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## Global stability of saddle-node bifurcation of a periodic orbit for vector fields<sup>(\*)</sup>

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**RÉSUMÉ.** — On étudie la stabilité d'une famille générique de champs de vecteurs ayant une orbite périodique de type col-nœud.

**ABSTRACT.** — In a bifurcation value, the global stability of families of vector fields which have a generically unfolding saddle-node periodic orbit is studied.

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### Introduction

In this paper we study the global stability of families of vector fields which have a saddle-node periodic orbit which unfolds generically. We recall that a generic characterization of stable families for the stability of one-parameter families of gradient vector fields was obtained by J. Palis and F. Takens [9]. We also recall that for one-parameter families of vector fields with simple recurrences and no-cycles, the global stability for those which have bifurcations due to quasi-transversal orbits was studied by R. Labarca [4] for the cases in which the bifurcation is due to a saddle-node (or Hopf) singularity or a flip periodic orbit, under generic conditions, the global stability follows from results of S. Newhouse, J. Palis and F. Takens [8]. When the bifurcation is due to a saddle-node periodic orbit, the global stability has only been studied in the case of two-dimensional manifolds by I. Malta and J. Palis [6]. Before stating our results, we recall some concepts and results on one-parameter families of vector fields.

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Let  $M$  be a  $C^\infty$  boundaryless compact manifold. We let  $\mathcal{X}^\infty(M)$  denote the space of  $C^\infty$  vector fields on  $M$  and  $\mathcal{X}_1^\infty(M)$  denote the space of  $C^\infty$  arcs,  $\xi : I = [-1, 1] \mapsto \mathcal{X}^\infty(M)$  both endowed with the  $C^\infty$  Whitney topology. We use the notation  $\xi = \{X_\mu\}$ , where for each  $\mu \in I$ ,  $\xi(\mu) = X_\mu$ . Now, let  $\{X_\mu\}, \{\tilde{X}_\mu\} \in \mathcal{X}_1^\infty(M)$  and  $\bar{\mu}, \tilde{\mu} \in I$ ; we say that  $\{X_\mu\}$  at  $\bar{\mu}$  is equivalent to  $\{\tilde{X}_\mu\}$  at  $\tilde{\mu}$  if there are homeomorphisms  $\rho : (I, \bar{\mu}) \mapsto (I, \tilde{\mu})$ , increasing, and  $H_\mu : M \mapsto M$ , which depends continuously on  $\mu$ , such that, for each  $\mu$  near  $\bar{\mu}$ ,  $H_\mu$  is a topological equivalence between  $X_\mu$  and  $\tilde{X}_{\rho(\mu)}$ , *i.e.*  $H_\mu$  sends orbits of  $X_\mu$  into orbits of  $\tilde{X}_{\rho(\mu)}$  preserving the sense of the trajectories. We say that  $\{X_\mu\}$  is stable at  $\bar{\mu}$  if there exists a neighborhood  $\mathcal{U} \subseteq \mathcal{X}_1^\infty(M)$  of  $\{X_\mu\}$  such that, for each  $\{\tilde{X}_\mu\} \in \mathcal{U}$ , there exists  $\tilde{\mu} \in I$  near  $\bar{\mu}$  such that  $\{X_\mu\}$  at  $\bar{\mu}$  is equivalent to  $\{\tilde{X}_\mu\}$  at  $\tilde{\mu}$ .

Let  $\Gamma \subseteq \mathcal{X}_1^\infty(M)$  be characterized by :

- for each  $\mu \in I$ , the non-wandering set of  $X_\mu$  is constituted by a finite number of critical elements (*i.e.* singularities and periodic orbits) of  $X_\mu$ ;
- there are no cycles among the critical elements of  $X_\mu$ , (*i.e.* there is no sequence  $\alpha_1, \dots, \alpha_k$ ,  $k \geq 1$ , of critical elements of  $X_\mu$  such that  $\alpha_k = \alpha_1$  and  $\mathcal{W}^u(\alpha_i) \cap \mathcal{W}^s(\alpha_{i+1}) \neq \emptyset$ ,  $i = 1, \dots, k - 1$ ).

Let  $\{X_\mu\} \in \Gamma$  and  $\bar{\mu} \in B(\{X_\mu\}) = \{\mu \in I \mid X_\mu \text{ is not stable}\}$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . Let  $\Sigma_q \subseteq M$  be a transversal section for  $X_{\bar{\mu}}$  at  $q \in \theta$ , and  $X(x, \mu) = (X_\mu(x), 0)$ . There exists an interval  $I_1 \subseteq I$ ,  $\bar{\mu} \in I_1$ , such that  $\Sigma = \Sigma_q \times I_1$  is a transversal section to  $X$  at  $(q, \bar{\mu})$ . We let  $\mathcal{P} : \Sigma \mapsto \Sigma$  denote the Poincaré map of  $X$  in the non-hyperbolic periodic orbit  $(\theta, \bar{\mu})$ , and  $\mathcal{P}_\mu = \mathcal{P}/\Sigma_q \times \{\mu\}$  denote the Poincaré map of  $X_\mu$ . Then  $\mathcal{P} = \{\mathcal{P}_\mu\}$  is an arc of saddle-node diffeomorphisms (see [8]). From the theory of invariant manifolds (see [3]) we have that there exists  $C^r$ ,  $1 \leq r < \infty$ ,  $\mathcal{P}$ -invariant manifolds  $\mathcal{W}^{cs}(q)$ ,  $\mathcal{W}^{cu}(q)$  and  $\mathcal{W}^c(q)$ . Set

$$\mathcal{W}_\mu^{cs}(q) = \mathcal{W}^{cs}(q) \cap \Sigma_q \times \{\mu\};$$

analogously for  $\mathcal{W}_\mu^{cu}(q)$  and  $\mathcal{W}_\mu^c(q)$ . Let

$$\mathcal{W}^s(q) = \{(x, \bar{\mu}) \in \Sigma_q \times \{\bar{\mu}\} \mid \mathcal{P}^n(x, \bar{\mu}) \mapsto (q, \bar{\mu}), n \mapsto +\infty\}$$

and

$$\mathcal{W}^u(q) = \{(x, \bar{\mu}) \in \Sigma_q \times \{\bar{\mu}\} \mid \mathcal{P}^{-n}(x, \bar{\mu}) \mapsto (q, \bar{\mu}), n \mapsto +\infty\}$$

be the stable and unstable manifolds for  $q$ , respectively. We have that

$$\mathcal{W}^s(q) \subseteq \mathcal{W}_{\bar{\mu}}^{cs}(q) \quad \text{and} \quad \mathcal{W}^u(q) \subseteq \mathcal{W}_{\bar{\mu}}^{cu}(q)$$

are  $C^r$ ,  $r \geq 1$ , injectively immersed manifolds of  $M$  with boundary

$$\partial\mathcal{W}^s(q) = \mathcal{W}^{ss}(q), \quad \partial\mathcal{W}^u(q) = \mathcal{W}^{uu}(q),$$

are called strong stable and strong unstable manifolds, respectively. In addition, in  $\mathcal{W}^s(q)$  there exists a (unique)  $\mathcal{P}$ -invariant  $C^1$  codimension one foliation  $\mathcal{F}^{ss}(q)$  with space of leaves  $\mathcal{W}_{\bar{\mu}}^c(q)$ ; analogously, there exists a foliation  $\mathcal{F}^{uu}(q)$  in  $\mathcal{W}^u(q)$ . Let  $\alpha$  be a hyperbolic critical element of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  intersects  $\mathcal{W}^s(q)$ , we say that  $\alpha$  is  $s$ -critical, or that  $\mathcal{W}^u(\alpha)$  has  $s$ -criticalities if  $\mathcal{W}^u(\alpha) \cap \Sigma_q \times \{\bar{\mu}\}$  has a non-transversal intersection with some leaf of  $\mathcal{F}^{ss}(q)$ ; analogously, we define  $u$ -criticalities. We say that the  $s$ -criticality between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  is generic if there exists a unique tangency orbit between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  along which they have a quasi-transversal contact.

Excepting certain additional conditions which will be specified later, our results on the stability of families  $\{X_{\mu}\} \in \Gamma$  which have saddle-node periodic orbits are the following.

PROPOSITION 1

- (a) *If there exists a hyperbolic critical element  $\alpha$  of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has non-generic  $s$ -criticalities with  $\mathcal{F}^{ss}(q)$ , then  $\{X_{\mu}\}$  is non-stable at  $\bar{\mu}$ .*
- (b) *If there exists two hyperbolic critical elements  $\alpha_1, \alpha_2$  of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha_1)$  and  $\mathcal{W}^u(\alpha_2)$  have generic  $s$ -criticalities with  $\mathcal{F}^{ss}(q)$ , then  $\{X_{\mu}\}$  is non-stable at  $\bar{\mu}$ .*

THEOREM 1

- (a) *If  $\alpha$  is a hyperbolic periodic orbit of  $X_{\bar{\mu}}$  such that there exists the weakest contraction in  $\alpha$  and  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality, then  $\{X_{\mu}\}$  is non-stable at  $\bar{\mu}$ . Moreover, in this case the weakest contraction in  $\alpha$  is a modulus of stability for  $X_{\bar{\mu}}$ .*
- (b) *If  $\alpha$  is a hyperbolic singularity with complex weakest contraction of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality, then  $\{X_{\mu}\}$  is non-stable at  $\bar{\mu}$ . Moreover, if  $a \pm ib$ ,  $a < 0$ ,  $b \neq 0$ , denotes the weakest contraction in  $\alpha$ , then  $\rho = a/b$  is modulus of stability for  $X_{\bar{\mu}}$ .*

**THEOREM 2**

- (a) *If the hyperbolic critical elements of  $X_{\bar{\mu}}$  have no criticalities, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$ .*
- (b) *If there exists a unique hyperbolic singularity  $\alpha$  which has real weakest contraction and  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality, and the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  have no criticalities, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$ .*
- (c) *If there exist hyperbolic singularities  $\alpha, \beta$  ( $\alpha \neq \beta$ ) of  $X_{\bar{\mu}}$  such that the weakest contraction in  $\alpha$  and the weakest expansion in  $\beta$  are real, and  $\mathcal{W}^u(\alpha)$ , respectively  $\mathcal{W}^s(\beta)$ , has a generic  $s$ -criticality with  $\mathcal{F}^{ss}(q)$ , respectively has a generic  $u$ -criticality with  $\mathcal{F}^{uu}(q)$  and the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  have no criticalities, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$ .*

From the methods developed in the proof of Theorem 2 (Section 3) and in [6], the following result follows.

**THEOREM 3.** — *If there exist hyperbolic singularities*

$$\alpha_1, \dots, \alpha_k, \quad \beta_1, \dots, \beta_\ell$$

*such that the weakest contractions in the  $\alpha_i$ 's and the weakest expansions in the  $\beta_j$ 's are real, and  $\mathcal{W}^u(\alpha_s)$  for some  $1 \leq s \leq k$  (resp.  $\mathcal{W}^s(\beta_r)$  for some  $1 \leq r \leq \ell$ ) has a generic  $s$ -criticality, respectively has a generic  $u$ -criticality, and the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  have no criticalities, then  $\{X_{\mu}\}$  has finite modulus of stability.*

### 1. Basic concepts

In this section we will recall some basic concepts and state the above results in greater detail. Let  $M$  be a  $C^\infty$  boundaryless compact manifold. We will denote by  $\mathcal{X}^\infty(M)$  the space of  $C^\infty$  vector fields on  $M$  endowed with the  $C^\infty$  Whitney topology. Let  $x \in M$  be a singularity of  $X \in \mathcal{X}^\infty(M)$ , i.e.  $X(x) = 0$ , we say  $x$  is hyperbolic if  $DX(x)$  has no eigenvalues with null real part. Let  $\sigma$  be a periodic orbit of  $X$  and  $\Sigma_q$  be a transversal section to  $X$  at  $q \in \sigma$  and  $\mathcal{P} : (\Sigma_q, q) \mapsto (\Sigma_q, q)$  be the Poincaré map for  $X$  at  $\sigma$ , we say  $\sigma$  is hyperbolic if  $D\mathcal{P}(q)$  has no eigenvalues with norm equaling one. The singularities and periodic orbits of  $X$  will be called critical elements of  $X$ .

Let  $\theta$  be a hyperbolic critical element of  $X$ , then

$$\mathcal{W}^s(\theta) = \{x \in M \mid X_t(x) \mapsto \theta, t \mapsto +\infty\}$$

and

$$\mathcal{W}^u(\theta) = \{x \in M \mid X_t(x) \mapsto \theta, t \mapsto -\infty\}$$

( $X_t$  denotes the flow of  $X$ ) are  $C^\infty$  submanifolds injectively immersed in  $M$  called stable and unstable manifolds of  $\theta$ , respectively.

Let  $x \in M$ , we say  $x$  is a non-wandering point of  $X$  if, for each neighborhood  $U$  of  $x$  and each  $t_0 > 0$ , there exists  $t_1 > t_0$  such that

$$X_{t_1}(U) \cap U \neq \emptyset.$$

We let  $\Omega(X)$  denote the set of non-wandering points of  $X$ . If  $\Omega(X)$  is constituted by a finite number of critical elements of  $X$ , we say  $X$  has simple recurrences. The interior of the set of the vector fields with simple recurrences will be denoted by  $WR^\infty(M)$ .

Let  $\alpha, \beta$  be critical elements of  $X$ , we say  $\mathcal{W}^u(\alpha)$  and  $\mathcal{W}^s(\beta)$  are transversal if

$$T_x\mathcal{W}^u(\alpha) + T_x\mathcal{W}^s(\beta) = T_xM,$$

for each  $x \in \mathcal{W}^u(\alpha) \cap \mathcal{W}^s(\beta)$ . A cycle for  $X$  is a sequence  $\alpha_1, \dots, \alpha_k$ ,  $k \geq 1$ ,  $\alpha_k = \alpha_1$ , of critical elements of  $X$  such that  $\mathcal{W}^u(\alpha_i) \cap \mathcal{W}^s(\alpha_{i+1}) \neq \emptyset$ ,  $i = 1, \dots, k-1$ . We let  $WC^\infty(M)$  denote the set of vector fields which do not have cycles among their critical elements.

Let  $\sigma \in M$  be a hyperbolic singularity (resp. periodic orbit) of  $X \in \mathcal{X}^\infty(M)$ , we say the weakest contraction at  $\sigma$  is defined if among the contractive eigenvalues of  $DX(\sigma)$  (resp.  $D\mathcal{P}(q)$ ,  $q \in \sigma$ ,  $\mathcal{P}$  the Poincaré map) the one with biggest real part (resp. norm) is simple.

We let  $\mathcal{X}_1^\infty(M)$  denote the space of  $C^\infty$  arcs,

$$\xi : I = [-1, 1] \mapsto \mathcal{X}^\infty(M)$$

endowed with the  $C^\infty$  Whitney topology, we use the notation  $\xi = \{X_\mu\}$ ,  $\xi(\mu) = X_\mu$ .

