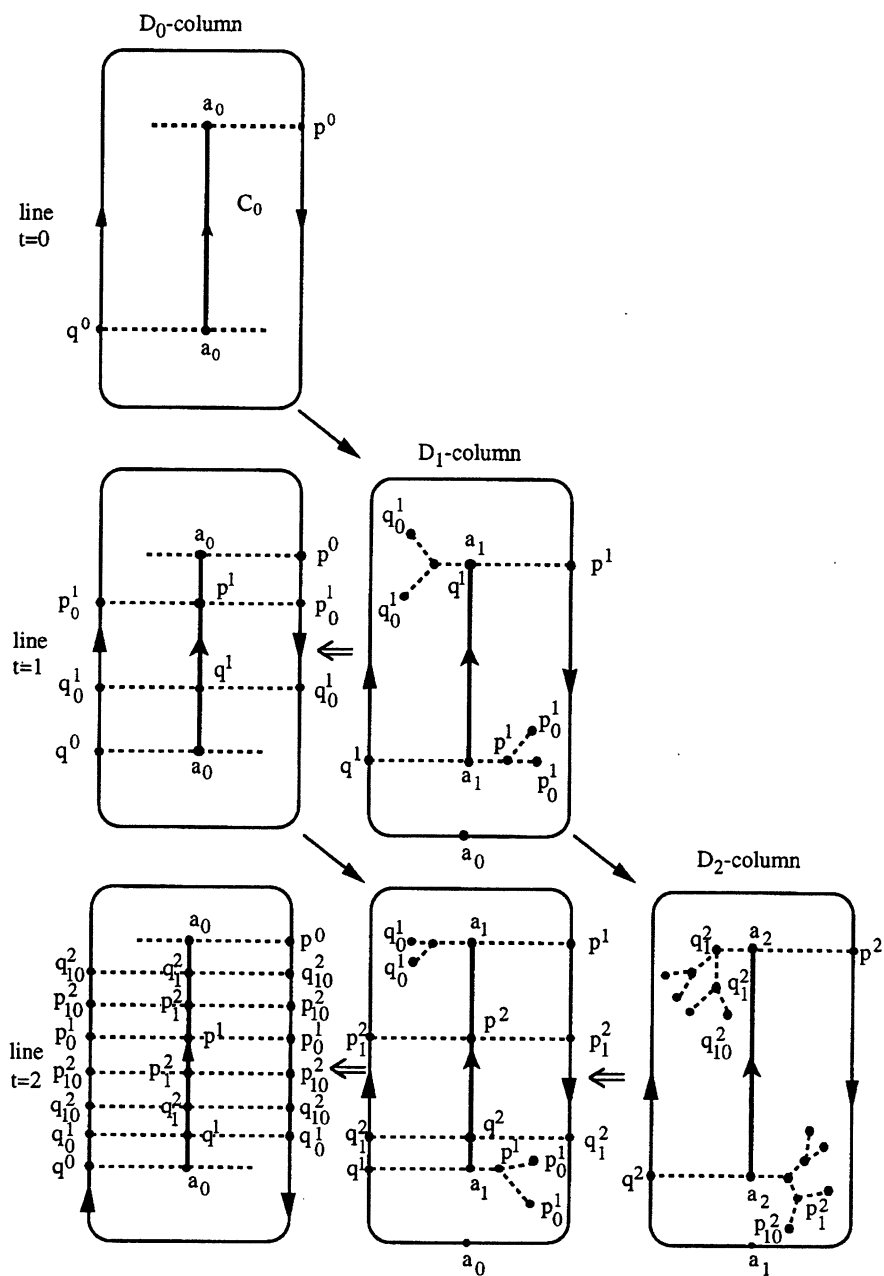
$$M_2(g) \cap D_0 - M_2(g|D_0). \quad (2.7)$$

Fig. 2.9 This figure (next page) and figure 2.9b) which continues it shows the recursive rules for building  $M_2(g_1) \subset K_1$ ; to simplify matters we made as if

$$a'_n = a''_n = (1/2)(a'_n + a''_n) \stackrel{\text{def}}{=} a_n, \quad n = 0, 1, 2, \dots;$$

the double points appear as dotted lines and we are really interested in what happens with  $\text{Int } C_0 \cap M_2(g_1) \approx \Lambda \cap M_2(g)$ .

