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On approximate controllability of the Stokes system(*)

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RÉSUMÉ. — On étudie un problème d'une contrôlabilité approximative dans des espaces fonctionnels divers pour un système non stationnaire de Stokes qui décrit un courant de fluide visqueux incompressible dans un domaine borné Ω. On prend, en guise de fonction de contrôle, la densité de forces extérieures concentrée dans un sous-domaine $\omega \subset \Omega$ la condition aux limites concentrées dans un sous-domaine de la frontière, ou d'autres fonctions. On démontre que l'équation de Burgers ne possède pas de contrôlabilité approchée par rapport à la fonction de contrôle indiquée ci-dessus.

ABSTRACT. — The problem of the approximate controllability in different functional spaces of the nonstationary Stokes system which describes a viscous incompressible fluid flow in a bounded domain Ω is investigated. We use as a control function the density of external forces concentrated in an arbitrary subdomain $\omega \subset \Omega$, or the boundary datum concentrated in a subdomain of the boundary, or some other functions. It is proved that the Burgers equation is not approximately controllable with respect to the control functions mentioned above.

1. Introduction

This paper is devoted to the problem of controllability of the Stokes system, defined in a boundary domain $\Omega \subset \mathbb{R}^d$ with a C^∞-boundary $\partial\Omega$:

$$\partial_t y(t,x) - \Delta y + \nabla p(t,x) = u(t,x)\,, \quad \operatorname{div} y = \sum_{i=1}^{d} \frac{\partial y}{\partial x_i} = 0\,. \tag{1.1}$$

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Here $t \in [0, T]$ is the time, $x = (x_1, \ldots, x_d)$ is the spatial variables, $y(t,x) = (y_1, \ldots, y_d)$ is the vector field of a fluid velocity, Δ is the Laplace operator, $\nabla p(t,x)$ is the pressure gradient, $u(t,x) = (u_1, \ldots, u_d)$ is the density of external forces. We impose the boundary and initial conditions for system (1.1):

$$y\big|_{[0,T]\times\partial\Omega} = 0 \tag{1.2}$$

$$y\big|_{t=0} = y_0(x)\,. \tag{1.3}$$

Suppose that $u(t,x)$ in (1.1) is a control and it runs through a fonctional space U. Besides for an arbitrary $u \in U$, we suppose that the unique solution (y,p) of problem (1.1) to (1.3) exists in some corresponding functional space and that the restriction $y(T,\cdot)$ of y at time moment T belongs to the functional space H.

Problem (1.1) to (1.3) is called H-approximately controllable with respect to the control set U, if $y(T,\cdot)$ runs through a H-dense set when u runs through U.

The case when U is a space of vector fields concentrated in a certain subdomain ω of domain Ω is the main one in this paper. The problem of approximate controllability of the Stokes system for this kind of control was formulated by J.-L. Lions in [7], [9]. This problem has been studied in section 3 below. To formulate our results we introduce the functional space

$$\begin{aligned} &X^0 = \left\{ y(x) = (y_1, \ldots, y_d) \in \big(L_2(\Omega)\big)^d \,:\, \operatorname{div} y = 0\,,\ (y,n)\big|_{\partial\Omega} = 0 \right\}\,,\\ &\|\cdot\|_{X^0} = \|\cdot\|_{(L_2(\Omega))^d}\,, \end{aligned} \tag{1.4}$$

where $n = n(x')$, $x' \in \partial\Omega$ is a field of external normals to $\partial\Omega$, and (y,n) is the normal to $\partial\Omega$ component of field y. For a natural number k, we set

$$\begin{aligned} &X^k = \left\{ y(x) \in \big(W_2^k(\Omega)\big)^d : \operatorname{div} y = 0\,,\ y\big|_{\partial\Omega} = 0 \right\}\,,\\ &\|\cdot\|_{X^k} = \|\cdot\|_{(W_2^k(\Omega))^d}\,, \end{aligned} \tag{1.5}$$

where $W_2^k(\Omega)$ is the Sobolev space of functions which are quadratic integrable together with its derivatives up to the order k. The $W_2^k(\Omega)$-norm is defined below.

In section 3, we prove that problem (1.1) to (1.3) is X^k-approximately controllable with respect to U if $k = 0,\ 1,\ 2$. Here U is the subspace of all functions belonging to $L_2(0,T\,;X^0)$ and vanishing on $x \in \Omega \setminus \omega$.

Note that some arguments connected with the smoothness of the solutions force us to suppose, in the case $k = 2$, that the controls from U are equal to zero for t belonging to some neighbourhood of T. The case $k \geq 3$ is considered under the same assumption. But if $k \geq 3$ then approximate uncontrollability of this problem is proved. Besides in section 3, we show a method of construction of a sequence $u_\nu \in U$ such that the corresponding $y_\nu(T\,,\,\cdot\,)$ approximate the prescribed vector field $\widehat{y}(x)$ in X^k, $k \leq 2$.

An analogous result on the approximate controllability is proved also for impulse control, *i.e.* for controls having the form $\delta(t-t_0)v(x)$, where $\delta(t-t_0)$ is the Dirac measure and $\operatorname{supp} v \subset \omega \subset \Omega$, and for initial control with support belonging to a subdomain of Ω (section 4). A similar result is proved for the case when the control is a density of external forces concentrated on a hypersurface $S \subset \Omega$ (section 6). Here, we consider the case of a closed surfaces S (for the Stokes system of an arbitrary dimension d) and the case when S has a boundary (when $d = 2$). In section 5, problem (1.1) with $u(t,x) = 0$, (1.3) and with the boundary condition

$$u\big|_{[0,T]\times\partial\Omega} = v \tag{1.6}$$

is considered, where v is a control. In this case some results on the approximate controllability, similar to the results of section 3, are obtained. We also study the problem of the exact controllability. For an arbitrary initial condition $y_0(x)$, we prove the existence of a control v having support on the whole boundary $[\,0\,,\,T\,] \times \partial\Omega$, such that the solution (y,p) of problem (1.1), (1.3), (1.6) satisfies the equation $y(T,x) = 0$ (*i.e.* control v transforms initial value y_0 to zero). This result is deduced from an analogous assertion for the heat equation (see G. Schmidt [11]).

We remark that the problem of the approximate controllability for the Navier-Stokes system with distributed control concentrated in a subdomain $\omega \subset \Omega$ is open at the present. E. Fernandez-Cara and J. Real [3] have proved that the linear cover of the set $\{y(T,\,\cdot\,)\}$ is dense in X^0 where (y,p) is the solution corresponding to a control u and u runs through U. Nevertheless, it is possible that because of nonlinear term the Navier-Stokes system rests approximately uncontrollable in the sense of the definition which was given above. Indeed, J. I. Diaz [2] illustrated with a semilinear parabolic equation having a boundary control, that the presence of suitable nonlinearity can imply the approximate uncontrollability of equation. In section 7, we consider the Burgers equation which is connected with the Navier-Stokes system more closely than any semilinear parabolic equation.

It is proved that the Burgers equation is not approximately controllable with respect to a distributed control concentrated in subinterval as well as with respect to a boundary control. This result is obtained by means of a new estimate on a solution of the Burgers equation. This estimate shows that the velocity of change of the positive part of the solution at a finite distance to the left at the source of influence is bounded by a constant which does not depend of the magnitude of the influence.

Note that one of results of section 3 was announced in [5].

2. Preliminaries

We recall the definition of functional spaces, used for the investigation of problem (1.1) to (1.3). Set

$$V = \left\{ v(x) \in \left(C_0^\infty(\Omega) \right)^d ;\ \operatorname{div} v = 0 \right\} , \tag{2.1}$$

$$H^0(\Omega) \text{ is the closure of } V \text{ in } \left(L_2(\Omega) \right)^d , \tag{2.2}$$

$$\| \cdot \|_{H^0(\Omega)} = \| \cdot \|_{(L_2(\Omega))^d} .$$

$H^1(\Omega)$ is the closure of V in $\left(W_2^1(\Omega)\right)^d$; $H^2(\Omega) = H^1(\Omega) \cap \left(W_2^2(\Omega)\right)^d$ where $W_2^k(\Omega)$ is the Sobolev space of functions with finite norm

$$\|w\|_{W_2^k(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega \left| D^\alpha w(x) \right|^2 \mathrm{d}x \right)^{1/2} ,$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_d) , \quad |\alpha| = \alpha_1 + \cdots + \alpha_d , \quad D^\alpha w = \frac{\partial^{|\alpha|} w}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} .$$