We say  $\bar{\mu} \in I$  is a bifurcation value for  $\{X_\mu\}$  if  $X_{\bar{\mu}}$  is non-stable. We set

$$B(\{X_\mu\}) = \{\bar{\mu} \in I \mid \bar{\mu} \text{ is a bifurcation value for } \{X_\mu\}\}.$$

**DEFINITION 1.1.** — Let  $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a non-hyperbolic periodic orbit  $\theta$ , we say  $\theta$  is a saddle-node periodic orbit which unfolds generically at  $\mu = \bar{\mu}$  if there exists a  $C^r$  ( $1 \leq r < \infty$ ),  $\mu$ -depending center manifold  $\mathcal{W}^c(q)$  through  $q \in \theta$  ( $\dim \mathcal{W}^c(q) = 2$ ) such that the Poincaré map  $\mathcal{P}(x, \mu) = (\mathcal{P}_\mu(x), \mu)$  associated to the vector field  $X(x, \mu) = (X_\mu(x), 0)$  in  $(\theta, \bar{\mu})$  restricted to  $\mathcal{W}^c(q)$  has the form

$$\begin{aligned} \mathcal{P}_\mu(x) &= b(\mu - \bar{\mu}) + v_1(\mu) + v_2(\mu)x^2 + O(x^2), \\ v_1(\bar{\mu}) &= 1, \quad \left. \frac{dv_1(\mu)}{d\mu} \right|_{\mu=\bar{\mu}} \neq 0, \quad b \neq 0, \quad v_2(\mu) \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

**DEFINITION 1.2.** — Let  $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a non-transversal intersection orbit,  $\gamma \subseteq \mathcal{W}^u(\alpha) \cap \mathcal{W}^s(\beta)$ ,  $\alpha, \beta$  hyperbolic critical elements of  $X_{\bar{\mu}}$ . We say  $\gamma$  is a quasi-transversal intersection orbit which unfolds generically at  $\mu = \bar{\mu}$  for  $\{X_\mu\}$  if for each  $x \in \gamma$  there exist  $C^\infty$ ,  $\mu$ -depending coordinates,  $\psi_\mu : (V, x) \mapsto (\mathbb{R}^n, 0)$ ,  $V$  a neighborhood of  $x$  ( $n = \dim M$ ) such that:

- (a)  $(\psi_\mu)_*(X_\mu) = \frac{\partial}{\partial y_n}$ ;
- (b)  $\psi_\mu(\mathcal{W}^u(\alpha_\mu) \cap V) = \{y_1 = \dots = y_{n-u} = 0\}$ ;
- (c) if  $n - u = s$ ,

$$\psi_\mu(\mathcal{W}^s(\beta_\mu) \cap V) = \{y_1 = \varepsilon(\mu), y_{s+1} = \dots = y_{n-1} = 0\};$$

if  $n - u \leq s - 1$ ,

$$\begin{aligned} \psi_\mu(\mathcal{W}^s(\beta_\mu) \cap V) &= \\ &= \{y_1 = Q(y_{n-u}, \dots, y_s) + \varepsilon(\mu), y_{s+1} = \dots = y_{n-1} = 0\}, \end{aligned}$$

where  $u = \dim \mathcal{W}^u(\alpha)$ ,  $s = \dim \mathcal{W}^s(\beta)$  and for each  $\mu$  near  $\bar{\mu}$ ,  $\alpha_\mu$  and  $\beta_\mu$  denote the hyperbolic critical elements of  $X_\mu$  near  $\alpha$  and  $\beta$ , respectively,  $Q$  is a Morse function and  $\varepsilon$  is a  $C^\infty$  function such that

$$\varepsilon(\bar{\mu}) = 0 \quad \text{and} \quad \left. \frac{d\varepsilon(\mu)}{d\mu} \right|_{\mu=\bar{\mu}} \neq 0.$$

Now let  $\Gamma_{sn} = \Gamma_{sn}(M) \subseteq \mathcal{X}_1^\infty(M)$  be the set of arcs  $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$  such that:

- $B(\{X_\mu\}) \subseteq ]-1, 1[$  is at most countable;
- for each  $\bar{\mu} \in B(\{X_\mu\})$ ,  $X_{\bar{\mu}}$  has a unique orbit  $\theta_{\bar{\mu}}$  along which it is non-stable in one of the following senses:
  - (a) if  $\theta_{\bar{\mu}}$  is a non-hyperbolic periodic orbit of  $X_{\bar{\mu}}$ , then  $\theta_{\bar{\mu}}$  is a saddle-node periodic orbit unfolding generically at  $\mu = \bar{\mu}$ ,
  - (b) if  $\theta_{\bar{\mu}}$  is a non-transversal intersection orbit of an unstable and a stable manifolds of hyperbolic critical elements of  $X_{\bar{\mu}}$ , then  $\theta_{\bar{\mu}}$  is a quasi-transversal intersection orbit unfolding generically at  $\mu = \bar{\mu}$ ;
- for each  $\mu \in I$ ,  $X_\mu \in WR^\infty(M) \cap WC^\infty(M)$ .

Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta_{\bar{\mu}}$ , we have the following theorem.

**THEOREM (I. Malta, J. Palis [6]).** — *Assume  $\dim M = 2$ . Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  as above. Then:*

- (a) *if there exists more than one saddle separatrix (stable or unstable manifold of a saddle) accumulating on the same side of  $\theta_{\bar{\mu}}$ , then  $\{X_\mu\}$  is non-stable at  $\bar{\mu}$ ; moreover, under generic conditions,  $X_{\bar{\mu}}$  has finite stability modulus;*
- (b) *if there exists at most one saddle separatrix accumulating at  $\theta_{\bar{\mu}}$ , then  $\{X_\mu\}$  is stable at  $\bar{\mu}$ ;*
- (c) *if there exists at most one unstable saddle separatrix and one stable saddle separatrix of  $X_{\bar{\mu}}$  accumulating at  $\theta_{\bar{\mu}}$  then  $\{X_\mu\}$  is stable at  $\bar{\mu}$ .*

In this work we generalize the above theorem for  $\dim M > 2$ .

Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has saddle-node periodic orbit  $\theta$ . Let  $\Sigma_q \subseteq M$  be a transversal section to  $X_{\bar{\mu}}$  at  $q \in \theta$  and  $I_1 \subseteq I$  be a neighborhood of  $\bar{\mu}$ . We let  $\mathcal{P} : \Sigma_q \times I_1 \mapsto \Sigma_q \times I_1$  be the Poincaré map of the vector field  $X(x, \mu) = (X_\mu(x), 0)$  at  $(\theta, \bar{\mu})$ . Set  $\mathcal{P}_\mu = \mathcal{P}/\Sigma_q \times \{\mu\}$ ; then  $\mathcal{P} = \{\mathcal{P}_\mu\}$  is an arc of saddle-node diffeomorphisms (see [8]).

In what follows, we will assume that for  $\mu < \bar{\mu}$  there exist two hyperbolic periodic orbits,  $\theta_{1,\mu}, \theta_{2,\mu}$ , of  $X_\mu$  which collapse at  $\mu = \bar{\mu}$  originating the saddle-node periodic orbit  $\theta = \theta_{1,\bar{\mu}} = \theta_{2,\bar{\mu}}$  of  $X_{\bar{\mu}}$ , and disappearing for  $\mu > \bar{\mu}$ . Thus  $\mathcal{P} = \{\mathcal{P}_\mu\}$  has, for  $\mu < \bar{\mu}$ , two hyperbolic fixed points  $q_{1,\mu}, q_{2,\mu}$ , which collapse at  $\mu = \bar{\mu}$  originating the saddle-node fixed point  $q = q_{1,\bar{\mu}} = q_{2,\bar{\mu}}$  of  $\mathcal{P}_{\bar{\mu}}$  and disappearing for  $\mu > \bar{\mu}$ .

From [8] it follows that for  $\mathcal{P} = \{\mathcal{P}_\mu\}$  we have, in  $\Sigma_q \times I_1$ , there are  $C^r$  ( $1 \leq r < \infty$ )  $\mathcal{P}$ -invariant submanifolds  $\mathcal{W}^c(q)$ ,  $\mathcal{W}^{cs}(q)$  and  $\mathcal{W}^{cu}(q)$ ,

$$\mathcal{W}^c(q) \subseteq \mathcal{W}^{cs}(q) \cap \mathcal{W}^{cu}(q),$$

called center, center-stable and center-unstable manifolds respectively. In addition, in  $\mathcal{W}^{cs}(q)$ , there exists a  $C^r$  ( $1 \leq r \leq \infty$ )  $\mathcal{P}$ -invariant, strong stable foliation  $\mathcal{F}^{ss}(q)$ . Analogously in  $\mathcal{W}^{cu}(q)$ , there exists a strong unstable foliation  $\mathcal{F}^{uu}(q)$ . Furthermore at  $\mu = \bar{\mu}$ ,  $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$  is unique and should be preserved by conjugations of arcs of saddle-node diffeomorphisms; similarly for  $\mathcal{F}^{uu}(q)/\mathcal{W}^u(q)$ . This is a necessary condition (rigidity) for the construction of conjugations of arcs of saddle-node diffeomorphisms and, therefore, a necessary condition (rigidity) for the construction of equivalences for arcs in  $\Gamma_{sn}$  for the parameter values in which there are saddle-node periodic orbits.

We next consider the arc of diffeomorphisms  $\mathcal{P}^c = \{\mathcal{P}_\mu^c\}$  where

$$\mathcal{P}_\mu^c = \mathcal{P}/\mathcal{W}_\mu^c(q) : \mathcal{W}_\mu^c(q) \mapsto \mathcal{W}_\mu^c(q).$$

Thus  $\mathcal{P}^c = \{\mathcal{P}_\mu^c\}$  is an arc of saddle-node diffeomorphisms in  $\mathbb{R}$ . Therefore, from [12], there exists a unique  $C^\infty$  vector field  $Z$  defined in a neighborhood of  $(q, \bar{\mu})$  in  $\mathcal{W}_{\bar{\mu}}^c(q)$  such that

$$\mathcal{P}_{\bar{\mu}}^c = Z_{t=1}. \quad (1)$$

And, in addition, if  $\{\tilde{X}_\mu\} \in \Gamma_{sn}$  and  $\tilde{\mu} \in I$  have the same characteristics as  $\{X_\mu\}$  and  $\bar{\mu}$ . Respectively, we denote by  $\tilde{\mathcal{P}}^c = \{\tilde{\mathcal{P}}_\mu^c\}$  the corresponding arc of saddle-node diffeomorphisms in  $\mathbb{R}$ , and by  $\tilde{Z}$  the unique  $C^\infty$  vector field defined in a neighborhood of  $(\tilde{q}, \tilde{\mu})$  in  $\mathcal{W}_{\tilde{\mu}}^c(\tilde{q})$ . If  $h : \mathcal{W}^c(q) \mapsto \mathcal{W}^c(\tilde{q})$  is a conjugation between  $\mathcal{P}^c = \{\mathcal{P}_\mu^c\}$  and  $\tilde{\mathcal{P}}^c = \{\tilde{\mathcal{P}}_\mu^c\}$ ,  $h = (h_\mu, \rho)$ ,  $\rho : (I, \bar{\mu}) \mapsto (I, \tilde{\mu})$  reparametrization,  $h_\mu$  conjugation between  $\mathcal{P}_\mu^c$  and  $\tilde{\mathcal{P}}_{\rho(\mu)}^c$ , then

$$h_{\bar{\mu}} : \mathcal{W}_{\bar{\mu}}^c(q) \mapsto \mathcal{W}_{\tilde{\mu}}^c(\tilde{q})$$

is a conjugation between the flows  $Z_t$  and  $\tilde{Z}_t$ ; this is another rigidity condition for the construction of equivalences for arcs in  $\Gamma_{sn}$  in the parameter values in which there are saddle-node periodic orbits.

**DEFINITION 1.3.** — Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has saddle-node periodic orbit  $\theta$ . We say  $\theta$  is  $s$ -critical if there exists a hyperbolic critical element  $\alpha$  of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has a non-transversal intersection with some leaf of  $\mathcal{F}^{ss}(q)$ , and we say this  $s$ -criticality is generic if there exists a unique tangency orbit between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  along which they have a quasi-transversal contact.

**PROPOSITION 1.** — Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has saddle-node periodic orbit  $\theta$ . Then:

- (a) if  $\alpha$  is a hyperbolic critical element of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has non-generic  $s$ -criticalities, then  $\{X_\mu\}$  is non-stable at  $\bar{\mu}$ ;
- (b) if  $\alpha_1, \alpha_2$  ( $\alpha_1 \neq \alpha_2$ ) are hyperbolic critical elements of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha_1)$  and  $\mathcal{W}^u(\alpha_2)$  have generic  $s$ -criticalities, then  $\{X_\mu\}$  is non-stable at  $\bar{\mu}$ .

**THEOREM 1.** — Let  $\{X_\mu\} \in \Gamma_{sn}$  and  $\bar{\mu} \in B(\{X_\mu\})$  be as in Proposition 1.

- (a) If  $\alpha$  is a hyperbolic periodic orbit of  $X_{\bar{\mu}}$  such that there exists the weakest contraction in  $\alpha$  and  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality, then  $\{X_\mu\}$  is non-stable at  $\bar{\mu}$ . Moreover, if  $A$  denotes the weakest contraction of  $X_{\bar{\mu}}$  in  $\alpha$ , then  $A$  is stability modulus for  $X_{\bar{\mu}}$ .
- (b) If  $\alpha$  is a hyperbolic singularity with complex weakest contraction of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality, then  $\{X_\mu\}$  is non-stable at  $\bar{\mu}$ . Moreover, if  $a \pm ib$ ,  $a < 0$ ,  $b \neq 0$ , denotes the weakest contraction in  $\alpha$ , then  $\rho = a/b$  is modulus of stability for  $X_{\bar{\mu}}$ .

We next impose some conditions on the families  $\{X_\mu\} \in \Gamma_{sn}$  we are considering:

**SN1** If  $\alpha$  is a saddle-type hyperbolic critical element of  $X_{\bar{\mu}}$  ( $\alpha \neq \theta$ ) and  $\mathcal{W}^u(\alpha) \cap \mathcal{W}^s(q) \neq \emptyset$  then  $\mathcal{W}^u(\alpha)$  is transversal to  $\mathcal{W}^{ss}(q)$  and  $\mathcal{W}^u(\alpha) \cap \mathcal{W}_{\bar{\mu}}^c(q) = \emptyset$ ; analogously for  $\mathcal{W}^s(\alpha)$ .

**SN2** Let  $\alpha$  be a hyperbolic critical element of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality with  $\mathcal{F}^{ss}(q)$ ; then:

- $\mathcal{W}^u(\alpha)$  is transversal to  $\mathcal{W}^s(q)$  in  $\Sigma_q \times \{\bar{\mu}\}$ ; moreover, there exists  $C^r$ -linearisations ( $r \geq 2$ ) coordinates in a neighborhood of  $\alpha$ ;
- the weakest contraction is defined in  $\alpha$  which is real;

- there exists a  $C^r$  ( $r \geq 1$ ) center-unstable manifold  $\mathcal{W}^{cu}(\alpha)$  which is transversal to  $\mathcal{F}^{ss}(q)$  in a neighborhood of the  $s$ -criticality; analogously for  $\mathcal{W}^s(\alpha)$ ,  $\mathcal{W}^{cs}(\alpha)$  and  $\mathcal{F}^{uu}(q)$ .

SN3 Let  $\alpha$  be a hyperbolic critical element of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has an  $s$ -criticality, and  $\mathcal{D}_{\bar{\mu}}^{cs}(q)$  be a fundamental domain for  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cs}(q)$ . Then the  $s$ -criticality between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  is generic and the tangency is a unique point.

We let  $\Gamma_{sn}^1 \subseteq \Gamma_{sn}$  denote the set characterized by  $\{X_{\mu}\} \in \Gamma_{sn}^1$  if and only if, for each  $\bar{\mu} \in B(\{X_{\mu}\})$ ,  $X_{\bar{\mu}}$  has a unique saddle-node periodic orbit, and the conditions SN1, SN2 and SN3 are satisfied.

**THEOREM 2.** — *Let  $\{X_{\mu}\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_{\mu}\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ , we have:*

- if  $\theta$  is not  $s$ -critical nor  $u$ -critical, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$  (in this case  $\bar{\mu}$  is isolated);*
- if there exists a unique hyperbolic singularity  $\sigma$  of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\sigma)$  is  $s$ -critical, and the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  do not have criticalities, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$  (in this case  $\bar{\mu}$  is isolated);*
- if there exist hyperbolic singularities  $\sigma_1, \sigma_2$  ( $\sigma_1 \neq \sigma_2$ ) of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\sigma_1)$  is  $s$ -critical and  $\mathcal{W}^s(\sigma_2)$  is  $u$ -critical, and if the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  do not have criticalities, then  $\{X_{\mu}\}$  is stable at  $\bar{\mu}$ ; in this case there exists a strictly monotone sequence  $(\bar{\mu}_n)$  of parameter values which converges to  $\bar{\mu}$  such that, for each  $n \in \mathbb{N}$ , the vector field  $X_{\bar{\mu}_n}$  has a unique orbit of quasi-transversal intersection between  $\mathcal{W}^s(\sigma_1, \bar{\mu}_n)$  and  $\mathcal{W}^s(\sigma_2, \bar{\mu}_n)$  which unfolds generically at  $\mu = \bar{\mu}$ .*

Now let  $\{X_{\mu}\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_{\mu}\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . Suppose that there exist hyperbolic critical elements  $\alpha_1, \dots, \alpha_m$  and  $\beta_1, \dots, \beta_k$  of  $X_{\bar{\mu}}$  such that, for each  $i = 1, \dots, m$  (resp.  $j = 1, \dots, k$ ),  $\mathcal{W}^u(\alpha_i)$  (resp.  $\mathcal{W}^s(\beta_j)$ ) has a generic  $s$ -criticality (resp.  $u$ -criticality). We let  $\pi^{ss}$  (resp.  $\pi^{uu}$ ) denote the projection on  $\mathcal{W}_{\bar{\mu}}^{cs}(q)$  via leaves of  $\mathcal{F}^{ss}(q)$  (resp.  $\mathcal{F}^{uu}(q)$ ) and  $x_1, \dots, x_m$  (resp.  $y_1, \dots, y_k$ ) denote the points of  $s$ -criticality (resp.  $u$ -criticality) between  $\mathcal{W}^u(\alpha_i)$  and  $\mathcal{F}^{ss}(q)$  (resp.  $\mathcal{W}^s(\beta_j)$  and  $\mathcal{F}^{uu}(q)$ ),  $i = 1, \dots, m$  (resp.  $j = 1, \dots, k$ ), in a fundamental domain for  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cs}(q)$  (resp.  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cu}(q)$ ). Let  $\pi^{ss}(x_i) = x_i^c$  (resp.  $\pi^{uu}(y_j) = y_j^c$ )  $i = 1, \dots, m$  (resp.  $j = 1, \dots, k$ ). We impose the following conditions.

