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## STRUCTURE OF THE SET OF STATIONARY SOLUTIONS OF VISCOUS HYDROMAGNETIC EQUATIONS WITH DIFFUSIVITY

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**Résumé :** Dans ce papier, à la suite de C. Foias et de R. Temam [ 4 ], nous démontrons les propriétés génériques de l'ensemble des solutions des équations stationnaires de l'hydromagnétisme avec viscosité et diffusivité magnétique. L'outil important utilisé est le théorème de Sard dans sa version en dimension infinie due à Smale.

**Summary :** In this paper, following C. