ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

MANUEL DE LEÓN
PAULO R. RODRIGUES

Dynamical connections and non-autonomous lagrangian systems

Annales de la faculté des sciences de Toulouse 5e série, tome 9, nº 2 (1988), p. 171-181

<http://www.numdam.org/item?id=AFST_1988_5_9_2_171_0>

Annales Faculté des Sciences de Toulouse Vol. IX, n°2, 1988

Dynamical connections and non-autonomous Lagrangian systems[(1)]

MANUEL DE LEON[(2)] and PAULO R. RODRIGUES[(3)]

RÉSUMÉ. — On montre que si ξ est une équation différentielle du deuxième ordre (semigerbe) sur le fibré des jets $J^1(\mathbf{R}, M)$ telle que les courbes intégrales sont des solutions de l'équation de Lagrange non-autonome alors il existe une connexion Γ sur $J^1(\mathbf{R}, M)$ dont les courbes intégrales sont aussi des solutions de la même équation. En plus, Γ est une connexion ayant comme semigerbe ξ. L'étude est une extension à la dynamique Lagrangienne non-autonome de quelques résultats de Grifone pour le cas autonome.

ABSTRACT. — We show that if ξ is a second-order differential equation (semispray) on the jet bundle $J^1(\mathbf{R}, M)$ whose paths are solutions of the non-autonomous Lagrange equations then there is a connection Γ on $J^1(\mathbf{R}, M)$ whose paths are also solutions of the same equations. Moreover, Γ is a connection whose associated semispray is precisely ξ. This is an extension to non-autonomous Lagrangian dynamics of a previous result due to Grifone for autonomous Lagrangians.

1. Introduction

The geometrical description of autonomous Lagrangian systems, started with GALLISOT [G], was elucidated by KLEIN [K1], [K2] (see also GODBILLON [GB]). He showed that the differential geometry of Lagrangian dynamics is intrinsically related to a (1.1) tensor field J, called *almost tangent structure*, defined on the tangent bundle of a manifold.

(1) Partially supported by Xunta de Galicia, Spain and CNPq-Brazil, Proc. 301115/79.
(2) CECIME Consejo Superior de Investigaciones Cientificas, Serrano 123, 28006 Madrid - Spain
(3) Departamento de Geometria Instituto de Matematica, Universidade Federal Fluminense, 24.000 Niteroi, RJ - Brazil

Since the early works of KLEIN some articles have been published showing that almost tangent geometry provides a natural framework in wich interesting generalizations of autonomous Lagrangian systems may be developed (see for instance, CRAMPIN [C], CRAMPIN et al. [CSC], de LEON and RODRIGUES [DLR1], [DLA2], GOTAY and NESTER [GN], SARLET et al. [SCC]). In particular, an extensive study about the theory of connections on tangent bundles in terms of the almost tangent geometry, including some aspects of autonomous Lagrangians, was proposed by GRIFONE [GR] around 1972.

As far as we know the non-autonomous case has been practically unknown in the literature (an exception, for example, is the recent paper of CRAMPIN and co-workers [CPT]). It is the purpose of this paper to stablish some intrinsical properties about almost tangent theory of connections and its relation with non-autonomous Lagrangian dynamics. We will see that the theory of connections on the jet manifold $J^1(\mathbf{R}, M)$ is surprinsingly more simpler than the theory of connections on TM, where M is a given manifold (the reader is invited to compare our results with GRIFONE).

2. Preliminaries

Throughout the text we shall keep in mind all results, definitions and notations previously introduced in [DLR 1] (see also [DLR 2]). All structures, functions, etc, are assumed to be smooth (C^∞). Let M be a manifold of dimension m (called *configuration* manifold) and Γ a connection on the tangent bundle TM of M. We recall here that a connection Γ on TM generates two projectors $h : T(TM) \to Hor(TM)$, $v : T(TM) \to Ver(TM)$ such that $T(TM) = Hor(TM) \oplus Ver(TM)$, where $Hor(TM)$ (resp. $Ver(TM)$) is the horizontal (resp. vertical) bundle over TM. If $\overline{\xi}$ is an arbitrary semispray (second-order differential equation) on TM then $\xi = h(\overline{\xi})$ is a semispray on TM which does not depend on the choice of $\overline{\xi}$. We call ξ the *associated* semispray of Γ. A connection Γ and its associated semispray have same paths.