Global stability of saddle-node bifurcation of a periodic orbit for vector fields

- $x_i^c \neq x_\ell^c$  and  $y_j^c \neq y_s^c$   $i, \ell = 1, \dots, m, j, s = 1, \dots, k, i \neq \ell, j \neq s$ .
- $x_i^c, y_j^c$  belong to the interior of  $D_{\bar{\mu}}^{cs}(q), D_{\bar{\mu}}^{cu}(q)$  respectively,  $i = 1, \dots, m, j = 1, \dots, k$ .

We now define the numbers  $t_i, s_j$  ( $i = 1, \dots, m, j = 1, \dots, k$ ) by the equations

$$Z_{t_i}(x_1^c) = x_i^c \quad \text{and} \quad Z_{s_j}(y_1^c) = y_j^c.$$

We furthermore suppose that the remaining hyperbolic critical elements of  $X_{\bar{\mu}}$  do not have criticalities.

**THEOREM 3.** — *Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  be as above. Suppose that*

$$|t_i - t_\ell| \neq |s_s - s_r| \quad \text{and that} \quad 1 - |t_i - t_j| \neq |s_s - s_r|,$$

*for each  $i, \ell = 1, \dots, m, i \neq \ell$  and each  $j, r = 1, \dots, k, j \neq r$ ; then  $\{X_\mu\}$  has finite stability modulus.*

*Remark.* — In the two-dimensional case Theorem 3 was proved by I. Malta and J. Palis [6]. For  $\dim M \geq 3$ , the proof of Theorem 3 follows from the proof of Theorem 2 and the two-dimensional case.

## 2. Proof of Proposition 1 and Theorem 1

### Proof of Proposition 1

Let  $\Sigma \subseteq M$  be a cross section to  $X_{\bar{\mu}}$  at  $q \in \theta$ . We let  $\pi^{ss} : \mathcal{W}^{cs}(q) \mapsto \mathcal{W}^c(q)$  denote the projection through leaves of  $\mathcal{F}^{ss}(q)$ .

*Case (a)* Let  $x$  a point of non-generic  $s$ -criticality. We now make a small perturbation of  $\mathcal{W}^u(\alpha)$  in a small neighborhood of  $x$  in  $\mathcal{W}^s(q)$  in such a way as to produce generic  $s$ -criticalities among  $\mathcal{W}^u(\alpha)$  and two or more leaves of  $\mathcal{F}^{ss}(q)$ .

*Case (b)* Let  $x_1, x_2$  be the points of generic  $s$ -criticalities of  $\mathcal{W}^u(\alpha_1)$  and  $\mathcal{W}^u(\alpha_2)$  with  $\mathcal{F}^{ss}(q)$ , respectively. Again, we make a small perturbation of  $\mathcal{W}^u(\alpha_2)$  in a small neighborhood of  $x_2$  in  $\mathcal{W}^s(q)$  such that the new  $s$ -criticality  $x'_2$  is in leaf  $\mathcal{F}^{ss}(x'_2)$ , with

$$\mathcal{F}^{ss}(x'_2) \neq \mathcal{F}^{ss}(x_2)$$

(we note that we may suppose  $\pi^{ss}(x_1) \neq \pi^{ss}(x_2)$ ).

In cases (a) and (b) these perturbations may be obtained as perturbations of  $\{X_\mu\}$ . Therefore, we obtain a family  $\{\tilde{X}_\mu\} \in \Gamma_{sn}$  near  $\{X_\mu\}$ ; it is clear that they are not equivalent by the rigidity condition on  $\mathcal{W}^c(q)$ .

### Proof of Theorem 1

In part (a) as well as in part (b) of the theorem, the unstability of  $\{X_\mu\}$  at  $\bar{\mu}$  follows from the existence of modulus of stability for  $X_{\bar{\mu}}$  (which we will prove). In what follows we will write  $Y$  for  $X_{\bar{\mu}}$ .

*Part (a)* Let  $\tilde{Y}$  be a vector field near  $Y$  with the same characteristics of  $Y$ . We let  $\tilde{A}$  denote the weakest contraction of the hyperbolic periodic orbit  $\tilde{\alpha}$  of  $\tilde{Y}$ . Each time we make a construction for  $Y$ , we will suppose it made for  $\tilde{Y}$ .

Let  $\Sigma_{q_1}$  be a cross section for  $Y$  at  $q_1 \in \alpha$  and  $\mathcal{P}_{q_1} : (\Sigma_{q_1}, q_1) \mapsto (\Sigma_{q_1}, q_1)$  be the Poincaré map of  $Y$  at  $\alpha$ . Similarly, let  $\Sigma_q$  be a cross section of  $Y$  at  $q \in \theta$  and  $\mathcal{P}_q : (\Sigma_q, q) \mapsto (\Sigma_q, q)$  be the Poincaré map of  $Y$  at  $\theta$ . We let  $D^{cs}(q)$  denote a fundamental domain for  $\mathcal{P}_q$  in  $\mathcal{W}^{cs}(q)$ . Let  $x$  be the point of generic  $s$ -criticality between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  in  $D^{cs}(q)$ . Suppose

$$x^c = \pi^{ss}(x) \in \text{Int}(D^{cs}(q) \cap \mathcal{W}^c(q)).$$

If  $H$  is a topological equivalence between  $Y$  and  $\tilde{Y}$  in a neighborhood  $V$  of the closure of  $O_Y(x)$  in  $M$ , then without loss of generality, we may suppose that  $H(\Sigma_{q_1}) = \Sigma_{q_1}^-$  and that  $H(\Sigma_q) = \Sigma_q^-$ . Then

$$(H/\Sigma_{q_1}) \circ \mathcal{P}_{q_1} = \mathcal{P}_{q_1}^- \circ (H/\Sigma_{q_1}) \quad \text{and} \quad (H/\Sigma_q) \circ \mathcal{P}_q = \mathcal{P}_q^- \circ (H/\Sigma_q).$$

We will write  $h_1 = H/\Sigma_{q_1}$  and  $h = H/\Sigma_q$ . Now, since  $H$  preserves  $\mathcal{F}^{ss}$  and  $H(\mathcal{W}^u(\alpha)) = \mathcal{W}^u(\tilde{\alpha})$ , we have that  $H(x) = \tilde{x}$  (see [1]). As a first case we suppose  $A$  is real and positive; similarly for  $\tilde{A}$ . Let  $\varphi = (y_1, \dots, y_{n-1})$  be  $C^2$  linearizing coordinates in a neighborhood  $U$  of  $q_1$  in  $\Sigma_{q_1}$  such that  $y_1$ -axis is the eigenspace corresponding to the weakest contraction. Let  $\mathcal{W}^{cu}(q_1)$  be a  $C^1$  center-unstable manifold of  $q_1$ ; it induces a  $C^1$  center-unstable manifold  $\mathcal{W}^{cu}(\alpha)$  of  $\alpha$ . Since  $\mathcal{W}^u(\alpha)$  is a codimension one submanifold of  $\mathcal{W}^{cu}(\alpha)$ , there exists a distance  $\tilde{d}$  in  $\Sigma_q^-$  and two real numbers,  $c > 0$  (large enough) and  $\tau > 1$  (close to 1) such that

$$h(\mathcal{W}^{cu}(\alpha) \cap \Sigma_q) \cap \tilde{V} \subseteq K^{cu}(\tilde{q}),$$

where  $\tilde{V}$  is a neighborhood of  $\tilde{x}$  in  $\Sigma_q^-$  and  $K^{cu}(\tilde{q})$  is the cone

$$K^{cu}(\tilde{q}) = \left\{ y \in \Sigma_q^- \mid \tilde{d}(y, \mathcal{W}^{cu}(\tilde{\alpha}) \cap \Sigma_q^-) \leq c \left( \tilde{d}(y, \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_q^-) \right)^\tau \right\}.$$

We now reparametrize the flows  $Y_t$  and  $\tilde{Y}_t$  in a such a way that

$$Y_T(V) \subseteq U, \quad \tilde{Y}_T(\tilde{V}) \subseteq \tilde{U}, \quad T < 0.$$

We now consider a sequence

$x_n \in K^{cu}(q_1) = \{y \in \Sigma_q \mid d(y, \mathcal{W}^{cu}(\alpha) \cap \Sigma_q) \leq c(d(y, \mathcal{W}^u(\alpha) \cap \Sigma_q))^r\}$ ,  
 $x_n \mapsto x$ , such that the sequence  $y_n = Y_T(x_n)$  satisfies  $\mathcal{P}_{q_1}^{-k_n}(y_n) \mapsto s$ ,  
 $s \in \mathcal{W}^s(q_1) \setminus \{q_1\}$ ; in fact,  $s \in \mathcal{W}^s(q_1) \setminus \mathcal{W}^{ss}(q_1)$ . Note that  $k_n \mapsto \infty$  as  
 $n \mapsto \infty$  and that

$$d(x_n, \mathcal{W}^u(\alpha)) \cong \sigma_{q_1} A^{k_n} |\pi_1(s)|, \quad (\text{A*})$$

where  $\pi_1(s)$  is the first coordinate of  $s$  and  $\sigma_{q_1} > 0$  is a constant (transition constant) which does not depend on  $x$  nor  $d$ .

*Notation.*— Let  $(a_i)$  and  $(b_i)$  be real number sequences. Then  $a_i \cong b_i$  means

$$\lim_{i \rightarrow \infty} \frac{a_i}{b_i} = 1.$$

Since the contact between  $\mathcal{W}^u(\alpha) \cap \Sigma_q$  and  $F^{ss}(x)$  is Morse type, in a neighborhood of  $x$  in  $\mathcal{W}^{cu}(\alpha) \cap \Sigma_q$  we may write  $\mathcal{W}^u(\alpha) \cap \Sigma_q$  as the graph of a Morse function  $Q$  with respect to the leaf  $F^{ss}(x)$  of  $\mathcal{F}^{ss}(q)$  which contains  $x$ . Under the above conditions we have that there are distances  $d, \tilde{d}$ , in  $\Sigma_q$  and  $\Sigma_{\tilde{q}}$  respectively, such that:

- (1) if, at  $x$ ,  $Q$  has at  $x$  a saddle type critical point, then for each small  $\delta > 0$  there exists a point  $x_\delta \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{W}^{cs}(q)$  such that

$$\delta = d(x_\delta, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong d(x_\delta, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha))$$

and that

$$\tilde{d}(h(x_\delta), \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}) \cong \tilde{d}(h(x_\delta), F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha}));$$

- (2) if, at  $x$ ,  $Q$  has at  $x$ , a maximal (minimal) type critical point, then for each small  $\delta > 0$ , there are points  $x_\delta^i \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{W}^{cs}(q)$ ,  $i = 1, 2$ , such that

$$\delta = d(x_\delta^i, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong d(x_\delta^i, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha)), \quad i = 1, 2,$$

and that

$$\tilde{d}(h(x_\delta^1), \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}^-) < \tilde{d}(h(x_\delta^1), F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})),$$

and that

$$\tilde{d}(h(x_\delta^2), \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}^-) > \tilde{d}(h(x_\delta^2), F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})).$$

*Notation.*— Let  $(a_i)$  and  $(b_i)$  be real number sequences. Then  $a_i \lesssim b_i$  means

$$0 < \sup_{i \in \mathbb{N}} \left\{ \frac{a_i}{b_i} \right\} \leq 1.$$

Under the conditions (1), setting  $\delta = 1/n$ ,  $n \in \mathbb{N}$  large enough, we obtain a sequence  $x_n \in \mathcal{W}^{cu}(\alpha) \cap \mathcal{D}^{cs}(q)$  such that

$$\frac{1}{n} = d(x_n, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong d(x_n, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha))$$

and that

$$\tilde{d}(h(x_n), \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}^-) \cong \tilde{d}(h(x_n), F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})).$$

Since  $h$  is  $C^1$  and preserves  $\mathcal{F}^{ss}$ , we have

$$\tilde{d}(h(x_n), F^{ss}(\tilde{x}) \cap \mathcal{W}^{cu}(\tilde{\alpha})) \cong \gamma d(x_n, F^{ss}(x) \cap \mathcal{W}^{cu}(\alpha)). \quad (\text{A1})$$

Now, by taking the sequences  $y_n = Y_T(x_n)$ ,  $\tilde{y}_n = \tilde{Y}_T(h(x_n))$ , we have that

$$\mathcal{P}_{q_1}^{-k_n}(y_n) \rightarrow s, \quad s \in \mathcal{W}^s(q_1) \setminus \mathcal{W}^{ss}(q_1)$$

$$\tilde{\mathcal{P}}_{\tilde{q}_1}^{-k_n}(\tilde{y}_n) \mapsto h_1(s), \quad h_1(s) \in \mathcal{W}^s(\tilde{q}_1) \setminus \mathcal{W}^{ss}(\tilde{q}_1)$$

and that

$$d(x_n, \mathcal{W}^u(\alpha) \cap \Sigma_q) \cong \sigma_{q_1} A^{k_n} |\pi_1(s)| \quad (\text{A*})$$

$$\tilde{d}(h(x_n), \mathcal{W}^u(\tilde{\alpha}) \cap \Sigma_{\tilde{q}}^-) \cong \tilde{\sigma}_{\tilde{q}_1} \tilde{A}^{k_n} |\tilde{\pi}_1(h_1(s))|. \quad (\text{A**})$$

Combining (A\*), (A\*\*) and (A1) we obtain

$$\gamma \sigma_{q_1} A^{k_n} |\pi_1(s)| \cong \tilde{\sigma}_{\tilde{q}_1} \tilde{A}^{k_n} |\tilde{\pi}_1(h_1(s))|.$$

Therefore, for  $n \in \mathbb{N}$  large enough, we have

$$\left( \frac{A}{\tilde{A}} \right)^{k_n} \cong K, \quad K \text{ non zero constant, thus } A = \tilde{A}.$$

Under the conditions (2), we analogously obtain asymptotic inequalities from which  $A = \tilde{A}$ .

In the case  $A$  is negative or complex, the result is obtained in a similar way [1].

Part (b) Let  $\tilde{Y}$  be a vector field near  $Y$  with the same characteristics of  $Y$ . We let  $\tilde{a} \pm i\tilde{b}$  ( $\tilde{a} < 0, \tilde{b} \neq 0$ ) denote the weakest contraction of the hyperbolic singularity  $\tilde{\alpha}$  of  $\tilde{Y}$ . Again, each time we make a construction for  $Y$ , we will suppose it made for  $\tilde{Y}$ .

We first suppose  $\dim M = 3$ .

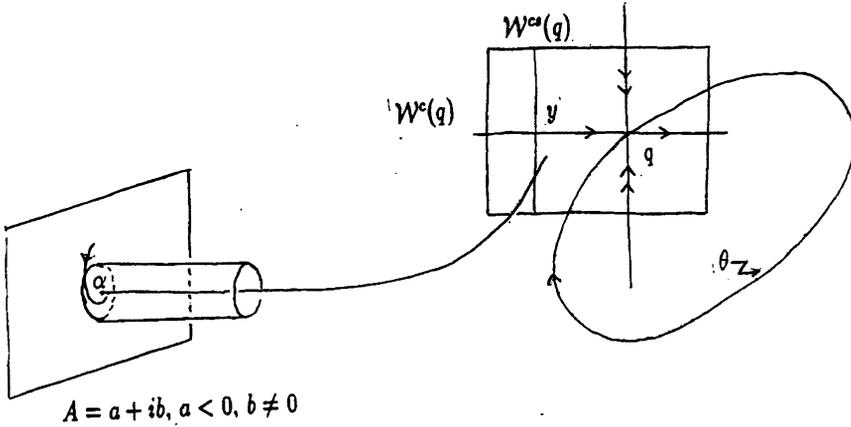


Fig. 1

Let  $H$  be an equivalence between  $Y$  and  $\tilde{Y}$  in a neighborhood  $U$  of the closure of  $O_Y(x)$  in  $M$ . We let  $\mathcal{P} : \Sigma_q \rightarrow \Sigma_q$  and  $\tilde{\mathcal{P}} : \Sigma_{\tilde{q}} \rightarrow \Sigma_{\tilde{q}}$  denote the Poincaré maps of the periodic orbits  $\theta$  and  $\tilde{\theta}$  in the transversal sections  $\Sigma_q$  ( $q \in \theta$ ),  $\Sigma_{\tilde{q}}$  ( $\tilde{q} \in \tilde{\theta}$ ), respectively. We may suppose  $H(\Sigma_q) = \Sigma_{\tilde{q}}$ ; thus  $(H/\Sigma_q) \circ \mathcal{P} = \tilde{\mathcal{P}} \circ (H/\Sigma_q)$ . Moreover, since  $H/\Sigma_q$  sends leaves of  $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$  into leaves of  $\mathcal{F}^{ss}(\tilde{q})/\mathcal{W}^s(\tilde{q})$  and

$$H(\mathcal{W}^u(\alpha) \cap \mathcal{W}^s(q)) = \mathcal{W}^u(\tilde{\alpha}) \cap \mathcal{W}^s(\tilde{q}),$$

we have that  $H$  is completely determined in a part of  $\mathcal{W}_{\mu}^c(q)$ .