# Representations of the Whitehead manifold $Wh^3$ and Julia sets



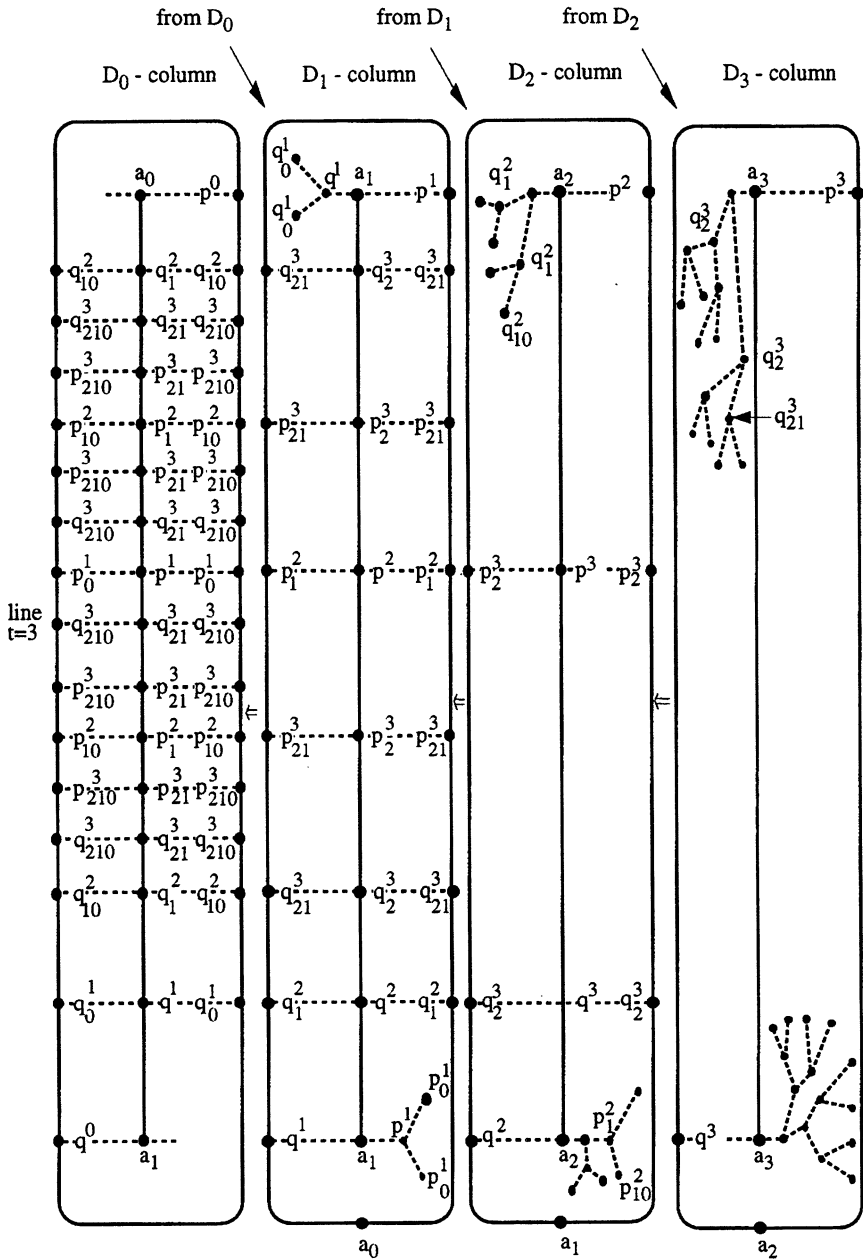


Fig. 2.9b) We see here  $t = 3$ .

We will investigate the set  $M_2(g) \cap \Lambda$ . Clearly (see remark (A) above) this is exactly the same thing as investigating  $M_2(g_1) \cap \Lambda = M_2(g) \cap \Lambda$ . So what we will do will be to pick up, with much more details the issues started by figure 2.6 with the help of which we generate figure 2.9. At each time  $t = n$  in this figure we have an isomorphism  $C_p \approx \partial D_{p+1}$  for all  $p+1 \leq n$ . Points on  $C_p$  and  $\partial D_{p+1}$  which are indicated by the same symbols, have the same image at the source (i.e. they are the same point in  $K$ ).

Here is the *recursive procedure* for constructing  $M_2(g_1) \subset K_1$ . At time  $t = 0$  we consider  $\overline{D}_0$  and  $M_2(g_1 | \overline{D}_0) \subset \overline{D}_0 \subset K_1$ . This is shown on the left upper corner (line  $t = 0$ , column  $D_0$ ) of figure 2.9. At time  $t = 1$  we consider both  $\overline{D}_0$  and  $\overline{D}_1$  and we look for  $M_2(g_1 | (\overline{D}_0 \cup \overline{D}_1))$ . The situation on (line  $t = 1$ , column  $D_1$ ) is forced by (line  $t = 0$ , column  $D_0$ ) which it is imitating (i.e.  $M_2(g_1 | \overline{D}_1)$  is isomorphic to  $M_2(g_1 | \overline{D}_0)$ ); this accounts for the diagonal arrow. But then the situation on (line  $t = 1$ , column  $D_1$ ) forces the situation of (line  $t = 1$ , column  $D_0$ ) which enriches what we already had at (line  $t = 0$ , column  $D_0$ ); this accounts for the horizontal double arrow. (Of course (line  $t = 1$ , column  $D_1$ ) contains some extra details with respect to  $(t = 0$ , column  $D_0$ ), like the lines  $(q^1, q_0^1)$ ,  $(p^1, p_0^1)$  which appear for  $(t = 1, D_1)$  but are absent for  $(t = 0, D_0)$ ; these details come from  $\{\text{the double points of } g_1 \text{ which involve both } D_1 \text{ and } D_0\} \cap D_1$ .) Figures 2.9 and 2.9b) (line  $t = 3$ ) suggests how this process continues recursively for all  $t$ 's. The situation  $(t = n, D_n)$  forces  $(t = n+1, D_{n+1})$ , which then in turn forces  $(t = n+1, D_n)$ ,  $(t = n+1, D_{n-1})$ , ...,  $(t = n+1, D_0)$ . Here again  $(t = n+1, D_{n+1})$  contains some extra details with respect to  $(t = n, D_n)$ . These come from  $\{\text{the double points of } g_1 \text{ involving both } D_{n+1} \text{ and } \sum_{i=0}^n D_i\} \cap D_{n+1}$ . But these kind of details are completely irrelevant for our main discussion, which focuses on  $M_2(g) \cap \Lambda$  and on the set of limit points of  $M_2(g) \cap \Lambda$ ; we will, generally speaking, ignore them hereafter.

For each  $t = n$  and  $p < n$  the situation on  $C_p$  and  $\partial D_{p+1}$  are canonically isomorphic, as one can see it in figures 2.9 and 2.9b). Also the various points of  $M_2(g_1)$  successively appearing on  $C_p$  and/or on  $\partial D_{p+1}$  at times  $t = 0, t = 1, \dots$  carry some lower indexes and we will define the *flavour* of the respective double point as being

$$\text{flavour} \stackrel{\text{def}}{=} (\text{the number of lower indexes}) + 1$$

(so that, for instance,  $\text{flavour}(p^0) = 1$ ,  $\text{flavour}(q_{21}^3) = 3, \dots$ ). If  $x \in M_2(g) \cap C_p$  and if the flavour of  $x$  is  $q$ , this means exactly that  $x$ , considered

as a point in  $C_p$ , is a double point involving  $D_p$  and  $D_{p+q}$ . Our double points also carry a letter (like  $p$  and  $q$ ) and an upper index, which together, will be called the *colour* of the corresponding double point. Notice that colour and flavour together determine completely the set of indices (upper and lower, with their order, and of course the letter  $p/q$  too) which accompany the double point. (But colour does not determine flavour, nor the other way around.) In other words any

$$x \in M_2(g) \cap \left( \sum_0^\infty C_i \right) \approx M_2(g) \cap \left( \sum_0^\infty \partial D_{i+1} \right)$$

is *completely identified* by its combinaison of flavor and colour. The geometric meaning of the lower indexes should also be clear, for instance

$$q_{210}^3 = \{ \text{the result of the } q^3 \in D_2 \cap \partial D_3 \text{ on } D_1 \text{ and then finally on } D_0 \}.$$

The upper index corresponds to the first time  $t$  when the double point in question (without its lower indices) appears.