If ξ is a semispray on TM then it can be shown that $\Gamma = -\mathcal{L}_\xi J$ is a connection on TM (here $\mathcal{L}_\xi$ is the Lie derivative and $\mathcal{L}_\xi J$ is defined by

$$(\mathcal{L}_\xi J)(Y) = \big([\xi, JY] - J[\xi, Y]\big).$$

When ξ is a spray (homogeneous second-order differential equation) then $\Gamma = -\mathcal{L}_\xi J$ is a connection on M such that its associated semispray is precisely ξ. For a semispray ξ there is a family of connections $\Gamma = -\mathcal{L}_\xi J + T$, where T is a semibasic tensor field of type (1.1) on TM in equilibrium with ξ (in fact T is the strong torsion of Γ) (see [GR]). In the non-autonomous situation the relation between connections and semisprays becomes much more simpler, as we will show below.

The jet manifold $J^1(\mathbf{R}, M)$ is fibred over $\mathbf{R}\times M$, $\mathbf{R}$ and M with projection maps π, π_1, and π_2. We notice that $J^1(\mathbf{R}, M)$ can be identified with $\mathbf{R}\times TM$ in a very natural way. Therefore we transport the geometric structures defined on TM to $J^1(\mathbf{R}, M)$ like the almost tangent structure J and the Liouville vector field C on TM. We may define a new tensor field $\widetilde{J}$ of type (1.1) on $J^1(\mathbf{R}, M)$ by

$$\widetilde{J} = J - C \otimes dt, \tag{1}$$

which is locally characterized by

$$\widetilde{J}(\partial/\partial t) = -C;\ \widetilde{J}(\partial/\partial x^i) = \partial/\partial x^i;\ \widetilde{J}(\partial/\partial y^i) = 0 \tag{2}$$

where (t, x, y) are local coordinates for $J^1(\mathbf{R}, M)$.

Hence $\widetilde{J}$ has rank m and satisfies $(\widetilde{J})^2 = 0$. We define the adjoint of $\widetilde{J}$, $\widetilde{J}^*$, as the endormorphism of the exterior algebra $\Lambda(J^1(\mathbf{R}, M))$ of $J^1(\mathbf{R}, M)$ locally given by

$$\widetilde{J}^*(dt) = 0,\ \widetilde{J}^*(dx^i) = 0,\ \widetilde{J}^*(dy^i) = dx^i - y^i\ dt. \tag{3}$$

Like in the autonomous situation we associate to $\widetilde{J}$ operators $i_{\widetilde{J}}$ and $d_{\widetilde{J}}$ on the algebra $\Lambda(J^1(\mathbf{R}, M))$ by

$$\left.\begin{aligned} &i_{\widetilde{J}}\omega(X_1, \cdots, X_r) = \sum_{\ell=1}^{r} \omega(X_1, \cdots, \widetilde{J}X_\ell, \cdots, X_r), \\ &d_{\widetilde{J}} = i_{\widetilde{J}}d - di_{\widetilde{J}}, \end{aligned}\right\} \tag{4}$$

and so we have

$$\left.\begin{aligned} &i_{\widetilde{J}}(df) = \widetilde{J}^*(df), \text{ for all } f \text{ on } J^1(R, M) \\ &i_{\widetilde{J}}(dt) = i_{\widetilde{J}}(dx^i) = 0;\ i_{\widetilde{J}}(dy^i) = dx^i - y^i\ dt \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &d_{\widetilde{J}}f = \frac{\partial f}{\partial y^i}\ (dx^i - y^i\ dt) \\ &d_{\widetilde{J}}(dt) = d_{\widetilde{J}}(dx^i) = 0; d_{\widetilde{J}}(dy^i) = -d(dx^i - y^i\ dt) = dy^i \Lambda dt. \end{aligned}\right\} \tag{6}$$

In the following we will set

$$\theta^i = dx^i - y^i\ dt,\ 1 \leq i \leq m. \tag{7}$$

Also, it is not hard to see that a vector field ξ on $J^1(\mathbf{R}, M)$ is a semispray iff $\theta^i(\xi) = 0$ and $dt(\xi) = 1$, $1 \leq i \leq m$. In such a case ξ is locally given by

$$\xi = \partial/\partial t + y^i \,\partial/\partial x^i + \xi^i \,\partial/\partial y^i. \tag{8}$$

Furthermore, a vector field ξ on $J^1(\mathbf{R}, M)$ is a semispray iff $J\xi = C$ and $\widetilde{J}\xi = 0$.

