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RÉSUMÉ. — Dans cet article, nous considérons l'existence d'inégalités variationnelles définies sur des domaines extérieurs. Nous obtenons deux solutions positives s'annulant à l'infini.

ABSTRACT. — In this paper, we consider the existence of variational inequalities defined exterior domains. Two decaying positive solutions are abtained.

1. Introduction

The aim of this paper is to establish the existence and asymptotic behaviour of positive solutions of the obstacle problem

$$u \in K, \quad a(u, v-u) := \int_\Omega \big(\nabla u \cdot \nabla(v-u) + u(v-u)\big)\, \mathrm{d}x \\ \geq \lambda \int_\Omega f(u)(v-u)\, \mathrm{d}x\,, \quad \forall\, v \in K\,, \tag{1.1$_\lambda$}$$

defined on an exterior domain $\Omega = \mathbb{R}^N \setminus \omega, N \geq 3$, where ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. The set K is defined by

$$K := \{v \in H_0^1 \mid v \geq \psi\} \tag{1.2}$$

which is convex; ψ is the obstacle function.

The extensive scientific applicability of the obstacle problem is well known (see [6], [11] and references therein), for example, in mechanics,

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engineering, mathematical programming, control and optimization. Mathematical interest in the problem $(1.1)_\lambda$ arises in part because first the nonlinearity f we consider here is superlinear, so the corresponding variational functional is neither bounded above nor below, we are unable to establish the existence of positive solutions in a usual way by finding the minimum of the functional in the convex set K. Instead, we shall try to obtain critical points of the corresponding functional of $(1.1)_\lambda$ in K; second because the problem is considered on unbounded domains, the injection $H^1(\Omega)$ in $L^2(\Omega)$ is not compact in the case, therefore the variational functional fails to satisfy the Palais-Smale (P.S.) condition.

The existence problem related to similar obstacle problems of $(1.1)_\lambda$ has been studied in [9], [10], [14] in bounded domains. A counterpart in unbounded domains can be found in [1]. Among other things, it is required in [1] that the positive part of the obstacle function has a compact support, hence the key steps are essentially infered by the techniques for semilinear elliptic problems. In present paper, more delicate results will be established under assumptions for the obstacle function ψ:

$$\begin{aligned} &\psi \in H^{2,q}(\Omega)\,, \quad q > N \\ &\psi\restriction_{\partial\Omega} \le 0\,, \quad \psi^+ \not\equiv 0\,, \quad \psi^+ \in H^1(\Omega) \end{aligned} \tag{1.3}$$

$$\left(-\Delta\psi + \psi\right)^+ \le C\exp\left(-\tau|x|\right) \quad \text{for } |x| \text{ large}\,, \tag{1.4}$$

where $\tau > 1$, $C > 0$ are constants. We shall show that there exists a positive number $\lambda^* < +\infty$ such that the problem $(1.1)_\lambda$ has no positive solutions for $\lambda > \lambda^*$, while it has at least a minimal positive decaying solution for all $\lambda \in (0, \lambda^*)$. Furthermore, we prove that there exists λ^{**}, $0 < \lambda^{**} \le \lambda^*$, such that $(1.1)_\lambda$ has at least two positive decaying solutions for all $\lambda \in (0, \lambda^{**})$. It will be interesting to know if $\lambda^{**} = \lambda^*$ and if λ^* is a turning point. Precisely, suppose $f(t)$ is a function satisfying:

(f1) $f \in C^2$, $f(t) > 0$ if $t > 0$; $f(t) = 0$ if $t \le 0$ and $f(0) = 0$, $f'(0) = 0$;

(f2) there exists a positive constant $C > 0$ such that

$$0 < f(t) \le C(t + t^p)\,, \quad \text{for } t > 0\,,\ 1 < p < \frac{N+2}{N-2}\,,\ N \ge 3\,;$$

(f3) $f(t)$ is strictly convex and increasing for $t > 0$;

(f4) there is a number $\theta \in (0,1)$ such that

$$\theta t f'(t) \ge f(t) > 0\,, \quad t > 0\,.$$

Our main result is the following theorem.

THEOREM 1.— *Suppose (1.3)-(1.4) and (f1)-(f4) hold, there exists* $\lambda^*, 0 < \lambda^* < +\infty$ *such that* $(1.1)_\lambda$ *has at least a positive solution in* $H^{2,q}(\Omega) \cap C^{1,\alpha}(\Omega)$ *for* $\lambda \in (0, \lambda^*)$, *but it admits no positive solutions for* $\lambda > \lambda^*$; *moreover, there exists* λ^{**}, $0 < \lambda^{**} < \lambda^*$, *such that* $(1.1)_\lambda$ *possesses at least two distinct positive solutions* $0 < u(\lambda) \leq U(\lambda)$, $u(\lambda), U(\lambda) \in H^{2,q}(\Omega) \cap C^{1,\alpha}(\Omega)$ *for each* $\lambda \in (0, \lambda^{**})$, *where* $u(\lambda)$ *is a minimal positive solution. Furthermore, if* u *is a positive solution of* $(1.1)_\lambda$ *in* $H_0^1(\Omega)$, *then* $u \in H^{2,q}(\Omega) \cap C^{1,\alpha}(\Omega)$, *both* $u(x)$ *and* $|\nabla u(x)|$ *have uniform limits zero as* $|x| \to \infty$, *and* u *satisfies*

$$C_1 \exp\bigl(-(1+\delta)\,|x|\bigr) \leq u(x) \leq C_2 \exp\bigl(-(1-\delta)\,|x|\bigr)\,, \quad |x| \geq R$$

for $R > 0$ *large enough and any* $\delta > 0$, *where* C_1, C_2 *are positive constants.*

A minimal positive solution of the problem $(1.1)_\lambda$ is defined to be the positive solution $u(\lambda)$ of the problem $(1.1)_\lambda$ such that $u \geq u(\lambda)$ for any positive solution u of $(1.1)_\lambda$.

Theorem 1 is deduced in section 1 and section 2 by several propositions. We first show in section 1 that there is a local minimum of the variational functional in K, then a barrier device enable us to obtain the minimal positive solution; the existence of a second positive solution is investigated in section 2 by a variant of the mountain pass theorem. We shall infer that the variational functional satisfies $(\mathrm{P.S.})_c$ condition for c in some intervals via the concentration-compactness principle in [8], therefore we may find nontrivial critical points of the functional.

2. Minimal positive solutions

In this section we show that there exists $\lambda^* > 0$ such that the problem $(1.1)_\lambda$ has a minimal positive solution for $\lambda \in (0, \lambda^*)$. Furthermore, λ^* is verified to be finite; asymptotic decay laws are also established for positive solutions of $(1.1)_\lambda$.

Let $H^1(\Omega)$, $H_0^1(\Omega)$ be the completions of the sets $C^\infty(\Omega)$, $C_0^\infty(\Omega)$ respectively in the norm

$$\|u\|_{H^1} = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The energy functional of $(1.1)_\lambda$ is defined by

$$I(u) = \frac{1}{2}\, a(u,u) - \lambda \int_\Omega F(u^+)\, \mathrm{d}x\,, \tag{2.1}$$

where $F(t) = \int_0^t f(s)\, \mathrm{d}s$. It is readily to verify that I is well defined and differentiable on $H_0^1(\Omega)$.

By a critical point u of I we mean that $u \in K$ and

$$\langle I'(u)\,,\ v - u \rangle \geq 0\,, \quad \forall\ v \in K \tag{2.2}$$

Critical points of $I(u)$ in K correspond to weak solutions of $(1.1)_\lambda$, the maximum principle implies the solutions are positive.

LEMMA 2.1.— *Suppose (f1)-(f3), (1.3), then there exists $\lambda_0 > 0$ such that the problem $(1.1)_\lambda$ has a positive solution for all $\lambda \in (0, \lambda_0)$, the solution is a local minimum of I in the convex set K.*

Proof.— It follows from (f2) that for $u \in H_0^1(\Omega)$

$$I(u) \geq \|u\|_{H^1}^2 \left(\frac{1}{2} - \lambda C_1 - \lambda C_2 \|u\|_{H^1}^{p-1} \right) .$$

We may choose $\lambda > 0$ small such that

$$\lambda < \lambda_0 := \min\left\{ (4C_1)^{-1}\,,\ \left(8C_2 \left(2\|\psi^+\|_{H^1}\right)^{p-1} \right)^{-1} \right\} .$$

Let

$$\rho = \rho(\lambda) = (8\lambda C_2)^{-1/(p-1)}\,, \quad B_\rho = \left\{ u \in H_0^1(\Omega) \;\middle|\; \|u\|_{H^1} < \rho \right\} .$$

Since $\|\psi^+\|_{H^1} < (1/2)\rho$, the set $K_\rho = K \cap B_\rho$ is not empty for $\lambda \in (0, \lambda_0)$. For any $u \in K_\rho$, $\lambda \in (0, \lambda_0)$, we have

$$I(u) \geq \frac{1}{8} \|u\|_{H^1}^2 \,. \tag{2.3}$$

Let $\{u_n\}$ be a minimizing sequence of the variational problem $L := \inf\{I(u) \mid u \in \overline{K}_\rho\}$, we know from (2.3) that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, hence we may assume that

$$\begin{aligned} &u_n \underset{n}{\rightharpoonup} u \quad \text{weakly in } H_0^1(\Omega), \\ &u_n \underset{n}{\rightarrow} u \quad \text{a.e. in } \Omega \end{aligned}$$

for some $u \in K$.