The canonical isomorphism  $C_n \approx \partial D_{n+1}$  (see for instance  $C_0$  and  $\partial D_1$  in figure 2.9b)) sends double points to double points with the same colour and flavour (like a  $p_{21}^3 \in C_0$  to a  $p_{21}^3 \in \partial D_1$  (fig. 2.9b)).

The important feature of this recursive procedure is the following. Consider two consecutive columns (we will focus from now on  $D_0$  and  $D_1$ ) after the inductive procedure  $(t = 0) \Rightarrow (t = 1) \Rightarrow \dots$  has been completed, i.e. at time  $t = \infty$ . Infinitely many horizontal dotted lines passing through points endowed with colour and flavour will be contained now both in  $C_0 \subset D_0$  and  $C_1 \subset D_1$  (the small irrelevant details like  $(q_0^1, q^1, q_0^1)$ ,  $(p_0^1, p^1, p_0^1)$  contained in  $\text{Int } D_1$  will be, of course, forgotten from now on). We make the following claims:

*Alg-0.* — Our whole *infinite* construction is invariant under the *unilateral shift*

$$(t = n, D_i) \xrightarrow{\text{shift}} (t = n + 1, D_{i+1}) \quad (2.8)$$

as long as colours are ignored. Both *the drawing and the flavours are shift invariant*. Of course, the shift can be accompanied by a precise, well-defined transformation of the colours which the reader can explicit easily. We also have a colour and flavour conserving monomorphic inclusion map

$$(t = n, D_i) \xrightarrow{\text{inclusion}} (t = n + 1, D_i). \quad (2.9)$$

*Alg-1.* — The situation of  $(\text{Int } C_0) \cap (\text{horizontal dotted lines})$  at the time  $t = \infty$  is exactly the  $M_2(g) \cap \Delta$  we are looking for.

*Alg-2.* — Before we state the next claim let us notice the following feature which is build into the detailed construction of the figure 2.9 and figure 2.9b. When for a given  $t = n$  (like for instance  $t = 3$  and for two consecutive columns  $i, i+1$  (like for instance  $D_0$  and  $D_1$ ) two horizontal lines are drawn exactly at the same level (like for instance  $[p_{10}^2, p_1^2, p_{10}^2] \subset D_0$  and  $[p_{21}^3, p_2^3, p_{21}^3] \subset D_1$ ) they correspond to the same flavours  $f$  for the middle point and  $f + 1$  for the two extreme points, i.e. these two lines go from a point of flavour  $f + 1$  to another point of flavour  $f + 1$  passing through a middle point of flavour  $f$ . On the column  $i$  the lines in question are in the image of the inclusion (2.9) and all this is consistent with the following “commutative” diagram (which is flavour preserving) but where the left horizontal translation is not colour preserving (see Diagram 1). Actually the  $p/q$  part of colour is preserved, but this is irrelevant. Our figures, as far as the horizontal alignment is concerned, have been drawn exactly so as to make the diagram above commutative. This has nothing to do with the  $C_i \approx \partial D_{i+1}$  isomorphism. On the other hand the two horizontal lines in question correspond, respectively, to the double points involving  $\{ \text{both } D_i \text{ and } D_{i+f} \}$  and/or  $\{ \text{both } D_{i+1} \text{ and } D_{i+f+1} \}$ .