Let ξ be a semispray on $J^1(\mathbf{R}, M)$. A curve s in M is called a *path* of ξ if its canonical prolongation is an integral curve of ξ.

Let s be a curve in M locally given by $\big(x^i(t)\big)$. Then $\widetilde{s}^1(t) = \big(t, x^i(t), \dot{x}^i(t)\big)$ and so s is a path of ξ if and only if satisfies the following non-autonomous system of differential equations

$$\frac{d^2x^i}{dt^2} = \xi^i\left(t, x, \frac{dx}{dt}\right), \quad 1 \leq i \leq m$$

where ξ is given by (8).

3. Semisprays and dynamical connections

The tensor fields J and $\widetilde{J}$ on $J^1(\mathbf{R}, M)$ permit us to give a characterization of a kind of connections for the fibration $\pi : J^1(\mathbf{R}, M) \to \mathbf{R} \times M$.

DEFINITION (1).— *By a dynamical connection on $J^1(\mathbf{R}, M)$ we mean a tensor field Γ of type (1.1) on $J^1(\mathbf{R}, M)$ satisfying*

$$J\Gamma = \widetilde{J}\Gamma = \widetilde{J}, \ \Gamma\widetilde{J} = -\widetilde{J}, \ \Gamma J = -J. \tag{9}$$

By a straightforward computation from (9) we deduce that the local expressions of Γ are

$$\left.\begin{aligned} \Gamma(\partial/\partial t) &= -y^i \,\partial/\partial x^i + \Gamma^i \,\partial/\partial y^i, \\ \Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \Gamma_i^j \,\partial/\partial y^i, \\ \Gamma(\partial/\partial y^i) &= -\partial/\partial y^i. \end{aligned}\right\} \tag{10}$$

The functions $\Gamma^i = \Gamma^i(t, x, y)$, $\Gamma_i^j = \Gamma_i^j(t, x, y)$ will be called the *components* of the connection Γ. From (10) we easily deduce that

$$\Gamma^3 - \Gamma = 0 \text{ and rank } (\Gamma) = 2m.$$

This type of polynomial structure is called $f(3,-1)$-*structure* in the literature (see [YI]). Now, we can associate to Γ two canonical operators $\underline{\ell}$ and $\underline{m}$ given by

$$\underline{\ell}=\Gamma^2,\ \underline{m}=-\Gamma^2+I.$$

Then we have

$$\underline{\ell}^2=\underline{\ell},\underline{m}^2=\underline{m},\underline{\ell}\underline{m}=\underline{m}\underline{\ell}=0,\ \underline{\ell}+\underline{m}=I, \tag{11}$$

and $\underline{\ell}$ and $\underline{m}$ are complementary projectors. From (11) we deduce that $\underline{\ell}$ and $\underline{m}$ are locally given by

$$\begin{aligned}&\underline{\ell}(\partial/\partial t)=-y^i\ \partial/\partial x^i-(\Gamma^i+y^j\Gamma^i_j)\partial/\partial y^i;\underline{\ell}(\partial/\partial x^i)=\\&=\partial/\partial x^i;\underline{\ell}(\partial/\partial y^i)=\partial/\partial y^i;\underline{m}(\partial/\partial t)=\\&=\partial/\partial t+y^i\ \partial/\partial x^i+(\Gamma^i+y^j\Gamma^i_j)\partial/\partial y^i;\underline{m}(\partial/\partial x^i)=\\&=\underline{m}(\partial/\partial y^i)=0.\end{aligned} \tag{12}$$

If we put $\mathcal{L}=Im\underline{\ell}$, $\mathcal{M}=Im\underline{m}$, then we have that $\mathcal{L}$ and $\mathcal{M}$ are complementary distributions on $J^1(\mathbf{R},M)$, that is,

$$T(J^1(R,M))=\mathcal{M}\oplus\mathcal{L}.$$

From (12) we deduce that $\mathcal{L}$ is $2m$-dimensional and is locally spanned by $\{\partial/\partial x^i,\partial/\partial y^i\}$. $\mathcal{M}$ is one-dimensional, globally spanned by the vector field $\xi=\underline{m}(\partial/\partial t)$. Taking into account the local expression of ξ, we deduce that ξ is a sempispray which will be called the *canonical semispray associated to the dynamical connection* Γ.