Let  $V$  be a neighborhood of  $\alpha$  and  $\tilde{V} = H(V)$  and let  $f : V \rightarrow \mathbb{R}$ ,  $\tilde{f} : \tilde{V} \rightarrow \mathbb{R}$  be  $C^\infty$  Lyapunov functions for  $Y$  and  $\tilde{Y}$  respectively. Given  $\varepsilon > 0$  small, let  $C_\varepsilon \subseteq f^{-1}(-\varepsilon)$  and  $\tilde{C}_\varepsilon \subseteq \tilde{f}^{-1}(-\varepsilon)$  be transversal cylinders to  $\mathcal{W}^s(\alpha)$  and  $\mathcal{W}^s(\tilde{\alpha})$  respectively. Since  $H$  is an equivalence between  $Y/V$  and  $\tilde{Y}/\tilde{V}$ , we have that  $H(C_\varepsilon)$  is a cylinder topologically transversal to  $\mathcal{W}^s(\tilde{\alpha})$ . We may suppose  $H(C_\varepsilon) \subseteq \tilde{C}_\varepsilon$ .

Now in  $C_\epsilon$ , we take a  $C^1$  one-dimensional foliation  $\mathcal{F}^c$  with leaves transversal to  $\mathcal{W}^s(\alpha) \cap C_\epsilon$ . The foliation  $\mathcal{F}^c$  induces a  $C^1$  center-unstable foliation  $\mathcal{F}^{cu}$  as follows

$$F^{cu}(z) = \bigcup_{t \in \mathbb{R}} Y_t(F^c(z)), \quad z \in \mathcal{W}^s(\alpha) \cap C_\epsilon.$$

If  $z \in \mathcal{W}^s(\alpha) \cap C_\epsilon$ , we have that  $H(F^c(z)) \subseteq \tilde{C}_\epsilon$  is a curve topologically transversal to  $\mathcal{W}^s(\tilde{\alpha}) \cap \tilde{C}_\epsilon$ . A curve of the form  $H(F^c(z))$  is denoted by  $\hat{F}^c(H(z))$  and we define

$$\hat{F}^{cu}(H(z)) = \bigcup_{t \in \mathbb{R}} \tilde{Y}_t(\hat{F}^{cu}(H(z))).$$

Under the above conditions (see [13]), there exist  $\delta = \delta(H(z)) > 0$  and a cone  $\tilde{K}_{H(z)} \subseteq \tilde{C}_\epsilon$  with a vertex in  $H(z)$  such that if  $\tilde{C}_\delta \subseteq \tilde{C}_\epsilon$  denotes a  $\delta$ -size cylinder, then:

- $\tilde{\mathcal{F}}^c(H(z)) \subseteq \tilde{K}_{H(z)}$ ,
- $\tilde{\mathcal{F}}^{cu}(H(z)) \cap \tilde{C}_\delta \subseteq \tilde{K}_{H(z)}$ .

Thus,

$$\tilde{\mathcal{F}}^{cu}(H(z)) \cap \tilde{f}^{-1}(\epsilon) \quad \text{and} \quad \hat{F}^{cu}(H(z)) \cap \tilde{f}^{-1}(\epsilon)$$

approach each other when we come near  $\mathcal{W}^u(\tilde{\alpha}) \cap \tilde{f}^{-1}(\epsilon)$ . In this case we say they are asymptotically undistinguishable. Furthermore, the curves

$$F^{cu}(z) \cap f^{-1}(\epsilon) \quad \text{and} \quad \tilde{F}^{cu}(\tilde{z}) \cap \tilde{f}^{-1}(\epsilon),$$

$z \in \mathcal{W}^s(\alpha) \cap C_\epsilon$  and  $\tilde{z} \in \mathcal{W}^s(\tilde{\alpha}) \cap \tilde{C}_\epsilon$ , are spirals which we may suppose linear with contraction coefficients  $\rho = a/b$  and  $\tilde{\rho} = \tilde{a}/\tilde{b}$  respectively, in linearization neighborhoods of  $\alpha$  and  $\tilde{\alpha}$  respectively.

Now, we have that the intersections

$$\mathcal{F}^{cu} \cap \Sigma_q, \quad \tilde{\mathcal{F}}^{cu} \cap \Sigma_{\tilde{q}} \quad \text{and} \quad \hat{F}^{cu} \cap \Sigma_{\tilde{q}}$$

are spirals (see [7] or [11]). We let  $\gamma$  denote the  $C^1$  curve given by the tangencies between the leaves of  $\mathcal{F}^{cu} \cap \Sigma_q$  and the leaves of  $\mathcal{F}^{ss}(q)$  in a neighborhood of the  $s$ -criticality  $x$  in  $\mathcal{W}^{cs}(q)$ . We define  $\tilde{\gamma}$  analogously. Note that  $H(\gamma)$  does not necessarily coincide with  $\gamma$ . But since

$$\pi^{ss}/\gamma : \gamma \mapsto \pi^{ss}(\gamma) \quad \text{and} \quad \tilde{\pi}^{ss}/\tilde{\gamma} : \tilde{\gamma} \mapsto \tilde{\pi}(\tilde{\gamma})$$

are  $C^1$  diffeomorphisms, we may suppose  $H(\gamma) = \tilde{\gamma}$ .

Let  $h = H/\gamma : \gamma \mapsto \tilde{\gamma}$ ,  $h$  is  $C^1$  and compatible with  $H^c = H/\mathcal{W}^c(q)$ . Using the spirals  $\mathcal{F}^{cu} \cap \Sigma_q$ ,  $\tilde{\mathcal{F}}^{cu} \cap \Sigma_{\tilde{q}}$  and  $\hat{\mathcal{F}}^{cu} \cap \Sigma_{\hat{q}}$ , we define the maps  $h^{cu} : \gamma \mapsto \gamma$ ,  $\hat{h}^{cu}, \tilde{h}^{cu} : \tilde{\gamma} \mapsto \tilde{\gamma}$  as follows: given  $y \in \gamma$ ,  $h^{cu}(y)$  is the first point where  $\mathcal{F}^{cu}(y) \cap \Sigma_q$  intersects the component of  $\gamma \setminus \{x\}$  which contains  $y$ . We define  $\tilde{h}^{cu}$  and  $\hat{h}^{cu}$  analogously. We have that

$$h^{cu}(x) = x, \quad \tilde{h}^{cu}(\tilde{x}) = \hat{h}^{cu}(\tilde{x}) = \tilde{x}, \quad \hat{h}^{cu}(h(y)) = h(h^{cu}(y))$$

and that, in a neighborhood of  $x$ , respectively of  $\tilde{x}$ ,

$$h^{cu}(y) = e^{2\pi\rho}y, \quad \tilde{h}^{cu}(z) = e^{2\pi\tilde{\rho}}z,$$

where  $\rho = a/b$ ,  $\tilde{\rho} = \tilde{a}/\tilde{b}$  (fig. 2).

We now fix  $z_1, z_2 \in \mathcal{W}^s(\alpha) \cap C_\varepsilon$ ,  $z_1$  near  $z_2$ . Let  $u_1 = H(z_1)$ ,  $u_2 = H(z_2)$ . Associated to each  $u_i$  ( $i = 1, 2$ ), we have a cone  $\tilde{K}_{u_i} \subseteq \tilde{C}_\varepsilon$  with vertex  $u_i$ , and a cylinder  $C_{\delta_i}$  ( $\delta_i = \delta(u_i)$ ). Let  $0 < \delta < \min\{\delta_1, \delta_2\}$ ,  $C_\delta$  denotes the corresponding cylinder. Let  $W$  be a neighborhood of  $u_1$  and  $u_2$  in  $\mathcal{W}^u(\tilde{\alpha}) \cap \tilde{C}_\varepsilon$ , in  $W$  we take the points  $v_1, v_2$  as in figure 2. From this construction, we have that the spirals  $\hat{\mathcal{F}}^{cu}(u_1) \cap \Sigma_{\hat{q}}$  and  $\tilde{\mathcal{F}}^{cu}(u_2) \cap \Sigma_{\tilde{q}}$  are contained in a spiral neighborhood limited by the spirals  $\tilde{\mathcal{F}}^{cu}(v_1) \cap \Sigma_{\tilde{q}}$  and  $\tilde{\mathcal{F}}^{cu}(v_2) \cap \Sigma_{\tilde{q}}$  as in figure 3. We take a neighborhood of  $x$  (resp.  $\tilde{x}$ ) in  $\mathcal{W}^{cs}(q)$  (resp.  $\mathcal{W}^{cs}(\tilde{q})$ ) small enough such that  $\tilde{x}_0 = h(x_0)$  and  $\hat{y}_0 = h(y_0)$  are near  $\tilde{x}$ . We now define the following sequences and intervals in  $\gamma$  (resp.  $\tilde{\gamma}$ ):

- $x_n = (h^{cu})^n(x_0)$ ,  $y_n = (h^{cu})^n(y_0)$ ,  $I_0 = [y_0, x_0]$ ,  $I_n = [y_n, x_n]$ ;
- $u_n = (\tilde{h}^{cu})^n(\tilde{u}_0)$ ,  $\tilde{v}_n = (\tilde{h}^{cu})^n(\tilde{v}_0)$ ,  $\tilde{I}_0 = [\tilde{v}_0, \tilde{u}_0]$ ,  $\tilde{I}_n = [\tilde{v}_n, \tilde{u}_n]$ ;
- $\hat{x}_n = (\hat{h}^{cu})^n(\hat{x}_0)$ ,  $\hat{y}_n = (\hat{h}^{cu})^n(\hat{y}_0)$ ,  $\hat{I}_0 = [\hat{y}_0, \hat{x}_0]$ ,  $\hat{I}_n = [\hat{y}_n, \hat{x}_n]$ .

For each  $n \in \mathbb{N}$ , we have that

$$l(I_n) = e^{2n\pi\rho}l(I_0), \quad l(\tilde{I}_n) = e^{2n\pi\tilde{\rho}}l(\tilde{I}_0) \quad \text{and} \quad \hat{I}_n \subseteq \tilde{I}_n$$

(where  $l(J)$  is the length of the interval  $J$ ). Therefore  $\tilde{d}(\hat{x}_n, \hat{y}_n) = l(\tilde{I}_n)$ . On the other hand, we have:

$$\begin{aligned} \tilde{d}(\hat{x}_n, \hat{y}_n) &= \tilde{d}\left((\hat{h}^{cu})^n(\hat{x}_0), (\hat{h}^{cu})^n(\hat{y}_0)\right) \\ &= \tilde{d}\left((\hat{h}^{cu})^n(h(x_0)), (\hat{h}^{cu})^n(h(y_0))\right) \\ &= \tilde{d}\left(h\left((h^{cu})^n(x_0)\right), h\left((h^{cu})^n(y_0)\right)\right) \\ &= \tilde{d}(h(x_n), h(y_n)) \\ &= cd(x_n, y_n) \quad (\text{since } h \text{ is } C^1) \\ &= cl(I_n) \\ &= ce^{2n\pi\rho}l(I_0), \end{aligned}$$

where  $0 < c e^{2n\pi\rho l(I_0)} \leq e^{2n\pi\tilde{\rho}l(\tilde{I}_0)}$ ; thus  $0 < K \leq e^{2n\pi(\tilde{\rho}-\rho)}$ ,  $n \in \mathbb{N}$  large enough.

Through an analogous argument for  $H^{-1}$ , we have  $0 < \tilde{K} \leq e^{2n\pi(\rho-\tilde{\rho})}$ ,  $n \in \mathbb{N}$  large enough.

Therefore, if  $n \in \mathbb{N}$  is large enough, we have  $0 < C \leq e^{2n\pi(\tilde{\rho}-\rho)} \leq \tilde{C} < \infty$  from which  $\rho = \tilde{\rho}$ .

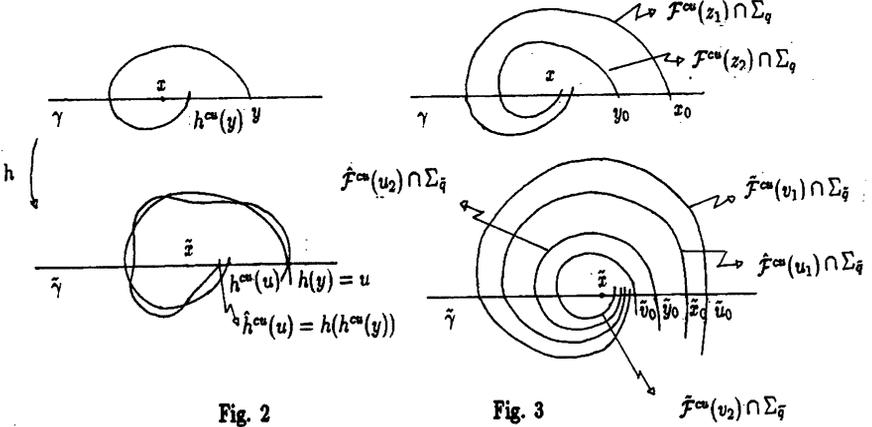


Fig. 2

Fig. 3

The  $n$ -dimensional case is reduced to the 3-dimensional case.

### 3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We only prove part (c) since the methods used can easily be adapted to parts (a) and (b). We first recall some definitions of invariant foliation associated to vector fields.

Let  $\{X_\mu\} \in \mathcal{X}_1^\infty(M)$ , and let  $\bar{\mu} \in I$ .

**DEFINITION 3.1.** — Let  $p \in M$  be a hyperbolic singularity of  $X_{\bar{\mu}}$  a (local) unstable foliation for  $\{X_\mu\}$  (or for the vector field  $X(x, \mu) = (X_\mu(x), 0)$ ) at  $(p, \bar{\mu})$ ,  $\mathcal{F}_p^u$  is a continuous foliation of  $U_p \times I_1$ , where  $U_p \subseteq M$ ,  $I_1 \subseteq I$ , are neighborhoods of  $p$  and  $\bar{\mu}$  respectively, such that:

- (a) the leaves of  $\mathcal{F}_p^u$  are  $C^r$  discs,  $r \geq 1$ , varying continuously in the  $C^r$  topology with distinguished leaf

$$F^u(p_\mu, \mu) = (W^u(p_\mu) \cap U_p) \times \{\mu\};$$

Global stability of saddle-node bifurcation of a periodic orbit for vector fields

(b) each leaf  $F^u(x, \mu)$  is contained in  $U_p \times \{\mu\}$ ;

(c)  $\mathcal{F}^u$  is invariant, i.e.,

$$F^u(X_{\mu,t}(x), \mu) \subset X_{\mu,t}(F^u(x, \mu)), \quad t \geq 0.$$

A global unstable foliation  $\mathcal{F}_p^u$  is just the positive saturation by the flow  $X_\mu$  ( $\mu \in I_1$ ) of the local unstable foliation. Similarly we define a stable foliation  $\mathcal{F}_p^s$ .

Now, let  $\sigma$  be a hyperbolic periodic orbit of  $X_{\bar{\mu}}$ , and let

$$\mathcal{P}_\mu : V_q \times I_1 \rightarrow \Sigma_q \times I_1$$

( $V_q$  is a neighborhood of  $q$  in  $\Sigma_q$ ) be the Poincaré map.

**DEFINITION 3.2.** — *A (local) unstable foliation for  $\{X_\mu\}$  (or for  $X(x, \mu)$ ) at  $(q, \bar{\mu})$ ,  $\mathcal{F}_\sigma^u$  is a continuous foliation of  $V_q \times I_1$  such that:*

(a) *the leaves of  $\mathcal{F}_\sigma^u$  are  $C^r$  discs,  $r \geq 1$ , varying continuously in the  $C^r$  topology with distinguished leaf*

$$F^u(q_\mu, \mu) = (\mathcal{W}^s(q_\mu) \cap V_q) \times \{\mu\};$$

(b) each leaf  $F^u(x, \mu)$  is contained in  $V_q \times \{\mu\}$ ;

(c)  $\mathcal{F}^u$  is invariant, i.e.,

$$F^u(\mathcal{P}_\mu(x), \mu) \subseteq \mathcal{P}(F^u(x, \mu)).$$

A global unstable foliation  $\mathcal{F}_\sigma^u$  is just the positive saturation by the flow  $X_\mu$  ( $\mu \in I_1$ ) of the local unstable foliation. Similarly we define a stable foliation  $\mathcal{F}_\sigma^s$ .