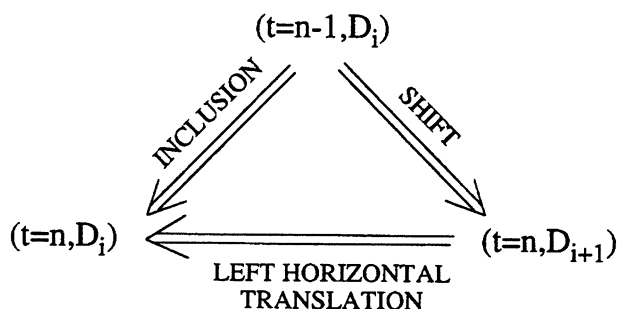


Diagram 1

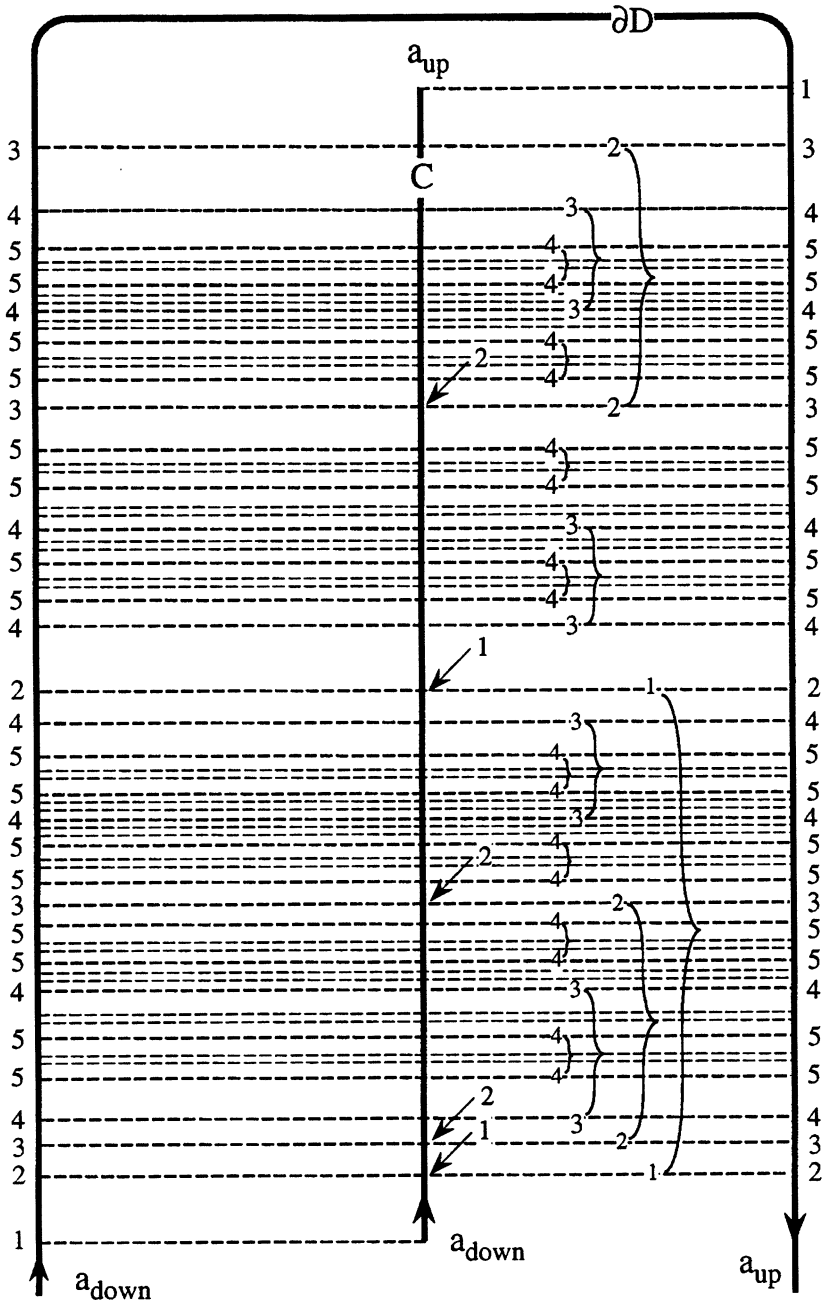


Fig. 2.10 The dynamical figure ( $t = \infty, D$ );

- - - (where no numbers are indicated) =  $[6, 5, 6]$ ; in order not to make the drawing too messy, most of the symbols 5 and all the 6's are lacking on  $C$  and  $\partial D$ ; for similar reasons, the numbers  $1, 2, \dots$ , for the points of  $C$  have been shifted to the right, on the same dotted horizontal line; the vertical parenthesis suggests the self similar pattern which this figure starts to build.

*Remark.* — As already said, the device of drawing at the same horizontal level lines which correspond to each other under (2.9) does not keep track of the flavour (and colour) conserving isomorphism

$$(t = n, C_i) \approx (t = n, \partial D_{i+1}).$$

But it makes possible what comes next.

As a consequence of all this (and this is finally our claim *Alg-2*, if we forget about the colours and keep only the flavours  $(1, 2, \dots)$  then the figures  $(t = \infty, D_0)$  and  $(t = \infty, D_1)$  are canonically isomorphic. Let us call this infinite object  $(t = \infty, D_0) \approx (t = \infty, D_1)$  “the figure  $(t = \infty, D)$ ”. The isomorphism  $(t = \infty, D_0) \approx (t = \infty, D_1)$  comes, of course, from the shift (2.8). At the level of  $(t = \infty, D)$  the isomorphism  $C_0 \approx \partial D_1$ , becomes  $C \approx \partial D$  and this isomorphism conserves flavour. We will draw the transversal  $\Lambda$  inside the figure  $(t = \infty, D)$  and consider

$$\begin{aligned} F_n &\stackrel{\text{def}}{=} \{\text{the double points of } \Lambda \cap M_2(g) \text{ which have flavour } n\} \\ &= \{\text{the double points of } \Lambda \cap M_2(g) \text{ which come from } gD_0 \cap gD_n\}. \end{aligned} \tag{2.10}$$

Figure 2.10 suggest how one builds, inductively our figure  $(t = \infty, D)$ . The important point is that now we are in the presence of a feedback  $C \Leftrightarrow \partial D$  which generates the following.

**Dynamical rule for construction  $(t = \infty, D)$**  (and hence the set  $\Lambda \cap M_2(g)$ )

We start by drawing the bare lines  $C$  and  $\partial D$  like in figure 2.10. “Bare” means without any details other than their orientations, in particular without any double point of  $g$  and/or  $g_1$ . Then, as an input we draw the (asymmetrical) horizontal lines  $(a_{\text{up}}, 1)$ ,  $(a_{\text{down}}, 1)$ . Now  $\partial D$  is no longer bare since it contains two points with flavour 1. The rule that there is a *flavour conserving isomorphism*  $C \approx \partial D$  forces then, by dynamical self

consistency, two points with flavour 1 on the line  $C$ . From these two points we draw horizontals *symmetrically* in both direction until we hit  $\partial D$  at four points with flavour 2. By dynamical self-consistency these points have to exist also on  $C$  so that

$$(C, \text{ with 1's and 2's}) \approx (\partial D, \text{ with 1's and 2's}).$$

From the points with flavours 2 on  $C$  we draw, symmetrically, horizontals which hit  $\partial D$  in eight points with flavour 3. By dynamical self-consistency these have to be drawn on  $C$  too, and so on.

By now this dynamical process of feedback between  $C$  and  $\partial D$  should be clear and hence, within the margin of freedom which this construction leaves, we know now our set  $\Lambda \cap M_2(g) \approx C_0 \cap M_2(g)$  which comes equipped with an obvious disjointed decomposition

$$\Lambda \cap M_2(g) = F_1 + F_2 + \cdots + F_n + \cdots \quad (2.10b)$$

where, like in (2.10), we have  $F_n = \{ \text{double points of flavour } n, \text{ i.e. double points which involve } D_0 \text{ and } D_n \}$

*Remark.* — Let us denote by  $\partial D \xrightarrow[\approx]{d} C$  the canonical homeomorphism  $\partial D \approx C$  and by  $\partial D \xrightarrow{h} C$  the not everywhere defined horizontal projection from  $\partial D$  to  $C$ , so that  $C \xrightarrow{h^{-1}} \partial D$  is the double-valued map which attaches to  $p \in C$  two points of  $\partial D$  sitting on the same horizontal line. With these notations our feedback loop via which the infinite figure 2.10 has been constructed is represented in Diagram 2.

$$\begin{array}{ccccc} \partial D & \xleftarrow{h^{-1}} & C & \xleftarrow{d} & \partial D. \\ & & \text{id} & & \end{array}$$

Diagram 2

If we denote generically by  $n \in \partial D$  the points with flavour  $n$ , then our dynamical rule for building up the  $F_1 + F_2 + \cdots$  (as a subset of  $\partial D$ ) is schematized by the formula

$$1 + \underbrace{h^{-1}d(1)}_2 + \underbrace{h^{-1}dh^{-1}d(1)}_3 + \cdots = 1 + W(1) + W^2(1) + \cdots \quad (2.10c)$$

Here  $\partial D \xrightarrow{W \stackrel{\text{def}}{=} h^{-1}d} \partial D$  and, by definition, formula (2.10c) is the "Whitehead algorithm".