Furthermore, we have $\Gamma^2\underline{\ell}=\underline{\ell}$ and $\Gamma\underline{m}=0$. Thus Γ acts on $\mathcal{L}$ as an *almost product structure* and trivially on $\mathcal{M}$. Since $\mathcal{M}=\ker\Gamma$, Γ is said to be an $f(3,-1)$-structure on $J^1(\mathbf{R},M)$ of *rank* $2m$ and *parallelizable kernel.* Moreover, $\Gamma/\mathcal{L}$ has eigenvalues $+1$ and -1. From (10) the eigenspaces corresponding to the eigenvalue -1 are the vertical subspaces V_z, $z\in J^1(\mathbf{R},M)$. Recall that for each $z\in J^1(\mathbf{R},M)$, V_z is the set of all tangent vectors to $J^1(\mathbf{R},M)$ at z which are projected to 0 by $T\pi$. Thus V is a distribution given by $z\mapsto V_z$. The eigenspace at $z\in J^1(\mathbf{R},M)$ corresponding to the eigenvalue $+1$ will be denoted by H_z and called the *strong-horizontal subspace* at z. We have a canonical decomposition

$$T_z(J^1(\mathbf{R},M))=\mathcal{M}_z\oplus H_z\oplus V_z,$$

and obviously,

$$T(J^1(\mathbf{R},M)) = \mathcal{M} \oplus H \oplus V, \tag{13}$$

where H is the distribution $z \mapsto H_z$.

Let us put $H'_z = \mathcal{M}_z \oplus H_z$; H'_z will be called the *weak-horizontal subspace at* z. Then we have the following decompositions

$$T_z(J^1(\mathbf{R},M)) = H'_z \oplus V_z, \quad z \in J^1(\mathbf{R},M)$$

and

$$T(J_1(\mathbf{R},M)) = H' \oplus V, \tag{14}$$

where $H' : x \to H'_z$ is the corresponding distribution.

We notice that $\mathcal{L}, \mathcal{M}, H$ and H' may be considered as vector bundles over $J^1(\mathbf{R},M)$; the bundles H and H' will be called *strong* and *weak-horizontal bundles*, respectively. Thus, from (14) Γ defines a connection on the fibration $\pi : J^1(\mathbf{R},M) \to \mathbf{R} \times M$ with horizontal bundle H' (see ROUX [R] and de LEON & RODRIGUES [DLR 1]). But *not every connection on the fibration* $\pi : J^1(\mathbf{R},N) \to \mathbf{R} \times M$ *arises in this way.*

A vector fieled X on $J^1(\mathbf{R},M)$ which belongs to H (resp. H') will be called a *strong* (resp. *weak*) horizontal vector field. From (14), we have that the canonical projection $\pi : J^1(\mathbf{R},M) \to \mathbf{R} \times M$ induces an isomorphism

$$\pi_* : H'_z \to T_{\pi(z)}(\mathbf{R} \times M), \quad z \in J^1(\mathbf{R},M).$$

Then, if X is a vector field on $\mathbf{R} \times M$, there exists a unique vector field $X^{H'}$ on $J^1(\mathbf{R},M)$ which is weak-horizontal and projects to X. The projection of $X^{H'}$ to H will be denoted by X^H.

From (10) and by a straightforward computation, we obtain

$$\begin{aligned} (\partial/\partial t)^{H'} &= \partial/\partial t + (\Gamma^j + \frac{1}{2}\, y^i \Gamma^j_i)\partial/\partial y^j \\ (\partial/\partial x^i)^{H'} &= \partial/\partial x^i + \frac{1}{2}\, \Gamma^j_i\, \partial/\partial y^j. \end{aligned} \tag{15}$$