Let

$$\Pi : \left(L_2(\Omega) \right)^d \longrightarrow H^0(\Omega) \tag{2.3}$$

be the orthogonal projection operator of space $\left(L_2(\Omega)\right)^d$ onto the space $H^0(\Omega)$. We consider the operator $\mathbf{A} = -\Pi\Delta$ in the space $H^0(\Omega)$. It is known (see V. A. Solonnikov [12]) that $\mathbf{A} : H^0(\Omega) \to H^0(\Omega)$ with domain $\mathbf{D}(\Omega) = H^2(\Omega)$ is a self-adjoint positive operator and its eigenfunctions $\{e_j\}$

form an orthonormal basis in $H^0(\Omega)$. We denote the eigenvalues of operator $\mathbf{A}$ by $0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots$. For arbitrary $\alpha \in \mathbb{R}$, we introduce the space

$$H^\alpha = \left\{ u = \sum_{j=1}^{\infty} u_j e_j \,,\ u_j \in \mathbb{R} : \|u\|_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha |u_j|^2 < \infty \right\} . \tag{2.4}$$

Denote by $R(f,g)$ the operator assigning to the functions f, g the solution z of the Neumann problem

$$\Delta z(x) = f(x)\,,\ x \in \Omega\,, \quad \left.\frac{\partial z}{\partial n}\right|_{\partial\Omega} = g \Leftrightarrow z(x) = R(f,g)\,, \tag{2.5}$$

where n is the external normal to $\partial\Omega$, f, g satisfy the condition

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\partial\Omega} g(x')\,\mathrm{d}x'$$

and $z(x)$ is the solution of problem (2.5) which satisfies the condition

$$\int_\Omega z(x)\,\mathrm{d}x = 0\,.$$

We relate the operator Π from (2.3) with the operator R from (2.5). For arbitrary $z \in \left(L_2(\Omega)\right)^d$ the Weyl decomposition holds (see O. A. Ladyzhenskaya [6], R. Temam [15]):

$$z = \Pi z + \nabla p\,, \tag{2.6}$$

where $\Pi z \in H^0(\Omega)$, $p \in W_2^1(\Omega)$. Applying the operator div to both sides of (2.6), we obtain that the function p is a solution of the Neumann problem (2.5) with $f(x) = \operatorname{div} z$, $g(x') = (z,n)\big|_{\partial\Omega}$. Hence,

$$\Pi z = z - \nabla R\left(\operatorname{div} z\,,\ (z,n)\big|_{\partial\Omega}\right)\,, \tag{2.7}$$

where $\operatorname{div} z$ and $(z,n)\big|_{\partial\Omega}$ can be understood in the sense of distribution theory. We shall use (2.7) only for smooth z, when $\operatorname{div} z$ and $(z,n)\big|_{\partial\Omega}$ may be understood in the classical sense.

We set

$$Y^\alpha = \left\{ y(t,x) \in L_2\big(0,T\,;\, H^{\alpha+1}(\Omega)\big) : \frac{\partial y}{\partial t} \in L_2\big(0,T\,;\, H^{\alpha-1}(\Omega)\big) \right\}\,,$$
$$P^\alpha = \left\{ p(t,x) \in \mathbf{D}\big((0,T)\times\Omega\big) : \nabla p \in L_2\left(0,T\,;\, \left(W_2^{\alpha-1}(\Omega)\right)^d\right) \right\}\,,$$

where $\mathbf{D}([0,T]\times\Omega)$ is the space of distributions defined on $[0,T]\times\Omega$. The following known assertions hold:

THEOREM 2.1.— *Let $\alpha\in\mathbb{R}$. Then for any $y_0\in H^\alpha(\Omega)$, $u\in L_2(0,T;H^{\alpha-1}(\Omega))$ there exists a unique solution $(y,p)\in Y^\alpha\times P^\alpha$ of problem (1.1) to (1.3).*

We apply to both sides of the first equation (1.1) the operator Π to prove the theorem in the case $\alpha\geq 0$. Since $\Pi\nabla p=0$, we obtain the equation

$$\partial_t y(t,x)-\Pi\Delta y=u(t,x)\,.$$

Theorem 2.1 is proved for this equation as in the book by M. J. Vishik, A. V. Fursikov [16, pp. 27-28]. If $\alpha<0$ then theorem 2.1 is proved by the methods of the book of J.-L. Lions, E. Magenes [10].

THEOREM 2.2.— *Let $\alpha\in\mathbb{R}$, $y_0\in H^\alpha(\Omega)$, $u\in L_2(0,T;H^{\alpha-1}(\Omega))$, $(y,p)\in Y^\alpha\times P^\alpha$ and a domain G belongs to $[0,T]\times\Omega$. Then:*

1) *if $u(t,x)$ is infinitely differentiable for $(t,x)\in G$, then $y(t,x)\in(C^\infty(G))^d$;*
2) *if we assume that $u\in L_2(t_0,T;H^\beta)$ where $\alpha<\beta$, then $y(t,x)\in C(t_0,T;H^\beta)$.*

The first assertion follows from the results of V. A. Solonnikov [13], and the second statement is proved as in M. J. Vishik and A. V. Fursikov's book [16, pp. 27-28].

3. Distributed control concentrated on a subdomain

We consider problem (1.1) to (1.3). Let ω be a fixed subdomain of the domain Ω and I be a fixed subinterval of $[0,T]$. We suppose that the right-hand-side $u(t,x)$ is a control and that

$$\begin{aligned}u(t,x)\in U(\omega,I)&=\\ &=\Big\{w(t,x)\in L_2(0,T;H^0(\omega)) : w(t,x)=0\\ &\qquad \text{for } (t,x)\in([0,T]\times(\Omega\setminus\omega))\cup\Big(([0,T]\setminus I)\times\Omega\Big)\Big\}\,.\end{aligned}\tag{3.1}$$

Obviously $U(\omega,I)\subset L_2(0,T;H^0(\Omega))$. We assume that a subinterval satisfies the condition

$$I=(\tau_1,\tau_2)\,,\quad 0<\tau_1<\tau_2<T\,.\tag{3.2}$$

DEFINITION 3.1.— *Let $\alpha \geq 0$. Problem (1.1) to (1.3) is called H^{α}-approximately controllable with respect to $U(\omega, I)$ if for any fixed $y_0 \in H^0(\Omega)$ the restriction $y(T, \cdot)$ at $t = T$ of a solution y of problem (1.1) to (1.3) runs through a dense set in $H^{\alpha}(\Omega)$ when the right-hand-side $u(t, x)$ runs through the space $U(\omega, I)$.*

Note that by (3.1), (3.2) and by the second assertion of theorem 2.2 the inclusion $y(T, \cdot) \in \bigcap_{\alpha \geq 0} H^{\alpha}(\Omega)$ holds and therefore definition 3.1 is correct. If $\alpha \in [0, 1]$ then condition (3.2) is unnecessary, because the inclusion $y(T, \cdot) \in H^{\alpha}$ follows from the assumption $u \in L_2(0, T; H^0(\Omega))$.

THEOREM 3.1.— *Let $I = [0, T]$ when $\alpha \in [0, 1]$ and for $\alpha > 1$ assume (3.2). Then problem (1.1) to (1.3) is H^{α}-approximately controllable with respect to $U(\omega, I)$.*

Proof.— Firstly we consider the case when initial condition is $y_0 = 0$. Let W be the closure in $H^{\alpha}(\Omega)$ of the set of restrictions $y(T, \cdot)$ at $t = T$ of solutions $(y, p) \in Y^0 \times P^0$ of problem (1.1) to (1.3), when the right-hand-side term $u(t, x)$ runs through $U(\omega, I)$. We have to prove that $W = H^{\alpha}(\Omega)$. Suppose the contrary. It follows from (2.4) that the space $H^{-\alpha}$ is the conjugate space of H^{α} with respect to the duality $\langle \cdot, \cdot \rangle$ generated by the scalar product in $H^0(\Omega)$. Therefore the assumption $W \neq H^{\alpha}(\Omega)$ implies the existence of $\phi_0(x)$ satisfying the conditions

$$\phi_0 \in H^{-\alpha}(\Omega), \quad \phi_0 \neq 0, \ \langle \phi_0, v \rangle = 0, \ \forall\, v \in W. \tag{3.3}$$

Applying the operator Π from (2.3) to the both sides of the first equation in (1.1) and taking into account that $\Pi \nabla p = 0$, we obtain the equality

$$\partial_t y(t, x) - \Pi \Delta y = u.$$

Using this equality and (2.7), we can write down (1.1) in the following way

$$\partial_t y(t, x) - \Delta y + \nabla R \left(0, (\Delta y, n)\big|_{\partial\Omega}\right) = u(t, x), \quad \operatorname{div} y = 0. \tag{3.4}$$

Let $t \in [0, T]$, $x \in \Omega$ and $\phi(t, x)$ be the solution of the problem

$$\partial_t \phi(t, x) + \Delta \phi = \nabla R \left(0, (\Delta \phi, n)\big|_{\partial\Omega}\right), \quad \operatorname{div} \phi = 0, \tag{3.5}$$

$$\phi\big|_{t=T} = \phi_0, \quad \phi\big|_{[0,T] \times \partial\Omega} = 0, \tag{3.6}$$

where ϕ_0 is the vector field (3.3). Comparing (1.1) with (3.4) it is easy to understand that (3.5)-(3.6) is the Stokes problem with the inverse time. Hence, from theorem 2.1 there exists a solution $\phi(t,x)$ of problem (3.5)-(3.6) and, by theorem 2.2, $\phi(t,x) \in C^\infty([0\,,\,T-\epsilon] \times \Omega)^d$ for arbitrary $\epsilon > 0$.

