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Bifurcation For Odd Nonlinear Elliptic Variational Inequalities

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RÉSUMÉ. — On étudie un problème de bifurcation variationnelle associé à des fonctionnelles paires non régulières. On prouve un théorème de multiplicité pour les branches de bifurcation. On montre une application aux équations de von Kármán avec contraintes symétriques.

ABSTRACT. — A problem of variational bifurcation associated with non-smooth even functionals is studied. A multiplicity theorem for bifurcation branches is proved. An application to von Kármán's equations with symmetric constraints is shown.

1. Introduction

The study of eigenvalue problems for variational inequalities of the form:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K} \\ \langle A(u), v-u\rangle \geq \lambda \langle Lu, v-u\rangle, \quad \forall\, v \in \mathbf{K}, \end{cases} \tag{1.1}$$

where $\mathbf{K}$ is a closed convex subset of a Hilbert space, A is a nonlinear operator and L is a symmetric bounded linear operator, has been the object of several papers in the recent years (see [2, 3, 7, 8, 12, 13, 14, 15, 17, 18, 24, 25, 26, 27, 30, 31, 32, 33, 34, 35, 38, 39, 41, 42, 43] and references therein).

Some of them concern the study of bifurcation, under the assumptions that $0 \in \mathbf{K}$ and $A(0) = 0$. The main problem is to characterize the values $\overline{\lambda} \in \mathbf{R}$ (bifurcation values) such that the pair $(\overline{\lambda}, 0)$ accumulates solutions (λ, u) of (1.1) with $u \neq 0$.

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Under reasonable assumptions, it is quite easy to see that every bifurcation value λ of (1.1) is an eigenvalue of the "linearized" problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathbf{K}_0 \\ \langle A'(0)u\,,\, v-u\rangle \geq \lambda\,\langle Lu\,,\, v-u\rangle\,, \quad \forall\, v \in \mathbf{K}_0\,, \end{cases} \tag{1.2}$$

namely there exists a solution (λ, u) of (1.2) with $u \neq 0$, where $\mathbf{K}_0$ is the closed convex cone defined as the closure of $\bigcup_{t>0} t\mathbf{K}$.

Now let us assume that A is a potential operator and let $\mathcal{A}$ be the potential of A such that $\mathcal{A}(0) = 0$. Then the converse is known to be true for equations [23]. For variational inequalities only partial results are available in the direction.

If $\mathbf{K}$ is a cone (so that $\mathbf{K}_0 = \mathbf{K}$) and λ is the minimum of $\mathcal{A}$ on $M^+ = \left\{u : \frac{1}{2}(Lu \mid u) = 1\right\}$ or $M^- = \left\{u : \frac{1}{2}(Lu \mid u) = -1\right\}$, then λ is an eigenvalue of (1.2) and a bifurcation value of (1.1) (see [17, 39]). The same result is generalized in [30], where $\mathbf{K}$ is supposed to verify a suitable intersection condition with $M^\pm$.

In [12, 13, 14, 15] we obtain the same conclusion without assuming any relation between the convex set $\mathbf{K}$ and $M^\pm$. Moreover it is shown that every eigenvalue of (1.2) is a bifurcation value of (1.1) in the case in which $\mathbf{K}_0$ is a linear space.

On the other hand multiplicity results are also known for equations (see [6, 21, 28, 36]). Some of them concern the case in which the operator A is odd.

The purpose of the present paper is just to extend a multiplicity result of [6, 28] to variational inequalities. More precisely, we assume that $-\mathbf{K} = \mathbf{K}$ and A is odd. In this situation $\mathbf{K}_0$ turns out to be a linear space, so that (1.2) becomes a linear problem and we can define the multiplicity of an eigenvalue λ of (1.2). Our main results (Theorems 3.23 and 4.8) assert that, if λ is an eigenvalue of (1.2) of multiplicity m, then there exist $\rho_0 > 0$,

$$\left\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{R}\,,$$
$$\left\{u_\rho^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{K}\,,$$

such that $(\lambda_\rho^{(i)}, u_\rho^{(i)})$ satisfies (1.1), $\frac{1}{2}\left|(Lu_\rho^{(i)} \mid u_\rho^{(i)})\right| = \rho^2$, $i \neq j \Rightarrow u_\rho^i \neq u_\rho^j$ and $\lim\limits_{\rho \to 0^+} (\lambda_\rho^{(i)}, u_\rho^{(i)}) = (\lambda, 0)$.

In the corresponding result for equations [6, 28, 37], the main tool was constituted by a finite-dimensional reduction based on the implicit function theorem. This technique is not extendable to variational inequalities, because of the loss of regularity caused by the unilatered constraint $\mathbf{K}$. Our approach relies on the techniques of critical point theory for nonsmooth functionals developed in [7, 9, 11, 16]. To obtain the multiplicity result, we take advantage of the theory of the relative cohomological index of [19, 20].

In the next section we recall some notions and results from [7, 9, 11, 16]. In §3 we prove the abstract bifurcation result (Theorem 3.23) and in §4 we show an application to elasticity (Theorem 4.8).

2. Some recalls of nonsmooth analysis

In this section we recall some notions and results of nonsmooth analysis [7, 9, 11, 16] which will be used later. For the notions of topology involved here, the reader is referred to [40].

Throughout this section H will denote a real Hilbert space. The scalar product, norm and metric of H will be denoted by $(\cdot \mid \cdot)$, $|\cdot|$ and d_H respectively, while $B(u, r)$ will denote the open ball of center u and radius r.

Let W be an open subset of H and:

$$f : W \to \mathbf{R} \cup \{+\infty\}$$

a function. We set:

$$D(f) = \{u \in W : f(u) < +\infty\}$$
$$\forall\, c \in \mathbf{R} \cup \{+\infty\}, \quad f^c = \{u \in D(f) : f(u) \leq c\}\,.$$

For every u in $D(f)$ let us denote by $\partial^- f(u)$ the (possibly empty) set of α's in H such that:

$$\varliminf_{v \to u} \frac{f(v) - f(u) - (\alpha \mid v - u)}{|v - u|} \geq 0\,.$$

We set also $\partial^- f(u) = \emptyset$, $\forall\, u \in W \setminus D(f)$ and:

$$D(\partial^- f) = \{u \in W : \partial^- f(u) \neq \emptyset\}\,.$$

Since $\partial^- f(u)$ is convex and closed, for every u in $D(\partial^- f)$ we can denote by $\operatorname{grad}^- f(u)$ the element of $\partial^- f(u)$ having minimal norm.

If $W = H$ and f is convex, the notion of $\partial^- f$ coincides with the usual notion of subdifferential in convex analysis.

If $g : W \to \mathbf{R}$ is Fréchet differentiable at $u \in W$, then $\partial^-(f + g) = \partial^- f(u) + \operatorname{grad} g(u)$.

DEFINITION 2.1.— *A point $u \in W$ is said to be critical from below for f, if $0 \in \partial^- f(u)$. A value $c \in \mathbf{R}$ is said to be critical from below for f, if there exists $u \in W$ such that $0 \in \partial^- f(u)$, $f(u) = c$.*

DEFINITION 2.2.— *The function f is said to have a φ-monotone subdifferential of order two, if there exists a continuous function $\chi : (D(f))^2 \times \mathbf{R}^2 \to \mathbf{R}^+$ such that:*

$$(\alpha - \beta \mid u - v) \geq -\chi(u, v, f(u), f(v)) \left(1 + |\alpha|^2 + |\beta|^2\right) |u - v|^2$$

whenever $u, v \in D(\partial^- f)$, $\alpha \in \partial^- f(u)$, $\beta \in \partial^- f(v)$.