If  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ ,  $\mathcal{P} = \{\mathcal{P}_\mu\}$  (the Poincaré map associated to the vector field  $X$  in the non-hyperbolic periodic orbit  $(\theta, \bar{\mu})$ ) is an arc of saddle-node diffeomorphisms. Thus we have that, in  $\mathcal{W}^{cs}(q)$  (resp.  $\mathcal{W}^{cu}(q)$ ), there is a strong stable (resp. strong unstable)  $C^r$  foliation,  $r \geq 1$ ,  $\mathcal{F}_q^{ss}(q)$  (resp.  $\mathcal{F}_q^{uu}(q)$ ).

**DEFINITION 3.3.** — *A strong unstable foliation for  $\mathcal{P} = \{\mathcal{P}_\mu\}$  at the saddle-node  $(q, \bar{\mu})$ ,  $\mathcal{F}_q^{uu}$  is a continuous foliation of  $\Sigma_q \times I_1$  such that:*

- (a) *the leaves of  $\mathcal{F}_q^{uu}$  are  $C^r$  discs,  $r \geq 1$ , varying continuously in the  $C^r$  topology with distinguished leaf*

$$F^{uu}(q, \bar{\mu}) = (\mathcal{W}^{uu}(q) \cap \Sigma_q) \times \{\bar{\mu}\};$$

- (b) *each leaf  $F^{uu}(x, \mu)$  is contained in  $\Sigma_q \times \{\mu\}$ ;*

- (c)  *$\mathcal{F}_q^{uu}$  is invariant, i.e.,*

$$F^{uu}(\mathcal{P}_\mu(x), \mu) \subseteq \mathcal{P}(F^{uu}(x, \mu)).$$

Similarly we define a strong stable foliation,  $\mathcal{F}_q^{ss}$ .

Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . Since for each  $\mu \in I$ ,  $X_\mu \in WR^\infty(M)$ , we have that  $\Omega(X_\mu)$  is the union of a finite number of critical elements of  $X_\mu$ . For  $\mu < \bar{\mu}$ , let  $\alpha_{1,\mu}, \dots, \alpha_{k,\mu}, \theta_{1,\mu}, \theta_{2,\mu}, \beta_{1,\mu}, \dots, \beta_{\ell,\mu}$  be the critical elements (hyperbolic) of  $X_\mu$ , where  $\theta_{1,\mu}$  and  $\theta_{2,\mu}$  are the periodic orbits which originate the saddle-node periodic orbit

$$\theta = \theta_{1,\bar{\mu}} = \theta_{2,\bar{\mu}} \quad \text{of } X_{\bar{\mu}}.$$

On the other hand, for each  $\mu \in I$ ,  $X_\mu \in WC^\infty(M)$ , then we may define a partial order  $\leq$  among the critical elements of  $X_\mu$ ,

$$\sigma_1 \leq \sigma_2 \quad \text{if and only if} \quad \mathcal{W}^u(\sigma_1) \cap \mathcal{W}^s(\sigma_2) \neq \emptyset.$$

This partial order is extended to a total order and we assume it is the following

$$\alpha_{1,\mu} \leq \dots \leq \alpha_{k,\mu} \leq \theta_{1,\mu} \leq \theta_{2,\mu} \leq \beta_{1,\mu} \leq \dots \leq \beta_{\ell,\mu}.$$

Clearly, this order holds for  $\mu = \bar{\mu}$  and even for  $\mu > \bar{\mu}$  near  $\bar{\mu}$ . Therefore, for each  $\mu \in I$ , we order the critical elements of  $X_\mu$  as follows

$$\begin{cases} \mu < \bar{\mu}: & \alpha_{1,\mu} \leq \dots \leq \alpha_{k,\mu} \leq \theta_{1,\mu} \leq \theta_{2,\mu} \leq \beta_{1,\mu} \leq \dots \leq \beta_{\ell,\mu} \\ \mu = \bar{\mu}: & \alpha_{1,\bar{\mu}} \leq \dots \leq \alpha_{k,\bar{\mu}} \leq \theta \leq \beta_{1,\bar{\mu}} \leq \dots \leq \beta_{\ell,\bar{\mu}} \\ \mu > \bar{\mu}: & \alpha_{1,\mu} \leq \dots \leq \alpha_{k,\mu} \leq \beta_{1,\mu} \leq \dots \leq \beta_{\ell,\mu}, \end{cases}$$

**DEFINITION 3.4.** — Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  be as above. We say the system of unstable and strong unstable foliations

$$\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_k), \mathcal{F}^{uu}(q)$$

is compatible:

- (a) if a leaf  $F$  of  $\mathcal{F}^u(\alpha_n)$  intersects a leaf  $E$  of  $\mathcal{F}^u(\alpha_i)$  ( $\alpha_n \leq \alpha_i \leq \alpha_k$ ) then  $F \supseteq E$  and the restriction of the foliation  $\mathcal{F}^u(\alpha_i)$  to each leaf of  $\mathcal{F}^u(\alpha_n)$  is a  $C^2$  foliation;
- (b) if a leaf  $F$  of  $\mathcal{F}^u(\alpha_i)$ ,  $\alpha_i \leq \alpha_k$ , intersects a leaf  $F^{uu}$  of  $\mathcal{F}^{uu}(q)$ , then  $F \supseteq F^{uu}$  and the restriction of the foliation  $\mathcal{F}^{uu}(q)$  to each leaf of  $\mathcal{F}^u(\alpha_n)$  is a  $C^1$  foliation.

Similarly, we define a compatible system of stable and strong stable foliations  $\mathcal{F}^{ss}(q), \mathcal{F}^s(\beta_1), \dots, \mathcal{F}^s(\beta_\ell)$ .

**LEMMA 1.** — (J. Palis and F. Takens [9]) Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . Then for  $\{X_\mu\}$  there exist compatible systems of foliations:

- (a)  $\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_k),$
- (b)  $\mathcal{F}^s(\beta_1), \dots, \mathcal{F}^s(\beta_\ell).$

Now let  $\{X_\mu\}, \{\tilde{X}_\mu\} \in \Gamma_{sn}^1$  be close families, and let  $\bar{\mu}, \tilde{\mu} \in I$  their first bifurcation values,  $\tilde{\mu}$  close to  $\bar{\mu}$ . We have that  $X_{\bar{\mu}}$  and  $\tilde{X}_{\tilde{\mu}}$  have close saddle-node periodic orbits  $\theta$  and  $\tilde{\theta}$ , respectively. If  $\rho : (I, \bar{\mu}) \rightarrow (I, \tilde{\mu})$  is a reparametrization then from the existence of compatible systems of foliations

$$\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_k), \quad \mathcal{F}^s(\beta_1), \dots, \mathcal{F}^s(\beta_\ell)$$

for  $\{X_\mu\}$  and of the respective one for  $\{\tilde{X}_\mu\}$ , we may suppose constructed the homeomorphisms:

$$H^s : \bigcup_{\mu \in I_1} \left( \bigcup_{i=1}^k (\mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\}) \right) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{i=1}^k (\mathcal{W}^s(\tilde{\alpha}_{i,\mu}) \times \{\mu\}) \right) \quad (1)$$

such that, for each  $i$  ( $1 \leq i \leq k$ ), the restrictions

$$H_{i,\mu}^s = H^s / \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} : \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \longrightarrow \mathcal{W}^s(\tilde{\alpha}_{i,\rho(\mu)}) \times \{\rho(\mu)\}$$

are conjugacies between the flows

$$X_{\mu,t}/\mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \quad \text{and} \quad \tilde{X}_{\rho(\mu),t}/\mathcal{W}^s(\tilde{\alpha}_{i,\rho(\mu)}) \times \{\rho(\mu)\};$$

$$H^u : \bigcup_{\mu \in I_1} \left( \bigcup_{j=1}^{\ell} (\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\}) \right) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{j=1}^{\ell} (\mathcal{W}^u(\tilde{\beta}_{j,\mu}) \times \{\mu\}) \right) \quad (2)$$

such that, for each  $j$  ( $1 \leq i \leq k$ ), the restrictions

$$H_{j,\mu}^u = H^u/\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} : \mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \longrightarrow \mathcal{W}^u(\tilde{\beta}_{j,\rho(\mu)}) \times \{\rho(\mu)\}$$

are conjugacies between the flows

$$X_{\mu,t}/\mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \quad \text{and} \quad \tilde{X}_{\rho(\mu),t}/\mathcal{W}^u(\tilde{\beta}_{j,\rho(\mu)}) \times \{\rho(\mu)\}.$$

*Remark.* — To construct a global equivalence between  $\{X_\mu\}$  and  $\{\tilde{X}_\mu\}$  as above, we first extend the definition of  $H^s$  to  $\mathcal{W}^{cs}(q)$  and that of  $H^u$  to  $\mathcal{W}^{cu}(q)$  in a such way as to conjugate  $\mathcal{P}/\mathcal{W}^{cs}(q)$  with  $\tilde{\mathcal{P}}/\mathcal{W}^{cs}(\tilde{q})$ , and  $\mathcal{P}/\mathcal{W}^{cu}(q)$  with  $\tilde{\mathcal{P}}/\mathcal{W}^{cu}(\tilde{q})$ , and afterwards extend the definitions given in  $\mathcal{W}^{cs}(q)$  and  $\mathcal{W}^{cu}(q)$  to a neighborhood of  $(q, \bar{\mu})$  in  $\Sigma_q \times I_1$  in order to obtain a homeomorphism  $H_q : \Sigma_q \times I_1 \rightarrow \Sigma_{\tilde{q}} \times \tilde{I}_1$  such that:

- it conjugates the Poincaré maps  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ ;
- it is compatible with the constructions done above;
- it is possible to extend the definition of  $H^s$  (or  $H^u$ ) to

$$\bigcup_{\mu \in I_1} \left( \bigcup_{j=1}^{\ell} (\mathcal{W}^s(\beta_{j,\mu}) \times \{\mu\}) \right) \quad \text{or} \quad \bigcup_{\mu \in I_1} \left( \bigcup_{i=1}^k (\mathcal{W}^u(\alpha_{i,\mu}) \times \{\mu\}) \right).$$

**DEFINITION 3.5.** — Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . Let  $\alpha_{\bar{\mu}}$  be a hyperbolic critical element of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha_{\bar{\mu}})$  has a generic  $s$ -criticality with  $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$ . We say the  $s$ -criticality between  $\mathcal{W}^u(\alpha_{\bar{\mu}})$  and  $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$  is of codimension  $p$ ,  $0 \leq p \leq \dim \mathcal{W}_{\bar{\mu}}^{cs}(q)$ , if  $\mathcal{W}^u(\alpha_{\bar{\mu}}) \times \{\bar{\mu}\} \cap \mathcal{W}_{\bar{\mu}}^{cs}(q)$  is a codimension  $p$  submanifold of  $\mathcal{W}_{\bar{\mu}}^{cs}(q)$ . If  $p = \dim \mathcal{W}_{\bar{\mu}}^{cs}(q)$ , we say the  $s$ -criticality is of zero dimension.

An analogous definition is given for  $u$ -criticalities.

*Remark 1.* — If the  $s$ -criticality is of zero dimension, without loss of generality, we may suppose that  $\alpha_{\bar{\mu}} = \alpha_{k, \bar{\mu}}$  in the order given above.

*Remark 2.* — If the  $s$ -criticality of codimension  $p \geq 1$ , since we are assuming a unique criticality in a fundamental domain for  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cs}(q)$ , we have that  $\mathcal{W}^u(\alpha_{\bar{\mu}}) \cap \mathcal{W}_{\bar{\mu}}^{cs}(q)$  is non-compact. Thus, there exists a hyperbolic critical element  $\rho$  ( $\rho \neq \alpha_{\bar{\mu}}$ ) such that  $\mathcal{W}^u(\rho) \cap \mathcal{W}_{\bar{\mu}}^{cs}(q)$  has no criticalities and  $\mathcal{W}^u(\alpha_{\bar{\mu}}) \cap \mathcal{W}^s(\rho)$  is non-empty. Without loss of generality, we suppose  $\rho = \alpha_{k, \bar{\mu}}$  and  $\alpha_{\bar{\mu}} = \alpha_{k-1, \bar{\mu}}$  in the order given above.

We will make a construction which is easily adapted to the general case.

Let  $\{X_{\mu}\} \in \mathcal{X}_1^{\infty}(M)$  and  $\alpha$  be a hyperbolic singularity with real weakest contraction of  $X_{\bar{\mu}}$ . Let  $I_1$  and  $U$  be neighborhoods of  $\bar{\mu}$  in  $I$  and of  $\alpha$  in  $M$ , respectively. Then there exists a  $C^r$  ( $r \geq 1$ ) center-unstable manifold  $\mathcal{W}^{cu}(\alpha_{\bar{\mu}})$  (not unique) for  $X(x, \mu) = (X_{\mu}(x), 0)$  at  $(\alpha_{\bar{\mu}}, \bar{\mu})$  such that for each  $\mu \in I_1$ ,

$$\mathcal{W}^{cu}(\alpha_{\mu}) = \mathcal{W}^{cu}(\alpha) \cap M \times \{\mu\}$$

is a  $C^r$  center-manifold for  $X_{\mu}$  at  $\alpha_{\mu}$ . A  $C^r$  center-manifold for  $X$  at  $(\alpha, \bar{\mu})$  (resp. for  $X_{\mu}$  at  $\alpha_{\mu}$ ) is given by

$$\mathcal{W}^c(\alpha) = \mathcal{W}^s(\alpha) \cap \mathcal{W}^{cu}(\alpha), \quad (\text{resp. } \mathcal{W}^c(\alpha_{\mu}) = (\mathcal{W}^s(\alpha_{\mu}) \cap \mathcal{W}^{cu}(\alpha_{\mu})) \times \{\mu\}).$$

Throughout, we will use  $C^r$  ( $r \geq 1$ ) center-unstable foliation  $\mathcal{F}^{cu}(\alpha)$  and  $\mathcal{F}_1^{cu}(\alpha)$  as constructed in [5].

**LEMMA 2.** — Under the above conditions, for  $\{X_{\mu}\} \in \Gamma_{sn}^1$ , we have:

(a) if  $\mathcal{W}^u(\alpha_{k, \bar{\mu}})$  has a zero dimensional  $s$ -criticality, then there exists a compatible system of foliations

$$\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_{k-1}), \mathcal{F}^{cu}(\alpha_k)(\mathcal{F}_1^{cu}(\alpha_k)), \mathcal{F}^{uu}(q)$$

for  $\{X_{\mu}\}$ ,  $\mu$  near  $\bar{\mu}$ ;

(b) if  $\mathcal{W}^u(\alpha_{k-1, \bar{\mu}})$  has a codimension  $p \geq 1$   $s$ -criticality, then there exists a compatible system of foliations

$$\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_{k-2}), \mathcal{F}^{cu}(\alpha_{k-1})(\mathcal{F}_1^{cu}(\alpha_{k-1})), \mathcal{F}^{cu}(\alpha_k), \mathcal{F}^{uu}(q)$$

for  $\{X_{\mu}\}$ ,  $\mu$  near  $\bar{\mu}$ .

*Remark.*— (Important) Due to a result of F. Takens [10], we have that if we construct a conjugacy  $H_q : \Sigma_q \times I_1 \rightarrow \Sigma_{\tilde{q}} \times \tilde{I}_1$  between  $\mathcal{P} = \{\mathcal{P}_\mu\}$  and  $\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_\mu\}$  as above so that:

- the homeomorphism  $H_q$  sends the intersections of the foliations  $\mathcal{F}^{cu}(\alpha)$ ,  $\mathcal{F}_1^{cu}(\alpha)$ ,  $\mathcal{F}^{cs}(\beta)$  and  $\mathcal{F}_1^{cs}(\beta)$  with  $\Sigma_q \times I_1$  for  $\{X_\mu\}$ , into the corresponding intersections of the foliations  $\mathcal{F}^{cu}(\tilde{\alpha})$ ,  $\mathcal{F}_1^{cu}(\tilde{\alpha})$ ,  $\mathcal{F}^{cs}(\tilde{\beta})$  and  $\mathcal{F}_1^{cs}(\tilde{\beta})$  with  $\Sigma_{\tilde{q}} \times \tilde{I}_1$  for  $\{\tilde{X}_\mu\}$  continuously on the parameter;
- $H_q$  is compatible with the homeomorphisms  $H^s$ ,  $H^u$ , and  $h^c : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  which will be constructed shortly;
- $H_q$  is  $C^1$ -close to the inclusion map.

Therefore, using Lyapunov functions, we extend the homeomorphism  $H_q$  to a neighborhood of  $\alpha$  or  $\beta$ , depending on the case, in such a way that, using the methods developed in [9] or [4]. We construct a global equivalence  $H : M \times I_1 \rightarrow M \times \tilde{I}_1$  between  $\{X_\mu\}$  at  $\bar{\mu}$  and  $\{\tilde{X}_\mu\}$  at  $\tilde{\mu}$ .