Scaling the first equation in (3.5) by the solution y of (3.4), (1.2), (1.3) in the space $L_2\left(0,T\,;\,(L_2(\Omega))^d\right)$, integrating by parts and doing other simple transformations, we obtain by (3.3) that

$$\begin{aligned} 0 &= \int_0^T \left(\partial_t\phi + \Delta\phi - \nabla R\left(0,(\Delta\phi,n)\big|_{\partial\Omega}\right)\,,\,y\right)_{(L_2(\Omega))^d} \mathrm{d}t \\ &= \left(\phi_0\,,\,y(T,\,\cdot)\right)_{(L_2(\Omega))^d} + \int_0^T \left(\phi\,,\,-\partial_t y + \Delta y\right)_{(L_2(\Omega))^d} \mathrm{d}t \\ &= \int_0^T \left(\phi\,,\,-u + \nabla R\left(0,(\Delta y,n)\big|_{\partial\Omega}\right)\right)_{(L_2(\Omega))^d} \mathrm{d}t \\ &= -\int_0^T \left(\phi\,,\,u\right)_{(L_2(\omega))^d} \mathrm{d}t\,. \end{aligned} \tag{3.7}$$

It follows from (3.7) that the restriction $\phi(t,x)$ at ω satisfies conditions $\phi(t,x) \in \left(L_2(\omega)\right)^d$, $\phi(t,x) \in \left(H^0(\omega)\right)^\perp$, for almost all $t \in I$. Therefore we obtain as in (2.6), (2.7) that

$$\phi(x,t) = \nabla R\left(0,(\phi,n)\big|_{\partial\omega}\right)\,, \quad x \in \omega\,,\ \text{a.a. } t \in I\,, \tag{3.8}$$

where R is operator (2.5).

Denote $w(t,x) = \Delta\phi(t,x)$. We apply the Laplace operator Δ to the both sides of the first equation in (3.5). Then taking into account that $\Delta R(0,g) = 0$ in virtue of (2.5), we obtain the equation

$$\partial_t w(t,x) + \Delta w = 0\,, \quad t \in [\,0\,,\,T\,]\,,\ x \in \Omega\,. \tag{3.9}$$

Applying Δ to (3.8), we obtain the equality

$$w(t,x) = 0\,, \quad x \in \omega\,,\ t \in I\,. \tag{3.10}$$

The solution $w(t,x)$ of parabolic equation (3.9) is analytic with respect to variables x, when $t \in (0,T)$. Hence it follows from (3.10) that

$$w(t,x) = \Delta\phi(t,x) = 0\,, \quad t \in [\,0\,,\,T\,]\,,\ x \in \Omega\,.$$

This equation and the second equality in (3.6) imply that $\phi(t,x) = 0$, $t \in [0, T]$, $x \in \Omega$. Therefore $\phi_0(x) = 0$ which contradicts (3.3).

Let now the initial condition $y_0 \in H^0(\Omega)$ be an arbitrary vector field. We write the solution of problem (1.1) to (1.3) as follows:

$$(y,p) = (y_1,p_1) + (z,q)$$

where (y_1,p_1) is the solution of (1.1) to (1.3) with $u = 0$, and (z,q) is the solution of (1.1) to (1.3) with $y_0 = 0$. We have proved earlier that $z(T,\,\cdot\,)$ runs through a dense set in $H^\alpha(\Omega)$. Hence the equality

$$y(t,\,\cdot\,) = y_1(T,\,\cdot\,) + z(T,\,\cdot\,)$$

shows that the theorem is also proved for $y_0 \neq 0$. □

We consider now the case, when control $u(t,x)$ runs through the space

$$U_1(\omega,I) = \Big\{X_\omega(x)u(t,x) \quad \text{where } u(t,x) \in L_2\big(0,T\,;\,H^0(\Omega)\big)\,,\ u(t,x) = 0 \text{ for } t \notin I\Big\} \tag{3.11}$$

Here $X_\omega(x)$ is characteristic function of subdomain $\omega \subset \Omega$:

$$X_\omega(x) = \begin{cases} 1\,, & \text{when } x \in \omega \\ 0\,, & \text{when } x \in \Omega\setminus\omega, \end{cases} \tag{3.12}$$

I is an interval satisfying (3.2) when $\alpha > 1$ and $I = [0, T]$ for $\alpha \in [0, 1]$.

COROLLARY 3.1.— *Let $\alpha \geq 0$. Then problem (1.1) to (1.3) is H^α-approximately controllable with respect to set (3.11).*

Indeed, since the embedding $U(\omega,I) \subset U_1(\omega,I)$ holds for spaces (3.11), (3.1) then corollary 3.1 follows from theorem 3.1.

LEMMA 3.1.— *Let X^k be space (1.4) or (1.5) and H^k be space (2.4). Then the following assertions hold:*

a) the identity $X^k = H^k$ holds when $k = 0, 1, 2$;

b) if $k \geq 3$ is natural then H^k is a closed subspace of X^k and H^k does not coincide with X^k.

The norms of spaces H^k and X^k are equivalent on H^k.

Proof.— The correctness of assertion *a)* is well known (see for instance R. Temam [15]). Assertion *b)* is proved by induction by means of the following chain of identities

$$\begin{gathered}
y \in H^k \quad \Leftrightarrow \quad \Pi\Delta y \in H^{k-2}\,, \\
y\big|_{\partial\Omega} = 0\,,\ \operatorname{div} y = 0 \quad \Leftrightarrow \quad \Pi\Delta y \in \left(W_2^{k-2}(\Omega)\right)^d, \\
(\Pi\Delta)^j y\big|_{\partial\Omega} = 0\,,\ 0 \leq j < \frac{k}{2}\,,\ \operatorname{div} y = 0 \quad \Leftrightarrow \quad y \in \left(W_2^k(\Omega)\right)^d, \\
(\Pi\Delta)^j y\big|_{\partial\Omega} = 0\,,\ 0 \leq j < \frac{k}{2}\,,\ \operatorname{div} y = 0 \quad \Leftrightarrow \\
\Leftrightarrow \quad y \in X^k\,,\ (\Pi\Delta)^j y\big|_{\partial\Omega} = 0\,,\ 0 < j < \frac{k}{2}\,.
\end{gathered}$$

Indeed, the first identity follows just from definition (2.4) of H^α, the second one is true by the inductive assumption and definition (1.5) of X^k. The theorem on smoothness of solutions of steady Stokes system (V. A. Solonnikov [14], R. Temam [15]) implies the third identity and the forth one follows from (1.5). The equivalence of the H^k and X^k-norms is proved in M. I. Vishik, A. V. Fursikov [16, p. 124]. □

The main result of this section follows from lemma 3.1.

THEOREM 3.2.— *Let $I = [0\,,T]$ when $k = 0, 1,$ and I satisfies (3.2) for $k \geq 2$. Then problem (1.1) to (1.3) is X^k-approximately controllable with respect to $U(\omega, I)$ when $k = 0, 1, 2$ and is not X^k-approximately controllable when $k \geq 3$. The analogous assertion holds if we change the control set $U(\omega, I)$ by $U_1(\omega, I)$.*

Proof.— The assertion follows from theorem 3.1 and lemma 3.1. □

We show how it is possible to construct solutions of an approximate controllability problem. Consider the following extremal problem.

To minimize the functional

$$J_\epsilon(y,u) = \left\|\widehat{y} - y(T\,\cdot)\right\|^2_{H^\alpha(\Omega)} + \epsilon \int_0^T \left\|u(\tau)\right\|^2_{H^0(\omega)} \mathrm{d}\tau \to \inf \tag{3.13}$$

when the pair

$$(y,u) \in Y^0 \times U(\omega, I) \tag{3.14}$$

satisfies conditions (3.4), (1.2), (1.3). Here $\widehat{y} \in H^\alpha$ is the datum which must be approached, $\epsilon > 0$, $\alpha > 0$ and $I = [\,0\,,\,T\,]$ when $\alpha \in [\,0\,,\,1\,]$ and satisfies (3.2) for $\alpha > 1$.

