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**L<sup>p</sup>-ESTIMATES FOR HYPERBOLIC OPERATORS  
APPLICATIONS TO  $\square u = u^k$**

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**Résumé :** Nous obtenons dans ce travail des estimations  $L^p - L^p$  pour l'équation des ondes non homogène. Le problème est complètement résolu en dimension d'espace  $n \leq 3$ . Dans le cas  $n \geq 4$  nous avons une réponse positive si  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$ . Finalement, nous utilisons ces estimations pour montrer l'existence et unicité de solutions faibles pour certains problèmes non linéaires.

**Summary :**  $L^p - L^p$  estimates are obtained in this paper for the non-homogeneous wave equation. The problem is completely solved for space dimension  $n \leq 3$ . A positive answer is given for dimension  $n \geq 4$  and  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$ . These estimates are used in order to prove existence and uniqueness of weak solutions for some non-linear problems.

**ABSTRACT**

We present some results on non-linear hyperbolic equations obtained by means of some information on the linear Cauchy problem. Those results for the linear case are essentially :

**THEOREM.** *Let*

$$\left\{ \begin{array}{l} \square u \equiv u_{tt} - \Delta_x u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{array} \right.$$

where  $f \in L^p(\mathbb{R}^n)$ ,  $g \in L^p_1(\mathbb{R}^n)$ , then for each  $t \in [0, \infty)$  we have

$$(1) \quad \|u(t,x)\|_{L^p(\mathbb{R}^n)} \leq C_p(t) (\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p_1(\mathbb{R}^n)})$$

if

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{n-1}$$

The result is sharp.

The homogeneity of Fourier multipliers implies  $C_p(t) \approx \max(1, |t|)$ .

Proof and extension to other kind of hyperbolic operators can be seen in J. C. Peral [3] and for related estimates of the type  $(L^p, L^q)$  Littman [1], [2] and Strichartz [8] [9].

In § 1 we will solve partially a problem posed by Littman in [1].

In § 2 we give existence and uniqueness results for certain non-linear hyperbolic equations  $\square u = F(u)$ . Those results are in the context of the  $L^p$  spaces.

Finally in § 3 we deal with the case  $|F(u)| = u^k$ .

Everything will be expressed in terms of either the wave equations of the Klein-Gordon equations.

It is not hard to generalize these results to some others more general hyperbolic equations.

### 1. - A PRIORI $L^p$ -ESTIMATES FOR THE NON-HOMOGENEOUS EQUATIONS

It has been proved by Littman in [1] that an estimates of the type

$$\int_{\mathbb{R}^n} \int_t |v(t,x)|^p dt dx \leq C_p(T) \int_{\mathbb{R}^n} \int_t |v(t,x)|^p dt dx$$

with  $v \in C^\infty_0(\mathbb{R}^{n+1})$  and say support  $(v) \subset (0,T) \times \mathbb{R}^n$  is false if  $p > \frac{2n}{n-3}$ .

For the same problem we get the following positive results.

THEOREM. Let

$$\left\{ \begin{array}{l} \square u(t,x) = F(t,x) \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{array} \right.$$

where  $F \in L^p([0, T] \times \mathbb{R}^n)$  and  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$ . Then

$$\|u\|_{L^p([0, T] \times \mathbb{R}^n)} \leq C_p(T) \|F\|_{L^p([0, T] \times \mathbb{R}^n)}$$

*Proof.* By the Duhamel principle we know that the solution of the problem is expressed as

$$u(t, x) = \int_0^t v(t, \tau, x) d\tau$$

where  $v(t, \tau, x)$  solves the following problem

$$\begin{cases} \square v(t, \tau, x) = 0 \\ v(\tau, \tau, x) = 0 \\ v_t(\tau, \tau, x) = F(\tau, x) \end{cases}$$

The estimates (1) gives for each  $t$

$$\left( \int_{\mathbb{R}^n} |v(t, \tau, x)|^p dx \right)^{1/p} \leq C_p(t-\tau) \left( \int_{\mathbb{R}^n} |F(\tau, x)|^p dx \right)^{1/p}$$

if  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$ . But,  $C_p(t-\tau) \leq C_p(t)$ ,  $0 \leq \tau \leq t$ .