DEFINITION 2.3.— *Let c be a real number. The function f is said to verify the Palais-Smale condition at level c (or, briefly, $(PS)_c$), if for every sequence (u_h) in $D(\partial^- f)$ with $\lim_h \operatorname{grad}^- f(u_h) = 0$, $\lim_h f(u_h) = c$, there exists a subsequence (u_{h_k}) converging to an element of W.*

Besides the metric d_H induced by H, it is convenient to consider on $D(f)$ also the graph metric d^* defined by:

$$d^*(u, v) = |u - v| + |f(u) - f(v)|. \tag{2.4}$$

However, when the metric is not specified, we mean that $D(f)$ is endowed with the metric d_H.

In the following we shall be concerned with metric spaces on which the group $\mathbf{Z}_2$ acts. If X is a symmetric (with respect to the origin) subset of some normed space, X will be considered as a $\mathbf{Z}_2$-space endowed with the usual action.

THEOREM 2.5.— *Let us suppose that f is lower semicontinuous and has a φ-monotone subdifferential of order two. Moreover let us assume that W is symmetric and f is even. Let $-\infty < a \leq b < +\infty$.*

Then the pair (f^b, f^a) endowed with the metric d^* is equivariantly homotopically equivalent to the pair (f^b, f^a) endowed with the metric d_H.

Proof.— See [11, Theorem 3.18]. □

Now let us recall the relative cohomological index of [19, 20]. More precisely, we shall consider a very special case.

Let S^∞ be the unit sphere in a real normed space of infinite dimension, $\mathbf{R}P^\infty = S^\infty/\mathbf{Z}_2$ the corresponding real projective space and $\alpha \in H^1(\mathbf{R}P^\infty; \mathbf{Z}_2) \setminus \{0\}$, where the functor H^* denotes Alexander-Spanier cohomology.

Let X be a metric space on which $\mathbf{Z}_2$ acts freely and let A be a closed invariant subset of X. Let $\widetilde{X} = X/\mathbf{Z}_2$ and $\widetilde{A} = A/\mathbf{Z}_2$ denote the corresponding orbit spaces. It is always possible to define an equivariant continuous map $X \to S^\infty$. Let $q_X : \widetilde{X} \to \mathbf{R}P^\infty$ be the corresponding map between the orbit spaces, which induces a homomorphism $q_X^* : H^*(\mathbf{R}P^\infty; \mathbf{Z}_2) \to H^*(\widetilde{X}; \mathbf{Z}_2)$ in cohomology. According to [19, Definition 7.4] and [20, Definition 2.3] (see also [21, 22] if $A = \emptyset$), we set:

$$\mathrm{Index}(X, A) = \inf\left\{k \in \mathbf{N} : q_X^*(\alpha^k) \cup \gamma = 0\,,\ \forall\, \gamma \in H^*(\widetilde{X}, \widetilde{A}; \mathbf{Z}_2)\right\}$$

with the conventions $\alpha^0 = 1 \in H^0(\mathbf{R}P^\infty; \mathbf{Z}_2)$, $\alpha^{k+1} = \alpha^k \cup \alpha$, $\inf \emptyset = +\infty$.

The index turns out to be well defined. Let us recall the properties that will be used in the following.

THEOREM 2.6.— *The following facts hold :*

i) if (X', A') is another pair which is equivariantly homotopically equivalent to (X, A), then:

$$\mathrm{Index}(X', A') = \mathrm{Index}(X, A)\,;$$

ii) if S^n $(n \geq 0)$ is the unit sphere in $\mathbf{R}^{n+1}$, then:

$$\mathrm{Index}(X \circ S^n\,,\, S^n) = \mathit{Index}(X, \emptyset)\,,$$
$$\mathrm{Index}(S^n, \emptyset) = n + 1\,,$$

where $X \circ S^n$ denotes the join of X and S^n.

Proof.— Property i) is obvious. By [20, Proposition 4.1] and [21, Proposition 3.13], also ii) follows. □

The relative index has applications to critical point theory (see [19, 20] where functions of class C^1 are considered). We are interested in a case involving a class of nonsmooth functions.

THEOREM 2.7. — *Let us assume that f is lower semicontinuous and has a φ-monotone subdifferential of order two. Moreover let W be symmetric and f even. Let $-\infty < a < b < +\infty$ and let us suppose that a and b are not critical from below for f, $0 \notin f^b$ and that $\forall\, c \in [\,a\,,\,b\,[$ the function f verifies $(PS)_c$ and f^c is closed in H.*

Then there exist at least $\mathrm{Index}(f^b, f^a)$ *pairs of antipodal points in $f^b \setminus f^a$ which are critical from below for f.*

Proof. — See [11, Theorem 4.13]. □

If A is a subset of H, we define a function $I_A : H \to \mathbf{R} \cup \{+\infty\}$ by:

$$I_A(u) = \begin{cases} 0 & u \in A \\ +\infty & u \in H \setminus A. \end{cases}$$

For every u in A, $\partial^- I_A(u)$ is a closed convex cone (in some sense, the outward normal cone to A at u).

Remark 2.8. — If M is a hypersurface in H of class C^1, we have for every u in M:

$$\partial^- I_M(u) = \{\lambda \nu(u) : \lambda \in \mathbf{R}\}$$

where $\nu(u)$ is a normal unit vector to M at u.

DEFINITION 2.9. — *Let A and B be two subsets of H and $u \in A \cap B$. Then A and B are said to be (outwardly) tangent at u, if:*

$$\partial^- I_A(u) \cap \left(-\partial^- I_B(u)\right) \neq \{0\}\,.$$

THEOREM 2.10. — *Let M be a hypersurface in W of class C^1 and let us assume that f is lower semicontinuous and, for some continuous function $q : D(f) \to \mathbf{R}$,*

$$f(v) \geq f(u) + (\alpha \mid v - u) - q(u)|v - u|^2$$

whenever $v \in W$, $u \in D(\partial^- f)$, $\alpha \in \partial^- f(u)$.

Let $u_0 \in D(f) \cap M$ and let us suppose that $D(f)$ and M are not tangent at u_0.

Then we have:

$$\partial^-(f + I_M)(u_0) = \partial^- f(u_0) + \partial^- I_M(u_0)\,.$$

Proof.— See [7, Theorem 1.13 and Remark 1.12$_b$]. □

Finally, let us recall the notion of variational convergence from [1, 10].

DEFINITION 2.11.— *Let X be a topological space and:*

$$g_h : X \to \mathbf{R} \cup \{+\infty\} \qquad (h \in \overline{\mathbf{N}} := \mathbf{N} \cup \{\infty\})$$

a sequence of functions. We say that:

$$g_\infty = \Gamma^-(X) \lim_h g_h$$

if the following facts hold:

i) for every u in X and for every sequence (u_h) in X converging to u, we have:

$$g_\infty(u) \leq \liminf_h g_h(u_h) ;$$

ii) for every u in X there exists a sequence (u_h) in X converging to u such that:

$$g_\infty(u) = \lim_h g_h(u_h) .$$

From now on in this section we shall consider a sequence of functions:

$$f_h : H \to \mathbf{R} \cup \{+\infty\} , \quad (h \in \overline{\mathbf{N}}) .$$

DEFINITION 2.12.— *The sequence (f_h) is said to be equicoercive, if for every real number c the closure of the set $\bigcup_{h \in \mathbf{N}} (f_h)^c$ is compact.*

Remark 2.13.— Let us suppose that (f_h) is equicoercive and that $f_\infty = \Gamma^-(H) \lim_h f_h$. Then for every real number c the set $(f_\infty)^c$ is compact. Therefore the closure of the set $\bigcup_{h \in \overline{\mathbf{N}}} (f_h)^c$ is compact.