PROPOSITION 3.3.— *There exists a unique solution $(y_\epsilon, u_\epsilon) \in Y^0 \times U(\omega, I)$ of problem (3.13), (3.14), (3.4), (1.2), (1.3).*

The proof is carried out by well know methods (see, for example J.-L. Lions [8], A. V. Fursikov [4]).

THEOREM 3.3.— *Let $(y_\epsilon, u_\epsilon) \in Y^0 \times U(\omega, I)$ be a solution of (3.13), (3.14), (3.4), (1.2), (1.3). Then*

$$\left\|\widehat{y} - y_\epsilon(T\,\cdot)\right\|_{H^\alpha(\Omega)} \to 0 \ \textit{as}\ \epsilon \to 0\,.$$

Proof.— In virtue of theorem 3.1 for any $\delta > 0$ there exists a pair $(y^\delta, u^\delta) \in Y^0 \times U(\omega, I)$ which satisfies equalities (3.4), (1.2), (1.3) and condition

$$\left\|\widehat{y} - y^\delta(T\,\cdot)\right\|_{H^\alpha(\Omega)} < \delta\,. \tag{3.15}$$

We choose $\epsilon > 0$ such that

$$\epsilon \int_0^T \left\|u^\delta(\tau,\,\cdot\,)\right\|^2 \mathrm{d}\tau < \delta\,. \tag{3.16}$$

Since (y_ϵ, u_ϵ) is the solution of the extremal problem, then it follows from (3.13), (3.15), (3.16) that

$$\left\|\widehat{y} - y_\epsilon(T\,\cdot)\right\|^2_{H^\alpha(\Omega)} < J_\epsilon(y_\epsilon, u_\epsilon) < J_\epsilon(y_\delta, u_\delta) < 2\delta\,. \ \square$$

4. Impulse and initial controls

We consider now the case of impulse control. Let us recall that a control $u(t,x)$ is called *impulse* if it is written as follows:

$$u(t,x) = \delta(t-t_0)v(x)\,, \tag{4.1}$$

where $\delta(t-t_0)$ is the Dirac measure concentrated at t_0. We suppose that

$$v(x) \in U_\omega = \left\{ u(x) \in H^0(\Omega) : u\big|_{x\in\omega} \in H^0(\omega)\,,\ u(x) = 0\,,\ x \in \Omega \setminus \omega \right\} \tag{4.2}$$

where ω is a subdomain of Ω.

PROPOSITION 4.1.— *Let $\alpha \geq 0$, $t_0 \in (0,T)$. Then problem (1.1) to (1.3) is H^α-approximately controllable with respect to the class of functions (4.1), (4.2).*

Proof.— The proof follows as in theorem 3.1. Instead of (3.7), we obtain equality

$$\big(\phi(t_0,\,\cdot\,)\,,\ v\big)_{(L_2(\omega))^d} = 0\,, \quad \forall\ v \in U_\omega\,. \tag{4.3}$$

Equality (3.10) where $t = t_0$ follows from (4.3). By means of (3.10), (3.9), we obtain the equation $\phi_0(x) = 0$ as in theorem 3.1. □

Proposition 4.1 and lemma 3.1 imply the following theorem.

THEOREM 4.1.— *Let $t_0 \in (0,T)$. Then problem (1.1) to (1.3) is X^k-approximately controllable with respect to the control set (4.1)-(4.2) when $k = 0,\,1,\,2$ and it is not X^k-approximately controllable with respect to the same control set when $k \geq 3$.*

Let us study the case of initial control when the control is included into the initial condition. This problem is similar to the case of impulse control. We consider the Stokes problem (1.1) with the following boundary and initial conditions

$$y\big|_{(0,T)\times\partial\Omega} = 0\,, \quad y\big|_{t=0} = y_0(x) + v(x)\,, \tag{4.4}$$

where $y_0(x) \in H^0(\Omega)$ is a fixed vector field and $v(x) \in U_\omega$ is a control (here U_ω is space (4.2)). We suppose about the right-hand-side $u(t,x)$ in (1.1) that $u \in L_2\big(0,T\,;\,H^0(\Omega)\big)$ is fixed and

$$u(t,x) = 0 \quad \text{for } t \in [\,T-\epsilon\,,\,T\,]\,. \tag{4.5}$$

We need the condition (4.5) to prove H^α-approximate controllability when $\alpha \geq 0$ is arbitrary. If $\alpha \in [0, 1]$ the condition (4.5) is unnecessary.

PROPOSITION 4.2.— *Let $\alpha \geq 0$, $y_0 \in H^0(\Omega)$, $u \in L_2(0,T; H^0(\Omega))$ and u satisfies (4.5). The problem (1.1), (4.4) is H^α-approximately controllable with respect to the class U_ω in (4.2).*

Proof.— We reduce the proof to the case $y_0(x) = 0$. By analogy with theorem 3.1, we obtain instead of (3.7) the equality

$$\big(\phi(0, \cdot), v\big)_{(L_2(\omega))^d} = 0, \quad \forall\, v \in U_\omega,$$

which is similar to (4.3). As in theorem 3.1, 4.1 it follows from this equality that $\phi_0(x) = 0$. □

Proposition 4.2 and lemma 3.1 imply the following theorem.

THEOREM 4.2.— *Let the conditions of proposition 4.2 hold. Then problem (1.1), (4.4) is X^k-approximately controllable with respect to U_ω when $k = 0, 1, 2$ and is not X^k-approximately controllable when $k \geq 3$.*

5. Boundary control

In this section we investigate approximate and exact boudary controllability of the Stokes system. Let Γ be an open set on the boundary $\partial\Omega$ of domain Ω. Firstly, we study the problem of the approximate controllability of the Stokes system when the Dirichlet boundary condition concentrated on Γ is taken as a control. Consider problem

$$\partial_t y(t,x) - \Delta y + \nabla p(t,x) = 0, \quad \operatorname{div} y = 0, \; y\big|_{t=0} = y_0(x), \tag{5.1}$$

$$y\big|_{[0,T]\times\partial\Omega} = w, \tag{5.2}$$

where $y_0(x) \in H^0(\Omega)$ is a fixed initial condition, w is a control and $\operatorname{supp} w \in [0, T] \times \Gamma$. The following identity is a necessary condition of solvability of problem (5.1)-(5.2):

$$\int_{\partial\Omega} \big(w(t,x'), n(x')\big)\, ds = 0, \quad t \in [0, T]. \tag{5.3}$$

Introduce the space of controls

$$W(\Gamma, I) = \Big\{ w(t,x) \in \big(C^2([\,0\,,\,T\,] \times \partial\Omega)\big)^d\ ,\ \operatorname{supp} w \in I \times \Gamma\,, \\ w \text{ satisfies } (5.3) \Big\}\,, \tag{5.4}$$

where $I \subset [\,0\,,\,T\,]$ is set (3.2). The proof of the existence and uniqueness of solutions of problem (5.1)-(5.2) for arbitrary $w \in W(\Gamma, I)$ is obtained easily by reducing it to the case of zero boundary conditions, *i.e.* theorem 2.1. □

THEOREM 5.1.— *For arbitrary $\alpha \geq 0$ problem (5.1)-(5.2) is H^α-approximately controllable with respect to controls set (5.4).*

Proof.— It is enough to consider the case $y_0(x) = 0$. Besides, we can suppose that the boundary $\partial\Gamma$ of Γ is C^∞-manifold. Indeed, if it is not so, we can change in (5.4) the set Γ by $\Gamma_1 \subset \Gamma$ with $\partial\Gamma_1 \in C^\infty$. Let $w \in W(\Gamma, I)$ be a control and (y,p) be the corresponding solution of problem (5.1)-(5.2). By virtue of (3.2), the pair (y,p) is a solution of (5.1) with the zero Dirichlet boundary condition when (τ_2, T), and therefore by theorem 2.2 on smoothness of solutions the inclusion $y(T,\,\cdot\,) \in H^\alpha(\Omega)$ holds. We set

$$W = \big\{ y(T,\,\cdot\,) : (y,p) \text{ is solution of (5.1)-(5.2) when } w \in W(\Gamma, I) \big\}\,.$$