In order to analyze  $\Lambda \cap M_2(g)$  we go back to the quadratic map from figure 2.1. On the real axis we can see four points  $0 < x_0 < y_0 < y_1 < x_1 < 1$  such that  $f_\lambda(x_0) = f_\lambda(x_1) = 0$  and  $f_\lambda(y_0) = f_\lambda(y_1) = 1$ . The following things happen.

### LEMMA 2.2

- (1) *Our dynamical rule for construction  $\Lambda \cap M_2(g)$  is exactly the same as the dynamical rule for constructing the set (see (2.1)):*

$$\mathcal{E} = E \cup f_\lambda^{-1}E \cup f_\lambda^{-2}E \cup \dots \cup f_\lambda^{-n}E \cup \dots \subset [0, 1], \quad (2.11)$$

where  $E = \{x_0, y_1\}$  and where  $f_\lambda$  is the quadratic map from figure 2.1.

- (2) *Whitin the margin or freedom allowed by our construction of the representation*

$$K \xrightarrow[g]{} \text{Wh}^3$$

we can choose  $g$  so that:

$$\begin{aligned} \{ \text{the set of limit point } \Lambda \cap M_2(g) \} = \\ = \{ \text{the Julia set of the quadratic map } f_\lambda \}. \end{aligned} \quad (2.12)$$

It is well-known that the Julia set in question is a Cantor set. But anyway we also have the following.

- (3) *For any choice of  $g$  (within the margin of freedom for the construction of  $K \xrightarrow[g]{} \text{Wh}^3$ ) the  $\{ \text{set of limit points of } \Lambda \cap M_2(g) \}$  is always uncountable (in fact it always contains a Cantor set).*

*Proof.* — Before we start constructing the set (2.11) we will redraw the graph of  $f_\lambda$  (or rather the interesting part of this graph) in figure 2.11, which the reader should compare with the bare figure 2.10 (i.e. that figure before the dynamical process starts). On the figure 2.11, we perceive two isomorphisms  $\partial D \approx \{\text{graph } f_\lambda\} \xrightarrow{j} A$  given by  $j(x, f_\lambda(x)) = x$  and  $A \xrightarrow{i} C$  going from  $p \in A$  vertically to the diagonal  $x = y$  and then to  $C$ , via the horizontal projection (see the figure 2.11). Let us denote generically by  $n$  the points in  $A$  which belong to  $f_\lambda^{-(n-1)}E$  (fig. 2.12).

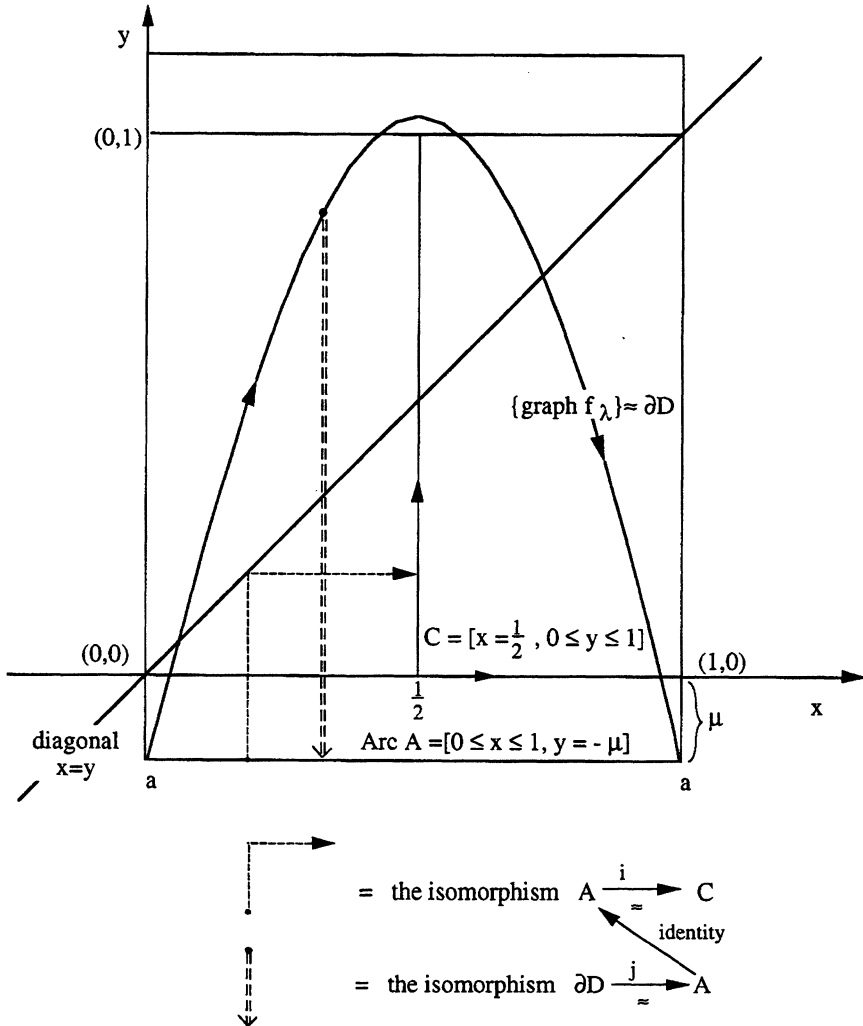
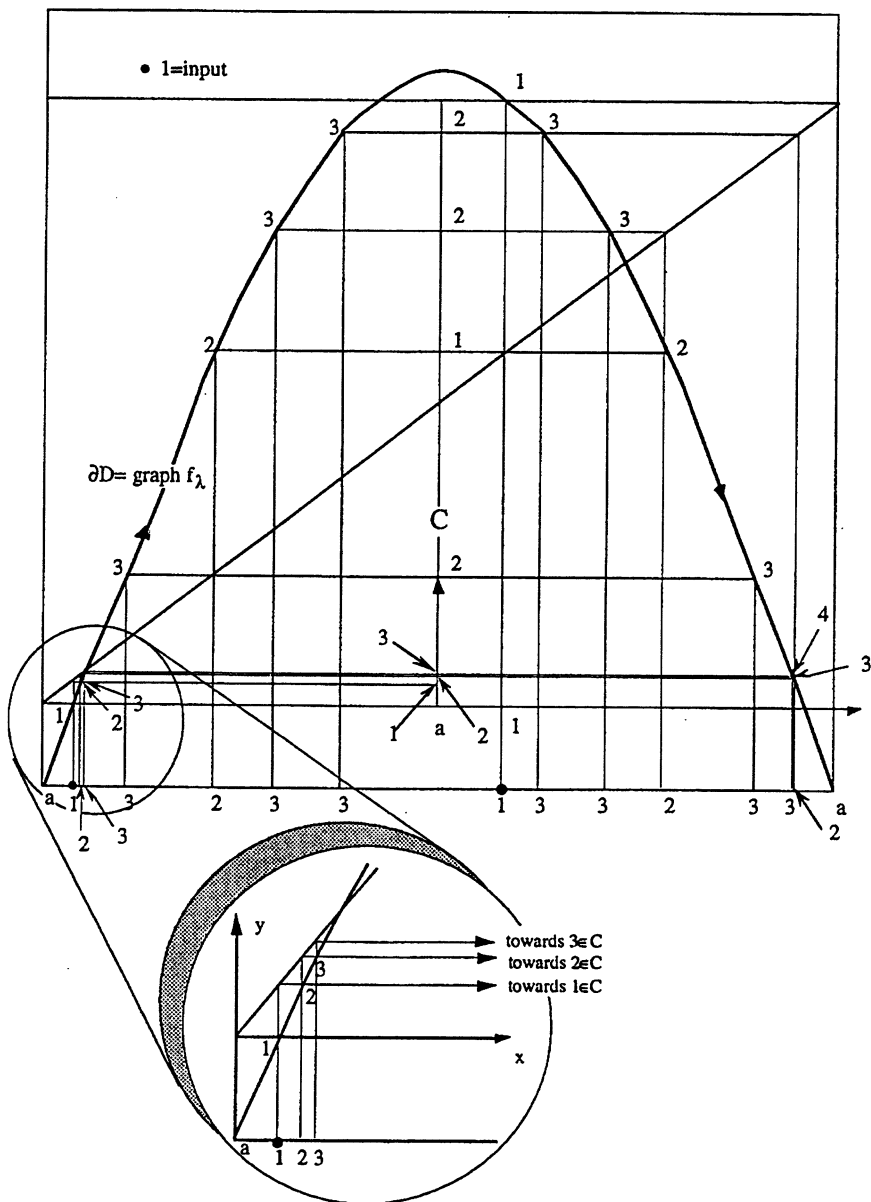