By (f3) and a lemma of Brezis and Lieb [5] we get

$$L = \lim_n I(u_n) = I(u) + \lim_n I(u_n - u) \geq L + \lim_n I(u_n - u),$$

it results

$$\lim_n I(u_n - u) \leq 0. \tag{2.4}$$

On the other hand

$$\begin{aligned}\lim_n a(u_n - u\,,\, u_n - u) &= \lim_n a(u_n, u_n) - 2\lim_n a(u_n, u) + a(u, u)\\ &\leq \lim_n a(u_n, u_n) \leq \rho^2,\end{aligned}$$

then (2.3) yields

$$\lim_n I(u_n - u) \geq 0.$$

This and (2.4) imply

$$\lim_n I(u_n - u) = 0. \tag{2.5}$$

By (2.3) we know that $u_n \underset{n}{\rightarrow} u$ strongly in $H_0^1(\Omega)$,and L is assumed by u. We claim that $u \in B_\rho$. In fact, by (2.3):

$$\frac{1}{8}\, a(u, u) \leq I(u) = L \leq I(\psi^+) \leq \frac{1}{2}\, a(\psi^+, \psi^+) < \frac{1}{8}\, \rho^2.$$

The proof is completed. □

Remark 2.2.— It follows from (2.3) and (2.5) that

$$I(u) \geq \frac{1}{8}\, a(u, u) > L \quad \text{if } a(u, u) = \rho^2.$$

Define

$$\lambda^* = \sup\{\lambda > 0 \mid (1.1)_\lambda \text{ has a positive solution}\}.$$

To obtain a minimal positive solution of $(1.1)_\lambda$, we collect some facts for the linear variational inequality

$$u \in K\,, \quad a(u\,, v - u) \geq \int_\Omega f(x)(v - u)\,\mathrm{d}x\,, \quad \forall\, v \in K. \tag{2.6}$$

LEMMA 2.3.— *Let ψ be a function satisfying (1.3), $f \in L^q(\Omega)$, then (2.6) has a unique solution $u \in H^{2,q}(\Omega)$ such that*

$$f \le Au \le \max\{A\psi\,,\, f\}\,, \quad a.e.\ in\ \Omega\,, \tag{2.7}$$

where $\langle Au\,,\, v\rangle := a(u,v)$.

Proof.— The existence and uniqueness of the solution to (2.6) are consequences of the Lions-Stampacchia theorem [11], the regularity of the solution and (2.7) are deduced in a standard way in [6] and [11], we omit the details. □

A function g is said to be a supersolution of $A - f$ if $g \in H^1(\Omega)$ such that

$$\langle Ag\,,\, \zeta\rangle = a(g,\zeta) \ge \int_\Omega f\zeta\,\mathrm{d}x\,, \quad \forall\, \zeta \in H_0^1(\Omega)\,,\ \zeta \ge 0\,.$$

Obviously, any solution of (2.6) is also a supersolution of $A - f$.

LEMMA 2.4.— *Let u be a solution of (2.6), $g \in H^1(\Omega)$ be a supersolution of $A - f$ satisfying $g \ge \psi$ and $g \ge 0$ on $\partial\Omega$ in sense of $H^1(\Omega)$, then*

$$u \le g\,, \quad in\ \Omega\,.$$

Proof.— We refer to [6] for the proof. □

LEMMA 2.5.— *Under the assumptions (1.3), (1.4), (f1) and (f2), if u is a positive solution of $(1.1)_\lambda$, then $u \in H^{2,q}(\Omega) \cap C^{1,\beta}(\Omega)$; moreover:*

(i) $u(x)$ and $|\nabla u(x)|$ have uniform limits zero as $|x| \to \infty$;

(ii) for any $\delta > 0$, there exist positive constants C_1, C_2 such that

$$\begin{aligned} C_1 \exp\bigl(-(1+\delta)\,|x|\bigr) &\le u(x) \\ &\le C_2 \exp\bigl(-(1-\delta)\,|x|\bigr)\,, \quad for\ |x| \ge R_1\,, \end{aligned} \tag{2.8}$$

where $R_1 > R_0$ large enough, R_0 denotes the smallest positive number such that $\omega \subset B_{R_0} = \bigl\{x \in \mathbb{R}^N \mid |x| < R_0\bigr\}$.

Proof.— We first prove (i). To this end, we claim that $u \in H^{2,q}(\Omega)$. In fact, by (f2) we have

$$f(u) \le C(u + u^p)\,.$$

Consider the following problems

$$u_1 \in K: \quad a(u_1\,,\, v - u_1) \geq \lambda C \int_\Omega u^p (v - u_1)\,\mathrm{d}x\,, \quad \forall\, v \in K \quad (2.9)$$

$$u_2 \in K: \quad a(u_2\,,\, v - u_2) \geq \lambda C \int_\Omega u (v - u_2)\,\mathrm{d}x\,, \quad \forall\, v \in K\,. \quad (2.10)$$

We remark that Lemma 2.3 still holds if $q \geq 2N/(N+2)$, hence (2.9) and (2.10) are unique solved by u_1 and u_2 in K. A bootstrap argument yields u_1, $u_2 \in H^{2,q}(\Omega)$ for $q > N$, therefore u_1 and u_2 belong to $C^{1,\beta}(\Omega)$ for some $\beta \in (0,1)$.

Since u_1 and u_2 are supersolution of $A - Cu^p$, $A - Cu$ respectively, $W = u_1 + u_2$ is a supersolution of $A - \lambda f(u)$. It implies by Lemma 2.4 that

$$u \leq W\,, \quad \text{in } \Omega\,,$$

hence $f(u) \in L^q(\Omega)$, Lemma 2.3 leads to $u \in H^{2,q}(\Omega)$.

Fixed $R > 0$, it is known from the Sobolev embedding that for $B_R(x) \subset \Omega$ there exists a positive constant C independent of $B_R(x)$ such that

$$\|u\|_{C^{1,\alpha}(B_R(x))} \leq C \|u\|_{H^{2,q}(B_R(x))}\,,$$

then (i) follows.

Since u is a supersolution of $Au - \lambda f(u)$, the inequality $u(x) \geq C_1 \exp\bigl(-(1+\delta)\,|x|\bigr)$, for $|x| > R_1$, can be established as [13] and [16].

On the other hand, $Au - \lambda f(u)$ is a positive linear functional on $H_0^1(\Omega)$, so there is a nonnegative Radon measure μ with $\operatorname{supp}\mu \subset \Gamma := \{x \in \Omega \mid u(x) = \psi(x)\}$ such that

$$Au = \lambda f(u) + \mu\,.$$

Furthermore, under our assumptions we obtain by (2.7) that

$$Au - \lambda f(u) \leq \bigl(A\psi - f(u)\bigr)^+ \leq \bigl(A\psi\bigr)^+ + \lambda f^-(u) \leq \bigl(A\psi\bigr)^+\,. \quad (2.11)$$

Let $\beta = \bigl(1 - \delta\bigr)^{1/2}$. Since $u(x) \to 0$ as $|x| \to +\infty$, by (f1) there exists $R_2 > R_0$ such that

$$1 - \lambda \frac{f(u)}{u} \geq 1 - \frac{1}{2}\delta\,, \quad \text{for } |x| \geq R_2\,. \quad (2.12)$$

The assumption (1.4) gives that there exist $\tau > 1$, $C > 0$ and $R_3 > 0$ large such that

$$(A\psi)^+ \le C \exp\bigl(-\tau|x|\bigr), \quad \text{for } |x| \ge R_3 . \tag{2.13}$$

Fixing $\delta_0 < \tau - 1$ by the first part of (ii), we may find $R_4 > 0$ such that

$$\frac{1}{2}\,\delta u(x) \ge \bigl(A\psi\bigr)^+, \quad \text{for } |x| \ge R_4 . \tag{2.14}$$

Let $v(x) = m\exp\bigl(-\beta(|x| - R_5)\bigr)$, where $R_5 = \max\{R_2, R_3, R_4\}$, $m = \max\{u(x) \mid |x| = R_5\} > 0$. For any $M > R_5$, set

$$\Omega(M) = \{x \in \Omega \mid R_5 < |x| < M \text{ and } u(x) > v(x)\} .$$

Then $\Omega(M)$ is open. By (2.11), (2.12) and (2.14) we have for $x \in \Omega(M)$:

$$\begin{aligned}
\Delta(v-u)(x) &\le \bigl(\beta^2 - \beta(N-1)\,|x|^{-1}\bigr)v(x) - \bigl(1 - \lambda f(u)u^{-1}\bigr)u + \bigl(A\psi\bigr)^+ \\
&\le \beta^2 v(x) - \left(1 - \frac{1}{2}\,\delta\right)u(x) + \bigl(A\psi\bigr)^+ \\
&= (1-\delta)\bigl(v(x) - u(x)\bigr) - \frac{1}{2}\,\delta u(x) + \bigl(A\psi\bigr)^+ \\
&\le (1-\delta)\bigl(v(x) - u(x)\bigr) < 0 .
\end{aligned}$$

By the maximum principle, we have for $x \in \Omega(M)$

$$\begin{aligned}
v(x) - u(x) &\ge \min\{(v-u)(x) \mid x \in \partial\Omega(M)\} \\
&= \min\bigl\{0, \min\{(v-u)(x) \mid |x| = M\}\bigr\} .
\end{aligned}$$

Since $\lim_{x\to+\infty} u(x) = \lim_{x\to+\infty} v(x) = 0$, it yields by letting $M \to +\infty$ that

$$v(x) \ge u(x), \quad \text{for } |x| \ge R_5 ,$$

hence the conclusion follows. □

PROPOSITION 2.6.— *Suppose (1.3), (f1)-(f3) hold, the problem* $(1.1)_\lambda$ *has a minimal positive solution for all* $\lambda \in (0, \lambda^*)$.