The existence of a nonzero vector field $\phi_0 \in W^\perp \subset H^{-\alpha}$ follows from the assumption that W is not dense in H^α. We consider problem (3.5)-(3.6) with this initial fonction ϕ_0. Scaling (5.1_1) by the solution ϕ of problem (3.5)-(3.6) in $\big(L_2([\,0\,,\,T\,] \times \Omega)\big)^d$ we obtain similarly to (3.7) that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (\phi\,,\,\partial_t y - \Delta y + \nabla p)\,\mathrm{d}x\,\mathrm{d}t \\ &= \big(\phi_0\,,\,y(T,\,\cdot\,)\big)_{H^0(\Omega)} - \int_0^T \big(\partial_t\phi + \Delta\phi\,,\,y\big)_{H_0(\Omega)}\,\mathrm{d}t + \\ &\quad + \int_{I\times\Gamma} \left(\frac{\partial\phi}{\partial n}\,,\,w\right)\mathrm{d}s\,\mathrm{d}t \\ &= -\int_0^T (\nabla q\,,\,y)_{H^0(\Omega)}\,\mathrm{d}t + \int_{I\times\Gamma}\left(\frac{\partial\phi}{\partial n}\,,\,w\right)\mathrm{d}s\,\mathrm{d}t\,, \end{aligned} \tag{5.5}$$

where we use the notation $q = R\big(0\,,\,(\Delta\phi, n)\big|_{\partial\Omega}\big)$ for brevity. It follows from (5.5) that

$$\int_{I\times\Gamma} \sum_{i=1}^d \left(\frac{\partial\phi_i}{\partial n} - n_i q\right) w_i\,\mathrm{d}t\,\mathrm{d}s = 0\,, \quad \forall\ w = (w_1,\,\ldots,\,w_d) \in W(\Gamma, I)\,.$$

This equality and (5.3) involve the existence of such constant λ, that

$$\frac{\partial \phi_i}{\partial n}(t,x') = n_i(t,x')\big(q(t,x')+\lambda\big)\,, \quad i=1,\ldots,d,\ (t,x)\in I\times\Omega\,. \tag{5.6}$$

Let $\omega \in \mathbb{R}^d$ be a domain which satisfies conditions :

$$\Omega\cap\omega = 0\,, \quad \overline{\Omega}\cap\overline{\omega} = \overline{\Gamma}\,, \tag{5.7}$$

where the line above means the closure of domain. We define the open set Ω_ω by formula

$$\Omega_\omega = \mathbb{R}\setminus\big(\overline{\mathbb{R}\setminus(\overline{\Omega}\cup\overline{\omega})}\big)\,. \tag{5.8}$$

By (5.7), this set is connected in $\mathbf{R}^d$ if Ω posseses this property. We suppose about ω that the boundary $\partial\Omega_\omega$ of Ω_ω is a C^∞ manifold. The existence of such domains ω follows from the assumption $\partial\Gamma\in C^\infty$. We define the function $z(t,x)$ in the cylinder $[\,0\,,\,T\,]\times\Omega_\omega$ by formula

$$z(t,x) = \begin{cases} \phi(t,x), & (t,x)\in[\,0\,,\,T\,]\times\Omega \\ 0 & (t,x)\in[\,0\,,\,T\,]\times\omega\,. \end{cases} \tag{5.9}$$

Let $\psi(t,x) \in \Big(C^2\big([\,0\,,\,T\,]\times\Omega_\omega\big)\Big)^d$ be an arbitrary function satisfying conditions

$$\psi(t,x) = 0 \text{ for } t\in[\,0\,,\,T\,]\setminus I\,, \quad \psi\big|_{[0,T]\times\partial\Omega_\omega} = 0\,.$$

Integrating by parts with respect to x and taking into account (5.9), (5.6), (3.5), we obtain that

$$\begin{aligned}
&\int_0^T \big(z\,,\,\partial_t\psi+\Delta\psi\big)_{(L_2(\Omega_\omega))^d}\,\mathrm{d}t = \\
&\quad = \int_0^T \big(\partial_t\phi+\Delta\phi\,,\,\psi\big)_{(L_2(\Omega))^d}\,\mathrm{d}t - \int_0^T\int_\Gamma (n,\psi)(q+\lambda)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = \int_0^T \big(\nabla(q+\lambda)\,,\,\psi\big)_{(L_2(\Omega))^d}\,\mathrm{d}t - \int_0^T\int_\Gamma (n,\psi)(q+\lambda)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = -\int_0^T \big(q+\lambda\,,\,\operatorname{div}\psi\big)_{(L_2(\Omega))^d}\,\mathrm{d}t\,.
\end{aligned}$$

These equations involve the identity

$$\partial_t z + \Delta z = \nabla \left(X_\Omega(x)\big(q(t,x) + \lambda\big)\right) , \quad x \in \Omega_\omega \tag{5.10}$$

which is understood in the sense of distributions. Here $X_\Omega(x)$ is the characteristic function of the set Ω (see (3.12)). It follows from (3.5_2), (3.6_2), (5.9) that $\operatorname{div} z(t,x) = 0$ when $x \in \Omega_\omega$. Applying the operator div to the both parts of (5.10), we obtain that

$$\Delta \left(X_\Omega(x)\big(q(t,x) + \lambda\big)\right) = 0 , \quad x \in \Omega_\omega . \tag{5.11}$$

In accordance with the definition of the function X_Ω the equation

$$X_\Omega(x)\big(q(t,x) + \lambda\big) = 0 , \quad x \in \omega$$

holds. It is easy to deduce from this identity and (5.11) that

$$X_\Omega(x)\big(q(t,x) + \lambda\big) = 0 , \quad x \in \Omega_\omega .$$

Hence by (5.10), the equation

$$\partial_t z + \Delta z = 0 , \quad (t,x) \in [\,0\,,\,T\,] \times \Omega_\omega \tag{5.12}$$

holds. Since by (5.9), the equation $z(t,x) = 0$ takes place for $x \in \omega$ then we have by (5.12) that $z(t,x) = 0$ and therefore the equality $\phi_0(x) = 0$ holds. □

By theorem 5.1 and lemma 3.1, we obtain such proposition:

COROLLARY 5.1.— *Problem (5.1)-(5.2) is X^k-approximately controllable with respect to (5.4) when $k = 0$, 1, 2 and it is not X^k-approximately controllable for $k \geq 3$ with respect to the same set of controls.*

Now we will prove a result on the exact boundary controllability of the Stokes problem (5.1)-(5.2) which is a simple corollary of an analogous result for the heat equation. The general problem of the exact controllability consists in the construction of a boundary control w such that the component y of the solution of problem (5.1)-(5.2) is equal to a given function $\widehat{y}(x)$ in the prescribed time $T : y(T,x) = \widehat{y}(x)$.

Since the Stokes system can not be inverted with respect to time, then the solution of the general problem of the exact controllability is very difficult, because of the necessity of describing precisely the space $\{y(T, \cdot)\}$ through which the solutions of problem (5.1)-(5.2) run. We solve this problem for a concrete function $\widehat{y}(x)$, which certainly belongs to $\{y(T, \cdot)\}$ namely, for $\widehat{y}(x) = 0$. We introduce the following functional spaces:

$$\widehat{H}^{\alpha}(\Omega) = \left\{u \in \left(W_2^{\alpha}(\Omega)\right)^d : \operatorname{div} u = 0\right\}$$

$$\widehat{Y} = \left\{y(t, \cdot) \in L_2\big(0, T\,;\, \widehat{H}^2(\Omega)\big) : \frac{\partial y}{\partial t} \in L_2\big(0, T\,;\, \widehat{H}^0(\Omega)\big)\right\}, \quad \widehat{W} = \gamma\widehat{Y},$$

where γ is the operator of restriction of a function $y(t, x) \in \widehat{Y}$ to $[\,0\,,\,T\,] \times \partial\Omega$.