Then, if we put $H_i = (\partial/\partial x^i)^{H'}$ and $V_i = \partial/\partial y^i$, one deduces that $\{\xi, H_i, Vi\}$ is a local basis of vector fields on $J^1(\mathbf{R},M)$. In fact, $\mathcal{M} =< \xi >$, $H =< H_i >$, and $V =< V_i >$; $\{\xi, H_i, V_i\}$ is called an *adapted basis* to the $f(3,-1)$-structure Γ. In terms of $\{\xi, H_i, V_i\}$ (15) becomes

$$(\partial/\partial t)^{H'} = \xi - y^i H_i, \; (\partial/\partial x^i)^{H'} = H_i.$$

Therefore, we obtain

$$(\partial/\partial t)^H = -y^i H_i, \ (\partial/\partial x^i)^H = H_i.$$

If $X = \tau\ \partial/\partial t + X^i\ \partial/\partial x^i$ is a vector field on $R \times M$, we have

$$X^H = (X^i - \tau y^i) H_i \tag{16}$$

(compare with CRAMPIN, PRINCE and THOMPSON [CPT]). Finally, we notice that the dual local basis of 1-forms of the adapted basis $\{\xi, H_i, V_i\}$ is given by $\{dt, \theta^i, \psi^i\}$, where $\theta^i = dx^i - y^i\ dt$, and $\psi^i = -(\Gamma^i + \frac{1}{2}\ y^j \Gamma^i_j) dt - \frac{1}{2}\ \Gamma^i_j\ dx^j + dy^i$. This fact can be shown by a straightforward computation.

Let ξ be a semispray on $J^1(\mathbf{R}, M)$ and suppose that ξ is locally expressed by

$$\xi = \partial/\partial t + y^i\ \partial/\partial x^i + \xi^i\ \partial/\partial y^i. \tag{17}$$

Then a simple computation in local coordinates shows that

$$\left.\begin{aligned} [\xi, \partial/\partial t] &= -\frac{\partial \xi^j}{\partial t}\ \partial/\partial y^j, \\ [\xi, \partial/\partial x^i] &= -\frac{\partial \xi^j}{\partial x^i}\ \partial/\partial y^j, \\ [\xi, \partial/\partial y^i] &= -\frac{\partial}{\partial x^i} - \frac{\partial \xi^j}{\partial y^i}\ \partial/\partial y^j. \end{aligned}\right\} \tag{18}$$

PROPOSITION (1).— *Let $\Gamma = -\mathcal{L}_\xi \widetilde{J}$. Then Γ is a dynamical connection on $J^1(\mathbf{R}, M)$ whose associated semispray is, precisely, ξ.*

Proof.— In fact from (18) we have

$$\left.\begin{aligned} \Gamma(\partial/\partial t) &= -y^i\ \partial/\partial x^i - \left(y^j \frac{\partial \xi^i}{\partial y^j} - \xi^i \right) \partial/\partial y^i, \\ \Gamma(\partial/\partial x^i) &= \partial/\partial x^i + \frac{\partial \xi^j}{\partial y^i}\ \partial/\partial y^j, \\ \Gamma(\partial/\partial y^i) &= -\partial/\partial y^i. \end{aligned}\right\} \tag{19}$$

Now, from (19) we easily deduce that Γ is a dynamical connection on $J^1(\mathbf{R}, M)$. Furthermore, taking into account (12), we have that the associated semispray to Γ is, precisely, ξ.

From Proposition (1) we may observe that the theory of dynamical connections on $J^1(\mathbf{R}, M)$ is more simpler than the theory of connections on TM.

Let Γ be a dynamical connection on $J^1(\mathbf{R}, M)$.

DEFINITION (2).— *A curve $u : \mathbf{R} \to M$ is called a path of Γ if the canonical prolongation j^1u of u to $J^1(\mathbf{R}, M)$ is a weak-horizontal curve.*

Now, we shall find the differential equations for the paths of Γ (the dots meaning time derivatives).

If $u : \mathbf{R} \to M$ is locally given by $t \mapsto (x^i(t))$, then we have $j^1u(t) = (t, x^i(t), \dot{x}^i(t))$. Hence,

$$\dot{\widehat{j^1}}\, u(t) = \frac{\partial}{\partial t} + \frac{dx^i}{dt}\,\frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2}\,\frac{\partial}{\partial y^i}.$$

Therefore, u is a path of Γ if and only if $\psi^i(\dot{\widehat{j^1}}\, u(t)) = 0$, $1 \leq i \leq m$, that is u satisfies the following system of differential equations :

$$\frac{d^2x^i}{dt^2} = \Gamma^i\left(t, x, \frac{dx}{dt}\right) + \Gamma^i_j\left(t, x, \frac{dx}{dt}\right)\,\frac{dx^j}{dt}. \tag{20}$$

Let ξ be the associated semispray of Γ. Then ξ is locally given by

$$\xi = \partial/\partial t + y^i\ \partial/\partial x^i + \xi^i\ \partial/\partial y^i,$$

where $\xi^i = \Gamma^i + y^j\Gamma^i_j$, $1 \leq i \leq m$.