Proof.— For any $\lambda \in (0, \lambda^*)$, we may find $\lambda < \lambda' < \lambda^*$ such that $(1.1)_{\lambda'}$ has a positive solution $u \in H^{2,q}(\Omega)$, then u is a supersolution of $A - \lambda' f(\cdot)$, that is for any $\zeta \in H_0^1(\Omega)$, $\zeta \ge 0$:

$$\langle Au\,, \zeta\rangle := a(u, \zeta) \ge \lambda' \int_\Omega f(u)\zeta\,\mathrm{d}x .$$

Thus u is also a supersolution of $A - \lambda f(\cdot)$.

We consider the problem

$$u_1 \in K\,, \quad a(u_1\,,\, v - u_1) \geq \lambda \int_\Omega f(u)(v - u_1)\,\mathrm{d}x\,, \quad \forall\, v \in K\,. \qquad (2.15)_1$$

It is deduced by Lemma 2.3 that $(2.15)_1$ has a unique positive solution $u_1 \in H^{2,q}(\Omega)$, $q > N$, $u_1 \in K$. By Lemma 2.3

$$u_1 \leq u\,, \quad \text{a.e. in } \Omega\,. \qquad (2.16)$$

Since f is increasing

$$-\Delta u_1 + u_1 \geq \lambda f(u) \geq \lambda f(u_1)\,,$$

i.e. u_1 is a supersolution of $A - \lambda f(\cdot)$.

Consider following problems

$$u_k \in K\,, \quad a(u_k\,,\, v - u_k) \geq \lambda \int_\Omega f(u_{k-1})(v - u_k)\,\mathrm{d}x\,, \quad \forall\, v \in K\,. \quad (2.15)_k$$

By (2.16), we know that $f(u_1) \in L^q(\Omega)$. Therefore, it yields by iterating that $(2.15)_k$ has a unique solution u_k satisfying

$$\psi \leq \cdots \leq u_k \leq u_{k-1} \leq \cdots \leq u_1 \leq u\,. \qquad (2.17)$$

Let $v = u$ in $(2.15)_k$, then by (2.17) and (f3)

$$\begin{aligned} a(u_k\,,\, u - u_k) &\geq \lambda \int_\Omega f(u_{k-1})(u - u_k)\,\mathrm{d}x \\ &\geq \lambda \int_\Omega f(u_k)(u - u_k)\,\mathrm{d}x\,. \end{aligned}$$

Therefore

$$\begin{aligned} a(u_k, u_k) &\leq a(u_k, u) + \lambda \int_\Omega u_k f(u_k)\,\mathrm{d}x - \lambda \int_\Omega u f(u)\,\mathrm{d}x \\ &\leq a(u_k, u) + \lambda \int_\Omega u f(u)\,\mathrm{d}x\,. \end{aligned}$$

By Young's inequality

$$\|u_k\|_{H^1}^2 = a(u_k, u_k) \leq C\left(\int_\Omega u f(u)\,\mathrm{d}x + \|u\|_{H^1}^2\right).$$

Then we may assume

$$u_k \to u_0 \quad \text{weakly in } H_0^1(\Omega)\,. \tag{2.18}$$

$$u_k \to u_0 \quad \text{a.e. in } \Omega\,. \tag{2.19}$$

We claim that u_0 is a solution of $(1.1)_\lambda$, then $(1.1)_\lambda$ is solvable for all $\lambda \in (0, \lambda^*)$. In fact, by $(2.15)_k$ one has

$$a(u_k, v) \geq a(u_k, u_k) + \lambda \int_\Omega f(u_{k-1}) v \, dx - \lambda \int_\Omega f(u_{k-1}) u_k \, dx\,. \tag{2.20}$$

By (2.17) and Lebesgue's monotone convergence theorem we get

$$\lim_{k\to\infty} \int_\Omega f(u_{k-1}) u_k \, dx = \int_\Omega f(u_0) u_0 \, dx\,. \tag{2.21}$$

The weak convergence implies

$$\lim_{k\to\infty} a(u_k, u_k) \geq a(u_0, u_0) \tag{2.22}$$

and

$$\lim_{k\to\infty} \int_\Omega f(u_{k-1}) v \, dx = \int_\Omega f(u_0) v \, dx\,. \tag{2.23}$$

From (2.21)-(2.22), it concludes in taking limit in (2.20) that

$$a(u_0, v) \geq a(u_0, u_0) + \lambda \int_\Omega v f(u_0) \, dx - \lambda \int_\Omega u_0 f(u_0) \, dx\,, \quad \forall\, v \in K\,,$$

hence u_0 is a solution of $(1.1)_\lambda$.

Denote by $Q(\lambda)$ the set of supersolutions of $A - \lambda f(\cdot)$, namely

$$Q(\lambda) := \left\{ u \in K \;\middle|\; a(u, v) \geq \lambda \int_\Omega v f(u) \, dx\,,\ \forall\, v \in H_0^1(\Omega)\,,\ v \geq 0 \right\}.$$

Since solutions of $(1.1)_\lambda$ are supersolutions of $A - \lambda f(\cdot)$, then $Q(\lambda)$ is not empty for $\lambda \in (0, \lambda^*)$. Define

$$W = \inf\{ u \mid u \in Q(\lambda) \}\,,$$

We claim that W is a supersolution of $A - \lambda f(\cdot)$.

We note that $\psi \le W \le u_0$, $f(W) \in L^q(\Omega)$, it is claritied by Lemma 2.3 that the problem

$$\zeta \in K\,, \quad a(\zeta\,,\, v-\zeta) \ge \lambda \int_\Omega f(W)(v-\zeta)\,\mathrm{d}x\,, \quad \forall\, v \in K \tag{2.24}$$

has a unique solution $\zeta \in H^{2,q}(\Omega)$.

For any $u \in Q(\lambda)$, it satisfies

$$W \le u \quad \text{and} \quad a(u,v) \ge \lambda \int_\Omega v f(u)\,\mathrm{d}x \ge \lambda \int_\Omega v f(W)\,\mathrm{d}x$$

for $v \in H_0^1(\Omega)$, $v \ge 0$. Therefore u is a supersolution of $A - \lambda f(W)$. Lemma 2.4 gives

$$\zeta \le u\,, \quad \forall\, u \in Q(\lambda)\,, \tag{2.25}$$

hence

$$\zeta \le W\,. \tag{2.26}$$

We remark that ζ is also a supersolution of $A - \lambda f(W)$, then for all $v \in H_0^1(\Omega)$, $v \ge 0$

$$a(\zeta, v) \ge \lambda \int_\Omega v f(W)\,\mathrm{d}x \ge \lambda \int_\Omega v f(\zeta)\,\mathrm{d}x\,,$$

this means $\zeta \in Q(\lambda)$; so $\zeta \ge W$, we conclude $\zeta = W$ and $W \in Q(\lambda)$.

We claim that W is the minimal positive solution of $(1.1)_\lambda$. In fact, replacing u in $(2.15)_1$ by W, we may construct a sequence of solutions $\{u_k\}$ of $(2.15)_k$ such that u_k tends to a solution $u(\lambda)$ of $(1.1)_\lambda$ as $k \to \infty$, the solution $u(\lambda)$ satisfies

$$\psi \le u(\lambda) \le W\,. \tag{2.27}$$

Since $Q(\lambda)$ includes all solutions of $(1.1)_\lambda$, (2.27) implies that $u(\lambda)$ is the minimal solution of $(1.1)_\lambda$. Because of $u(\lambda) \in Q(\lambda)$, we have $u(\lambda) = W$. □

To show $\lambda^* < \infty$, we consider the linear eigenvalue problem

$$\begin{cases} -\Delta v + v = \nu g(x) v & x \in \Omega, \\ v = 0 & x \in \partial\Omega,\ v \in H_0^1(\Omega)\,. \end{cases} \tag{2.28}$$

LEMMA 2.7.— *Suppose $g \in C_b(R)$, $g(x) \geq 0$ and $g(x) \to 0$ as $|x| \to \infty$, then the problem (2.28) has a positive solution u such that*

$$\|u\|_{H_0^1}^2 = \nu := \inf\left\{ \|v\|_{H_0^1}^2 \;\middle|\; v \in H_0^1(\Omega)\,,\ \int_\Omega g(x)v^2\,\mathrm{d}x = 1 \right\}.$$

Proof.— It is obvious that $0 < \nu < +\infty$, let $\{v_n\}$ be a minimizing sequence of ν, *i.e.*

$$\int_\Omega g v_n^2\,\mathrm{d}x = 1\,, \quad \|v_n\|_{H_0^1}^2 \to \nu \text{ as } n \to \infty\,,$$

we may assume $v_n \geq 0$. Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, it may be assumed that

$$\begin{aligned} v_n \underset{n}{\rightharpoonup} v &\quad \text{weakly in } H_0^1(\Omega) \\ v_n \underset{n}{\to} v &\quad \text{a.e. in } (\Omega)\,. \end{aligned}$$