THEOREM 2.14.— *Let us suppose that:*

i) for every h in $\overline{\mathbf{N}}$, f_h is lower semicontinuous and even;

ii) there exists a continuous function:

$$\chi : \left(\bigcup_{h \in \overline{\mathbf{N}}} D(f_h) \right)^2 \times \mathbf{R}^2 \to \mathbf{R}^+$$

such that:

$$(\alpha - \beta \mid u - v) \geq$$
$$\geq -\chi\left(u, v, f_h(u), f_h(v)\right)\left(1 + |\alpha|^2 + |\beta|^2\right)|u - v|^2$$

whenever $h \in \overline{\mathbf{N}}$, $u, v \in D(\partial^- f_h)$, $\alpha \in \partial^- f_h(u)$, $\beta \in \partial^- f_h(v)$*;*

iii) $f_\infty = \Gamma^-(H)\lim_h f_h$ *;*

iv) the sequence (f_h) *is equicoercive.*

Let $-\infty < a \leq b < +\infty$ *and let us assume that* a *and* b *are not critical from below for* f_∞.

Then there exists $h_0 \in \mathbf{N}$ *such that for every* $h \geq h_0$, a *and* b *are not critical from below for* f_h *and the pair* (f_h^b, f_h^a) *is equivariantly homotopically equivalent to the pair* (f_∞^b, f_∞^a).

Proof. — By Remark 2.13 we can apply [11, Theorem 5.12 and Remark 5.13]. □

Remark 2.15. — In the previous theorem it is understood that all the pairs (f_h^b, f_h^a) are endowed with the metric d_H. However, by Theorem 2.5 nothing changes if some of these pairs is endowed with the corresponding graph metric d_h^*.

THEOREM 2.16. — *Let* M *be a symmetric hypersurface in* H *of class* $C^{1,1}_{\text{loc}}$. *Let us suppose that:*

a) M *is a closed subset of* H *and for every* h *in* $\overline{\mathbf{N}}$, f_h *is lower semicontinuous and even;*

b) there exists q *in* $\mathbf{R}^+$ *such that for every* h *in* $\overline{\mathbf{N}}$ *the function* $\left\{y \mapsto f_h(u) + q|u|^2\right\}$ *is convex;*

c) $f_\infty = \Gamma^-(H)\lim_h f_h$ *;*

d) the sequence $(f_h + I_M)$ *is equicoercive;*

e) for every u *in* $D(f_\infty) \cap M$, $D(f_\infty)$ *and* M *are not tangent at* u.

Then there exists a sequence $(\widehat{f}_h)$ *of functions* $\widehat{f}_h : H \to \mathbf{R} \cup \{+\infty\}$ $(h \in \overline{\mathbf{N}})$ *with the following properties.*

i) $\widehat{f}_\infty = f_\infty$ *;* $\forall\, h \in \mathbf{N}$, $\widehat{f}_h \geq f_h$ *;*

ii) $\forall\, h \in \overline{\mathbf{N}}$, $\widehat{f}_h$ *is lower semicontinuous and even;*

iii) $\forall\, h \in \overline{\mathbf{N}}$, *the function* $\left\{u \mapsto \widehat{f}_h(u) + q|u|^2\right\}$ *is convex;*

iv) $\widehat{f}_\infty = \Gamma^-(H) \lim\limits_h \widehat{f}_h$;

v) $\forall\, h \in \overline{\mathbf{N}}$, $\forall\, u \in D(\widehat{f}_h) \cap M$, $D(\widehat{f}_h)$ *and* M *are not tangent at* u;

vi) if (u_h) *is a sequence in* M *with* $\limsup_h \widehat{f}_h(u_h) < +\infty$, *we have:*

$$\widehat{f}_h(u_n) = f_h(u_h)\,, \quad \partial^- \widehat{f}_h(u_n) = \partial^- f_h(u_h)$$

eventually as $h \to \infty$.

Moreover, if we set $\widetilde{f}_h = \widehat{f}_h + I_M$, $\forall\, h \in \overline{\mathbf{N}}$, *the following facts hold:*

vii) $\forall\, h \in \overline{\mathbf{N}}$, $\widetilde{f}_h$ *is lower semicontinuous and even;*

viii) there exists a continuous function:

$$\chi : \left(\bigcup_{h \in \overline{\mathbf{N}}} D(\widetilde{f}_h) \right)^2 \times \mathbf{R}^2 \to \mathbf{R}^+$$

such that:

$$(\alpha - \beta \mid u - v) \geq$$
$$\geq -\chi\left(u, v, \widetilde{f}_h(u), \widetilde{f}_h(v)\right)\left(1 + |\alpha|^2 + |\beta|^2\right)|u - v|^2\,;$$

whenever $h \in \overline{\mathbf{N}}$, $u, v \in D(\partial^- \widetilde{f}_h)$, $\alpha \in \partial^- \widetilde{f}_h(u)$, $\beta \in \partial^- \widetilde{f}_h(v)$;

ix) $\widetilde{f}_\infty = \Gamma^-(H) \lim_h \widetilde{f}_h$;

x) the sequence $(\widetilde{f}_h)$ *is equicoercive.*

Proof.— See [12, Theorem 2.17]. □

3. Variational bifurcation for even nonsmooth functions

Throughout this section we keep the notations of §2. We shall consider a real Hilbert space H, a convex open subset W of H with $0 \in W$, $-W = W$, an even function:

$$f : \overline{W} \to \mathbf{R} \cup \{+\infty\}$$

such that:

$$f(0) = 0\,, \quad 0 \in \partial^- f(0) \tag{3.1}$$

and a symmetric bounded linear operator:

$$L : H \to H .$$

Our purpose is to study the set of the pairs (λ, u) such that:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times W \\ \lambda\, Lu \in \partial^- f(u) . \end{cases} \tag{3.2}$$

Because of (3.1), for every λ in $\mathbf{R}$ the pair $(\lambda, 0)$ satisfies (3.2). Moreover, since f is even, solutions of (3.2) always occur in pairs (λ, u) and $(\lambda, -u)$.

DEFINITION 3.3.— *A real number λ is said to be of bifurcation for (3.2), if there exists a sequence $((\lambda_h, u_h))$ of solutions of (3.2) with $u_h \neq 0$ and:*

$$\lim_h (\lambda_h, u_h) = (\lambda, 0) .$$

As in the case of smooth functions f (see [23]), a first problem is to compare the bifurcation values with the eigenvalues of some "linearized" problem. This question has been the object of [12, 13, 14, 15]. Here we are interested in a multiplicity result, corresponding to that of [6, 28], which is connected with the eveness hypothesis on f.

Let us make the following further assumptions of f:

(A1) the function f is lower semicontinuous on $\overline{W}$ and there exists $q \in \mathbf{R}^+$ such that the function $\left\{u \mapsto f(u) + q|u|^2\right\}$ is convex on $\overline{W}$;

(A2) there exists a function $f_0 : H \to \mathbf{R} \cup \{+\infty\}$ such that for every sequence (ρ_h) in $]\,0\,,\,1\,]$ converging to zero, we have:

$$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$$

where $\forall\, \rho \in \;]\,0\,,\,1\,]$, $f_\rho : H \to \mathbf{R} \cup \{+\infty\}$ is defined by:

$$f_\rho(u) = \begin{cases} \rho^{-2} f(\rho u) & \rho u \in \overline{W} \\ +\infty & \rho u \in H \setminus \overline{W} . \end{cases}$$

Let us summarize the main properties of f_ρ $(0 \leq \rho \leq 1)$.