Fig. 2.11 The isomorphism  $\partial D \xrightarrow{\approx} C$ .

**Fig 2.12** In this figure (next page), on the segment  $A$ , points marked with 1 are in  $E$ , those with 2 in  $f_\lambda^{-1}E$ , those with 3 in  $f_\lambda^{-2}E$ , and so one; the dynamical feedback between  $C$  and  $\partial D$ , in this figure, is exactly the same as in figure 2.10; for typographical reasons this figure differs from 2.1 by a homothety which brings the diagonal closer to the  $x$  axis.

## Representations of the Whitehead manifold $Wh^3$ and Julia sets



In that figure we also have denoted with the same letter  $n$  the corresponding points in  $\partial D$  via  $j^{-1}$ , or in  $C$  via  $i$ . The dynamics of the transformation

$$n = f_{\lambda}^{-(n-1)} E \implies f_{\lambda}^{-n} E = (n+1)$$

can be expressed by the following graphical rules which we can see displayed in figure 2.12.

Go from  $n \in A$  to  $i(n) \in C$ , then draw the horizontal line through  $i(n)$ , take its two intersection points with  $\partial D \approx \text{graph } f_{\lambda}$  and finally take the images of these intersection points via  $j$  back to  $A$ . This defines the  $\{n+1\} \subset A$ . So we can define

$$\mathcal{J} \stackrel{\text{def}}{=} jh^{-1}i \\ A \longrightarrow A$$

and passing from  $\partial D$  to  $A$  via  $j$ , we have the obvious commutative diagram

$$\begin{array}{ccccc} \partial D & \xrightarrow{d=ioj} & C & \xrightarrow{h^{-1}} & \partial D \\ & & \text{Whitehead} & & \\ j \downarrow & & & & \downarrow j \\ A & \xrightarrow{i} & C & \xrightarrow{h^{-1}} \partial D \xrightarrow{j} & A \\ & & \text{Julia} & & \end{array}$$

which means  $\mathcal{J}(j(x)) = jW(x)$  and hence also  $\mathcal{J}^p(j(x)) = jW^p(x)$  for all  $p$ 's and  $x \in \partial D$ . This shows that if we start with  $j(1) = E$  (where "1" is the same now as in formula (2.10c)) then the "Whitehead algorithm" (2.10c) and the "Julia algorithm"

$$j(1) + \mathcal{J}(j(1)) + \mathcal{J}^2(j(1)) + \dots$$

are conjugate via  $\partial D \xrightarrow{j} A$ . In other words via the isomorphism  $A \xrightarrow{i} C$ , the rule we just have for building (2.11) is exactly the dynamical rule for constructing  $(t = \infty, D)$  and hence  $\Lambda \cap M_2(g)$ ; figures 2.10 and 2.12 are basically the same, and with this our (1) is proved.

Notice now that in passing from  $t = n-1$  to  $t = n$  via the

$$(t = n-1, D_0) \xrightarrow{\text{inclusion}} (t = n, D_0) \quad (2.13)$$

(see also (2.9)) one has to add at level  $(t = n)$  all the horizontal lines of the form

$$[\ell_{i_1 i_2 \dots i_n}^n, \ell_{i_1 i_2 \dots i_{n-1}}^n, \ell_{i_1 i_2 \dots i_n}^n], \quad (2.13n)$$

with  $\ell = p$  or  $q$ , which correspond exactly to

$$\{\text{the double points of } g_1 \text{ involving both } D_0 \text{ and } D_n\} \cap D_0, \quad (2.14)$$