THEOREM 5.2.— *Let $y_0 \in H^0(\Omega)$. Then there exists a boundary value $w \in \widehat{W}$ such that the component y of the solution $(y, p) \in \widehat{Y} \times P^0$ of problem (5.1)-(5.2) satisfies condition*

$$y(T, x) = 0\,. \tag{5.13}$$

Proof.— Let $\Omega_1 \in \mathbb{R}^d$ be a domain with C^1-boundary $\partial\Omega_1$ which contains $\Omega : \Omega \subseteq \Omega_1$. We prolong $y_0(x)$ onto $\Omega_1 \backslash \Omega$ by the equality $y_0(x) = 0$ for $x \in \Omega_1 \setminus \Omega$. It is known (see, G. Schmidt [11]) that there exists a vector-function $u(t, x) \in L_2\left(0, T\,;\, \left(W_2^2(\Omega)\right)^d\right)$ satisfying the equations

$$\partial_t u(t, x) - \Delta u = 0\,, \quad u(0, x) = y_0(x)\,, \quad u(T, x) = 0\,. \tag{5.14}$$

For every $t \in [\,0\,,\,T\,]$ the Weyl decomposition for the function $u(t, x)$ gives

$$u = y + \nabla q\,, \tag{5.15}$$

where $y = \Pi_1 u$ and $\Pi_1 : \left(L_2(\Omega)\right)^d \to H^0(\Omega_1)$ is the operator of orthogonal projection on $H^0(\Omega_1)$. Substituting the decomposition (5.15) into the first of equations (5.14), we obtain that (5.1_1) holds with $\nabla p = \nabla(\partial_t q - \Delta q)$. Besides, (5.13) follows from (5.14_3), (5.15). Thus if we take $w = \gamma y$ then all the assertions of theorem will be fulfilled. □

6. Control on a hypersurface

We develop the results of section 3 narrowing the support of the control $u(t,x)$. Les us consider a hypersurface S as a support of u and begin from the case when $S \subset \Omega \subset \mathbb{R}^d$ is a closed C^∞-manifold. Suppose that in (1.1) to (1.3) the control u has a form

$$u = \delta(S, \beta_1) + \frac{\partial \delta(S, \beta_2)}{\partial n}, \tag{6.1}$$

where $\delta(S, \cdot)$, $\partial\delta(S, \cdot)/\partial n$ are Dirac measure concentrated on the surface S and its derivative with respect to the external normal to S. The value of distribution (6.1) at test function $\phi \in \left(C^\infty([0, T] \times \overline{\Omega})\right)^d$ is defined by the formula

$$\begin{aligned} \langle u, \phi \rangle = \int_0^T \int_S (\phi(t,s), \beta_1(t,s)) \, \mathrm{d}s \, \mathrm{d}t \, + \\ - \int_0^T \int_S \left(\frac{\partial \phi(t,s)}{\partial n}, \beta_2(t,s) \right) \mathrm{d}s \, \mathrm{d}t \end{aligned} \tag{6.2}$$

where β_1, β_2 are given vector-functions and

$$\begin{aligned} &\beta_i(t,s) \in \left(L_2([0, T] \times S)\right)^d, \\ &\operatorname{supp} \beta_i \in I \times S, \quad i = 1, 2, \quad I = [\tau_1, \tau_2] \subseteq [0, T]. \end{aligned} \tag{6.3}$$

We denote by $U(S, I)$ the set of controls having form (6.1), (6.2), (6.3).

THEOREM 6.1.— *The Stokes problem (1.1) to (1.3) is H^α-approximately controllable with respect to $U(S,I)$ for arbitrary $\alpha \geq 0$.*

Proof.— As in previous theorems we can limit ourselves to the case $y_0 = 0$. Similarly to theorem 3.1, we denote by W the closure in $H^\alpha(\Omega)$ of the set $\{y(T, \cdot)\}$ if $(y,p) \in Y^0 \times P^0$ is a solution of problem (1.1) to (1.3) with right-hand-side running through $U(S,I)$. Assuming that the relation $W \neq H^\alpha(\Omega)$ holds, we denote by ϕ_0 a vector function satisfying (3.3) and by $\phi(t,x)$ the solution of (3.5)-(3.6). We obtain by (3.7), (6.1), (6.2) that

$$\int_0^T \int_S (\phi(t,s), \beta_1(t,s)) \, \mathrm{d}s \, \mathrm{d}t - \int_0^T \int_S \left(\frac{\partial \phi(t,s)}{\partial n}, \beta_2(t,s) \right) \mathrm{d}s \, \mathrm{d}t = 0$$

for any β_i satisfying (6.3). Therefore

$$\phi\big|_{I\times S} = 0\,, \quad \frac{\partial \phi}{\partial n}\Big|_{I\times S} = 0\,. \tag{6.4}$$

Denote by ω a domain, bounded by a closed surface S. We set

$$\begin{aligned} z(t,x) &= \begin{cases} \phi(t,x) & x \in \Omega \setminus \omega, \\ 0 & x \in \omega, \end{cases} \\ h(t,x) &= \begin{cases} \nabla q(t,x) & x \in \Omega \setminus \omega \\ 0 & x \in \omega, \end{cases}, \quad \nabla q = \nabla R\left(0\,,\, (\Delta\phi, n)\big|_{\partial\Omega}\right) \end{aligned} \tag{6.5}$$

and R is operator from (2.5). By (3.5), (6.4), (6.5) the following equalities hold in the sense of distributions:

$$\partial_t z(t,x) + \Delta z = h\,, \quad \operatorname{div} z = 0\,, \quad (t,x) \in I \times \Omega\,. \tag{6.6}$$

Applying the operator div to both sides of (6.6_1), we obtain by (6.6_2) that

$$\operatorname{div} h = 0\,, \quad (t,x) \in I \times \Omega$$

and therefore the restriction of the normal component of vector function h on to the surface S is defined and by (6.5_2) the equality

$$(h,n)\big|_S = 0 \tag{6.7}$$

holds, where n is the vector field of normals to S. It follows from (6.5_3) that $\Delta q(t,x) = 0$, $x \in \Omega$ and in virtue of (6.5_2), (6.7) the identity $(\partial q/\partial n)\big|_S = 0$ holds. Thus $q(t,x) = \text{const}$ for $(t,x) \in I \times \omega$ and hence by $\Delta q = 0$ we obtain that $\nabla q(t,x) = 0$ for $(t,x) \in I \times \Omega$.

Therefore identity $h(t,x) = 0$ follows from (6.5_2), and this means that (6.6_1) is a heat equation when $(t,x) \in I \times \Omega$. This fact and (6.5_1) imply the identity

$$z(t,x) = 0\,, \quad (t,x) \in I \times \Omega\,.$$

Since $\nabla q = 0$ for such (t,x) then $\phi(t,x) = 0$ also. The equality $\phi_0 = 0$ follows from the last identity. □

We consider now the case when the hupersurfaces S is not a closed manifold and we suppose for simplicity that the dimension of the Stokes system equals two. Thus let $\Omega \in \mathbb{R}^2$ be a bounded domain with C^∞-curve S which is placed inside the domain Ω.

As it turns out, it is more convenient in this situation to take as unknown function the current function $F(t,x)$ which is connected with velocity (y_1, y_2) by the relations

$$\partial_{x_2} F(t,x) = y_1(t,x)\,, \quad \partial_{x_1} F(t,x) = -y_2(t,x)\,. \tag{6.8}$$

Assume for simplicity that Ω is a simply connected domain. In this case the current function F is defined by solenoidal vector field up to an arbitrary constant. Let $v(t, x_1, x_2)$ be a current function corresponding to the density of external forces u:

$$\partial_{x_2} v(t,x) = u_1(t,x)\,, \quad \partial_{x_1} v(t,x) = -u_2(t,x)\,. \tag{6.9}$$

The first of the equalities (1.1) is in reality a system of two equations. Applying $\partial/\partial x_2$ to the first equation of this system and $-\partial/\partial x_1$ to the second one, adding the resulting equations and taking (6.8), (6.9) into account we obtain the equation for the current function:

$$\partial_t \Delta F(t,x) - \Delta^2 F(t,x) = \Delta v(t,x)\,. \tag{6.10}$$

Let us deduce the boundary conditions for (6.10). It follows from (6.8), (1.2) that

$$\frac{\partial F}{\partial n}\Big|_{\partial\Omega} = 0\,, \quad \frac{\partial F}{\partial \tau}\Big|_{\partial\Omega} = 0\,,$$

where n is the field of external normals to $\partial\Omega$, τ is the tangent toward $\partial\Omega$ field. The second equality implies that $F\big|_{\partial\Omega} = \text{const}$ and since F is determined up to a constant then we can take $F\big|_{\partial\Omega} = 0$. Thus we have the following boundary and initial conditions for (6.10):

$$F\big|_{\partial\Omega} = 0\,, \quad \frac{\partial F}{\partial n}\Big|_{\partial\Omega} = 0\,, \quad F\big|_{t=0} = F_0(x) \tag{6.11}$$

where F_0 is an initial condition for the current function.

Let $[\,0\,,\,1\,] \ni \lambda \to S(\lambda) = S$ be a smooth curve disposed in Ω. We assume that the current function of external forces density u has the form (6.1) and belongs to the controls set $U(S,I)$ which is defined by (6.1), (6.2), (6.3) (when $d=1$ in (6.3)).