PROPOSITION 3.4.— *Let $\rho \in [0, 1]$. Then the following facts hold:*

i) *f_ρ is lower semicontinuous on H, even and the function:*

$$\left\{u \mapsto f_\rho(u) + q|u|^2\right\}$$

is convex on H;

ii) $\forall\, v \in H$, $\forall\, u \in D(\partial^- f_\rho)$, $\forall\, \alpha \in \partial^- f_\rho(u)$, *we have:*

$$f_\rho(v) \geq f_\rho(u) + (\alpha \mid v - u) - q|v - u|^2\,;$$

iii) $f_\rho(0) = 0$, $0 \in \partial^- f_\rho(0)$;

iv) *for every sequence (ρ_h) in $[0, 1]$ converging to ρ, we have:*

$$f_\rho = \Gamma^-(H) \lim_h f_{\rho_h}\,.$$

v) *if $\rho > 0$, $\forall\, u, \alpha \in H$ we have:*

$u \in D(\partial^- f_\rho)$ *and* $\alpha \in \partial^- f_\rho(u) \Leftrightarrow \rho u \in D(\partial^- f)$ *and* $\rho\alpha \in \partial^- f(\rho u)$;

vi) $\forall\, s > 0$, $\forall\, u \in H$, $f_0(su) = s^2 f_0(u)$;

vii) $\forall\, s > 0$, $\forall\, u \in D(\partial^- f_0)$, $\forall\, \alpha \in \partial^- f_0(u)$, $s\alpha \in \partial^- f_0(su)$;

viii) $\forall\, u \in D(\partial^- f_0)$, $\forall\, \alpha \in \partial^- f_0(u)$, $(\alpha \mid u) = 2 f_0(u)$.

Proof.— See [12, Propositions 3.4 and 3.6]. □

The function $f_0(u)$ introduced in (A2) plays the role of the quadratic form $\frac{1}{2} f''(0)(u, u)$ in the smooth case. In the following it will be convenient to consider also the "linearized" problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times H \\ \lambda\, Lu \in \partial^- f_0(u)\,. \end{cases} \tag{3.5}$$

Remark 3.6.— For every λ in $\mathbf{R}$ the set:

$$\left\{u \in H : \lambda\, Lu \in \partial^- f_0(u)\right\}$$

is a closed cone.

Proof.— It is a consequence of (i), (ii) and (vii) of Proposition 3.4. □

DEFINITION 3.7.— *A real number λ is said to be an eigenvalues of (3.5), if the pair (λ, u) satisfies (3.5) for some $u \neq 0$.*

Remark 3.8.— If (λ, u) satisfies (3.5), then:

$$f_0(u) = \frac{1}{2}\,\lambda(Lu \mid u)\,.$$

Proof.— If follows from Proposition 3.4_{viii}. □

In order to give a variational characterization of the eigenvalues of (3.5), let us introduce the sets:

$$M^+ = \left\{u \in H : \frac{1}{2}\,(Lu \mid u) = 1\right\}\,,$$
$$M^- = \left\{u \in H : \frac{1}{2}\,(Lu \mid u) = -1\right\}\,,$$

which are symmetric hypersurfaces in H of class C^∞ and closed subsets of H.

According to Remark 2.8, we have for every u in $M^\pm$,

$$\partial^- I_{M^\pm}(u) = \{\lambda L u : \lambda \in \mathbf{R}\}\ .$$

PROPOSITION 3.9.— *For every $u \in D(f_0) \cap M^\pm$, $D(f_0)$ and $M^\pm$ are not tangent at u.*

Proof.— See [12, Proposition 3.11]. □

PROPOSITION 3.10.— *Let $\lambda \in \mathbf{R}$ and let us consider the following facts:*

i) (λ, u) *satisfies (3.5) for some u with* $(Lu \mid u) > 0$;

ii) λ *is critical from below for* $(f_0 + I_{M^+})$;

iii) (λ, u) *satisfies (3.5) for some u with* $(Lu \mid u) < 0$;

iv) $-\lambda$ *is critical from below for* $(f_0 + I_{M^-})$.

Then we have:

$$i) \Leftrightarrow ii)\,;$$
$$iii) \Leftrightarrow iv)\,.$$

Proof.— See [12, Proposition 3.12].□

THEOREM 3.11.— *Let us assume that for every sequence (u_h) in $W\backslash\{0\}$ with:*

$$\lim_h u_h = 0\,, \quad \sup_h |u_h|^{-2} f(u_h) < +\infty\,,$$

the sequence $\left(u_h/|u_h|\right)$ *has a convergent subsequence.*

Then every λ *of bifurcation for (3.2) is an eigenvalue of (3.5).*

Proof.— See [12, Theorem 3.14]. □

The converse is not true, in general. Let us consider $H = \mathbf{R}^2$, $W = B(0,1)$, $f : \overline{W} \to \mathbf{R} \cup \{+\infty\}$ defined by:

$$f(x,y) = \begin{cases} x^3y^3(x^2+y^2)^{-2} + xy(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and L the identity map. Then all the assumptions of Theorem 3.11 are satisfied with:

$$f_0(x,y) = \begin{cases} x^3y^3(x^2+y^2)^{-2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

However $\lambda = 0$ is an isolated eigenvalue of (3.5) which is not of bifurcation for (3.2). The feature of this example is that f_0 does not behave like a quadratic form. On the contrary, if f_0 is a quadratic form in a generalized sense that will be precised later, then the converse of Theorem 3.11 holds [12, Theorem 3.29]. Our purpose is to show that the eveness of f allows to prove also a multiplicity result, as it is done in [6, 28] for smooth functions. Actually, we shall consider only the eigenvalues λ of (3.5) such that there exists a solution (λ, u) of (3.5) with $(Lu \mid u) \neq 0$. Since we can change L in $-L$, this is equivalent to consider the eigenvalues λ of (3.5) such that there exists a solution (λ, u) of (3.5) with $(Lu \mid u) > 0$.

First of all let us consider the following compactness assumptions:

($A3^+$) for every sequence (u_h) in $\overline{W}$ with $\frac{1}{2}(Lu_h \mid u_h) \in \left]0\,,\,1\right]$ and:

$$\sup_h |(Lu_h \mid u_h)|^{-1} f(u_h) < +\infty\,,$$

the sequence $\left(|(Lu_h \mid u_h)|^{-1/2}\, u_h\right)$ has a convergent subsequence;

($A3^-$) for every sequence (u_h) in $\overline{W}$ with $\frac{1}{2}(Lu_h \mid u_h) \in \left[-1\,,\,0\right[$ and:

$$\sup_h |(Lu_h \mid u_h)|^{-1} f(u_h) < +\infty\,,$$

the sequence $\left(|(Lu_h \mid u_h)|^{-1/2}\, u_h\right)$ has a convergent subsequence.

If L is the identity map of H, hypothesis (A3$^+$) implies the compactness assumption made in Theorem 3.11.

PROPOSITION 3.12.— *Condition (A3$^+$) (resp. (A3$^-$)) holds if and only if for every sequence (ρ_h) in $]0,1]$ the sequence $(f_{\rho_h} + I_{M^+})$ (resp. $(f_{\rho_h} + I_{M^-})$) is equicoercive.*

Proof.— See [12, Proposition 3.15]. □

PROPOSITION 3.13.— *Let (A3$^+$) (resp. (A3$^-$)) hold. Then for every c in $\mathbf{R}$ the set $(f_0 + I_{M^+})^c$ (resp. $(f_0 + I_{M^-})^c$) is compact.*

Proof.— See [12, Proposition 3.16]. □

Let us denote by $\mathcal{C}$ the set of the c's in $\mathbf{R}$ such that the function $\left\{u \mapsto f_0(u) + (c - \epsilon/2)\,|u|^2\right\}$ is convex for some $\epsilon > 0$ (in particular, $f_0(u) + (c-\epsilon/2)\,|u|^2 \geq 0$, $\forall\, u \in H$ by Proposition 3.4$_{\text{iii}}$).