By applying the Minkowski integral inequality, for each  $t$  we get

$$\begin{aligned} \left( \int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{1/p} &\leq \left( \int_0^t \int_{\mathbb{R}^n} |v(t, \tau, x)|^p dx \right)^{1/p} d\tau \leq \\ &\leq \int_0^t C_p(t) \left( \int_{\mathbb{R}^n} |F(\tau, x)|^p dx \right)^{1/p} d\tau \leq C_p(t) t^{1/q} \left( \int_0^t \int_{\mathbb{R}^n} |F(\tau, x)|^p dx d\tau \right)^{1/p} \end{aligned}$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ .

Then we get

$$\int_0^T \int_{\mathbb{R}^n} |u(t, x)|^p dx dt \leq \left( \int_0^T |C_p(t)|^p t^{p/q} dt \right) \|F\|_{L^p([0, T] \times \mathbb{R}^n)}$$

This implies

$$\|u\|_{L^p([0, T] \times \mathbb{R}^n)} \leq C'_p(T) \|F\|_{L^p([0, T] \times \mathbb{R}^n)}$$

where

$$C_p^1(T) \cong T^2$$

because  $C_p(t) \approx |t|$  is the norm of Fourier multiplier  $\frac{\sin t |\xi|}{|\xi|}$ .

*Remark.* For  $n \leq 3$  our result gives the range  $1 \leq p \leq \infty$ . For  $n \geq 4$  there is a range where the problem is open, that is  $\frac{2(n-1)}{n-3} \leq p \leq \frac{2n}{n-3}$  and the conjugates.

We can prove the same kind of estimates for more general equations such a Klein-Gordon equation for example. See J.C. Peral [3].

## 2. - NON-LINEAR CAUCHY PROBLEM

Assume

$$F : [0, \infty) \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$$

- and
- i)  $F(t, x, 0) = 0$
  - ii)  $|F(t, x, u) - F(t, x, v)| \leq K |u - v|$

if we consider

$$\begin{cases} \square u = F(t, x, u) \\ u(0, x) = g(x) \\ u_t(0, x) = f(x) \end{cases}$$

where  $f \in L^p(\mathbb{R}^n)$ ;  $g \in L^p_1(\mathbb{R}^n)$  and  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$  then we can give the following theorem.

**THEOREM 1.** *Let F be satisfying i) and ii) then given any  $T > 0$ , The Cauchy problem*

$$\begin{cases} \square u = F(t, x, u) \\ u(0, x) = g(x) \\ u_t(0, x) = f(x) \end{cases}$$

where  $g \in L^p_1(\mathbb{R}^n)$ ,  $f \in L^p(\mathbb{R}^n)$  and  $|\frac{1}{p} - \frac{1}{2}| \leq \frac{1}{n-1}$ , has unique weak solution  $u \in L^p([0, T] \times \mathbb{R}^n)$  and

$$\|u\|_{L^p([0, T] \times \mathbb{R}^n)} \leq C_p(T) (\|g\|_{p,1} + \|f\|_p)$$

*Proof.* With fixed  $T > 0$  consider  $u_1$  solution of the problem

$$\begin{cases} \square u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

and, in general for  $k \in \mathbb{N}$   $u_k$  solution of the problem

$$\begin{cases} \square u = F(t,x,u_{k-1}) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

As a consequence of the result in § 1, for each  $k \in \mathbb{N}$  we have

$$u_k \in L^\infty((0,T), L^p(\mathbb{R}^n)) \text{ and } u_k \in L^p((0,T) \times \mathbb{R}^n).$$

we will prove that  $\{u_k\}_{k \in \mathbb{N}}$  is a Cauchy sequence in the first space.