From (20) it is clear that the paths of Γ and ξ satisfy the same system of differential equations. Then we have

PROPOSITION (2).— *A dynamical connection and its associated semispray on $J^1(\mathbf{R}, M)$ have the same paths.*

4. Dynamical connections and non-autonomous regular Lagrangian equations

Suppose that a non-autonomous regular Lagrangian L is given, that is, L is a non-degenerate real function on $J^1(\mathbf{R}, M) = \mathbf{R} \times TM$. Then it is well-known that an extremal for L is a curve $s : \mathbf{R} \to M$ (or a section of $(\mathbf{R} \times M, p, \mathbf{R})$) such that

$$(\widetilde{s})^*(i_X \, dL \Lambda dt) = 0 \tag{21}$$

for all vertical vector fields on $\mathbf{R} \times TM$. Also, it is known that (21) is equivalent to

$$(\widetilde{s}^2)^*(i_X \, d\Omega_L) = 0, \tag{22}$$

for all π_1-vertical vector fields on $J^1(\mathbf{R}, M)$. In (22) Ω_L is the POINCARE-CARTAN canonical form on $J^1(\mathbf{R}, M)$ locally given by

$$\Omega_L = L(t, x, y)dt + \frac{\partial L}{\partial y^i} \, \theta^i,$$

where θ^i is defined in (7) of section 2.

In terms of the tensor field $\widetilde{J}$ and J and the Liouville vector field C on $J^1(\mathbf{R}, M)$, the POINCARE-CARTAN form takes the following expression :

$$\Omega_L = L \, dt + \frac{\partial L}{\partial y^i} \, \theta^i = L \, dt + d_{\widetilde{J}} L,$$

or equivalently,

$$\begin{aligned} \Omega_L &= L \, dt + \frac{\partial L}{\partial y^i} \, dx^i - y^i \frac{\partial L}{\partial y^i} \, dt = \left(L - y^i \frac{\partial L}{\partial y^i} \right) dt + d_J L \\ &= (L - CL)dt + d_J L = d_J L - E_L dt; E_L = CL - L. \end{aligned}$$

Thus

$$\Theta_L = d\Omega_L = dd_{\widetilde{J}} L + dL \Lambda dt$$

or

$$\Theta_L = dd_J L - dE_L \Lambda dt.$$

A straightforward computation in local coordinates shows that

$$\Theta_L \Lambda \cdots \Lambda \Theta_L = \pm det \left(\frac{\partial^2 L}{\partial y^j \partial y^i} \right) dx^1 \Lambda \cdots \Lambda dx^m \Lambda dy^1 \Lambda \cdots \Lambda dy^m$$

and if L is a non-autonomous regular Lagrangian we deduce that Θ_L is a contact form on $J^1(\mathbf{R}, M)$. Consequently, the *characteristic* bundle of Θ_L

$$R_{\Theta_L} = \left\{ v \in T(J^1(\mathbf{R}, M)); i_v \Theta_L = 0 \right\}$$

has one-dimensional fibers, that is, they are a line-bundle over $J^1(\mathbf{R}, M)$. Let us recall here that a vector field X on $J^1(\mathbf{R}, M)$ is *characteristic* if X is a section of R_{Θ_L}, that is, $i_X\Theta_L = 0$. The following result can be compared with the corresponding one for autonomous Lagrangian [see [DLR 1]].

PROPOSITION (3). — *Let L be a non-autonomous regular Lagrangian on $J^1(\mathbf{R}, M)$ and ξ a characteristic vector field which satisfies $i_\xi dt = 1$. Then ξ is a semispray on $J^1(\mathbf{R}, M)$ whose paths are the solutions of the Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \ 1 \leq i \leq m.$$

We call ξ the Lagrange vector field for L.