But once  $g_1 \mid (D_1 + D_2 + \cdots + D_{n-1})$  has been fixed (and hence also all the lines (2.13) for  $i \leq n-1$  we can fix arbitrarily the position of (2.3i) in a compatible way with the already existing situation. This is achieved by an obvious isotopy of  $g_1 \mid D_n$ , keeping  $g_1(\partial D_n)$  fixed. In other words within our allowed margin of maneuver we can realize a homeomorphism  $\Lambda \rightarrow [0, 1]$  sending  $\Lambda \cap M_2(g)$  onto  $\mathcal{E}$  and preserving the graduation by the  $n$ 's. This proves (2). Notice that there is no dynamical feedback at the level of this discussion. For the proof of (3) we start by noticing that the points of  $\Lambda \cap M_2(g)$ , as presented by the dynamical process which accompanies figure 2.10, come naturally in *pairs of points with the same flavour*. (See the vertical curved brackets parallel to  $C$ , in figure 2.10.) So we find in  $C$  one segment  $I_1$  with (endpoints of) flavour 1, two disjointed segments  $I_2$  with (endpoints of) flavour 2 and generally speaking  $2^{n-1}$  two-by-two disjointed segments  $I_n$  with (endpoints of) flavour  $n$ . The point (3) from lemma 2.2 is now a consequence of the following.

**LEMMA 2.3**

(1) *The nested system of segments  $I_n \subset C$  satisfies the following conditions.*

(a) *The arc  $I_1$  contains exactly one of the arcs  $I_2$ .*

(b) *Each arc  $I_2$  contains exactly one of the arcs  $I_3$  and as far as the two remaining  $I_3$  arcs are concerned, one is contained in  $I_1 - I_2$ , the other one is disjointed from  $[I_1, I_2]$ .*

$\vdots$

(n) *From the  $2^n = \sum_{i=0}^{n-1} 2^i + 1$  arcs  $I_{n+1}$ , the first  $2^{n-1}$  are contained each one in an  $I_n$  arc, the next  $2^{n-2}$  each one in an  $I_{n-1}$  arc but not in  $I_n$ , ..., the next  $2^i$  each one in an  $I_{i+1}$  arc but not in  $I_{i+2} + \cdots + I_n$ , ..., and finally, the last one is outside  $I_1 + \cdots + I_n$ .*

$\vdots$

(2) *As an immediate consequence of (1), the set of limit points of*

$$\Lambda \cap M_2(g) = \{\text{the set of endpoints of the various } I_n\}$$

*is uncountable.*

Before proving our lemma let us notice that our dynamical rule building the figure 2.10 produces for each  $n$  a collection of  $2^{n-1}$  details like the one in figure 2.13a). In passing from  $n$  to  $n+1$  each of the arcs  $I_{n+1}$  gives itself rise to a detail like the one in figure 2.13b). This kind of *self-similarity* which is expressed by (1) is, of course, a well-known feature of *chaotic behaviour*. This remark enhances on (1) in lemma 2.3. The reader will not find it hard to work out in detail the combinatorics of the nested system  $\{I_n\}$ , including the orientations of the details like the one in figure 2.13a) with respect to  $C$ . But the only thing we need in order to get (3) from lemma 2.2 is the numerical estimate from (1) in our present lemma.

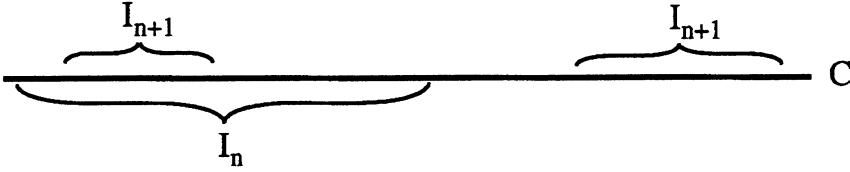


Fig. 2.13a)

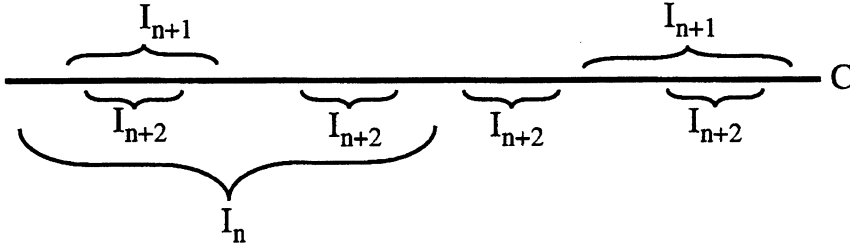


Fig. 2.13b)

*Proof of the lemma 2.3.* — We go back to the context of figure 2.1 and we consider now  $E' = \{x_0, y_0\}$  instead of  $E = \{x_0, y_1\}$ . Our first claim is that the dynamical rule for constructing our  $\Lambda \cap M_2(g)$  is exactly the same as the one for producing

$$E' \cup f_{\lambda}^{-1} E' \cup f_{\lambda}^{-2} E' \cup \dots \cup f_{\lambda}^{-n} E' \cup \dots \quad (2.15)$$

This can be shown exactly like in the proof of (1) from lemma 2.2. We will deduce the combinatorics of (2.15) (and hence the combinatorics of (2.11), i.e. of our  $\Lambda \cap M_2(g)$ ) from the much easier combinatorics of

$$E'' \cup f_{\lambda}^{-1} E'' \cup f_{\lambda}^{-2} E'' \cup \dots \cup f_{\lambda}^{-n} E'' \cup \dots \quad (2.16)$$

where  $E'' = \{x_0, y_0, y_1, x_1\}$ . Just like  $\Lambda \cap M_2(g)$ , i.e. like (2.11) or (2.15), the set (2.16) can be described by a system of nested intervals  $\dots, J_n, J_{n+1}, \dots$ . If we denote by  $A_1$  and  $A_2$  the arcs  $A_1 = f_\lambda[x_0, y_0]$ ,  $A_2 = f_\lambda[y_1, x_1]$  then a typical  $J_n$  has the form

$$J_n = f^{-n}(A_{i_n}) \cap f^{-n+1}(A_{i_{n-1}}) \cap f^{-1}(A_{i_1}) \cap A_{i_0}, \quad (2.17)$$

where  $i_1, i_2, \dots, i_n \in \{1, 2\}$ . This means that now there are  $2^n$  arcs  $J_n$  and each  $J_n$  contains exactly two  $J_{n+1}$ 's. One obtains the system  $\{I'_n\}$  attached to (2.15) from the system  $\{J_n\}$  by eliminating, individually, each  $J_n$  which is such that  $i_n = 2$  (see (2.17)). This changes something like figure 2.13c) into 2.13a). This proves lemma 2.3 and hence also finishes the proof of lemma 2.2.