In fact  $u_k - u_{k-1}$  verifies :

$$\begin{cases} \square (u_k - u_{k-1}) = F(t,x,u_{k-1}) - F(t,x,u_{k-2}) \\ (u_k - u_{k-1})(0,x) = 0 \\ (u_k - u_{k-1})_t(0,x) = 0 \end{cases}$$

then if

$$d_k(t) = \left( \int_{\mathbb{R}^n} |u_k(t,x) - u_{k-1}(t,x)|^p dx \right)^{1/p}$$

we get

$$d_k(t) \leq C_p(t)K \int_0^t d_{k-1}(s)ds \leq C_p(T)k \int_0^t d_{k-1}(s)ds = M(T) \int_0^t d_{k-1}(s)$$

Therefore

$$d_k(t) \leq \frac{TM(T)^k}{k!} \sup_{s \in [0,T]} d_0(s)$$

and, if  $k < \ell$ , for each  $t \in (0,T)$  we have

$$\|u_k(t) - u_\ell(t)\|_{L^p(\mathbb{R}^n)} \leq \sum_{j=k}^{\ell} d_j(t) \leq \left( \sum_{k=\ell}^{\infty} \frac{(M(T)T)^k}{k!} \right) \sup_{s \in (0,T)} d_0(s)$$

Then  $u_k \rightarrow u \in L^\infty((0,T), L^p(\mathbb{R}^n))$  and as  $A$  is a continuous operator,  $u$  is a solution of our problem.

By using the theorem locally is not hard to prove that  $u$  is the unique solution. In others words, we consider the operator

$$L^p((0,T) \times \mathbb{R}^n) \rightarrow L^p((0,T) \times \mathbb{R}^n)$$

such that  $Av = u$ , is solution of the linear problem

$$\begin{cases} \square u = F(t,x,v) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for small  $T$ ,  $A$  is contractive. Also it is clear that the convergence is in the space

$$L^p((0,T) \times \mathbb{R}^n)$$

and this finishes the proof.

An important example included in the preceding theorem is the sine-Klein-Gordon equations

$$\square u + m^2 u = \sin u$$

and for the case  $n = 3$ , there exists a unique solution for every  $p$ ,  $1 \leq p \leq \infty$ .

The following Gronwall lemma, shows the behaviour of the constant  $C(T)$ .

LEMMA. Let  $\alpha(t)$  and  $\beta(t)$  be positive and continuous functions on some interval  $[0,T]$ . If

$$y(t) \leq \alpha(t) + \beta(t) \int_0^t y(\tau) d\tau$$

then

$$y(t) \leq \alpha(t) + \beta(t) \int_0^t \alpha(s) \exp\left(\int_s^t \beta(u) du\right) ds$$

Then if we consider

$$y(t) = \left( \int_{\mathbb{R}^n} |u(t,x)|^p dx \right)^{1/p}$$

where  $u$  is the solution for the non-linear Cauchy problem and

$$\alpha(t) = C_p(t) ( \|g\|_{p,1} + \|f\|_p )$$

$$\beta(t) = KC_p(t)$$

we get for each  $t$

$$\|u(t, \cdot)\|_p \leq C_p(t) \exp \int_0^t KC_p(s) (\|f\|_p + \|g\|_{p,1})$$

where  $C_p(t) \approx \max(1, t)$  as in the linear case. Then

$$\|u\|_{L^p((0,T) \times \mathbb{R}^n)} \leq (\|f\|_p + \|g\|_{p,1}) \left( \int_0^T |C_p(t)|^p \exp \left\{ pk \int_0^t C_p(s) ds \right\} dt \right)^{1/p}$$

Observe that in the case  $|F(t,x,u)| \leq H$ ,  $C_p(t)$  behaves like  $t^\alpha$  for some  $\alpha(p)$ . This is the case of the sine-Klein-Gordon equations.

### 3. - A RESULT OF EXISTENCE AND UNIQUENESS IN $L^\infty$ FOR $\square u + m^2 u = u^k$

For the problem

$$\begin{cases} \square u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \quad x \in \mathbb{R}^3 \end{cases}$$

existence results are known in the case of finite energy ; see references in Segal [5] and Strauss [6] .