The set $\mathcal{C}$ is not empty. In fact $]2q\,,\,+\infty[\ \subset \mathcal{C}$ by Proposition 3.4$_{\text{i}}$).

If $c \in \mathcal{C}$ and $u \in H$, let us denote by $i_c^* u$ the minimum point of the function $\left\{v \mapsto f_0(v) + c/2\,|v|^2 - (v \mid u)\right\}$. Evidently:

$$\forall\, u \in H\,, \quad (u - c i_c^* u) \in \partial^- f_0(i_c^* u)\,; \tag{3.14}$$

$$\forall\, u \in D(\partial^- f_0)\,, \quad \forall\, \alpha \in \partial^- f_0(u)\,, \quad u = i_c^*(\alpha + cu)\,. \tag{3.15}$$

LEMMA 3.16.— *Let X be a real Hilbert space and $B, K : X \to X$ two symmetric bounded linear operators such that $BK = KB$. Let:*

$$M = \left\{u \in X : \frac{1}{2}(Bu \mid u) = 1\right\},$$
$$Q_0(u) = \frac{1}{2}(Ku \mid u)\,, \quad Q(u) = \frac{1}{2}\,|u|^2 - Q_0(u)$$

and let us assume that M is nonempty and sequentially weakly closed in X, $Q_{0|M}$ is sequentially weakly continuous and:

$$\lim_{|u| \to +\infty} Q_{|M}(u) = +\infty\,,$$

if M is not bounded. Let λ be a critical value of $Q_{|M}$.

Then the following facts hold:

i) the linear space $\{u \in X : u - Ku = \lambda Bu\}$ *has finite dimension, say* m*;*

ii) there exists $\epsilon_0 > 0$ *such that* $Q_{|M}$ *has no critical values in* $[\lambda - \epsilon_0\,,\, \lambda + \epsilon_0\,] \setminus \{\lambda\}$*;*

iii) for every $\epsilon \in \,]\,0\,,\,\epsilon_0\,]$*, the pair* $\left((Q_{|M})^{\lambda+\epsilon}, (Q_{|M})^{\lambda-\epsilon}\right)$ *is equivariantly homotopically equivalent to the pair* (S^{m+n-1}, S^{n-1}) *for some integer* $n \geq 0$ *(with the convention* $S^{-1} = \emptyset$*).*

Proof.— For properties i) and ii), we refer the reader to [12, Lemma 3.22]. In order to prove iii), let us recall some steps in the proof of [12, Lemma 3.22].

Let $A = I - K$ and let

$$\Lambda = \{\mu \in \mathbf{R} : Au = \mu Bu \text{ for some } u \in X \text{ with } (Bu \mid u) > 0\},$$

which is also the set of the critical values of $Q_{|M}$. Then $\Lambda \cap \,]-\infty\,,\,\lambda\,]$ is finite, say $\Lambda \cap \,]-\infty\,,\,\lambda\,] = \{\lambda_1, \ldots, \lambda_h\}$ with $\lambda_1 < \cdots < \lambda_h = \lambda$. If we set $V_i = \{u \in X : Au = \lambda_i Bu\}$ for $i = 1, \ldots, h$, then the subspaces V_i are finite-dimensional and pairwise orthogonal. Let $V' = V_1 \oplus \cdots \oplus V_{h-1}$, $V = V' \oplus V_h$, $Y = V^{\perp}$, $m = \dim(V_h)$ and $n = \dim(V')$, ($n = 0$ if $h = 1$). The subspaces V', V and Y are invariant for A and B. Moreover $(Bu \mid u) > 0$, $\forall\, u \in V \setminus \{0\}$ and:

$$\begin{aligned}\lambda_h &= \max\{Q(u) : u \in M \cap V\} = \\ &= \min\{Q(u) : u \in M \cap (V_h \oplus Y)\} < \min\{Q(u) : u \in M \cap Y\}\,, \\ \lambda_{h-1} &= \max\{Q(u) : u \in M \cap V'\}\end{aligned}$$

Finally let ϵ_0 be such that:

$$\lambda_{h-1} < \lambda - \epsilon_0 < \lambda + \epsilon_0 < \min_{M \cap Y} Q\,.$$

To prove iii), let us take $\epsilon \in \,]\,0\,,\,\epsilon_0\,]$.

Let $P_{V'}$ be the orthogonal projection on V', $\theta = \mathbf{R} \to [\,0\,,\,1\,]$ a continuous function such that $\theta(s) = 1$ for $s \leq \lambda - \epsilon$, $\theta(s) = 0$ for $s \geq \lambda - \epsilon/2$ and:

$$\mathcal{H}_1 : \left((Q_{|M})^{\lambda+\epsilon}, (Q_{|M})^{\lambda-\epsilon}\right) \times [\,0\,,\,1\,] \to \left((Q_{|M})^{\lambda+\epsilon}, (Q_{|M})^{\lambda-\epsilon}\right)$$

the map defined by:

$$\mathcal{H}_1(u,t) = \frac{\psi_1(u,t)}{\left\{\dfrac{1}{2}\,(B\psi_1(u,t) \mid \psi_1(u,t))\right\}^{1/2}}\,,$$

where $\psi_1(u,t) = P_{V'}\,u + \big(1 - t\theta(Q(u))\big)\,(u - P_{V'}u)$.

It is readily seen that $\mathcal{H}_1(-u,t) = -\mathcal{H}_1(u,t)$, $\mathcal{H}_1(u,0) = u$,

$$\forall\, u \in \left(Q_{|M}\right)^{\lambda-\epsilon}, \quad \mathcal{H}_1(u,1) \in M \cap V',$$
$$\forall\, u \in M \cap V', \forall\, t \in [0,1], \quad \mathcal{H}_1(u,t) = u.$$

Therefore the pair $\left((Q_{|M})^{\lambda+\epsilon}, (Q_{|M})^{\lambda-\epsilon}\right)$ is equivariantly homotopically equivalent to the pair $\left((Q_{|M})^{\lambda+\epsilon}, M \cap V'\right)$.

Now let us consider the map:

$$\mathcal{H}_2 : \left((Q_{|M})^{\lambda+\epsilon}, M \cap V'\right) \times [0,1] \to \left((Q_{|M})^{\lambda+\epsilon}, M \cap V'\right)$$

defined by:

$$\mathcal{H}_2(u,t) = \frac{\psi_2(u,t)}{\left\{\frac{1}{2}\left(B\psi_2(u,t) \mid \psi_2(u,t)\right)\right\}^{1/2}},$$

where $\psi_2(u,t) = P_V u + (1-t)(u - P_V u)$ and P_V is the orthogonal projection on V.

Then $\mathcal{H}_2(-u,t) = -\mathcal{H}_2(u,t)$, $\mathcal{H}_2(u,0) = u$,

$$\forall\, u \in M \cap V, \forall\, t \in [0,1], \quad \mathcal{H}_2(u,t) = u,$$
$$\forall\, u \in \left(Q_{|M}\right)^{\lambda+\epsilon}, \quad \mathcal{H}_2(u,1) \in M \cap V.$$

Therefore the pair $\left((Q_{|M})^{\lambda+\epsilon}, M \cap V'\right)$ is equivariantly homotopically equivalent to the pair $(M \cap V, M \cap V')$.

Finally, since B is positive definite on V, it is readily seen that the pair $(M \cap V, M \cap V')$ is equivariantly homeomorphic to the pair (S^{m+n-1}, S^{n-1}). □

THEOREM 3.17. — *Let $(A\mathcal{S}^+)$ hold and let us assume that:*

$$\forall\, u,v \in H, \quad f_0(u+v) + f_0(u-v) = 2f_0(u) + 2f_0(v) \tag{3.18}$$

and that:

$$ci_c^* L = L ci_c^* \tag{3.19}$$

for some $c \in \mathcal{C}$. Let (λ, u) be a solution of (3.5) with $(Lu \mid u) > 0$.