### Construction of equivalences in $\mathcal{W}^c$ for arcs in $\Gamma_{sn}^1$

Let  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  such that  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . We let  $\mathcal{P} = \{\mathcal{P}_\mu\}$  denote the Poincaré map associated to  $X(x, \mu) = (X_\mu(x), 0)$  at  $(q, \bar{\mu})$ . Let  $\{\tilde{X}_\mu\} \in \Gamma_{sn}^1$  be near  $\{X_\mu\}$  with first bifurcation value  $\tilde{\mu}$  near  $\bar{\mu}$ , and such that  $\tilde{X}_{\tilde{\mu}}$  has a saddle-node periodic orbit  $\tilde{\theta}$  near  $\theta$ . We let  $\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_\mu\}$  denote the Poincaré map associated to  $\tilde{X}(x, \mu) = (\tilde{X}_\mu(x), 0)$  at  $(\tilde{q}, \tilde{\mu})$ . Then  $\tilde{\mathcal{P}}$  and  $\mathcal{P}$  are close diffeomorphism arcs.

Let  $\alpha_{\bar{\mu}}$  and  $\beta_{\bar{\mu}}$  be critical elements (hyperbolic) of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha_{\bar{\mu}})$  is  $s$ -critical and  $\mathcal{W}^s(\beta_{\bar{\mu}})$  is  $u$ -critical. Let  $\mathcal{P}_\mu^c = \mathcal{P}/\mathcal{W}_\mu^c(q)$  and

$$\pi_1^{uu} : \Sigma_q \times I_1 \rightarrow \mathcal{W}^{cs}(q), \quad \pi_1^{ss} : \Sigma_q \times I_1 \rightarrow \mathcal{W}^{cu}(q)$$

be the projections via leaves of  $\mathcal{F}^{uu}(q)$  and  $\mathcal{F}^{ss}(q)$ , respectively, and let

$$\pi^{ss} : \mathcal{W}^{cs}(q) \rightarrow \mathcal{W}^c(q), \quad \pi^{uu} : \mathcal{W}^{cu}(q) \rightarrow \mathcal{W}^c(q)$$

be the projections via leaves of  $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}(q)$  and  $\mathcal{F}^{uu}(q)/\mathcal{W}^{cu}(q)$ , respectively. We have

$$\pi^{ss} \circ \pi_1^{uu} \circ \mathcal{P}_\mu = \mathcal{P}_\mu^c \circ \pi^{uu} \circ \pi_1^{ss}.$$

We let  $x(\bar{\mu})$  denote the point of  $s$ -criticality between  $\mathcal{W}^u(\alpha_{\bar{\mu}})$  and  $\mathcal{F}^{ss}(q)$  in a fundamental domain  $D_{\bar{\mu}}^{cs}(q)$  of  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cs}$ . Similarly,  $y(\bar{\mu})$  is the point of  $u$ -criticality between  $\mathcal{W}^s(\beta_{\bar{\mu}})$  and  $\mathcal{F}^{uu}(q)$  in a fundamental domain  $D_{\bar{\mu}}^{cu}(q)$  of  $\mathcal{P}/\mathcal{W}_{\bar{\mu}}^{cu}(q)$ . Set

$$x^c(\bar{\mu}) = \pi^{ss}(x(\bar{\mu})) \quad \text{and} \quad y^c(\bar{\mu}) = \pi^{uu}(y(\bar{\mu})).$$

We have the following lemma.

**LEMMA 3.** — *Under the above conditions, there are  $\varepsilon > 0$  and a strictly monotone sequence  $(\bar{\mu}_n)$  of parameter values such that:*

- (a)  $\bar{\mu} < \bar{\mu}_n < \varepsilon$ ,  $\bar{\mu}_n \rightarrow \bar{\mu}$ ,
- (b) for each  $\mu \in ]\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon[$ , we have
  - if  $\mu \neq \bar{\mu}_n$  for all  $n \in \mathbb{N}$ , then  $X_\mu$  has no quasi-transversal intersection orbits,
  - for each  $n \in \mathbb{N}$ ,  $\bar{\mu}_n \in B(\{X_\mu\})$  and  $X_{\bar{\mu}_n}$  has a unique quasi-transversal intersection orbit between  $\mathcal{W}^u(\alpha_{\bar{\mu}_n})$  and  $\mathcal{W}^s(\beta_{\bar{\mu}_n})$ .

*Proof.* — We use the equation  $\pi^{ss} \circ \pi_1^{uu} \circ \mathcal{P}_\mu = \mathcal{P}_\mu^c \circ \pi^{uu} \circ \pi_1^{ss}$  and Lemma 1 of [6, p. 21] to obtain the result.

In what follows, we choose  $I_1 = ]\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon[$  as a neighborhood of  $\bar{\mu}$  in  $I$ ,  $\varepsilon$  as in the lemma.

**LEMMA 4.** — (I. Malta and J. Palis [6]) *Given  $\{X_\mu\} \in \Gamma_{sn}^1$  and  $\bar{\mu} \in B(\{X_\mu\})$  as above. Let  $x_1, y_1 \in \mathcal{W}_{\bar{\mu}}^c(q)$  be near  $q$  such that*

$$x_1 < x^c(\bar{\mu}) < \mathcal{P}_{\bar{\mu}}^c(x_1) < q < y_1 < y^c(\bar{\mu}) < \mathcal{P}_{\bar{\mu}}^c(y_1).$$

Let  $\mathcal{Z}$  be a  $C^5$  saddle-node vector field adapted to  $\{\mathcal{P}_\mu^c\}$  defined in a neighborhood of  $(q, \bar{\mu})$  in  $\mathcal{W}^c(q)$ . Then there exists a local conjugacy,  $H = (h, \eta)$ , between  $\{\mathcal{P}_\mu^c\}$  and  $\mathcal{Z}_{t=1}$  such that:

- (a)  $h_\mu(x^c(\mu)) = x^c(\mu)$ ,  $h_\mu(y^c(\mu)) = y^c(\mu)$ ;
- (b)  $(\mathcal{P}_\mu^c)^n(x_c(\mu)) = y^c(\mu)$  if and only if  $\mathcal{Z}_1^n(x^c(\mu), \eta(\mu)) = y^c(\mu)$ , where  $x^c(\mu) = \pi^{ss}(x(\mu))$ ,  $y^c(\mu) = \pi^{uu}(y(\mu))$  and  $x(\mu)$  (resp.  $y(\mu)$ ) is the point of  $s$ -criticality (resp.  $u$ -criticality) between  $\mathcal{W}^u(\alpha_\mu)$  and  $\mathcal{F}^{ss}(q)$  (resp.  $\mathcal{W}^s(\beta_\mu)$  and  $\mathcal{F}^{uu}(q)$ ) in a fundamental domain  $D_\mu^{cs}(q)$  (resp.  $D_\mu^{cu}(q)$ ) for  $\mathcal{P}/\mathcal{W}_\mu^{cs}(q)$  (resp. for  $\mathcal{P}/\mathcal{W}_\mu^{cu}(q)$ ) for  $\mu$  near  $\bar{\mu}$ .

For the definition of a saddle-node vector field adapted to saddle-node diffeomorphism arcs (see [8]).

Let  $\{X_\mu\} \{\tilde{X}_\mu\} \in \Gamma_{sn}^1$  be close families, and  $\bar{\mu}, \tilde{\mu} \in I$  be their first bifurcation values. If we let  $\mathcal{Z}, \tilde{\mathcal{Z}}$  denote the  $C^5$  saddle-node vector fields adapted to the saddle-node diffeomorphism arcs  $\{\mathcal{P}_\mu^c\}$  and  $\{\tilde{\mathcal{P}}_\mu^c\}$  respectively, then:

- (a) there exists a conjugacy  $h_2^c : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  between  $\mathcal{Z}$  and  $\tilde{\mathcal{Z}}$ ; this conjugacy induces a conjugacy between  $\mathcal{Z}_{t=1}$  and  $\tilde{\mathcal{Z}}_{t=1}$ , which we will denote by  $h_2 = (h_{2,\mu}, k)$  where  $k : (I_1, \bar{\mu}) \rightarrow (\tilde{I}_1, \tilde{\mu})$  is a reparametrization and  $h_{2,\mu} : \mathcal{W}_\mu^c(q) \rightarrow \mathcal{W}_{k(\mu)}^c(\tilde{q})$  is a conjugacy between  $\mathcal{Z}_{t=1}/\mathcal{W}_\mu^c(q)$  and  $\tilde{\mathcal{Z}}_{t=1}/\mathcal{W}_{k(\mu)}^c(\tilde{q})$  (see [8]);
- (b) there are conjugacies:

- $h_1 : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  between  $\{\mathcal{P}_\mu^c\}$  and  $\mathcal{Z}_{t=1}$ ,

$$h_1 = (h_{1,\mu}, \eta), \quad \eta : (I_1, \bar{\mu}) \rightarrow (\tilde{I}_1, \tilde{\mu})$$

reparametrization;

- $h_3 : \mathcal{W}^c(\tilde{q}) \rightarrow \mathcal{W}^c(q)$  between  $\{\tilde{\mathcal{P}}_\mu^c\}$  and  $\tilde{\mathcal{Z}}_{t=1}$ ,

$$h_3 = (h_{3,\mu}, \tilde{\eta}), \quad \tilde{\eta} : (\tilde{I}_1, \tilde{\mu}) \rightarrow (I_1, \bar{\mu})$$

reparametrization.

Thus  $h^c : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  given by

$$h^c(x, \mu) = (h_{3,\mu}^{-1} \circ h_{2,\mu} \circ h_{1,\mu}(x), \tilde{\eta}^{-1} \circ k \circ \eta(\mu))$$

is a conjugacy between  $\{\mathcal{P}_\mu^c\}$  and  $\{\tilde{\mathcal{P}}_\mu^c\}$  such that the reparametrization  $\rho = \tilde{\eta}^{-1} \circ k \circ \eta$  sends the parameter values  $\bar{\mu}_n$ , for which there exists a quasi-transversal intersection orbit between  $\mathcal{W}^u(\alpha_{\bar{\mu}_n})$  and  $\mathcal{W}^s(\beta_{\bar{\mu}_n})$ , into the corresponding parameter values  $\tilde{\mu}_{m(n)}$  for which there exists a quasi-transversal intersection orbit between  $\mathcal{W}^u(\tilde{\alpha}_{\tilde{\mu}_{m(n)}})$  and  $\mathcal{W}^s(\tilde{\beta}_{\tilde{\mu}_{m(n)}})$ .

### Construction of equivalences in $\mathcal{W}^{cs}$ ( $\mathcal{W}^{cu}$ )

Let  $\alpha = \alpha_{\bar{\mu}}$  be a singularity (hyperbolic) of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha)$  has a generic  $s$ -criticality with  $\mathcal{F}^{ss}(q)/\mathcal{W}^s(q)$  in a fundamental domain  $D_{\bar{\mu}}^{cs}(q)$  for  $\mathcal{P}_{\bar{\mu}}/\mathcal{W}_{\bar{\mu}}^{cs}(q)$ .

LEMMA 5. — Under the above conditions, there exists a  $C^r$  ( $1 \leq r \leq \infty$ ) vector field  $Y$  defined in a neighborhood of  $\mathcal{W}^u(\alpha) \times \{\bar{\mu}\} \cap \mathcal{W}_{\bar{\mu}}^{cs}(q)$  in  $D_{\bar{\mu}}^{cs}(q)$  which satisfies:

- (a) the only singularities of  $Y$  are the tangency points between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  in  $D_{\bar{\mu}}^{cs}(q)$ ;
- (b) if the  $s$ -criticality between  $\mathcal{W}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$  is of codimension  $p$  ( $p \geq 1$ ), then  $Y$  is of the saddle-node type, where a center-manifold is the  $C^1$  curve given by the tangencies among the leaves of  $\mathcal{F}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)$ ;
- (c)  $Y$  is transversal to the leaves of  $\mathcal{F}^{ss}(q)$  except at the singularity;
- (d)  $Y$  is tangent at the leaves of  $\mathcal{F}^{cu}(\alpha)$  and  $\mathcal{F}_1^{cu}(\alpha)$  in a neighborhood of the  $s$ -criticality.

*Proof.* — We let  $p$  denote the codimension of the  $s$ -criticality.

Case 1.  $p = 1$

Let  $(y_1, \dots, y_s, x)$  be a coordinate in  $\mathcal{W}_{\bar{\mu}}^{cs}(q)$ , where  $(y_1, \dots, y_s)$  are the coordinates in  $\mathcal{W}^{ss}(q)$ , and  $x$  is the coordinate in  $\mathcal{W}_{\bar{\mu}}^c(q)$ . In these coordinates,  $\mathcal{F}^{ss}(q)$  is given by a  $C^r$  ( $r \geq 2$ ) projection  $\pi^{ss} : \mathcal{W}_{\bar{\mu}}^{cs}(q) \rightarrow \mathcal{W}_{\bar{\mu}}^c(q)$ ,

$$\pi^{ss}(y_1, \dots, y_s, x) = x + Q(y_1, \dots, y_s, x)$$

where  $Q$  only has terms of degree  $\geq 2$ . In  $D_{\bar{\mu}}^{cs}(q)$ ,  $\mathcal{W}^u(\alpha) \cap D_{\bar{\mu}}^{cs}(q)$  is given by

$$x = F(y), \quad y = (y_1, \dots, y_s),$$

where  $F$  is a Morse function with a unique critical point  $x_0$  which corresponds to the tangency between  $\mathcal{W}^{cu}(\alpha) \cap D_{\bar{\mu}}^{cs}(q)$  and  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\bar{\mu}}^{cs}(q)$ . Let  $\gamma$  be a  $C^1$  curve given by the tangencies among the leaves of  $\mathcal{F}^u(\alpha)$  and the ones of  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\bar{\mu}}^{cs}(q)$ . Restricting  $\pi^{ss}$  to  $\mathcal{W}^u(\alpha) \cap D_{\bar{\mu}}^{cs}(q)$ , the tangency condition between  $\mathcal{F}^u(\alpha)$  and  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\bar{\mu}}^{cs}(q)$  is given by the differential equation

$$\frac{\partial F}{\partial y} + \frac{\partial Q}{\partial y} + \frac{\partial Q}{\partial x} \cdot \frac{\partial F}{\partial y} = 0.$$

Therefore, in a neighborhood of  $\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q)$  in  $\mathcal{W}_{\mu}^{cs}(q)$ , we consider the vector field  $Y$ :

$$Y : \begin{cases} \dot{x} = (x - F(y))^2 + \frac{\partial F}{\partial y} \dot{y} \\ \dot{y}_i = \frac{\partial \pi^{ss}}{\partial x}(x, y) \cdot \frac{\partial F}{\partial y_i}(y) + \frac{\partial \pi^{ss}}{\partial y_i}(y, x), \quad i = 1, \dots, s. \end{cases}$$

The singularities of  $Y$  are given by  $\dot{y} = (\dot{y}_1, \dots, \dot{y}_s)$  and  $F(y) - x = 0$ : that is the tangency point  $x_0$ . On the other hand,  $Y/(\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q))$  has the form

$$\begin{cases} \dot{x} = \frac{\partial F}{\partial y}(y) \dot{y} \\ \dot{y}_i = \frac{\partial \pi^{ss}}{\partial x}(y, x) \frac{\partial F}{\partial y_i}(y) + \frac{\partial \pi^{ss}}{\partial y_i}(y, x), \quad i = 1, \dots, s. \end{cases}$$

Thus,  $Y$  is tangent to  $\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q)$  and  $x_0$  is a hyperbolic singularity of  $Y/(\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q))$ . In addition

$$Y/\gamma(y, x) = \left( (x - F(y))^2, 0 \right);$$

thus  $x_0$  is a saddle-node singularity for  $Y/\gamma$ .

We have

$$\begin{aligned} \frac{\partial \pi^{ss}}{\partial x} \dot{x} + \frac{\partial \pi^{ss}}{\partial y} \dot{y} &= \frac{\partial \pi^{ss}}{\partial x} (x - F(y))^2 + \frac{\partial \pi^{ss}}{\partial x} \frac{\partial F^{ss}}{\partial y} \dot{y} + \frac{\partial \pi^{ss}}{\partial y} \dot{y} \\ &= \frac{\partial \pi^{ss}}{\partial x} (x - F(y))^2 + \left( \frac{\partial \pi^{ss}}{\partial x} \frac{\partial F^{ss}}{\partial y} + \frac{\partial \pi^{ss}}{\partial y} \right) \dot{y} \\ &= \frac{\partial \pi^{ss}}{\partial x} (x - F(y))^2 + \|\dot{y}\| > 0, \quad \text{if } x \neq x_0. \end{aligned}$$

Then  $Y$  is transversal to the leaves of  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\mu}^{cs}(q)$ .