Let us recall that

$$\overset{\circ}{W}{}_2^2(\Omega) = \left\{ u(x) \in W_2^2(\Omega) : u\big|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \right\}.$$

THEOREM 6.2.— *Let $F_0 \in \overset{\circ}{W}{}_2^2(\Omega)$ and let v be function (6.1). Then problem (6.10)-(6.11) has a unique solution $F(t,x) \in L_2(0,T;\, W_2^{-\beta+2}(\Omega))$, $(\partial F/\partial t) \in L_2(0,T;\, W_2^{-\beta}(\Omega))$, $\beta > 3/2$, and $F(t,x)$ is infinitely differentiable when $x \in \Omega \setminus S$, $t \in I$ and when $x \in \Omega$, $t \notin I$. Besides, the vector field y defined in (6.8) belongs to $H^\alpha(\Omega)$ with arbitrary $\alpha \geq 0$.*

Proof.— We write the solution F of problem (6.3), (6.11) in the form

$$F = F_1 + F_2\,,$$

where F_1 is the solution of problem

$$\partial_t F_1(t,x) - \Delta F_1(t,x) = \Delta g(t,x)\,, \quad F_1\big|_{t=0} = 0 \tag{6.12}$$

which is defined for $(t,x) \in [\,0\,,\,T\,] \times \mathbb{R}^2$,

$$g(t,x) = \begin{cases} v(t,x) & \text{when } x \in \Omega, \\ 0 & \text{when } x \in \mathbb{R}^2 \setminus \Omega \end{cases}$$

and F_2 is the solution of problem

$$\begin{aligned} &\partial_t \Delta F_2 - \Delta^2 F_2 = 0\,, \quad F_2\big|_{\partial\Omega} = -F_1\big|_{\partial\Omega}\,, \\ &\frac{\partial F_2}{\partial n}\Big|_{\partial\Omega} = -\frac{\partial F_1}{\partial n}\Big|_{\partial\Omega}\,, \quad F_2\big|_{t=0} = F_0\,. \end{aligned} \tag{6.13}$$

We apply the Fourier transformation to both sides of (6.12) with respect to x:

$$-\partial_t |\xi|^2 \widehat{F}_1(t,\xi) - |\xi|^4 \widehat{F}_1(t,\xi) = -|\xi|^2 \widehat{g}(t,\xi)\,, \quad \widehat{F}_1\big|_{t=0} = 0\,, \tag{6.14}$$

where

$$\widehat{g}(t,\xi) = \int \mathrm{e}^{-i\langle x,\xi\rangle} g(t,x)\,\mathrm{d}x$$

is the Fourier transformation of g and $\widehat{F}_1$ is the Fourier transformation of F_1. Since $g(t,x)$ has a compact support with respect to x, then $\widehat{g}(t,\xi)$ is an analytic function with respect to ξ. Hence, after dividing (6.14) by $-|\xi|^2$, we obtain the Fourier transformation of heat equation

$$\partial_t \widehat{F}_1 - |\widehat{\xi}|^2 F_1 = \widehat{g}\,, \quad \widehat{F}_1\big|_{t=0} = 0\,.$$

Solving this problem, applying the inverse Fourier transformation and restricting the obtained function to $[0, T] \times \Omega$, we shall have that

$$F_1 \in L_2\big(0, T;\, W_2^{-s+2}(\Omega)\big), \quad \left(\frac{\partial F_1}{\partial t}\right) \in L_2\big(0, T;\, W_2^{-s}(\Omega)\big)$$

and F_1 is infinitely differentiable outside the support of u and, particularly, in a neighborhood of $[0, T] \times \partial\Omega$.

Problem (6.13) has a unique smooth solution. It is possible to prove this assertion using the methods of [1] for example. □

We denote by $\mathbf{G}$ the set of fields $u = (u_1, u_2)$ defined in (6.9) where v is a function of form (6.1) to (6.3) and prove H^α-approximate controllability of Stokes problem (6.10)-(6.11) with respect to $\mathbf{G}$.

THEOREM 6.3. — *The Stokes problem (6.10)-(6.11) is H^α-approximately controllable with respect to $\mathbf{G}$ in the case of dimension $d = 2$ and arbitrary $\alpha > 0$.*

Proof. — It is sufficient to consider the case $y_0 = 0$. Let W be closure in H^α of the set $\{y(T, \cdot)\}$, where (y, p) is the solution of problem (1.1) to (1.3) with density of external forces $u \in \mathbf{G}$. Assuming that $W \neq H^\alpha$ as in previous theorems, we choose a non zero function $\phi^0 \in H^{-\alpha}$ such that $\big(\phi^0, y(T, \cdot)\big)_{H_0} = 0$, $\forall\, y(T, \cdot) \in W$. In a vector notation $\phi^0 = (\phi_1^0, \phi_2^0)$. We construct the differential form $\phi_2^0 \, \mathrm{d}x_1 + \phi_1^0 \, \mathrm{d}x_2$ which is closed because ϕ^0 is solenoidal vector field. Since Ω is a simply connected domain then this form is exact, and there exists such distribution $\Psi^0(x)$ that

$$\mathrm{d}\Psi^0 = -\phi_2^0 \, \mathrm{d}x_1 + \phi_1^0 \, \mathrm{d}x_2 \,. \tag{6.15}$$

The function Ψ^0 is defined up an arbitrary constant and is a current function for the vector field ϕ^0. We consider the problem

$$\begin{gathered} \partial_t \Delta\Psi(t, x) + \Delta^2 \Psi(t, x) = 0\,, \quad \Psi\big|_{\partial\Omega} = \frac{\partial \Psi}{\partial n}\Big|_{\partial\Omega} = 0\,, \\ \Psi\big|_{t=T} = \Psi^0 \,. \end{gathered} \tag{6.16}$$

which is adjoint to (6.10)-(6.11) and prove that it has a solution.

Note that problem (3.5)-(3.6) has a unique solution $\phi(t,x)$ for $\phi^0 \in H^{-\alpha}$, and $\phi(y,x)$ is an infinitely differentiable function for $t < T$.

By means of ϕ we construct another function $\Psi(t,x) \in C^\infty(\Omega)$ when $t < T$, such that

$$\mathrm{d}\Psi(t,x) = -\phi_2(t,x)\,\mathrm{d}x_1 + \phi_1(t,x)\,\mathrm{d}x_2\,, \quad \Psi\big|_{\partial\Omega} = \frac{\partial\Psi}{\partial n}\Big|_{\partial\Omega} = 0\,. \tag{6.17}$$

Obviously, the function

$$\Psi = \int_a^x (\phi_1\,\mathrm{d}x_2 - \phi_2\,\mathrm{d}x_1)$$

satisfies (6.17), where $a \in \partial\Omega$ and the integral is taken along an arbitrary curve $\gamma \subset \Omega$ jointing the points a and x. Defining Ψ^0 we can choose a constant such that $\lim_{t\to T}\Psi(t,x) = \Psi^0(x)$ in the sense of distributions. The constructed function $\Psi(t,x)$ is, obviously, the solution of problem (6.16).

Scaling (6.16) in $L_2\big([\,0\,,T\,]\times\Omega\big)$ by function F which is a solution of (6.10)-(6.11), we obtain similarly to (3.7) that

$$\begin{aligned} 0 &= \int_0^T \int_S F(\partial_t\Delta\Psi + \Delta^2\Psi)\,\mathrm{d}x\,\mathrm{d}t \\ &= \big(\phi^0, y(T,\,\cdot\,)\big)_{(L_2(\Omega))^2} + \int_0^T \int_\Omega \Psi(-\partial_t\Delta F + \Delta^2 F)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega (-\Delta\Psi)v\,\mathrm{d}x\,\mathrm{d}t = 0\,. \end{aligned} \tag{6.18}$$

It follows from (6.18), (6.2) that

$$\Delta\Psi\big|_{I\times S} = \frac{\partial\Delta\Psi}{\partial n}\Big|_{I\times S} = 0\,. \tag{6.19}$$

Besides, by (6.16) $\Delta\Psi$ is a solution of the inverse heat equation. Using the analyticity of $\Delta\Psi(t,x)$ with respect to x when $t \in I$, we can deduce from (6.19) that $\Delta\Psi(t,x) = 0$. Hence, taking into account (6.16_2) we obtain that $\Psi(t,x) = 0$ and by (6.16_3), (6.15) we have $\phi_0 = 0$. □

7. On the approximate uncontrollability of the Burgers equation

In this section, we show that the Burgers equation is not approximately controllable on arbitrary bounded time intervals. Let us consider the Burgers equation

$$\begin{gathered}\partial_t y(t,x) - \partial^2_{x^2} y(t,x) + y(t,x)\partial_x y(t,x) = u(t,x)\,, \\ x \in [\,0\,,\,a\,]\,,\; t \in [\,0\,,\,T\,]\,,\end{gathered} \tag{7.1}$$

where $a > 0$, and $T > 0$ are arbitrary fixed numbers. We suppose that a solution $y(t,x)$ satisfies zero boundary and initial conditions

$$y(t,0) = y(t,a) = 0\,, \quad y(0,x) = 0\,. \tag{7.2}$$

Assume that $u(t,x) \in L_2\big([\,0\,,\,T\,] \times [\,0\,,\,a\,]\big)$ and that for any $t \in [\,0\,,\,T\,]$

$$\operatorname{supp} u(t,x) \in [\,b\,,\,c\,]\,, \quad 0 < b < c < a\,. \tag{7.3}$$

It is well-known that for an arbitrary $u \in L_2\big([\,0\,,\,T\,] \times [\,0\,,\,a\,]\big)$ there exists a unique solution $y(t,x) \in L_2\big(0,T\,;\, W^2_2(0,a)\big)$ of problem (7.1)-(7.2). It is possible to see, that $\partial y(t,x)/\partial t \in L_2\big((0,T) \times (0,a)\big)$. We deduce one estimate for the solution $y(t,x)$ of problem (7.1)-(7.2) which simply implies the uncontrollability of this problem.