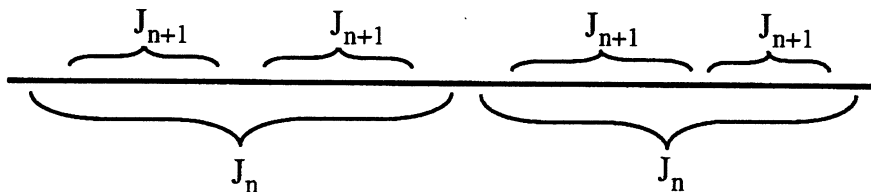


Fig. 2.13c)

### Comments

(I) We make the following conjecture.

CONJECTURE. — Consider any arborescent representation of the form

$$X \xrightarrow{\varphi} \text{Wh}^3.$$

for the Whitehead manifold. One can always find smooth tight transversals  $\Lambda \subset X$  to  $M_2(\varphi)$  such that the set of limit points of  $\Lambda \cap M_2(\varphi)$  is uncountable.

We conjecture actually that the situation we have just analyzed is the best possible as far as  $\Lambda \cap M_2(\varphi)$  is concerned.

Of course, if we move from  $\text{Wh}^3$  to a highly non-aspherical

$$(\text{Wh})_h^3 = \text{Wh}^3 - \bigcup_{n=1}^{\infty} C_n$$

(see formula (1.3)) then the representation theorem provides us with almost arborescent representations

$$Y \xrightarrow{\psi} (\text{Wh})_h^3,$$

where the set of limit points of  $\Lambda \cap M_2(\psi)$  is finite. In the other words, via redundancy we can get rid of the chaotic behaviour.

(II) It has been speculated by several people that there is a connection between exotic low-dimensional topology and chaotic dynamical systems (see for instance the first lines in chapter 6 of [FL]). Our previous discussion is a hint of possible connections between wild open 3-manifolds and real 1-dimensional dynamics. This might be part of a more general picture: John Hubbard conjectures that complex 2-dimensional (or already complex 1-dimensional?) dynamics can be connected to wild behaviour in 4-dimensional differential topology, in particular to the fake  $R^4$ 's [FL]. Here is a more precise conjecture of John Hubbard.

Consider the Hénon map  $\mathbb{C}^2 \xrightarrow{H} \mathbb{C}^2$ :

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{H} \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

with precisely tuned values of the parameters  $a, c$  (like for instance  $c = 0$ ,  $a = 0, 3$ ) so that  $H$  has an attractive fixed point with a basin  $U$  for which there is a biholomorphic surjection  $\mathbb{C}^2 \xrightarrow{L} U \subset \mathbb{C}^2$  conjugating  $H$  to the linear part at the fixed point).

CONJECTURE . — *The manifold  $\varinjlim (\mathbb{C}^2, L)$ , produced by glueing the  $n$ 'th  $\mathbb{C}^2$  to the  $(n + 1)$ 'th  $\mathbb{C}^2$  via  $\mathbb{C}^2 \xrightarrow{L} U \subset \mathbb{C}^2$  is a nonstandard  $\mathbb{C}^2$  (from the analytic standpoint).*

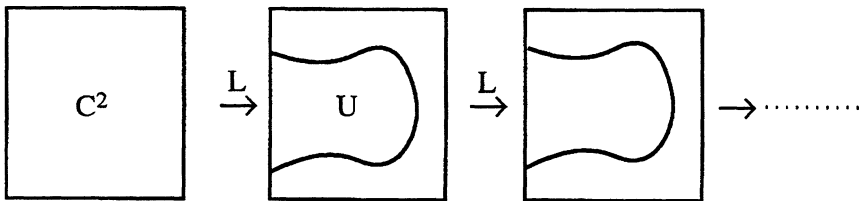


Fig. 2.14

(III) The kind of dynamics which appears in this paper is the relatively simple dynamics  $z \rightarrow z^2 + c$  with  $c$  outside the Mandelbrot set (which is also akin to the Smale horseshoe, see figure 2.15).

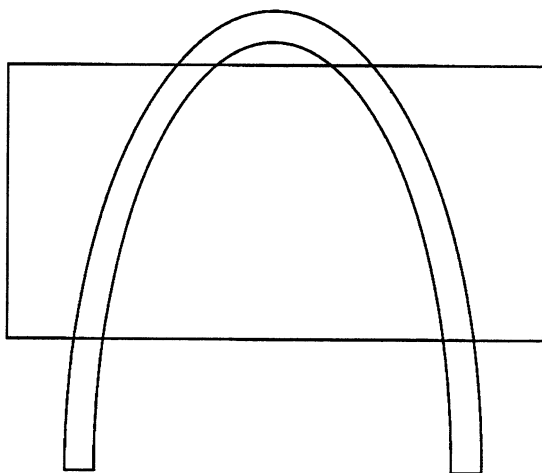


Fig. 2.15

Is there also a topological role for the more subtle dynamics  $z \rightarrow z^2 + c$  with  $c$  inside the Mandelbrot set and/or for something like the Hénon map?

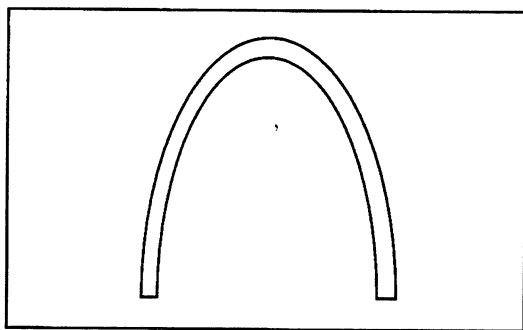


Fig. 2.16 Schematical view of the Hénon map.

(IV) Here is an explicit conjecture of John Hubbard. We consider the following map  $\mathbb{C}^2 \rightarrow \mathbb{C}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x^2 - cy^2 + x^3 \\ y^2 \end{bmatrix}$$

which has 0 as an attractive fixed point.

**CONJECTURE (John Hubbard)** .— *For any  $c$  on the boundary of the Mandelbrot set the basin of 0 is a nonstandard  $R^4$ . Moreover, for distinct  $c$ 's these manifolds are distinct.*

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