*Case 2.*  $1 < p \leq \dim \mathcal{W}_{\mu}^{cs}(q)$

Let  $\mathcal{W}^{cu}(\alpha)$  be a  $C^r$  ( $r \geq 2$ ) center-unstable manifold of  $\alpha$ . Since  $\mathcal{W}^u(\alpha)$  is a codimension 1 submanifold of  $\mathcal{W}^{cu}(\alpha)$ , by Case 1, there exists a  $C^1$  vector field  $Y_1$  defined in a neighborhood of  $\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q)$  in  $\mathcal{W}^{cu}(\alpha) \cap D_{\mu}^{cs}(q)$ . Now as in [11], we extend  $Y_1$  to a neighborhood of  $\mathcal{W}^u(\alpha) \cap D_{\mu}^{cs}(q)$  in  $\mathcal{W}_{\mu}^{cs}(q)$ . The only condition we require for this extension

is that the extended vector field be tangent to the leaves of  $\mathcal{F}^{cu}(\alpha)$  and  $\mathcal{F}_1^{cu}(\alpha)$ .

As above: let  $\{X_\mu\} \in \Gamma_{sn}^1$ ; let  $\bar{\mu} \in B(\{X_\mu\})$ ; and let  $\mathcal{P} = \{\mathcal{P}_\mu\}$ . Let

$$x^c(\mu) \in (\mathcal{W}^c(q) \cap \Sigma_q \times \{\mu\})$$

vary continuously with  $\mu \in I_1$  such that the continuous curve  $\mu \rightarrow (x^c(\mu), \mu)$  is transversal to  $\mathcal{W}_\mu^c(q)$ . Let  $A_\mu^s \subseteq \mathcal{F}^{ss}(x^c(\mu))$  be a continuous family of closed discs centered at  $x^c(\mu)$ . We now let  $C \subseteq \mathcal{W}_\mu^{cs}(q)$  be a transversal cylinder to  $\mathcal{W}^{ss}(q)$  such that

$$C \times \{\mu\} \cap F^{ss}(x^c(\mu)) = \partial A_\mu^c$$

and, for each  $\mu \in [\bar{\mu} - \varepsilon, \bar{\mu}]$ ,  $C \times \{\mu\}$  is transversal to  $\mathcal{W}^{ss}(q_{1,\mu})$  and to  $\mathcal{W}^{ss}(q_{2,\mu})$ , and  $C \cap \mathcal{W}^{cu}(\alpha) = \emptyset$  for each critical element  $\alpha$  of  $X_{\bar{\mu}}$  such that  $\mathcal{W}^u(\alpha) \cap \mathcal{W}^{ss}(q) = \emptyset$ . Now we define a fundamental domain  $D_\mu^{cs}$  for  $(\mathcal{P}_\mu, \bar{\mu})$  in  $\mathcal{W}_\mu^{cs}(q)$  as: the external boundary of  $D_\mu^{cs}$  is  $C \cup A_\mu^c$  and the internal boundary is  $\mathcal{P}_\mu(C \cup A_\mu^c)$ .

A fundamental domain for  $\mathcal{P}/W^{cs}(q)$  is defined as: the external boundary is  $A^s \cup C \times I_1$  and the internal boundary is  $\mathcal{P}(A^s \cup C \times I_1)$ ,  $A^s = \bigcup_{\mu \in I_1} A_\mu^s$ .

If  $\{\tilde{X}_\mu\} \in \Gamma_{sn}^1$  is close to  $\{X_\mu\}$ , we assume the same hypotheses and constructions for  $\{\tilde{X}_\mu\}$  as have been made for  $\{X_\mu\}$ .

### Construction of $h^{cs}$ ( $h^{cu}$ )

The construction of the homeomorphism  $h^{cs}$  ( $h^{cu}$ ) is carried out in several steps which depend on the  $s(u)$ -criticality.

#### Step 1. Codimension one critical

We assume  $\mathcal{W}^u(\alpha_{k-1,\bar{\mu}})$  has a codimension one  $s$ -criticality. Let  $x_0(\mu) \in D_\mu^{cs}(q)$  be the point of  $s$ -criticality. Since  $x_0(\bar{\mu})$  is generic,  $\mathcal{W}^u(\alpha_{k-1,\bar{\mu}}) \cap D_\mu^{cs}(q)$  may be expressed as the graph of Morse function

$$\mathcal{F}_\mu : \mathcal{W}^{ss}(q) \rightarrow \mathcal{W}_\mu^c(q).$$

Case 1.  $\mathcal{F}_{\bar{\mu}}$  has a minimum (or maximum) critical point at  $x_0(\bar{\mu})$

Let  $\{Y_{2,\mu}\}$  be family of  $C^1$  vector fields defined in a neighborhood of

$$\bigcup_{\mu \in I_1} (\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q)$$

as in Lemma 5. We may assume the continuous curve  $\mu \rightarrow (x_0(\mu), \mu)$  is transversal to  $\mathcal{W}_{\mu}^{cs}(q)$ . Since  $\mathcal{W}^u(\alpha_{i,\mu}) \times \{\mu\}$  is transversal to  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\mu}^{cs}(q)$ ,  $i \neq k-1$ , we may construct a family of  $C^1$  vector fields  $\{Y_{1,\mu}\}$ , such that restricted to leaves of  $\mathcal{F}^u(\alpha_i)$ ,  $i \neq k-1$ , it is also  $C^1$ ,  $Y_{1,\mu}$  has no singularities in  $D_{\mu}^{cs}(q)$ , its trajectories are transversal to the leaves of  $\mathcal{F}^{ss}(q)/\mathcal{W}_{\mu}^{cs}(q)$ , and the cylinder  $C \times \{\mu\}$  is  $Y_{1,\mu}$ -invariant. We use the families  $\{Y_{1,\mu}\}$  and  $\{Y_{2,\mu}\}$ , and construct a family of  $C^1$  vector fields  $\{Y_{\mu}\}$  such that, in a neighborhood of  $\bigcup_{\mu \in I_1} (\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q)$  it coincides with  $\{Y_{2,\mu}\}$ , and on the outside it does so with  $\{Y_{1,\mu}\}$ .

Now let  $h^c : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  be a conjugacy between  $\{\mathcal{P}_{\mu}^c\}$  and  $\{\tilde{\mathcal{P}}_{\mu}^c\}$ , and let  $\rho(I_1, \bar{\mu}) \rightarrow (\tilde{I}_1, \tilde{\mu})$  be the reparametrization determined by  $h^c$ , and

$$H^s : \bigcup_{\mu \in I_1} \left( \bigcup_{i=1}^k \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \right) \rightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{i=1}^k \mathcal{W}^s(\tilde{\alpha}_{i,\mu}) \times \{\mu\} \right)$$

be the conjugacy constructed as before. We now consider a family  $\{D_{1,\mu}\}$  (resp.  $\{\tilde{D}_{1,\mu}\}$ ) of  $s$ -dimensional discs  $D_{1,\mu} \subseteq A_{\mu}^s$  (resp.  $\tilde{D}_{1,\mu} \subseteq \tilde{A}_{\mu}^s$ ) which varies continuously with the parameter as in figure 3.1. For each  $z \in D_{1,\mu}$ , we have that the  $\omega$ -limit of  $z$  is  $x_0(\mu)$ . Since  $\{X_{\mu}\}$  and  $\{\tilde{X}_{\mu}\}$  are close, for each  $\mu$  near  $\bar{\mu}$ , there exists a  $C^1$  diffeomorphism

$$h_{1,\mu} : D_{1,\mu} \rightarrow \tilde{D}_{1,\rho(\mu)}$$

$C^1$ -close to the inclusion map which varies continuously with  $\mu$ , and such that  $h_{1,\mu}(\partial D_{1,\mu}) = \partial \tilde{D}_{1,\rho(\mu)}$ , and that  $h_{1,\mu}(r_{\mu}) = \tilde{r}_{\rho(\mu)}$  where  $r_{\mu} = \mathcal{W}_{Y_{\mu}}^c(x_0(\mu)) \cap D_{1,\mu}$  (resp.  $\tilde{r}_{\mu} = \mathcal{W}_{\tilde{Y}_{\mu}}^c(\tilde{x}_0(\mu)) \cap \tilde{D}_{1,\rho(\mu)}$ ).

Let  $D_{2,\mu} = \overline{\bigcup_{t \geq 0} Y_{t,\mu}(D_{1,\mu})}$  and let  $D_2 = \bigcup_{\mu \in I_1} D_{2,\mu} \times \{\mu\}$ ; similar definitions for  $\tilde{D}_{2,\mu}$  and  $\tilde{D}_2$ . Thus, for each  $z \in D_{2,\mu} \setminus \{x_0(\mu)\}$ , there exists a unique  $t(z) < 0$  such that  $Y_{\mu,t(z)}(z) \in D_{1,\mu}$ . Since we know

$$h_{1,\mu}(Y_{\mu,t(z)}(z)) \in \tilde{D}_{1,\rho(\mu)}$$

and

$$h^c(\pi^{ss}(z)) \in \mathcal{W}^c(\tilde{q}) \cap \Sigma_{\tilde{q}} \times \{\rho(\mu)\},$$

we define

$$h_{1,\mu}(z) = \left( \bigcup_{t \geq 0} \tilde{Y}_{\rho(\mu),t} \left( h_{1,\mu}(Y_{\mu,t}(z)) \right) \right) \cap F_{\tilde{q}}^{ss} \left( h^c(\pi^{ss}(z)) \right)$$

and

$$h_{1,\mu}(x_0(\mu)) = \tilde{x}_0(\rho(\mu)).$$

This defines a homeomorphism  $h_1 : D_2 \rightarrow \tilde{D}_2$  compatible with  $h^c$  and the reparametrization  $\rho$ . Now, let  $\{T_\mu\}$  be a continuous family of  $s$ -dimensional open discs centered at  $r_\mu$  such that  $T_\mu \supseteq D_{1,\mu}$  and that  $T_\mu \setminus \bar{D}_{1,\mu}$  is an open annulus. On  $\partial D_{1,\mu}$  we raise a radial one-dimensional foliation  $S_\mu^c$  whose leaves are contained in  $T_\mu \setminus D_{1,\mu}$ . The construction is performed continuously with  $\mu$  and in a compatible way with the leaves of  $\mathcal{F}^u(\alpha_i)$ ,  $i \neq k-1$ . Given  $x \in \partial D_{1,\mu}$ , let

$$\overset{\circ}{S}_\mu^c(x) = S_\mu^c(x) \setminus \{x\},$$

and let

$$F^c(x) = \bigcup_{t \geq 0} Y_{\mu,t}(\overset{\circ}{S}_\mu^c(x)).$$

Clearly  $F^c(x)$  is a two-dimensional surface. Let  $I^c$  be the one-dimensional foliation induced by  $\mathcal{F}^c$  in a neighborhood of  $i_\mu = \mathcal{W}_{Y_\mu}^c(x_0(\mu)) \cap \mathcal{P}(A_\mu^s)$ ; we extend  $I^c$  to a singular foliation by including the point  $i_\mu$ . Using  $H^s$ , we extend  $h_1$  to a homeomorphism

$$H_1 : T = \bigcup_{\mu \in I_1} T_\mu \times \{\mu\} \longrightarrow \tilde{T} = \bigcup_{\mu \in \tilde{I}_1} \tilde{T}_\mu \times \{\mu\}$$

such that, for each  $\mu \in I_1$ ,  $H_{1,\mu} = H_1/T_\mu \times \{\mu\}$ :

$$T_\mu \times \{\mu\} \longrightarrow \tilde{T}_{\rho(\mu)} \times \{\rho(\mu)\}$$

is a diffeomorphism  $C^1$ -close to the inclusion map; this  $H_1$  induces a homeomorphism  $H_1 : \mathcal{P}(T \setminus D_1) \longrightarrow \tilde{\mathcal{P}}(\tilde{T} \setminus \tilde{D}_1)$  where

$$D_1 = \bigcup_{\mu \in I_1} D_{1,\mu} \times \{\mu\}, \quad \tilde{D}_1 = \bigcup_{\mu \in \tilde{I}_1} \tilde{D}_{1,\mu} \times \{\mu\},$$

such that, for each  $\mu \in I_1$ ,

$$H_{1,\mu} : \mathcal{P}_\mu(T_\mu \setminus D_{1,\mu}) \longrightarrow \tilde{\mathcal{P}}_{\rho(\mu)}(\tilde{T}_{\rho(\mu)} \setminus \tilde{D}_{1,\rho(\mu)}) \quad \text{and} \quad H_{1,\mu}(i_\mu) = \tilde{i}_{\rho(\mu)}.$$

To extend  $H_1$  to a homeomorphism

$$\begin{aligned} H_1 : \bigcup_{\mu \in I_1} \left( \bigcup_{t \geq 0} (Y_{\mu,t}(T_\mu \setminus D_{1,\mu}) \times \{\mu\}) \right) \\ \longrightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{t \geq 0} (\tilde{Y}_{\mu,t}(\tilde{T}_\mu \setminus \tilde{D}_{1,\mu}) \times \{\mu\}) \right), \end{aligned}$$

we use the homeomorphisms  $h^c$  and  $H_1$ , the foliations  $\mathcal{F}^{ss}(q)$  and  $\mathcal{F}^{ss}(\tilde{q})$ , and the trajectories of  $Y_\mu$  and  $\tilde{Y}_\mu$  (fig. 3.2). This defines a homeomorphism  $H_1 : V \rightarrow \tilde{V}$ , where  $V$  (resp.  $\tilde{V}$ ) is a neighborhood of

$$\bigcup_{\mu \in I_1} (\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q)$$

(resp.  $\bigcup_{\mu \in \tilde{I}_1} (\mathcal{W}^u(\tilde{\alpha}_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(\tilde{q})$  in  $D^{cs}(q)$  (resp.  $D^{cs}(\tilde{q})$ ), which satisfies:

- it is compatible with  $h^c$ ,  $H^s$  and  $\rho$ ,
- it conjugates  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ .

To obtain a homeomorphism  $h^{cs} : D^{cs}(q) \rightarrow D^{cs}(\tilde{q})$ , we extend the definition of  $H_1$  to  $C \times I_1 \cup A^s$  using the leaves of  $\mathcal{F}^{ss}(q)$  and of  $\mathcal{F}^{ss}(\tilde{q})$  as well as the trajectories of  $Y$  and of  $\tilde{Y}$ , the definition of  $H^s$  and the Isotopy Extension Theorem as is done in [2]. Using  $H_1$  we define a homeomorphism

$$H_1 : \mathcal{P}(C \times I_1 \cup A^s) \longrightarrow \tilde{\mathcal{P}}(\tilde{C} \times \tilde{I}_1 \cup \tilde{A}^s).$$

Finally, we extend these homeomorphisms to  $D^{cs}$  using the trajectories of  $Y_\mu$  and of  $\tilde{Y}_\mu$ , the leaves of  $\mathcal{F}^{ss}(q)$  and of  $\mathcal{F}^{ss}(\tilde{q})$ , and the homeomorphisms  $h^c$  and the one defined in  $(C \times I_1 \cup A^s) \cup \mathcal{P}(C \times I_1 \cup A^s)$ . We denote by  $h^{cs} : \mathcal{W}^{cs}(q) \rightarrow \mathcal{W}^{cs}(\tilde{q})$  the resulting homeomorphism. By construction it is clear that  $h^{cs}$  verifies

Global stability of saddle-node bifurcation of a periodic orbit for vector fields

- it is compatible with  $h^c$ ,  $H^c$  and  $\rho$ ,
- it sends leaves of  $\mathcal{F}^{ss}(q)$  into leaves of  $\mathcal{F}^{ss}(\tilde{q})$ ,
- $h^{cs}(x_0(\mu)) = \tilde{x}(\rho(\mu))$ ,  $h_{\bar{\mu}}^{cs}(q) = \tilde{q}$ ,  $h_{\mu}^{cs}(q_{j,\mu}) = \tilde{q}_{j,\rho(\mu)}$ ,  $j = 1, 2$ .

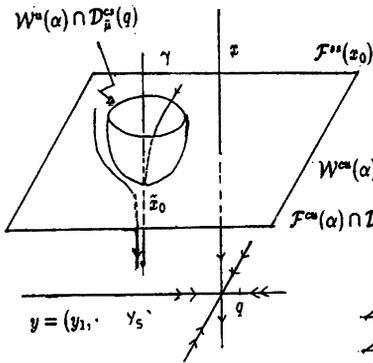


Fig. 3.1

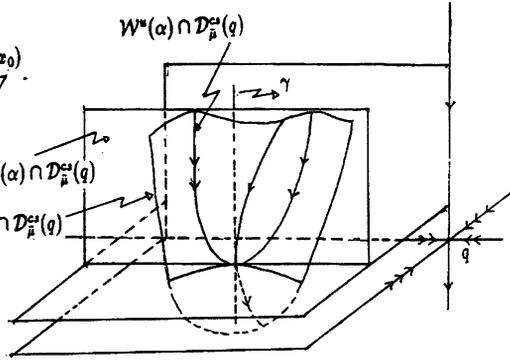


Fig. 3.2

Case 2.  $\mathcal{F}_{\bar{\mu}}$  has a saddle type singularity at  $x_0(\bar{\mu})$

In this case the family  $\{Y_{2,\mu}\}$  of  $C^1$  vector fields given by Lemma 5 has a center-unstable and a center-stable manifolds; we construct homeomorphisms on these manifolds as in Case 1. The extension of these homeomorphisms to  $h^{cs} : \mathcal{W}^{cs}(q) \rightarrow \mathcal{W}^{cs}(\tilde{q})$  is also done as in Case 1.