LEMMA 7.1.— *Let $u \in L_2\big([\,0\,,\,T\,] \times [\,0\,,\,a\,]\big)$ satisfy condition (7.3), and let $y(t,x)$ be the solution of problem (7.1)-(7.2). Denote $y_+(t,x) = \max\big(y(t,x),0\big)$. Then for arbitrary $N > 5$ the estimate*

$$\frac{\partial}{\partial t} \int_0^b (b-x)^N y_+^4(t,x)\,\mathrm{d}x < \alpha(N) b^{N-5} \tag{7.4}$$

holds where b is the constant from (7.3) and $\alpha(N) > 0$ is a constant, depending on N only.

Proof.— We multiply both sides of (7.1) by $(b-x)^N y_+^3(t,x)$ and integrate them with respect to x from 0 to b. Integrating by part in the second term of the left side of the obtained identity we shall have

$$\begin{gathered}\int_0^b (b-x)^N (\partial_t y) y_+^3\,\mathrm{d}x + \int_0^b (b-x)^N 3y_+^2\,\partial_x y_+\,\partial_x y\,\mathrm{d}x\,+ \\ -\int_0^b N(b-x)^{N-1} y_+^3\,\partial_x y\,\mathrm{d}x + \int_0^b (b-x)^N y_+^4\,\partial_x y\,\mathrm{d}x = 0\,.\end{gathered} \tag{7.5}$$

It follows from the theorem on the smoothness of a solution of the Burgers equation that $y(t,x) \in C^\infty\big((0,T)\times(0,a)\big)$. Denote $y_- = \min(y,0)$. Then

$$y_+^3 \frac{\partial y}{\partial x} = y_+^3 \left(\frac{\partial y_+}{\partial t} + \frac{\partial y_-}{\partial t}\right) = y_+^3 \frac{\partial y_+}{\partial t} = \frac{1}{4}\frac{\partial y_+^4}{\partial t}\,.$$

The following identities are proved in an analogous way

$$y_+^2 \frac{\partial y_+}{\partial x}\frac{\partial y}{\partial x} = y_+^2 \left(\frac{\partial y_+}{\partial x}\right)^2, \quad y_+^k \frac{\partial y_+}{\partial x} = \frac{1}{k+1}\frac{\partial y_+^{k+1}}{\partial x}\,.$$

Using these equalities and integrating by parts in the last two terms of equation (7.5), we obtain

$$\begin{aligned}\int_0^b (b-x)^N \frac{1}{4}\,\partial_t y_+^4\,\mathrm{d}x + \int_0^b (b-x)^N 3y_+^2(\partial_x y_+)^2\,\mathrm{d}x + \\ -\int_0^b \frac{N}{4}(N-1)(b-x)^{N-2}\,y_+^4\,\mathrm{d}x + \int_0^b \frac{N}{5}(b-x)^{N-1}\,y_+^5\,\mathrm{d}x = 0\,.\end{aligned} \tag{7.6}$$

By the Hölder inequality

$$\begin{aligned}&\int_0^b (b-x)^{N-2} y_+^4\,\mathrm{d}x \leq \\ &\leq \left(\int_0^b (b-x)^{N-6}\,\mathrm{d}x\right)^{1/5}\left(\int_0^b (b-x)^{N-1} y_+^5\,\mathrm{d}x\right)^{4/5} = \\ &= \frac{b^{(N-5)/5}}{(N-5)^{1/5}}\left(\int_0^b (b-x)^{N-1} y_+^5\,\mathrm{d}x\right)^{4/5}.\end{aligned} \tag{7.7}$$

Using the Young inequality, we shall have

$$\begin{aligned}\frac{N}{5}\int_0^b (b-x)^{N-1}y_+^5\,\mathrm{d}x - \frac{N(N-1)}{4(N-5)^{1/5}}\,b^{\frac{N-5}{5}}\left(\int_0^b (b-x)^{N-1} y_+^5\,\mathrm{d}x\right)^{4/5} \geq \\ \geq -\alpha(N)b^{N-5}\,,\end{aligned} \tag{7.8}$$

where $\alpha(N)$ is a positive constant, depending on $N > 5$ only. Substituting (7.7)-(7.8) into (7.6) we obtain (7.4). □

THEOREM 7.1.— *Let $T > 0$ be an arbitrary finite number. Then problem (7.1)-(7.2) is not $L_2(0,a)$-approximately controllable with respect to set of controls $u \in L_2\big((0,T)\times(0,a)\big)$ satisfying (7.3).*

Proof.— Let $\widehat{y}(x) \in L_2(0,a)$, $\widehat{y}(x) \geq 0$, y be a solution of problem (7.1)-(7.2) and $T > 0$. Then

$$\left(\int_0^a \left|\widehat{y}(x) - y(T,x)\right|^2 \mathrm{d}x\right)^{1/2} > \left(\int_0^{b/2} \left|\widehat{y}(x) - y_+(T,x)\right|^2 \mathrm{d}x\right)^{1/2} \geq \\ \geq \left\|\widehat{y}\right\|_{L_2(0,b/2)} - \left\|y_+(T,\,\cdot\,)\right\|_{L_2(0,b/2)} . \tag{7.9}$$

By the Cauchy-Bunyakovskii inequality, we have:

$$\begin{aligned} &\left\|y_+(T,\,\cdot\,)\right\|_{L_2(0,b/2)} \leq \\ &\leq \left(\int_0^{b/2} (b-x)^{-N}\,\mathrm{d}x\right)^{1/2} \left(\int_0^{b/2} (b-x)^N \left|y_+(T,x)\right|^4 \mathrm{d}x\right)^{1/2} \\ &\leq \left(\frac{b^{1-N}(2^{N-1}-1)}{N-1}\right)^{1/2} \left(\int_0^{b/2} (b-x)^N \left|y_+(T,x)\right|^4 \mathrm{d}x\right)^{1/2} . \end{aligned} \tag{7.10}$$

In virtue of (7.4) for any $T > 0$ the inequality

$$\int_0^b (b-x)^N \left|y_+(T,x)\right|^4 \mathrm{d}x \leq T\alpha(N) n^{N-5} \tag{7.11}$$

holds. Let $T > 0$ be fixed and $\widehat{y}(x) \in L_2(0,a)$ satisfies condition

$$\left\|\widehat{y}\right\|_{L_2(0,b/2)} > \left(\frac{b^{1-N}(2^{N-1}-1)}{N-1}\, T\alpha(N) b^{N-5}\right)^{1/2} + 1 . \tag{7.12}$$

Then it follows from (7.9) to (7.12) that for any control $u \in L_2\big((0,T)\times(0,a)\big)$ satisfying (7.3), the solution y of problem (7.1)-(7.2) satisfies inequality

$$\left\|\widehat{y} - y(T,\,\cdot\,)\right\|_{L_2(0,a)} > 1 .$$

The inequality implies the approximate uncontrollability of problem (7.1)-(7.2). □

Now we consider the Burgers equation with boundary control u:

$$\partial_t y(t,x) - \partial^2_{x^2} y + y\partial_x y = 0\,, \quad x \in [\,0\,,\,b\,]\,,\; t \in [\,0\,,\,T\,] \tag{7.13}$$

$$y(t,0) = 0\,, \quad y(t,b) = u(t)\,, \quad y\big|_{t=0} = 0\,, \quad u \in L_2(0,T)\,. \tag{7.14}$$

THEOREM 7.2.— *Problem (7.13)-(7.14) is not $L_2(0,b)$-approximately controllable with respect to the control space $L_2(0,T)$ for arbitrary $T > 0$.*

Proof.— Estimate (7.4) holds for solution y of problem (7.13)-(7.14) and its proof does not differ from the proof of lemma 7.1. We obtain the assertion of the theorem by means of this estimate after repeating the proof of theorem 7.1 word by word. □

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