Fatou's lemma implies $\|v\|_{H_0^1}^2 \leq \underline{\lim}_n \|v_n\|_{H_0^1}^2$. We claim that

$$1 = \lim_n \int_\Omega g v_n^2\,\mathrm{d}x = \int_\Omega g v^2\,\mathrm{d}x\,. \tag{2.29}$$

By the assumptions of g, there exists a constant $C > 0$ such that $g(x) \leq C$ for all $x \in \Omega$; for any $\varepsilon > 0$, we may find $R > 0$ large enough such that $g(x) \leq \varepsilon$, for $|x| \geq R$, thus

$$\begin{aligned} \int_\Omega g|v_n - v|^2\,\mathrm{d}x &= \int_{\Omega\cap B_R} g|v_n - v|^2\,\mathrm{d}x + \int_{\Omega\setminus B_R} g|v_n - v|^2\,\mathrm{d}x \\ &\leq C\int_{\Omega\cap B_R} |v_n - v|^2\,\mathrm{d}x + \varepsilon C\,. \end{aligned} \tag{2.30}$$

By the Sobolev embedding, it follows that for $n \geq n_0$, $n_0 \geq 1$ large

$$\int_{\Omega\cap B_R} |v_n - v|^2\,\mathrm{d}x < \varepsilon\,. \tag{2.31}$$

(2.30), (2.31) and the arbitrariness of ε imply (2.29). Hence ν is attained by v, the maximum principle yields $v > 0$. □

Let $u(\lambda)$ be the minimal positive solution corresponding to $\lambda \in (0, \lambda^*)$ obtained in Proposition 2.6, we remark that for λ, $\lambda' \in (0, \lambda^*)$, correspondingly we have $u(\lambda) < u(\lambda')$ (here and below "$<$" means "$\leq$" and "$\not\equiv$"). Define

$$\nu(\lambda) = \inf \left\{ \|v\|^2_{H_0^1} \;\Big|\; \int_\Omega f'\big(u(\lambda)\big) v^2 \,\mathrm{d}x = 1 \right\} ,$$

then we have the following proposition.

PROPOSITION 2.8.— *λ^* is finite.*

Proof.— First we claim that $\nu(\lambda)$ is nonincreasing in λ for $\lambda \in (0, \lambda^*)$.

Let $\lambda < \lambda'$, λ, $\lambda' \in (0, \lambda^*)$, $u(\lambda)$ and $u(\lambda')$ are corresponding minimal positive solutions with $u(\lambda) < u(\lambda')$. By Lemma 2.5 and Lemma 2.7, $\nu(\lambda)$ is attained by some $v_\lambda \in H_0^1(\Omega)$, $v_\lambda > 0$. Since $f'(t)$ is increasing for $t > 0$

$$\int_\Omega f'\big(u(\lambda')\big) v_\lambda^2 \,\mathrm{d}x \geq \int_\Omega f'\big(u(\lambda)\big) v_\lambda^2 \,\mathrm{d}x = 1 ,$$

so there exists $0 < t < 1$ such that

$$\int_\Omega f'\big(u(\lambda')\big) t^2 v^2 \,\mathrm{d}x = 1 ,$$

therefore

$$\nu(\lambda') \leq t^2 \|v_\lambda\|^2_{H_0^1} \leq \|v_\lambda\|^2_{H_0^1} = \nu(\lambda) .$$

Next we show $\lambda^* < +\infty$.

Fixing $\lambda_0 \in (0, \lambda^*)$, for any $\lambda \in (\lambda_0, \lambda^*)$ there exists $\varepsilon > 0$ such that $\lambda_0 < \lambda_1 := \lambda_0 + \varepsilon < \lambda$; correspondingly, we have minimal positive solutions $u(\lambda_0)$, $u(\lambda_1)$ and $u(\lambda)$ verifying

$$u(\lambda_0) < u(\lambda_1) < u(\lambda), \tag{2.32}$$

and nonnegative Radon measures $\mu(\lambda_0)$, $\mu(\lambda_1)$ and $\mu(\lambda)$ satisfying

$$\mu(\lambda_0) \,,\ \mu(\lambda_1) \,,\ \mu(\lambda) \leq (A\psi)^+ . \tag{2.33}$$

Therefore

$$\begin{aligned} -\Delta\big(u(\lambda) - u(\lambda_0)\big) &+ \big(u(\lambda) - u(\lambda_0)\big) \\ &= \lambda f\big(u(\lambda)\big) - \lambda_0 f\big(u(\lambda_0)\big) + \mu(\lambda) - \mu(\lambda_0) \\ &\geq \lambda \left(f\big(u(\lambda)\big) - f\big(u(\lambda_0)\big)\right) - \mu(\lambda_0) \\ &\geq \lambda f'\big(u(\lambda_0)\big)\big(u(\lambda) - u(\lambda_0)\big) - \mu(\lambda_0) . \end{aligned}$$

Multipling both sides by v_{λ_0} and integrating by part, we deduce

$$\begin{aligned}\int_\Omega (A\psi)^+ v_{\lambda_0}\,dx + \nu(\lambda_0)\int_\Omega f'\big(u(\lambda_0)\big)\big(u(\lambda)-u(\lambda_0)\big)v_{\lambda_0}\,dx \geq \\ \geq \lambda\int_\Omega f'\big(u(\lambda_0)\big)\big(u(\lambda)-u(\lambda_0)\big)v_{\lambda_0}\,dx\,.\end{aligned}\tag{2.34}$$

If $\lambda \leq \nu(\lambda_0)$ for any $\lambda \in (\lambda_0, \lambda^*)$, then we have done; if $\lambda > \nu(\lambda_0)$ for $\lambda \in (\lambda_0, \lambda^*)$, by (2.34)

$$\begin{aligned}\int_\Omega (A\psi)^+ v_{\lambda_0}\,dx &\geq \big(\lambda-\nu(\lambda_0)\big)\int_\Omega f'\big(u(\lambda_0)\big)\big(u(\lambda)-u(\lambda_0)\big)v_{\lambda_0}\,dx \\ &\geq \big(\lambda-\nu(\lambda_0)\big)\int_\Omega f'\big(u(\lambda_0)\big)\big(u(\lambda_1)-u(\lambda_0)\big)v_{\lambda_0}\,dx\,.\end{aligned}\tag{2.35}$$

Since

$$\int_\Omega f'\big(u(\lambda_0)\big)\big(u(\lambda_1)-u(\lambda_0)\big)v_{\lambda_0}\,dx > 0\,,$$

it follows from (2.35) that there exists a positive constant C such that $\lambda \leq C$ for all $\lambda^* > \lambda > \max\{\lambda_1\,,\ \nu(\lambda_0)\}$. Thus the proof is completed. □

3. Existence of a second positive solution

The existence of a second positive solution of $(1.1)_\lambda$ will be established in this section. We shall find a critical point of the functional

$$J(u) = \frac{1}{2}a(u,u) - \lambda\int_\Omega F(u)\,dx + \delta(u) = I(u) + \delta(u) \tag{3.1}$$

in the convex set

$$\begin{aligned}\mathcal{K}_\lambda = \big\{v \in H_0^1(\Omega) \mid v \geq u(\lambda)\,, \\ u(\lambda) \text{ is the minimum positive solution of } (1.1)_\lambda\big\}\end{aligned}$$

other than the minimal positive solution $u(\lambda)$, where δ is the indicator function of $\mathcal{K}_\lambda$, *i.e.* $\delta(u) = 0$ if $u \in \mathcal{K}_\lambda$ and $\delta(u) = +\infty$ otherwise.

We need show that a critical point u of $J(u)$ in $\mathcal{K}_\lambda$ is a solution of $(1.1)_\lambda$.

Let u be a critical point of $J(u)$ in $\mathcal{K}_\lambda$, it satisfies

$$a(u\,,\ v-u) \geq \lambda\int_\Omega f(u)(v-u)\,dx\,,\quad \forall\ v \in \mathcal{K}_\lambda \tag{3.2}$$

Denote by $\Gamma = \{x \in \Omega \mid u(x) = u_\lambda(x)\}$ the coincidence set, it is standard to verify

$$a(u, \varphi) = \lambda \int_\Omega f(u)\varphi \,\mathrm{d}x\,, \quad \forall\, \varphi \in C_0^\infty(\Omega \setminus \Gamma)\,.. \tag{3.3}$$

Furthermore, since

$$a(u, \varphi) \geq \lambda \int_\Omega f(u)\varphi \,\mathrm{d}x\,, \quad \forall\, \varphi \in H_0^1(\Omega)\,,\ \varphi \geq 0\,,$$

by the Riesz-Schwartz theorem there exists a nonnegative Radon measure μ such that

$$a(u, \varphi) = \lambda \int_\Omega \varphi f(u) \,\mathrm{d}x + \int_\Omega \varphi \,\mathrm{d}\mu\,, \quad \forall\, \varphi \in H_0^1(\Omega)\,. \tag{3.4}$$

From (3.3) we derive supp $\mu \subset \Gamma$.

LEMMA 3.1.— *Suppose u satisfies (3.2). Then u is a solution of $(1.1)_\lambda$.*

Proof.— We remark that (3.3) is valid for the minimal positive solution u_λ for all $v \in C_0^\infty(\{u_\lambda > \psi\})$. For any $\varphi \in C_0^\infty(\{u > \psi\})$, by (3.3) and above remark we deduce

$$\begin{aligned}
&a(u, \varphi) = \\
&= \int_{\{u>u_\lambda\}} (\nabla u \cdot \nabla\varphi + u\varphi) \,\mathrm{d}x + \int_{\{u=u_\lambda\}\cap\{u_\lambda>\psi\}} (\nabla u \cdot \nabla\varphi + u\varphi) \,\mathrm{d}x \\
&= \lambda \int_{\{u>u_\lambda\}} \varphi f(u) \,\mathrm{d}x + \int_{\{u=u_\lambda\}\cap\{u_\lambda>\psi\}} (\nabla u_\lambda \cdot \nabla\varphi + u\varphi) \,\mathrm{d}x \\
&= \lambda \int_{\{u>u_\lambda\}} \varphi f(u) \,\mathrm{d}x + \int_{\{u=u_\lambda\}\cap\{u_\lambda>\psi\}} \varphi f(u_\lambda) \,\mathrm{d}x \\
&= \lambda \int_\Omega \varphi f(u) \,\mathrm{d}x\,.
\end{aligned} \tag{3.5}$$

It implies supp $\mu \subset \widetilde{\Gamma} = \{x \in \Omega \mid u_\lambda(x) = \psi(x)\}$. Then for any given $v \in K$, we obtain

$$\begin{aligned}
\int_\Omega (v - u) \,\mathrm{d}\mu &= \int_{\{v<u\}} (v - u) \,\mathrm{d}\mu + \int_{\{v\geq u\}} (v - u) \,\mathrm{d}\mu \\
&= \int_{\{v\geq u\}} (v - u) \,\mathrm{d}\mu \geq 0\,.
\end{aligned}$$

Therefore

$$a(u\,,\,v-u)=\lambda\int_\Omega f(u)(v-u)\,\mathrm{d}x+\int_\Omega (v-u)\,\mathrm{d}\mu$$
$$\geq \lambda\int_\Omega f(u)(v-u)\,\mathrm{d}x\,.\ \square$$

LEMMA 3.2.— *Suppose (f1), (f3) and (f4), then:*

(i) there exists $\varepsilon\in\big(0\,,\,(1/2)\big)$ *such that* $\varepsilon t f(t)\geq F(t)$ *for* $t>0$*;*

(ii) $t^{-1/\theta}f(t)$ *is monotone nondecreasing for* $t>0$ *and* $t^{-1}f(t)$ *is strictly monotone increasing for* $t>0$*;*

(iii) for $s,\ t\in(0,+\infty)$

$$f(s+t)\geq f(s)+f(t)\quad \textit{and}\quad f(s+t)\not\equiv f(s)+f(t)\,.$$

For the proof of Lemma 3.2 we refer to that of Lemma 2.1 in [16].

A sequence $\{u_n\}\subset\mathcal{K}_\lambda$ is known as a (P.S.)$_c$ sequence for $c\in\mathbb{R}$ if $I(u_n)\underset{n}{\to} c$ and

$$\big\langle I'(u_n)\,,\,v-u_n\big\rangle\geq\big\langle z_n\,,\,v-u_n\big\rangle\,,\quad \forall\, v\in\mathcal{K}_\lambda\,,$$

where $z_n\to 0$ as $n\to\infty$.

The functional J satisfies (P.S.)$_c$ if and only if each (P.S.)$_c$ sequence of J has a convergence subsequence.

Let $u,\ v\in L^p$, we denote $u\vee v=\max\{u,v\}$, $u\wedge v=\min\{u,v\}$.

LEMMA 3.3.— *Let* $\{u_n\}$ *be a (P.S.)*$_c$ *sequence of* J*. Suppose* u_n *weakly converges to* u*, then* u *is a solution of (1.1)*$_\lambda$*.*

Proof.— By Lemma 3.1 we only need show u satisfies (3.2).

By the assumptions, $\{u_n\}$ satisfies

$$I(u_n)=c+\mathrm{o}(1)\quad\text{as } n\to\infty\,,\tag{3.6}$$

$$a(u_n\,,\,v-u_n)-\lambda\int_\Omega f(u_n)(v-u_n)\,\mathrm{d}x\geq\big\langle\xi_n\,,\,v-u_n\big\rangle\quad\text{for } v\in\mathcal{K}_\lambda\,,\tag{3.7}$$

where $\xi_n\to 0$ as $n\to\infty$.

Let $v = 2u_n$ in (3.7) we obtain

$$a(u_n, u_n) - \lambda \int_\Omega u_n f(u_n)\, \mathrm{d}x \geq \langle \xi_n\,,\, u_n \rangle\,. \tag{3.8}$$

Lemma 3.2 and (3.6) yield

$$\frac{1}{2}\, a(u_n, u_n) - \varepsilon\lambda \int_\Omega u_n f(u_n)\, \mathrm{d}x \leq c + \mathrm{o}(1) \quad \text{as } n \to \infty\,. \tag{3.9}$$

By (3.8) and (3.9)

$$\left(\frac{1}{2} - \varepsilon\right) \|u_n\|^2_{H_0^1} \leq \|\xi_n\|_{H^{-1}} \|u_n\|_{H_0^1} + c + \mathrm{o}(1) \tag{3.10}$$

so $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we may assume $u_n \underset{n}{\rightharpoonup} u$ weakly in $H_0^1(\Omega)$ and $u_n \underset{n}{\to} u$, a.e. in Ω.

Denote $u_\lambda = u(\lambda)$, let $v_n = u_n + (u_n - u - u_\lambda)^+$ in (3.7), setting $\varphi_n = u_n - u$ we get $(\varphi_n - u_\lambda)^+ = \varphi_n - \varphi_n \wedge u_\lambda$, thus

$$\begin{aligned} &a(u_n, \varphi_n) - a(u_n\,,\, \varphi_n \wedge u_\lambda) \geq \\ &\quad \geq \lambda \int_\Omega f(u_n)(\varphi_n - \varphi_n \wedge u_\lambda)\, \mathrm{d}x + \mathrm{o}(1) \quad \text{as } n \to \infty\,. \end{aligned} \tag{3.11}$$

We claim that

$$\int_\Omega f(u_n)(\varphi_n \wedge u_\lambda)\, \mathrm{d}x \underset{n}{\to} 0\,, \tag{3.12}$$

$$\int_\Omega u_n(\varphi_n \wedge u_\lambda)\, \mathrm{d}x \underset{n}{\to} 0\,, \tag{3.13}$$

$$\int_\Omega \nabla u_n \cdot \nabla(\varphi_n \wedge u_\lambda)\, \mathrm{d}x \underset{n}{\to} 0\,. \tag{3.14}$$

In fact for $R > 0$

$$\begin{aligned} \int_\Omega f(u_n)(\varphi_n \wedge u_\lambda)\, \mathrm{d}x = &\int_{B_R} f(u_n)(\varphi_n \wedge u_\lambda)\, \mathrm{d}x\, + \\ &+ \int_{\Omega \setminus B_R} f(u_n)(\varphi_n \wedge u_\lambda)\, \mathrm{d}x\,. \end{aligned}$$

Since $\varphi_n \wedge u_\lambda \underset{n}{\to} 0$ a.e. in Ω, we infer by Lebesgue's dominated convergence theorem that

$$\int_{B_R} f(u_n)(\varphi_n \wedge u_\lambda)\, \mathrm{d}x \underset{n}{\to} 0 \tag{3.15}$$

In addition by (f2)

$$
\begin{aligned}
\int_{\Omega\setminus B_R} f(u_n)(\varphi_n \wedge u_\lambda)\,\mathrm{d}x \le \\
&\le C\left\{\int_{\Omega\setminus B_R} u_n u_\lambda\,\mathrm{d}x + \int_{\Omega\setminus B_R} u_n^p u_\lambda\,\mathrm{d}x\right\} \\
&\le C\left(\int_\Omega u_n^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\Omega\setminus B_R} u_\lambda^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \\
&\quad + C\left(\int_\Omega u_n^{p+1}\,\mathrm{d}x\right)^{\frac{p}{p+1}}\left(\int_{\Omega\setminus B_R} u_\lambda^{p+1}\,\mathrm{d}x\right)^{\frac{1}{p+1}} \\
&\le C\left(\int_{\Omega\setminus B_R} u_\lambda^2\,\mathrm{d}x\right)^{\frac{1}{2}} + C\left(\int_{\Omega\setminus B_R} u_\lambda^{p+1}\,\mathrm{d}x\right)^{\frac{1}{p+1}}.
\end{aligned}
\tag{3.16}
$$

By (3.15) and (3.16), letting $n \to \infty$ then $R \to \infty$, we obtain (3.12). Similarly, we can prove (3.13).

We follow the argument in [10] to deduce (3.14). Let $v = u + (\varphi_n - u_\lambda)^+$ in (3.7) then

$$
a\left(u_n\,,\ -\varphi_n + (\varphi_n - u_\lambda)^+\right) \ge \lambda\left\langle f(u_n)\,,\ -\varphi_n + (\varphi_n - u_\lambda)^+\right\rangle + \mathrm{o}(1)
$$

that is

$$
\begin{aligned}
a(u_n, \varphi_n) \le a\left(u_n\,,\ (\varphi_n - u_\lambda)^+\right) + \\
&+ \lambda\int_\Omega \left(f(u_n)\varphi_n - f(u_n)(\varphi_n - u_\lambda)^+\right)\,\mathrm{d}x + \mathrm{o}(1)\,.
\end{aligned}
$$

It results by noting $\varphi_n = \varphi_n \wedge u_\lambda + (\varphi_n - u_\lambda)^+$ that

$$
a(u_n\,,\ \varphi_n \wedge u_\lambda) \le \lambda\int_\Omega f(u_n)(\varphi_n \wedge u_\lambda)\,\mathrm{d}x\,.
$$

By (3.12)

$$
\overline{\lim_n}\, a(u_n\,,\ \varphi_n \wedge u_\lambda) \le \lambda\,\overline{\lim_n}\int_\Omega f(u_n)(\varphi_n \wedge u_\lambda)\,\mathrm{d}x = 0\,. \tag{3.17}
$$

On the other hand, $\varphi_n \wedge u_\lambda \underset{n}{\rightharpoonup} 0$ weakly in $H_0^1(\Omega)$, it implies

$$a(u_n\,,\,\varphi_n \wedge u_\lambda) = a(\varphi_n\,,\,\varphi_n \wedge u_\lambda) + a(u\,,\,\varphi_n \wedge u_\lambda) + o(1)\,, \quad n \to \infty\,.$$

By (3.13)

$$\underline{\lim}_n \int_\Omega \nabla u_n \cdot \nabla(\varphi_n \wedge u_\lambda)\, dx = \underline{\lim}_n \int_\Omega \nabla \varphi_n \cdot \nabla(\varphi_n \wedge u_\lambda)\, dx\,. \tag{3.18}$$

Let χ_n be the characteristic function of the set $\{x \in \Omega \mid \varphi_n(x) \geq u_\lambda(x)\}$, then

$$\lim_n \left| \int_\Omega (\nabla \varphi_n \cdot \nabla u_\lambda) \chi_n \, dx \right| \leq C \lim_n \left(\int_\Omega \left|\nabla u_\lambda\right|^2 \chi_n \, dx \right)^{\frac{1}{2}} = 0$$

since $\chi_n \underset{n}{\rightarrow} 0$ almost for every x for which $u_\lambda(x) > 0$. Therefore (3.18) gives

$$\underline{\lim}_n \int_\Omega \nabla u_n \cdot \nabla(\varphi_n \wedge u_\lambda)\, dx \geq 0\,. \tag{3.19}$$

Hence (3.14) follows by (3.17) and (3.19).

We deduce by (3.11)-(3.14) that

$$\begin{aligned} \|u_n\|^2_{H_0^1} - \int_\Omega (\nabla u_n \cdot \nabla u + u_n u)\, dx \geq \\ \geq \lambda \int_\Omega u_n f(u_n)\, dx - \lambda \int_\Omega u f(u_n)\, dx + o(1) \end{aligned} \tag{3.20}$$

therefore

$$\begin{aligned} \lim_n a(u_n, u_n) - \lambda \lim_n \int_\Omega f(u_n) u_n \, dx \geq \\ \geq \lim_n a(u_n, u) - \lambda \lim_n \int_\Omega u f(u_n)\, dx\,. \end{aligned} \tag{3.21}$$

The weak convergence yields

$$\lim_n a(u_n, u) = a(u, u)\,. \tag{3.22}$$

By Strauss lemma [12] for $R > 0$

$$\int_{B_R} (f(u_n) - f(u)) v \, dx \underset{n}{\rightarrow} 0 \quad \forall\, v \in \mathcal{K}_\lambda\,.$$

Furthermore, it can be proved as (3.16) that

$$\int_{\Omega\setminus B_R} f(u_n)v\,\mathrm{d}x \to 0\,, \quad \text{as } R \to \infty\,.$$

Consequently

$$\lim_n \int_\Omega f(u_n)v\,\mathrm{d}x = \int_\Omega f(u)v\,\mathrm{d}x\,. \tag{3.23}$$

Hence (3.21)-(3.23) give

$$\lim_n \|u_n\|^2_{H^1_0} - \lambda \lim_n \int_\Omega u_n f(u_n)\,\mathrm{d}x \geq a(u,u) - \lambda \int_\Omega u f(u)\,\mathrm{d}x\,. \tag{3.24}$$

From (3.7) and (3.24) for any $v \in \mathcal{K}_\lambda$

$$\begin{aligned} a(u,v) - \lambda \int_\Omega v f(u)\,\mathrm{d}x &\geq \lim_n a(u_n,u_n) - \lambda \lim_n \int_\Omega u_n f(u_n)\,\mathrm{d}x \\ &\geq a(u,u) - \lambda \int_\Omega f(u) u\,\mathrm{d}x\,, \end{aligned}$$

i.e. u satisfies (3.2). □

A ground state $w(x) > 0$ of the problem

$$\begin{cases} -\Delta u + u = \lambda f(u) & \text{for } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) & N \geq 3 \end{cases} \tag{3.25}$$

will be used in our proofs. By a ground state w of (3.25) we mean a solution of (3.25) such that the minimum of the energy functional of (3.25)

$$I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x - \lambda \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x \tag{3.26}$$

is attained by it among all solutions of (3.25), that is

$$S^\infty := I^\infty(w) = \inf\{I^\infty(u) \mid \text{ for all solutions } u \not\equiv 0 \text{ of } (3.25)\}\,. \tag{3.27}$$

It is well known ([4], [7], [12]) that there exists a ground state $w(x) = w(|x|) > 0$ of (3.25) under the assumptions (f1)-(f2), and the ground state w satisfies following estimates

$$w(x)\,|x|^{(N-1)/2} \exp(|x|) \to C > 0\,. \tag{3.28}$$

and

$$w'(r)\, r^{(N-1)/2} \exp(r) \to -C \tag{3.29}$$

as $r = |x| \to \infty$.

The $(\mathrm{P.S.})_c$ sequence is precisely described by Proposition 3.4.

PROPOSITION 3.4.— *Under the assumptions (f1)-(f4), suppose that $\{u_n\}$ is a $(P.S.)_c$ sequence for J. Then there exists a subsequence (still denoted by $\{u_n\}$) for which the following holds : there exist an integer $m \geq 0$, sequences $\{x_n^i\} \subset \mathbb{R}^N$ for $1 \leq i \leq m$, a solution u of $(1.1)_\lambda$ and solutions u^i $(1 \leq i \leq m)$ of (3.25) such that*

$$u_n \underset{n}{\rightarrow} u \quad \textit{weakly in } H_0^1(\Omega)\,, \tag{3.30}$$

$$I(u_n) \underset{n}{\rightarrow} I(u) + \sum_{i=1}^{m} I^{\infty}(u^i)\,, \tag{3.31}$$

$$u_n - \left(u + \sum_{i=1}^{m} u^i(x - x_n^i) \right) \underset{n}{\rightarrow} 0\,, \tag{3.32}$$

$$x_n^i \underset{n}{\rightarrow} +\infty\,, \quad \left| x_n^i - x_n^j \right| \underset{n}{\rightarrow} +\infty \quad \textit{for } 1 \leq i \neq j \leq m\,, \tag{3.33}$$

where we agree that in the case $m = 0$ the above holds without u^i, x_n^i.

Proof.— The boundedness of $\{u_n\}$ can be demonstrated as (3.10), then we way assume

$$u_n \underset{n}{\rightarrow} u \quad \text{weakly in } H_0^1(\Omega) \text{ for some } u \in \mathcal{K}_\lambda\,, \tag{3.34}$$

$$u_n \underset{n}{\rightarrow} u \quad \text{weakly in } L^q(\Omega) \text{ for } 2 \leq q \leq 2N/(N-2)\,, \tag{3.35}$$

$$u_n \underset{n}{\rightarrow} u \quad \text{a.e. in } \Omega\,. \tag{3.36}$$

It is known from lemmas 3.1 and 3.3 that u is a solution of $(1.1)_\lambda$.

From (3.34)-(3.36) for $\varphi \in H_0^1(\Omega)$

$$a(u_n, \varphi) \underset{n}{\rightarrow} a(u, \varphi) \tag{3.37}$$

$$\int_\Omega f(u_n)\varphi \,\mathrm{d}x \underset{n}{\rightarrow} \int_\Omega f(u)\varphi \,\mathrm{d}x\,. \tag{3.38}$$

By Brezis and Lieb lemma [5]

$$\lim_n \int_\Omega f(u_n - u)\varphi \,\mathrm{d}x = \lim_n \int_\Omega \big(f(u_n) - f(u)\big)\varphi \,\mathrm{d}x = 0\,. \tag{3.39}$$

Let $\Psi_n^1(x) = (u_n - u)(x)$ if $x \in \Omega$ and $\Psi_n^1(x) = 0$ if $x \in \mathbb{R}^N \setminus \Omega$. Then by (3.34)-(3.36) $\Psi_n^1 \underset{n}{\rightharpoonup} 0$ weakly in $H_0^1(\Omega)$ and $L^q(\Omega)$. Since f is convex, by applying Brezis-Lieb lemma and by taking account of (3.37) and (3.39) we obtain for n large

$$\begin{aligned} I^\infty(\Psi_n^1) &= J(u_n) - J(u) + o(1)\,, \\ I^{\infty\prime}(\Psi_n^1) &= o(1) \in H^{-1}(\Omega)\,. \end{aligned} \tag{3.40}$$

The rest part of the proof is the same as that of Lemma 3.1 in [3], we outline the proof. Suppose Ψ_n^1 does not converges strongly to 0 in H_0^1 (otherwise we shall have done), then we may show that there is a sequence $\{x_n^1\} \subset \mathbb{R}^N$ such that $\Psi_n^1(x + x_n^1) \underset{n}{\rightharpoonup} u^1 \not\equiv 0$ weakly in $H^1(\mathbb{R}^N)$ and u^1 solves (3.25). Iterating this procedure we obtain sequences

$$\Psi_n^j(x) = \Psi_n^{j-1}(x + x_n) - u^{j-1}(x)\,, \quad j \geq 2\,,$$

and $\{x_n^j\} \subset \mathbb{R}^N$ such that $|x_n^j| \underset{n}{\rightarrow} \infty$, $\Psi_n^j(x + x_n^j) \underset{n}{\rightharpoonup} u^j(x)$ weakly in $H^1(\mathbb{R}^N)$, u^j solves (3.25). The proof then follows by induction. □

Recall that $R_0 = \inf\{R > 0 \mid \omega \subset B_R\}$, let $\bar{\xi} : \mathbb{R}^+ \cup \{0\} \to [0\,,1]$ be a C^∞ nondecreasing function such that

$$\bar{\xi}(t) = \begin{cases} 0 & \text{for } t \leq 1, \\ 1 & \text{for } t \geq 2. \end{cases}$$

Set $\xi(x) = \bar{\xi}(|x|/R_0)$, $w_\beta(x) = w(x + \beta e)$, $\beta \in (0, +\infty)$, e is a fixed unit vector in $\mathbb{R}^N$, w is the ground state of (3.25), then $\xi w_\beta \in H_0^1(\Omega)$.

LEMMA 3.5.— *If (1.3), (1.4) and (f1)-(f4) are satisfied, there exists a* $v \in H_0^1(\Omega)$, $v \geq 0$, $v \not\equiv 0$ *such that*

$$\sup_{t \geq 0} J\big(u(\lambda) + tv\big) < I\big(u(\lambda)\big) + S^\infty\,,$$

where $u(\lambda)$ *is the minimal positive solution of* $(1.1)_\lambda$, S^∞ *is defined in (3.27).*

Proof.— We follow the arguments of [16]. For the simplity, $u(\lambda)$ is abbreviated to u

$$\begin{aligned} J(u + t\xi w_\beta) = \frac{1}{2}\, a(u,u) + a(t\xi w_\beta\,,\, t\xi w_\beta) + 2a(u, t\xi w_\beta) + \\ - \lambda \int_\Omega F(u + t\xi w_\beta)\, \mathrm{d}x\,. \end{aligned} \tag{3.41}$$

Since u is a solution of $(1.1)_\lambda$, there exists a nonnegative Radon measure μ such that

$$-\Delta u + u = \lambda f(u) + \mu\,, \tag{3.42}$$

$$\operatorname{supp}\mu \subset \Gamma\,, \quad \mu \le (A\psi)^+\,. \tag{3.43}$$

Therefore

$$a(u\,,\, t\xi w_\beta) = \lambda \int_\Omega t\xi w_\beta f(u)\,\mathrm{d}x + \langle \mu\,,\, t\xi w_\beta\rangle\,. \tag{3.44}$$

So we may rewrite (3.41) as

$$\begin{aligned} J(u + t\xi w_\beta) &= J(u) + J(t\xi w_\beta) + \lambda \int_\Omega f(u)(t\xi w_\beta)\,\mathrm{d}x + \\ &\quad + \langle \mu\,,\, t\xi w_\beta\rangle - \lambda \int_\Omega \big(F(u + t\xi w_\beta) - F(u) - F(t\xi w_\beta)\big)\,\mathrm{d}x \\ &= J(u) + J(t\xi w_\beta) + \langle \mu\,,\, t\xi w_\beta\rangle + \\ &\quad - \lambda \int_\Omega \int_0^{t\xi w_\beta} \big(f(u+s) - f(s) - f(u)\big)\,\mathrm{d}s\,\mathrm{d}x\,. \end{aligned}$$

It is clear that $a(w_\beta, w_\beta)$ is uniformly bounded in β. By Lemma 3.2

$$f(u+s) - f(s) - f(u) \ge 0\,,$$

thus by the continuity of J we find that there exists $t_1 > 0$ such that

$$J(u + t\xi w_\beta) < J(u) + S^\infty \quad \text{for } t \in [\,0\,,\, t_1)\,. \tag{3.45}$$

It can be showed by (f4) as in [16] that there exists $t_2 > 0$ such that for $t > t_2$

$$\begin{aligned} &F(t\xi w_\beta) \ge C(t\xi w_\beta)^\gamma \\ &\qquad \text{for } \beta > 2R_0 + 1\,,\ x \in B(\beta e, 1) = \Big\{ y - \beta e \;\Big|\; \|y\|_{H_0^1} < 1 \Big\}\,, \end{aligned}$$

where $\gamma = 1 + \theta^{-1} > 2$. In addition

$$\langle \mu\,,\, \xi w_\beta\rangle \le \int_\Omega (A\psi)^+ \xi w_\beta\,\mathrm{d}x < +\infty\,.$$

Thus for $t > t_2$, $\beta > 2R_0 + 1$

$$\begin{aligned}
J(u + t\xi w_\beta) &\leq J(u) + J(t\xi w_\beta) + \langle \mu\,,\, t\xi w_\beta \rangle \\
&= J(u) + \frac{1}{2}\, t^2 a(\xi w_\beta\,,\, \xi w_\beta) - \lambda \int_\Omega F(t\xi w_\beta)\, \mathrm{d}x + \langle \mu\,,\, t\xi w_\beta \rangle \\
&\leq J(u) + \frac{1}{2}\, t^2 a(\xi w_\beta\,,\, \xi w_\beta) - \lambda \int_{\Omega \setminus B_{2R_0}} F(t w_\beta)\, \mathrm{d}x \\
&\leq J(u) + \frac{1}{2}\, t^2 a(\xi w_\beta\,,\, \xi w_\beta) - C\lambda t^\gamma \int_{\Omega \setminus B_{R_0}} w_\beta^\gamma\, \mathrm{d}x + \\
&\quad + t \int_\Omega \big(A\psi\big)^+ w_\beta\, \mathrm{d}x\,,
\end{aligned}$$

thus we may choose $t_2 > 0$ large enough such that $t_2 > t_1$ and

$$J(u + t\xi w_\beta) < J(u) \quad \text{for } t \geq t_2\,,\ \beta > 2R_0 + 1\,. \tag{3.46}$$

The assertion then follows once we show

$$\sup_{t_1 \leq t \leq t_2} J(u + t\xi w_\beta) < J(u) + S^\infty\,. \tag{3.47}$$

It is known that

$$\frac{1}{2}\, t^2 \int_{\mathbb{R}^N} \big(|\nabla w_\beta|^2 + w_\beta^2\big)\, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} F(t w_\beta)\, \mathrm{d}x \leq S^\infty \tag{3.48}$$

with "=" holds if and only if $t = 1$. Hence for $\beta > 2R_0 + 1$, $t_1 \leq t \leq t_2$:

$$\begin{aligned}
J(t\xi w_\beta) &= \frac{1}{2}\, a(t\xi w_\beta\,,\, t\xi w_\beta) - \lambda \int_{\mathbb{R}^N} F(t\xi w_\beta)\, \mathrm{d}x \\
&= \frac{1}{2}\, t^2 \int_{\mathbb{R}^N} \xi^2 \big(|\nabla w_\beta|^2 + w_\beta^2\big)\, \mathrm{d}x + \frac{1}{2}\, t^2 \int_{\mathbb{R}^N} |\nabla \xi|^2 w_\beta^2\, \mathrm{d}x + \\
&\quad + t^2 \int_{\mathbb{R}^N} |\nabla \xi|\, |\nabla w_\beta| w_\beta\, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} F(t\xi w_\beta)\, \mathrm{d}x \\
&\leq \sup_{t>0} I^\infty(t w_\beta) + \frac{1}{2}\, t^2 \int_{\mathbb{R}^N} |\nabla \xi|^2 w_\beta^2\, \mathrm{d}x + \\
&\quad + t^2 \int_{\mathbb{R}^N} |\nabla \xi|\, |\nabla w_\beta| w_\beta\, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} \int_{t\xi w_\beta}^{t w_\beta} f(s)\, \mathrm{d}s\, \mathrm{d}x \\
&\leq S^\infty + C \int_{B_{2R_0}} \big(w_\beta^2 + w_\beta |\nabla w_\beta| + |w_\beta|^{p+1}\big)\, \mathrm{d}x\,.
\end{aligned}$$

By (3.28) and (3.29)

$$\int_{B_{2R_0}} \left(w_\beta^2 + w_\beta|\nabla w_\beta| + |w_\beta|^{p+1}\right) \mathrm{d}x \leq$$
$$\leq C \int_{\{|x|<2R_0\}} |x+\beta e|^{-(N-1)} \exp\bigl(-2|x+\beta e|\bigr)\, \mathrm{d}x$$
$$\leq C \int_{\{|x+\beta e|>\beta-2R_0\}} |x+\beta e|^{-(N-1)} \exp\bigl(-2|x+\beta e|\bigr)\, \mathrm{d}x$$
$$\leq C \int_{\{|y|>\beta-2R_0\}} |y|^{-(N-1)} \exp\bigl(-2|y|\bigr)\, \mathrm{d}y$$
$$\leq C \int_{\beta-2R_0}^{+\infty} \left(r^{-(N-1)} \exp(-2r)\right) r^{N-1}\, \mathrm{d}x \leq C\, e^{-2\beta}.$$

Hence

$$J(t\xi w_\beta) \leq S^\infty + C\, e^{-2\beta}. \tag{3.49}$$

Since u is a solution of $(1.1)_\lambda$, (2.8) holds for u. We may infer as [16] that for any $\delta > 0$ there exist $C_1, C_2, \beta_1 > 0$ such that if $\beta > \max\{\beta_1, R_1\}$,

$$\begin{aligned} \int_{B(\beta e,1)} \int_0^{t_1\xi w_\beta} \bigl(f(u+s) - f(u) - f(s)\bigr)\, \mathrm{d}s\, \mathrm{d}x &\geq \\ \geq C_1 \int_{B(\beta e,1)} t_1 u\, \mathrm{d}x & \\ \geq C_1 \int_{B(\beta e,1)} u\, \mathrm{d}x \geq C_2 \exp\bigl(-(1+\delta)\beta\bigr). & \end{aligned} \tag{3.50}$$

By (1.4) and (3.44):

$$\begin{aligned} \langle \mu\,,\, t\xi w_\beta\rangle &\leq t_2 \int_{\mathbb{R}^N} (A\psi)^+ \xi w_\beta\, \mathrm{d}x \\ &\leq C \int_{B(\beta e,1)} (A\psi)^+ w_\beta\, \mathrm{d}x \leq C_0\, e^{-\tau\beta}. \end{aligned} \tag{3.51}$$

Using (3.49)-(3.51) we obtain

$$\begin{aligned} J(u + t\xi w_\beta) = J(u) &+ J(t\xi w_\beta) + \langle \mu\,,\, t\xi w_\beta\rangle + \\ &- \int_{\mathbb{R}^N} \int_0^{t\xi w_\beta} \lambda\bigl(f(u+s) - f(s) - f(u)\bigr)\, \mathrm{d}s\, \mathrm{d}x \\ &\leq S^\infty + C\, e^{-2\beta} + C_0\, e^{-\tau\beta} - C_2\, e^{-(1+\delta)\beta}. \end{aligned}$$

Choosing $\delta < \tau - 1$ and β large enough, we obtain the result. □

An application of the deformation lemma due to Szulkin [14] for functionals of the form C^1 + convex-proper-lower semicontinous enables us to give a variant of the mountain pass theorem in ([2], [14]), it clarifies that if

(i) there exists an open set $\mathcal{B}$ such that $u(\lambda) \in \mathcal{B}$

$$J(u) \geq J\big(u(\lambda)\big)\,,\ \forall\ u \in \mathcal{B} \quad \text{and} \quad \inf_{\partial\mathcal{B}} J(u) > J\big(u(\lambda)\big)\ ;$$

(ii) there exists $e \overline{\in} \mathcal{B}$ such that $J(e) \leq J\big(u(\lambda)\big)$.

Then there exists a (P.S.)$_c$ sequence $\{u_n\} \subset \mathcal{K}_\lambda$ such that

$$I(u_n) \underset{n}{\rightarrow} c\,,$$

where $c := \inf_{\mathcal{J}} \sup_{0 \leq t \leq 1} J\big(\gamma(t)\big)$,

$$\mathcal{J} = \Big\{\gamma \in C\big([\,0\,,1\,]\,,\, H_0^1(\Omega)\big)\ \Big|\ \gamma(0) = u(\lambda)\,,\ J\big(\gamma(1)\big) < J\big(u(\lambda)\big)\Big\}\,.$$

For the functional $I_\lambda(u)$ defined by (2.1) we remark that if the variational problem

$$m_\lambda = \inf\big\{I_\lambda(u) \mid u \in K_\rho := K \cap B_\rho\big\} \tag{3.52}$$

has a minimizer u for some $\lambda_0 > 0$ with $u \in K_\rho$, then we know from the proof of Lemma 2.1 that it has a minimizer for all $\lambda \leq \lambda_0$ in $K_{\rho(\lambda)}$. Define

$$\lambda^{**} = \sup\big\{\lambda > 0 \mid \exists\ \rho(\lambda) > 0 \\ \text{and } u \text{ such that } I_\lambda(u) = m_\lambda \text{ with } u \in K_{\rho(\lambda)}\big\}\,.$$

According to Lemma 2.1 we know that $\lambda^{**} > 0$ and the minimizer of m_λ is a solution of $(1.1)_\lambda$, thus $\lambda^{**} \leq \lambda^*$. It will be interesting to know if λ^{**} equals to λ^*.

Proof of theorem 1

According to Lemma 2.5, Propositions 2.6 and 2.8, Theorem 1 will be proved if we can show that there exists another positive solution of $(1.1)_\lambda$ different from the minimal positive solution $u(\lambda)$.

For any $\lambda \in (0, \lambda^{**})$, let u be the minimizer of m_λ.

If $u(\lambda) < u$, the assertion follows.

If $u(\lambda) = u$, since u is a local minimum of I, then (i) of the mountain pass theorem is valid.

Set $e = tu$, then $e \in \mathcal{K}_\lambda$ and $e \overline{\in} \mathcal{K}_\lambda \cap B_\rho$ for $t > 0$ large enough. Moreover

$$J(tu) \to -\infty \quad \text{as } t \to +\infty .$$

Hence by the variant of the mountain pass theorem there exists a (P.S.)$_c$ sequence $\{u_n\} \subset \mathcal{K}_\lambda$ such that

$$I(u_n) \underset{n}{\longrightarrow} c \quad \text{as } n \to -\infty ,$$

where $c = \inf_{\mathcal{J}} \sup_{0 \le t \le 1} J\big(\gamma(t)\big)$,

$$\mathcal{J} = \Big\{ \gamma \in C\big([0,1], H_0^1(\Omega)\big) \;\Big|\; \gamma(0) = u(\lambda),\ J\big(\gamma(1)\big) < J\big(u(\lambda)\big) \Big\} .$$

Obviously we have $c > J(u)$.

By Lemma 3.1, Lemma 3.3 and Proposition 3.4, we may assume $u_n \underset{n}{\rightharpoonup} u_0$ weakly in $H_0^1(\Omega)$ and

$$c = I(u_0) + \sum_{i=1}^{m} I^\infty(u^i) , \tag{3.53}$$

where u^i, $1 \le i \le m$, are solutions of (3.25), u_0 is a solution of $(1.1)_\lambda$, $u_0 \ge u(\lambda)$.

We claim that $u_0 > u(\lambda)$.

Suppose by contradiction that $u_0 = u(\lambda)$. If $m = 0$, then $c = J\big(u(\lambda)\big)$ we get a contradiction; if $m > 0$, by (3.53) and Lemma 3.5

$$J(u) + S^\infty > c \ge S^\infty + J(u) ,$$

a contradiction. Hence the conclusion follows. □

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References

[1] AI JUN and ZHU XIPING .— *Positive solutions of elliptic obstacle problems*, Preprint.

[2] AMBROSETTI (A.) and RABINOWITZ (P. H.) .— *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), pp. 349-381.

[3] BENCI (V.) and CERAMI (G.) .— *Positive solutions of some nonlinear elliptic problems in unbounded domains*, Arch. Rational Mech. and Anal. **99** (1987), pp. 283-300.

[4] BERESTYCKI (H.) and LIONS (P. L.) .— *Nonlinear scalar field equations*, I and II, Arch. Rational Mech. and Anal. **82**, No 4 (1983), pp. 313-375.

[5] BREZIS (H.) and LIEB (E. H.) .— *A relation between pointwise convergence of functions and convergence of integrals*, Proc. Amer. Math. Soc. **88** (1983), pp. 486-490.

[6] KINDERLEHRER (D.) and STAMPACCHIA (G.) .— *An introduction to variational inequalities and their applications*, Academic Press, New York (1980).

[7] KWONG (M. K.) .— *Uniqueness of positive solution of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. and Anal. **105** (1989), pp. 243-266.

[8] LIONS (P. L.) .— *The concentration-compactness principle in the calculus of variations, the locally compact case, part 1 and part 2*, Ann. Inst. H.-Poincaré Anal. Non linéaire **1** (1984), pp. 109-145, 223-283.

[9] MANCINI (G.) and MUSINA (R.) .— *A free boundary problem involving limiting Sobolev exponents*, Manuscripta Math. **58** (1987), pp. 77-93.

[10] MANCINI (G.) and MUSINA (R.) .— *Holes and obstacles*, Ann. Inst. H.-Poincaré Anal. Non linéaire **5** (1988), pp. 323-345.

[11] RODRIGUES (J. F.) .— *Obstacle problems in mathematical physics*, Mathematics Studies **134**, The Netherlands (1987).

[12] STRAUSS (W.) .— *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), pp. 149-162.

[13] STUART (C. A.) .— *Bifurcation in* $L^p(\mathbb{R}^N)$ *for a semilinear elliptic equation*, Proc. London Math. Soc. **57** (1988), pp. 511-541.

[14] SZUKIN (A.) .— *Minimax principle for lower semicontinous functions and applications to nonlinear boundary value problems*, Ann. Inst. H.-Poincaré Anal. Non linéaire **3** (1986), pp. 77-109.

[15] YANG (J.F.) .— *Positive solutions of semilinear elliptic problems in exterior domains*, J. Diff. Equas. **106** (1993), pp. 40-69.

[16] ZHU (X. P.) and ZHOU (H. S.) .— *Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domains*, Proc. Royal Soc. Edinburg **115** A (1990), pp. 301-318.