## Step 2. Codimension $p > 1$ criticality

In this case, we take a center-unstable manifold  $\mathcal{W}^{cu}(\alpha_{k-1,\bar{\mu}})$  and, in

$$\bigcup_{\mu \in I_1} (\mathcal{W}^{cu}(\alpha_{k-1,\mu}) \times \{\mu\}) \cap D^{cs}(q),$$

we construct a homeomorphism as in Step 1, the extension to a homeomorphism  $h^{cs} : \mathcal{W}^{cs}(q) \rightarrow \mathcal{W}^{cs}(\tilde{q})$  is done as above.

### Global stability of families in $\Gamma_{sn}^1$

Let  $\{X_\mu\} \in \Gamma_{sn}^1$  such that, in its first bifurcation value  $\bar{\mu} \in I$ ,  $X_{\bar{\mu}}$  has a saddle-node periodic orbit  $\theta$ . In what follows we will suppose the critical elements of  $\{X_\mu\}$  ordered as follows:

$$\begin{cases} \mu < \bar{\mu}: & \alpha_{1,\mu} \leq \dots \leq \alpha_{k,\mu} \leq \theta_{1,\mu} \leq \theta_{2,\mu} \leq \beta_{1,\mu} \leq \dots \leq \beta_{\ell,\mu} \\ \mu = \bar{\mu}: & \alpha_{1,\bar{\mu}} \leq \dots \leq \alpha_{k,\bar{\mu}} \leq \theta \leq \beta_{1,\bar{\mu}} \leq \dots \leq \beta_{\ell,\bar{\mu}} \\ \mu > \bar{\mu}: & \alpha_{1,\mu} \leq \dots \leq \alpha_{k,\mu} \leq \beta_{1,\mu} \leq \dots \leq \beta_{\ell,\mu} \end{cases}$$

where for  $\mu < \bar{\mu}$ ,  $\theta_{1,\mu}$  and  $\theta_{2,\mu}$  are the periodic orbits of  $X_\mu$  which produce the saddle-node periodic orbit  $\theta$ .

Let  $\{\tilde{X}_\mu\} \in \Gamma_{sn}^1$  be close to  $\{X_\mu\}$  which has its first bifurcation value  $\tilde{\mu}$  near  $\bar{\mu}$  at which  $\tilde{X}_{\tilde{\mu}}$  has a saddle-node periodic orbit  $\tilde{\theta}$  near  $\theta$ . We let  $\tilde{\alpha}_{i,\mu}$ ,  $\tilde{\beta}_{j,\mu}$ ,  $\tilde{\theta}_{1,\mu}$ ,  $\tilde{\theta}_{2,\mu}$ ,  $\tilde{\theta}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, \ell$ , denote the critical elements of  $\tilde{X}_\mu$  near the respective critical elements of  $X_\mu$ . As before, each time we make an assumption for  $\{X_\mu\}$ , we will suppose it done for  $\{\tilde{X}_\mu\}$ .

The proof of the global stability of families in  $\Gamma_{sn}^1$  will be done in several steps according to the types of  $s$ -criticalities. We will prove the stability in the codimension one  $s$ -criticality,  $u$ -criticality case; the remaining cases may be proved with similar arguments.

In what follows we will assume that  $\mathcal{W}^u(\alpha_{k-1,\bar{\mu}})$  as a codimension one  $s$ -criticality and that  $\mathcal{W}^s(\beta_{2,\bar{\mu}})$  has a codimension one  $u$ -criticality. Let  $x_0(\mu)$  (resp.  $y_0(\mu)$ ) be the point of  $s$ -criticality (resp.  $u$ -criticality) in  $D_\mu^{cs}(q)$  (resp.  $D_\mu^{cu}(q)$ ), where  $D_\mu^{cs}(q)$  (resp.  $D_\mu^{cu}(q)$ ) is constructed in such a way that

$$\pi^{ss}(x_0(\mu)) = x_0^c(\mu) \quad (\text{resp. } \pi^{uu}(y_0(\mu)) = y_0^c(\mu))$$

is in the interior of  $\mathcal{W}^c(q) \cap D_\mu^{cs}(q)$  (resp.  $\mathcal{W}^c(q) \cap D_\mu^{cu}(q)$ ). In addition we suppose the continuous curves  $\mu \rightarrow (x_0(\mu), \mu)$  and  $\mu \rightarrow (y_0(\mu), \mu)$  are transversal to  $\mathcal{W}_\mu^{cs}(q)$ ,  $\mathcal{W}_\mu^{cu}(q)$ , respectively. Recall that we already have the homeomorphisms

$$h^{cs} : \mathcal{W}^{cs}(q) \longrightarrow \mathcal{W}^{cs}(\tilde{q}) \quad (1)$$

$$h^{cu} : \mathcal{W}^{cu}(q) \longrightarrow \mathcal{W}^{cu}(\tilde{q}) \quad (2)$$

$$H^s : \bigcup_{\mu \in I_1} \left( \bigcup_{i=1}^k \mathcal{W}^s(\alpha_{i,\mu}) \times \{\mu\} \right) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{i=1}^k \mathcal{W}^s(\tilde{\alpha}_{i,\mu}) \times \{\mu\} \right) \quad (3)$$

$$H^u : \bigcup_{\mu \in I_1} \left( \bigcup_{j=1}^{\ell} \mathcal{W}^u(\beta_{j,\mu}) \times \{\mu\} \right) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} \left( \bigcup_{j=1}^{\ell} \mathcal{W}^u(\tilde{\beta}_{j,\mu}) \times \{\mu\} \right) \quad (4)$$

$$h^c : \mathcal{W}^c(q) \longrightarrow \mathcal{W}^c(\tilde{q}). \quad (5)$$

Now, by the transversality of

- (a)  $\mathcal{W}^u(\alpha_i)$ ,  $\mathcal{W}^s(\beta_j)$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, \ell$ , with  $\mathcal{W}^{cs}(q)$ ,  $\mathcal{W}^{cu}(q)$ , respectively,
- (b)  $\mathcal{W}^u(\alpha_i)$ ,  $\mathcal{W}^s(\beta_j)$ ,  $i \neq k-1$ ,  $j \neq 2$ , with  $\mathcal{F}^{ss}(q)/\mathcal{W}^{cs}(q)$ ,  $\mathcal{F}^{uu}(q)/\mathcal{W}^{cu}(q)$ , respectively,

in  $\Sigma_q \times I$ , we may construct  $C^1$ ,  $\mathcal{P}$ -invariant foliations  $\mathcal{F}^{uu}(q)$  and  $\mathcal{F}^{ss}(q)$  compatible with the systems of foliations  $\mathcal{F}^u(\alpha_1), \dots, \mathcal{F}^u(\alpha_k)$ ,  $\mathcal{F}^s(\beta_1), \dots, \mathcal{F}^s(\beta_\ell)$ . Moreover, we may construct a codimension two  $C^1$ ,  $\mathcal{P}$ -invariant foliation  $\mathcal{F}^{su}(q)$  in  $\Sigma_q \times I_1$  which is compatible with the above foliations. We let  $\{Y_\mu^{cs}\}$  and  $\{Y_\mu^{cu}\}$  denote the families of  $C^1$  vector fields constructed in  $\mathcal{W}^{cs}(q)$  and  $\mathcal{W}^{cu}(q)$  as in Lemme 5, respectively. Using  $\{Y_\mu^{cs}\}$  and  $\{Y_\mu^{cu}\}$ , we construct the singular foliations  $\mathcal{F}^{cs}(q)$  and  $\mathcal{F}^{cu}(q)$  in  $\Sigma_q \times I_1$  as follows:

Each leaf  $F^{cs}$  of  $\mathcal{F}^{cs}(q)$  is a union of leaves of  $\mathcal{F}^{uu}(q)$ , where the union is taken over a trajectory of  $Y_\mu^{cs}$ . Note that each leaf of  $\mathcal{F}^{cs}(q)$  intersects transversally  $\mathcal{W}^{cu}(q)$  along a trajectory of  $\{Y_\mu^{cs}\}$ , and that  $\mathcal{F}^{cs}(q)$  has distinguished leaves:

- one of them is  $F^{cs}(x_0(\mu))$  which is raised over  $\mathcal{W}_{Y_\mu^{cs}}^c(x_0(\mu))$ ;
- the others are those raised over the trajectories of

$$Y_\mu^{cs} / ((\mathcal{W}^s(\alpha_{k-1,\mu}) \cap \Sigma_q) \times \{\mu\}).$$

Analogously we construct  $\mathcal{F}^{cu}(q)$ . Without loss of generality, we will suppose that, for each  $\mu > \bar{\mu}$ ,

$$L_\mu = F^{cs}(x_0(\mu)) \cap F^{cu}(y_0(\mu))$$

contains the tangencies between the leaves of  $\mathcal{F}^u(\alpha_{k-1})$  and  $\mathcal{F}^s(\beta_2)$ . We set  $L = \bigcup_{\mu \in I_1} L_\mu \times \{\mu\}$  ( $\dim L = 2$ ).

In  $\bigcup_{\mu \in I_1} (F^{cs}(x_0(\mu)) \times \{\mu\})$  (resp.  $\bigcup_{\mu \in I_1} (F^{cu}(y_0(\mu)) \times \{\mu\})$ ), we consider the induced foliations:

- $\mathcal{F}^{uu} = \mathcal{F}^{su}(q) \cap F^{cs}(x_0(\mu))$  (resp.  $\mathcal{F}^{ss} = \mathcal{F}^{su}(q) \cap F^{cu}(y_0(\mu))$ ) whose space of leaves is  $L$ ;

- $\mathcal{F}_1 = \mathcal{F}^{cu}(q) \cap F^{cs}(x_0(\mu))$  (resp.  $\mathcal{F}_2 = \mathcal{F}^{cs}(q) \cap F^{cu}(y_0(\mu))$ ),

and the homeomorphism  $h_L^c : L \rightarrow \tilde{L}$  induced by  $h^c : \mathcal{W}^c(q) \rightarrow \mathcal{W}^c(\tilde{q})$  via leaves of  $\mathcal{F}^{su}(q)$ . Note that, for each  $\mu \in I_1$ ,

$$h_{L,\mu}^c = h_L^c/L_\mu \times \{\mu\} : L_\mu \times \{\mu\} \rightarrow \tilde{L}_{\rho(\mu)} \times \{\rho(\mu)\}.$$

We now use the homeomorphisms  $h_L^c : L \rightarrow \tilde{L}$ ,  $h^{cs} : \mathcal{W}^{cs}(q) \rightarrow \mathcal{W}^{cs}(\tilde{q})$ , and  $h^{cu} : \mathcal{W}^{cu}(q) \rightarrow \mathcal{W}^{cu}(\tilde{q})$  to construct homeomorphisms

$$H_1 : \bigcup_{\mu \in I_1} (F^{cs}(x_0(\mu)) \times \{\mu\}) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} (F^{cs}(\tilde{x}_0(\mu)) \times \{\mu\})$$

and

$$H_2 : \bigcup_{\mu \in I_1} (F^{cu}(y_0(\mu)) \times \{\mu\}) \longrightarrow \bigcup_{\mu \in \tilde{I}_1} (\tilde{F}^{cu}(\tilde{y}_0(\mu)) \times \{\mu\})$$

such that:

- $H_1$  and  $H_2$  are compatible with  $h_L^c$ ;
- for each  $\mu \in I_1$ ,

$$\begin{aligned} H_{1,\mu} &= H_1/F^{cs}(x_0(\mu)) \times \{\mu\} : \\ &F^{cs}(x_0(\mu)) \times \{\mu\} \longrightarrow \tilde{F}^{cs}(\tilde{x}_0(\rho(\mu))) \times \{\rho(\mu)\} \end{aligned}$$

and

$$\begin{aligned} H_{2,\mu} &= H_2/F^{cu}(y_0(\mu)) \times \{\mu\} : \\ &F^{cu}(y_0(\mu)) \times \{\mu\} \longrightarrow \tilde{F}^{cu}(\tilde{y}_0(\rho(\mu))) \times \{\rho(\mu)\}; \end{aligned}$$

- $H_1$  (resp.  $H_2$ ) sends leaves of  $\mathcal{F}^{uu}$  and  $\mathcal{F}_1$  (resp.  $\mathcal{F}^{ss}$  and  $\mathcal{F}_2$ ) into the respective ones of  $\tilde{\mathcal{F}}^{uu}$  and  $\tilde{\mathcal{F}}_1$  (resp.  $\tilde{\mathcal{F}}^{ss}$  and  $\tilde{\mathcal{F}}_2$ ); clearly, this construction is compatible with the ones above.

To construct the homeomorphism  $H_q : \Sigma_q \times I_1 \rightarrow \Sigma_{\tilde{q}} \times \tilde{I}_1$  which conjugates the Poincaré maps  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$ , we use the homeomorphisms  $h^{cs}$ ,  $h^{cu}$ ,  $H_1$ ,  $H_2$ ,  $h$ ,  $h_L^c$ ,  $H^s$ ,  $H^u$  and the foliations  $\mathcal{F}^{ss}(q)$ ,  $\mathcal{F}^{uu}(q)$ ,  $\mathcal{F}^{su}(q)$ ,  $\mathcal{F}^{cs}(q)$ ,  $\mathcal{F}^{cu}(q)$  for  $\{X_\mu\}$ , and the respective ones for  $\{\tilde{X}_\mu\}$ . The construction is carried out preserving the later foliations (see [8]). Finally,

we extend the homeomorphism  $H^s$  to neighborhoods of the critical elements  $\beta_1, \beta_2, \dots, \beta_\ell$  thus obtaining a global equivalence  $H = (H_\mu, \rho) : M \times I_1 \rightarrow M \times \tilde{I}_1$  between  $\{X_\mu\}$ , and  $\{\tilde{X}_\mu\}$ .

*Remark 1.* — Since  $\mathcal{W}^s(\beta_{2,\bar{\mu}})$  is  $u$ -critical of codimension one, we have that  $\dim \mathcal{W}^u(\beta_{2,\bar{\mu}}) = 1$ ; therefore, to extend the definition of the homeomorphism  $H^s$  to a neighborhood of  $\beta_2$  via Lyapunov functions, the preservation of  $\mathcal{F}^s(\beta_2)$  is not required.

*Remark 2.* — The construction in

$$\bigcup_{\mu \in I_1} ((F^{cs}(x_0(\mu)) \cup F^{cu}(y_0(\mu))) \times \{\mu\})$$

is essential to guarantee that, for each  $\bar{\mu}_n \in I_1$  given by Lemma 3, we are sending the orbits of quasi-transversal intersection between

$$(\mathcal{W}^u(\alpha_{k-1,\bar{\mu}_n}) \cap \Sigma_q) \times \{\bar{\mu}_n\} \quad \text{and} \quad (\mathcal{W}^s(\beta_{2,\bar{\mu}_n}) \cap \Sigma_q) \times \{\bar{\mu}_n\}$$

for  $X_{\bar{\mu}_n}$  into the respective orbits of quasi-transversal intersection between

$$(\mathcal{W}^u(\tilde{\alpha}_{k-1,\rho(\bar{\mu}_n)}) \cap \Sigma_{\tilde{q}}) \times \{\rho(\bar{\mu}_n)\} \quad \text{and} \quad (\mathcal{W}^s(\tilde{\beta}_{2,\rho(\bar{\mu}_n)}) \cap \Sigma_{\tilde{q}}) \times \{\rho(\bar{\mu}_n)\}$$

for  $\tilde{X}_{\rho(\bar{\mu}_n)}$ .

*Remark 3.* — Only on the surface  $L$ , the foliations  $\mathcal{F}^{cs}(q)$  and  $\mathcal{F}^{cu}(q)$  are not transverse, that is  $L$  contains the tangency points (quasi-transversal) between

$$\bigcup_{\mu \in I_1} (\mathcal{W}^u(\alpha_{k-1,\mu}) \times \{\mu\}) \cap \Sigma_q \times I_1 \quad \text{and} \quad \bigcup_{\mu \in I_1} (\mathcal{W}^s(\beta_{2,\mu}) \times \{\mu\}) \cap \Sigma_q \times I_1$$

at the parameter values  $\mu = \bar{\mu}_n$ .

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