Then the following facts hold:

i) *the set* $E_\lambda := \{v \in H : \lambda L v \in \partial^- f_0(u)\}$ *is a linear subspace of* H *of finite dimension;*

ii) *there exists* $\epsilon_0 > 0$ *such that every* $\mu \in [\lambda - \epsilon_0 , \lambda + \epsilon_0] \setminus \{\lambda\}$ *is not critical from below for* $(f_0 + I_{M^+})$*;*

iii) *for every* $\epsilon \in \,]0, \epsilon_0]$*, the pair* $\left((f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon}\right)$*, endowed with the graph metric* d^**, is equivariantly homotopically equivalent to the pair* $\left(S^{m+n-1}, S^{n-1}\right)$*, where* $m = \dim(E_\lambda)$ *and* $n \geq 0$ *a suitable integer;*

iv) *for every* $\epsilon \in \,]0, \epsilon_0]$*, we have:*

$$\mathrm{Index}\left((f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon}\right) = \dim(E_\lambda).$$

Proof.— The proof follows the lines of the proof of [12, Theorem 3.26]. We sketch it for reader's convenience.

Let $c \in \mathcal{C}$ be such that (3.19) holds. Combining Proposition 3.4 with (3.18), we deduce that $D(f_0)$ is a linear subspace of H. Moreover:

$$((v \mid w)) = \frac{1}{4}\left(2f_0(v+w) + c|v+w|^2 - 2f_0(v-w) - c|v-w|^2\right)$$

defines a scalar product on $D(f_0)$ such that the embedding

$$i : \big(D(f_0), ((\cdot \mid \cdot))\big) \to \big(H, (\cdot \mid \cdot)\big)$$

is continuous. The space $\big(D(f_0), ((\cdot \mid \cdot))\big)$ is complete, hence a Hilbert space, and $i_c^* : H \to D(f_0)$ is just the adjoint map of the embedding i.

We can define two symmetric bounded linear operators $K, B : D(f_0) \to D(f_0)$ by $K = c i_c^* i$, $B = i_c^* L i$. We have $BK = KB$ by (3.19) and:

$$\forall\, v \in D(f_0), \quad \frac{1}{2}\,(((I-K)v \mid v)) = f_0(v), \quad ((Bv \mid v)) = (Lv \mid v).$$

If we set:

$$M := \left\{v \in D(f_0) : \frac{1}{2}\,((Bv \mid v)) = 1\right\} = M^+ \cap D(f_0),$$
$$Q_0(v) := \frac{1}{2}\,((Kv \mid v)) = \frac{1}{2}\,|v|^2,$$
$$Q(v) = \frac{1}{2}\,((v \mid v)) - Q_0(v),$$

the assumptions of Lemma 3.16 are satisfied. Moreover:

$$\forall\,(\mu, v) \in \mathbf{R} \times D(f_0)\,, \quad \mu L v \in \partial^- f_0(v) \Leftrightarrow v - Kv = \mu B v\,.$$

Therefore λ is a critical value of $Q_{|M}$ so that i) and ii) follows from Proposition 3.10 and the corresponding i) and ii) of Lemma 3.16.

On the other hand it is readily seen that the scalar product $((\cdot \mid \cdot))$ and the graph metric d^* associated with $(f_0 + I_{M^+})$ induce the same topology on $D(f_0 + I_{M^+})$ and $\forall\, b \in \mathbf{R}$, $(f_0 + I_{M^+})^b = \left(Q_{|M}\right)^b$. Then iii) follows from iii) of Lemma 3.16.

Finally, if $n = 0$ iv) is an immediate consequence of iii) and Theorem 2.6. If $n \geq 1$, the pair $\left(S^{m+n-1}\,, S^{n-1}\right)$ is equivariantly homeomorphic to the pair $\left(S^{m-1} \circ S^{n-1}\,, S^{n-1}\right)$. Combining again iii) with Theorem 2.6, iv) follows also in this case.□

Remark 3.20.— Assumption (3.19) is satisfied in each of the following situations:

i) L is the identity map of H;

ii) the map $\{v \mapsto f_0(v) - \epsilon |v|^2\}$ is convex for some $\epsilon > 0$ (so that we can choose $c = 0$).

Remark 3.21.— We shall see in the proof of Theorem 3.23 that the function $(f_0 + I_{M^+})$ satisfies the assumptions of Theorem 2.5. Therefore the precisation that $\left((f_0 + I_{M^+})^{\lambda+\epsilon}\,, (f_0 + I_{M^+})^{\lambda-\epsilon}\right)$ is endowed with the metric d^* is not essential.

LEMMA 3.22.— *Let $(A\mathcal{S}^+)$ holds and let $\lambda \in \mathbf{R}$, $m \in \mathbf{N}$ $m \geq 1$. Then the following facts are equivalent:*

i) there exists $\rho_0 > 0$, $\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0\,, 1 \leq i \leq 2m\} \subset \mathbf{R}$, $\{u_\rho^{(i)} : 0 < \rho \leq \rho_0\,, 1 \leq i \leq 2m\} \subset W$ such that:

$$\lambda_\rho^{(i)} L u_\rho^{(i)} \in \partial^- f(u_\rho^{(i)})\,, \quad \frac{1}{2}\left(L u_\rho^{(i)} \mid u_\rho^{(i)}\right) = \rho^2\,;$$

$$i \neq j \Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)}\,;$$

$$\lim_{\rho \to 0^+} \left(\lambda_\rho^{(i)}\,, u_\rho^{(i)}\right) = (\lambda, 0)\,, \quad \lim_{\rho \to 0^+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda\,;$$

ii) for every $\epsilon > 0$ *and for every sequence* (ρ_h) *in* $]0,1]$ *converging to zero, there exists* $h_0 \in \mathbf{N}$, $\left\{\lambda_h^{(i)} : h \geq h_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{R}$, $\left\{u_h^{(i)} : h \geq h_0\,,\ 1 \leq i \leq 2m\right\} \subset M^+$ *such that:*

$$\lambda_h^{(i)} L u_h^{(i)} \in \partial^- f_{\rho_h}(u_h^{(i)})\,,$$
$$i \neq j \Rightarrow u_h^{(i)} \neq u_h^{(j)}\ ;$$
$$\lambda - \epsilon \leq f_{\rho_h}(u_h^{(i)}) \leq \lambda + \epsilon\,.$$

Proof.— The case in which $2m$ is substituted by 1 is proved in [12, Lemma 3.19]. The adaptation to the present case is straightforward. □

Now we can prove the main abstract result.

THEOREM 3.23.— *Let us assume that (A3⁺), (3.18) and (3.19) hold. Let* (λ, u) *be a solution of (3.5) with* $(Lu \mid u) > 0$ *and let* $m = \dim\left(\{v \in H : \lambda L v \in \partial^- f_0(v)\}\right)$, *which is a linear subspace of* H *of finite dimension by Theorem 3.17.*

Then there exist $\rho_0 > 0$,

$$\left\{\lambda_\rho^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{R}\,,$$
$$\left\{u_\rho^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset W$$

such that:

$$\lambda_\rho^{(i)} L u_\rho^{(i)} \in \partial^- f(u_\rho^{(i)})\,, \quad \frac{1}{2}\left(L u_\rho^{(i)} \mid u_\rho^{(i)}\right) = \rho^2\ ;$$
$$i \neq j \Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)}\ ;$$
$$\lim_{\rho \to 0^+} \left(\lambda_\rho^{(i)}\,,\ u_\rho^{(i)}\right) = (\lambda, 0)\,, \qquad \lim_{\rho \to 0^+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda\ ;$$

Proof.— Let ϵ_0 be as in Theorem 3.17. To prove the thesis, it is sufficient to verify condition ii) of Lemma 3.22. Therefore let $\epsilon > 0$ and let (ρ_h) be a sequence in $]0,1]$ converging to zero. Without loss of generality, we can assume $\epsilon \leq \epsilon_0$.

If we set $\rho_\infty = 0$, by Propositions 3.4, 3.9, 3.12 and (A2) we can apply Theorem 2.16 to the sequence (f_{ρ_h}) and the hypersurface M^+. Let $(\widehat{f}_h)$ and $(\widetilde{f}_h)$ be the sequences given by Theorem 2.16.

Since $\widetilde{f}_\infty = \widehat{f}_\infty + I_{M^+} = f_0 + I_{M^+}$, by Theorem 3.17, $\lambda - \epsilon$ and $\lambda + \epsilon$ are not critical from below for $\widetilde{f}_\infty$. Therefore Theorem 2.16 allows to apply Theorem 2.14 to the sequence $(\widetilde{f}_h)$ with $a = \lambda - \epsilon$ and $b = \lambda + \epsilon$. Let $h_0 \in \mathbf{N}$ be such that $\forall\, h \geq h_0$ $\lambda - \epsilon$ and $\lambda + \epsilon$ are not critical from below for $\widetilde{f}_h$ and the pair $\left(\widetilde{f}_h^{\lambda+\epsilon}, \widetilde{f}_h^{\lambda-\epsilon}\right)$ is equivariantly homotopically equivalent to the pair $\left((f_0 + I_{M^+})^{\lambda+\epsilon}, (f_0 + I_{M^+})^{\lambda-\epsilon}\right)$.

Combining Theorem 2.6 with Theorem 3.17, we deduce that:

$$\forall\, h \geq h_0\,, \quad \mathrm{Index}\left(\widetilde{f}_h^{\lambda+\epsilon}, \widetilde{f}_h^{\lambda-\epsilon}\right) = m\,.$$

By Theorem 2.16_{x} it is readily seen that $\widetilde{f}_h$ satisfies $(PS)_c$ for every $c \in \mathbf{R}$. Therefore we can apply Theorem 2.7 to $\widetilde{f}_h$, obtaining the existence of $u_h^{(i)} \in M^+$ $(1 \leq i \leq 2m)$ with:

$$0 \in \partial^- \widetilde{f}_h(u_h^{(i)})\,,$$
$$i \neq j \Rightarrow u_h^{(i)} \neq u_h^{(j)}\,,$$
$$\lambda - \epsilon < \widetilde{f}_h(u_h^{(i)}) < \lambda + \epsilon\,.$$

By Theorem 2.16_{iii} we have:

$$\widehat{f}_h(w) \geq \widehat{f}_h(v) + (\alpha \mid w - v) - q|w - v|^2$$

whenever $h \in \mathbf{N}$, $w \in H$, $v \in D(\partial^- \widehat{f}_h)$, $\alpha \in \partial^- \widehat{f}_h(v)$. Therefore we can apply Theorem 2.10 to $\widehat{f}_h$ and M^+, obtaining $\lambda_h^{(i)} L u_h^{(i)} \in \partial^- \widehat{f}_h(u_h^{(i)})$ for some $\lambda_h^{(i)} \in \mathbf{R}$. Since $\widehat{f}_h(u_h^{(i)}) < \lambda - \epsilon$, we conclude that:

$$\widehat{f}_h(u_h^{(i)}) = f_{\rho_h}(u_h^{(i)})\,, \quad \partial^- \widehat{f}_h(u_h^{(i)}) = \partial^- f_{\rho_h}(u_h^{(i)})$$

eventually as $h \to \infty$, hence the thesis. □

4. On the buckling of a thin elastic plate subjected to symmetric constraints

We wish to show an application of the results of the previous section to the buckling of a thin elastic plate subjected to symmetric unilateral constraints.

Let Ω be a bounded open subset of $\mathbf{R}^2$, F_0 (the initial Airy stress function) an element of $W^{2,2}(\Omega)$ and $\varphi : \Omega \to [0, +\infty]$ a lower semicontinuous function. If we set:

$$\mathsf{K} = \left\{ u \in W_0^{2,2}(\Omega) : -\varphi \le u \le \varphi \right\} ,$$

we have to study, according to [4, 5, 29, 30, 39], the problem:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathsf{K} \\ \langle \Delta^2 u , v - u \rangle + \big\langle [J[u,u] , u] , v - u \big\rangle \ge \lambda \big\langle [F_0, u] , v - u \big\rangle , \ \forall\, v \in \mathsf{K} \end{cases} \tag{4.1}$$

where:

$$[v , w] := D^2_{xx} v \, D^2_{yy} w - 2 D^2_{xy} v \, D^2_{xy} w + D^2_{yy} v \, D^2_{xx} w$$

and for every $\eta \in W^{-2,2}(\Omega)$, $J\eta \in W_0^{2,2}(\Omega)$ is characterized by $\Delta^2 (J\eta) = \eta$. Let us remark that for every $v, w \in W^{2,2}(\Omega)$,

$$[v , w] \in L^1(\Omega) \hookrightarrow W^{-2,2}(\Omega) , \text{ as } W_0^{2,2}(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) .$$

Therefore the problem is meaningful.

It is readily seen that for every λ in $\mathbf{R}$, the pair $(\lambda, 0)$ is a solution of (4.1).

DEFINITION 4.2. — *A real number λ is said to be of bifurcation for (4.1), if there exists a sequence $((\lambda_h, u_h))$ of solutions of (4.1) with $u_h \ne 0$,*

$$\lim_h \lambda_h = \lambda , \quad \lim_h u_h = 0 \quad \textit{in} \quad W_0^{2,2}(\Omega) .$$

Let K_0 be the closure in $W_0^{2,2}(\Omega)$ of the set $(\bigcup_{t>0} t\mathsf{K})$.

Since $-\mathsf{K} = \mathsf{K}$, K_0 is a closed linear subspace of $W_0^{2,2}(\Omega)$ and:

$$\begin{cases} (\lambda, u) \in \mathbf{R} \times \mathsf{K}_0 \\ \left\langle \Delta^2 u , v \right\rangle = \lambda \big\langle [F_0, u] , v \big\rangle , \quad \forall\, v \in \mathsf{K}_0 \end{cases} \tag{4.3}$$

can be regarded as the linearized problem associated with (4.1).

DEFINITION 4.4. — *A real number λ is said to be an eigenvalue of (4.3), if the pair (λ, u) satisfies (4.3) for some $u \ne 0$.*

THEOREM 4.5. — *Let λ be a real number. Then λ is of bifurcation for (4.1) if and only if λ is an eigenvalue of (4.3).*

Proof. — Since K_0 is a linear space, $(\lambda, u) \in \mathbf{R} \times \mathsf{K}_0$ satisfies (4.3) if and only if:

$$\left\langle \Delta^2 u \,,\, v-u \right\rangle \geq \lambda \left\langle \left[F_0, u \right] , v-u \right\rangle , \quad \forall\, v \in \mathsf{K}_0 .$$

Then the result is contained in [13, Theorem 3.5 and 3.13]. □

Our purpose is to show a multiplicity result, taking advantage of the symmetry of K.