THEOREM (1). — *Let L be a non-autonomous regular Lagrangian on $J^1(\mathbf{R}, M)$ and let ξ be a Lagrange vector field for L. Then there exists a dynamical connection Γ on $J^1(\mathbf{R}, M)$ whose paths are the solutions of the Lagrange equations. This connection is given by $\Gamma = -\mathcal{L}_\xi \widetilde{J}$.*

Proof. — From Proposition (1) we deduce that $\Gamma = -\mathcal{L}_\xi \widetilde{J}$ is a dynamical connection whose associated semispray is precisely ξ. Thus the theorem follows directly from Proposition (2) and (3).

Finally, let ut remark that the results of CRAMPIN, PRINCE and THOMPSON [CPT] can be re-obtained in terms of Γ. In fact, with the notation of Section 3 we have a local basis of vector fields on $J^1(\mathbf{R}, M)$ given by $\{\xi, H_i, V_i\}$ where H_i is given by

$$H_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\,\frac{\partial \xi^j}{\partial y^i}\,\frac{\partial}{\partial y^j}.$$

Thus the corresponding dual basis is $\{dt, \theta^i, \psi^i\}$, where

$$\psi^i = -\left(\xi^i - \frac{1}{2}\, y^j \frac{\partial \xi^i}{\partial y^j}\}\right) dt - \frac{1}{2}\,\frac{\partial \xi^i}{\partial y^j}\, dx^j + dy^i.$$

The significance of this dual basis is that the form Θ_L can be re-written as follows

$$\Theta_L = \frac{\partial^2 L}{\partial y^i \partial y^j}\, \theta^i \Lambda \psi^j$$

and so the semispray ξ is uniquely determined by the equations

$$i_\xi \theta^i = i_\xi \psi^i = 0, \ i_\xi dt = 1.$$

References

[C] CRAMPIN (M.). — Tangent bundle geometry for Lagrangian dynamics, *J. Phys. A : Math. Gen.*, t. **16**, 1983, p. 3755-3772.

[CPT] CRAMPIN (M.), PRINCE (G.), THOMPSON (G.). — A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics, *J. Phys. A : Math. Gen.*, t. **17**, 1984, p. 1437-1447.

[DLR 1] DE LEON (M.), RODRIGUES (P.R.). — *Generalized Classical Mechanics and Field Theory*. — North-Holland Math. Studies, t. **112**, 1985.

[DLR 2] DE LEON (M.), RODRIGUES (P.R.). — Formalisme hamiltonien symplectique sur les fibrés tangents d'ordre supérieur, *C.R. Acad. Sc. Paris*, t. **301,ser.II,7**, 1985, p. 455- 458.

[G] GALLISSOT (T.). — Les formes extérieures en Mécanique, *Ann. Inst. Fourier, Grenoble*, t. **4**, 1952, p. 145-297.

[GN] GOTAY (M.), NESTER (J.). — Presymplectic Lagrangian Systems, I/II. Ann. Inst. Henri Poincaré **3O**,(2), 1979, p. 129-142; ibidem, **32**,(1), 1980, p. 1-13.

[GR] GRIFONE (J.). — Structure presque-tangente et connexions, I/II.Ann. Inst. Fourier (Grenoble), **22**,(3), 1972, p. 287-334; ibidem, **22**,(4), p. 291-338.

[K 1] KLEIN (J.). — Espaces variationnels et Mécanique, *Ann. Inst. Fourier (Grenoble)*, t. **12**, 1962, p. 1-124.

[K 2] KLEIN (J.). — Opérateurs différentiels sur les variétés presque- tangentes, *C.R. Acad. Sc. Paris*, t. **257 A**, 1963, p. 2392-2394.

[R] ROUX (A.). — Jets et connexions, *Publ. Math. Univ. Lyon*, t. **7**, 1970, p. 1-42.

[SCC] SARLET (W.), CANTRIJN (F.), CRAMPIN (M.). — A new look at second-order equations and Lagrangian mechanics, *J. Phys. A: Math. Gen.*, t. **17**, 1984, p. 1999-2009.

[YI] YANO (K.), ISHIHARA (S.). — *Tangent and Cotangent bundles*. — Pure and Appl. Math. Ser. **16**, Marcel Dekker, N.Y., 1973.

(Manuscrit reçu le 27 janvier 1987)