Uniqueness is known in the case  $k < 5$ . See also [7] . We get results of existence and uniqueness based in the estimates of § 1 using the fact that for  $n = 3$  such estimates are valid for  $1 \leq p \leq \infty$ .

Consider

$$\begin{cases} \square u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where  $g \in L_1^\infty(\mathbb{R}^3)$  and  $f \in L^\infty(\mathbb{R}^3)$  and in addition assume

$$\|f\|_\infty + \|g\|_{\infty,1} \leq \frac{1}{4}$$

Define  $u_0$  as the solution of the linear problem

$$\begin{cases} \square u + m^2 u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for each  $t$  we have

$$\|u_0(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq t \|f\|_\infty + \max(t, 1) \|g\|_{\infty, 1}.$$

Besides if  $t \in [0, 1]$  we get

$$\|u_0(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq 1/4$$

On the other hand let  $u_n$  be the solution of the linear problem

$$\begin{cases} \square u + m^2 u = \pm u_{n-1}^k \\ u(0, x) = g(x) \\ u_t(0, x) = f(x) \end{cases}$$

The result of § 1 implies

$$\|u_n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{4} + \left(\frac{1}{2}\right)^{k-1} \int_0^t \|u_{n-1}(\tau, \cdot)\|_{L^\infty(\mathbb{R}^3)} d\tau$$

if  $t \in [0, 1]$  and by recursion we deduce :

$$\|u_n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{2} \text{ if } t \in [0, 1]$$

Let's define

$$d_n(t) = \|u_n(t, \cdot) - u_{n-1}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}, t \in [0, 1]$$

then we obtain

$$d_n(t) \leq C_k \int_0^t d_{n-1}(\tau) d\tau$$

where  $C_k$  is the bound of the following expression

$$P_{k-1}(u, v) = u^{k-1} + u^{k-2}v + \dots + v^{k-1}$$

for

$$u = \|u_{n-1}\|_{L^\infty([0, 1] \times \mathbb{R}^3)}$$

and

$$v = \|u_{n-2}\|_{L^\infty([0, 1] \times \mathbb{R}^3)}$$

Therefore we have  $C_k \leq k \left(\frac{1}{2}\right)^k < 1$  if  $k \geq 2$ .

Obviously

$$d_n(t) \leq \frac{(C_k t)^n}{n} \frac{1}{2},$$

and then  $\{u_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in

$$L^\infty([0,1], L^\infty(\mathbb{R}^3)) \text{ and in } L^\infty([0,1] \times \mathbb{R}^3).$$

Since  $C_k \leq 1$  the following existence and uniqueness theorem has been proved

**THEOREM 2.** *Let  $f \in L^\infty(\mathbb{R}^3)$  and  $g \in L_1^\infty(\mathbb{R}^3)$  such that  $\|f\|_\infty + \|g\|_{\infty,1} \leq \frac{1}{4}$  then*

$$\left\{ \begin{array}{l} \square u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{array} \right.$$

*has unique solution in  $B_{1/2}(0) = \left\{ u \in L^\infty([0,1] \times \mathbb{R}^3) ; \|u\|_\infty \leq \frac{1}{2} \right\}$ .*

A classical theorem of uniqueness in the following (see Strauss [6]).

**THEOREM 3.** *Consider*

$$\left\{ \begin{array}{l} \square u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{array} \right.$$

*where  $g \in L_1^\infty(\mathbb{R}^3)$  and  $f \in L^\infty(\mathbb{R}^3)$ . If the problem has a solution  $u \in L^\infty((0,T) \times \mathbb{R}^3)$  then it is the unique solution in such space.*

Several observations should be made at this point. The theorem 3 gives a result on uniqueness with independence of how big or small are the  $L^\infty$ -norms of the data. However for the theorem 2 we need the data to be small.

From both theorems we get the following corollary :

**COROLLARY.** *Under the hypothesis of the Theorem 2, then there is a unique solution in  $L^\infty([0,1] \times \mathbb{R}^3)$ .*

Finally if  $\alpha = \|f\|_\infty + \|g\|_{\infty,1} > \frac{1}{4}$  results on existence and uniqueness can be given locally in  $t$ .

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