For every $v \in W^{2,2}(\Omega)$, $w \in W^{1,4}(\Omega)$, let us set:

$$[v\,,\,w]_1 = D_x \left(D^2_{yy} v D_x w - D^2_{xy} v D_y w \right) + D_y \left(D^2_{xx} v D_y w - D^2_{xy} v D_x w \right)$$

$$[w]_2 = - \left(D^2_{xx} (D_y w)^2 - 2 D^2_{xy} (D_x w D_y w) + D^2_{yy} (D_x w)^2 \right) .$$

It is readily seen that the maps:

$$[\,\cdot\,,\,\cdot\,]_1 : W^{2,2}(\Omega) \times W^{1,4}(\Omega) \to W^{-1,\,4/3}(\Omega) ,$$
$$[\,\cdot\,]_2 : W^{1,4}(\Omega) \to W^{-2,2}(\Omega)$$

are of class C^∞. Moreover (see [4, 29]) we have for every $v, w \in W^{2,2}_0(\Omega)$,

$$[v, w]_1 = [v, w] \ , \ [w]_2 = [w, w]$$

as elements of $W^{-2,2}(\Omega)$.

Let $3/2 < s < 2$, $H = W^{s,2}_0(\Omega)$ and let us define $A, f_0 : H \to \mathbf{R} \cup \{+\infty\}$, $B, C : H \to \mathbf{R}$ by:

$$A(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\Delta u|^2 \, dx & u \in \mathsf{K} \\ +\infty & u \in H \setminus \mathsf{K} , \end{cases}$$

$$f_0(u) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega |\Delta u|^2 \, dx & u \in \mathsf{K}_0 \\ +\infty & u \in H \setminus \mathsf{K}_0 , \end{cases}$$

$$B(u) = \frac{1}{2} \left\langle [F_0, u]_1 \ , u \right\rangle ,$$

$$C(u) = \frac{1}{4} \left\langle \left[J \left[u \right]_2 , u \right]_1 \ , u \right\rangle ,$$

which is meaningful, as $W^{s,2}_0(\Omega) \hookrightarrow W^{1,4}_0(\Omega)$.

The functionals B and C are of class C^∞ and homogenuous of degree 2 and 4, respectively. The functionals A and f_0 are convex and lower semicontinuous. Let us denote by $L : H \to H$ the gradient of B and let us set $f = A + C$. Finally, let W be the open unit ball centred at the origin in H.

PROPOSITION 4.6.— *The following facts hold:*

i) f *is lower semicontinuous, even and* $\left\{u \mapsto f(u) + q|u|_H^2\right\}$ *is convex on* $\overline{W}$ *for some* $q \in \mathbf{R}^+$*;*

ii) $f(0) = 0$, $0 \in \partial^- f(0)$*;*

iii) L *linear, bounded and symmetric;*

iv) for every sequence (ρ_h) *in* $]0, +\infty[$ *converging to zero, we have:*

$$f_0 = \Gamma^-(H) \lim_h f_{\rho_h}$$

where $f_\rho(u) = \rho^{-2} f(\rho u)$ *if* $\rho u \in \overline{W}$, $f_\rho(u) = +\infty$ *elsewhere;*

v) for every sequence (ρ_h) *in* $]0, +\infty[$*, the sequence* $\left(f_{\rho_h}\right)$ *is equicoercive;*

vi) $\forall\, u, v \in H$, $f_0(u+v) + f_0(u-v) = 2f_0(u) + 2f_0(v)$*;*

vii) the function $\left\{u \mapsto f_0(u) - \epsilon|u|_H^2\right\}$ *is convex for some* $\epsilon > 0$*;*

viii) for every λ *in* $\mathbf{R}$*,* u *in* $\mathbf{K}$*, we have* $\lambda L u \in \partial^- f(u)$ *if and only if:*

$$\left\langle \Delta^2 u \,,\, v - u \right\rangle + \langle [J[u,u] \,,\, u] \,,\, v - u\rangle \geq \lambda \left\langle [F_0, u] \,,\, v - u\right\rangle \,, \quad \forall\, v \in \mathbf{K};$$

ix) for every λ *in* $\mathbf{R}$*,* u *in* $\mathbf{K}_0$*, we have* $\lambda L u \in \partial^- f_0(u)$ *if and only if:*

$$\left\langle \Delta^2 u \,,\, v \right\rangle = \lambda \left\langle [F_0, u] \,,\, v\right\rangle \,, \quad \forall\, v \in \mathbf{K}_0 \,.$$

Proof.— The eveness of f is obvious. Then i), ii), iii), iv), v), viii) and ix) are contained in [13, Proposition 3.7]. Since $\mathbf{K}_0$ is a linear subspace of H, vi) follows immediately. Finally, vii) is a consequence of the continuous embedding $W_0^{2,2}(\Omega) \hookrightarrow H$. □

Remark 4.7.— Let (λ, u) be a solution of (4.3) with $u \neq 0$. Then $\lambda \left\langle [F_0, u] \,,\, u\right\rangle > 0$ and the set $\{u : (\lambda, u) \text{ satisfies } (4.3)\}$ is a linear subspace of $W_0^{2,2}(\Omega)$ of finite dimension.

Now we restrict our attention to positive eigenvalues of (4.3). By changing F_0 in $-F_0$, we can always reduce ourselves to this case. Let us state our main result.

THEOREM 4.8.— *Let λ be a positive eigenvalue of (4.3) and let $m = \dim\{u : (\lambda, u)$ satisfies (4.3)$\}$.*

Then there exist $\rho_0 > 0$,

$$\left\{\lambda_0^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{R}\,,$$
$$\left\{u_\rho^{(i)} : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\} \subset \mathbf{K}$$

such that $(\lambda_\rho^{(i)}, u_\rho^{(i)})$ satisfies (4.1) and:

$$\frac{1}{2}\int_\Omega \left[F_0, u_\rho^{(i)}\right] u_\rho^{(i)}\, dx = \rho^2\,,$$
$$i \neq j \Rightarrow u_\rho^{(i)} \neq u_\rho^{(j)}\,,$$
$$\lim_{\rho\to 0+} \left(\lambda_\rho^{(i)}, u_\rho^{(i)}\right) = (\lambda, 0) \text{ in } \mathbf{R} \times W_0^{2,2}(\Omega)\,,$$
$$\lim_{\rho\to 0+} \frac{1}{2\rho^2}\int_\Omega \left|\Delta u_\rho^{(i)}\right|^2 dx = \lambda\,.$$

Proof.— Combining Proposition 4.6$_\text{v}$ with Proposition 3.12, we deduce that assumption (A3$^+$) is satisfied. By Proposition 4.6 and Remark 3.20 we can apply Theorem 3.23. To conclude the proof, we have only to show that:

$$\lim_{\rho\to 0+} \frac{1}{2\rho^2}\int_\Omega \left|\Delta u_\rho^{(i)}\right|^2 dx = \lambda\,,$$

so that $\lim_{\rho\to 0+} u_\rho^{(i)} = 0$ in $W_0^{2,2}(\Omega)$ and not only in H.

Combining the inequality:

$$\forall\, u \in W_0^{2,2}(\Omega)\,,\quad f_\rho(u) \geq \frac{1}{2}\int_\Omega |\Delta u|^2\, dx$$

with:

$$\lim_{\rho\to 0+} \frac{f(u_\rho^{(i)})}{\rho^2} = \lambda\,,$$

we deduce that the set $\left\{u_\rho^{(i)}/\rho : 0 < \rho \leq \rho_0\,,\ 1 \leq i \leq 2m\right\}$ is bounded in $W_0^{2,2}(\Omega)$. Therefore:

$$\lim_{\rho\to 0+} \rho^{-2} C(u_\rho^{(i)}) = \lim_{\rho\to 0+} \rho^2 C\left(\frac{u_\rho^{(i)}}{\rho}\right) = 0\,,$$

so that:

$$\lim_{\rho\to 0+} \frac{1}{2\rho^2} \int_\Omega \left|\Delta u_\rho^{(i)}\right|^2 \mathrm{d}x = \lambda\,. \square$$

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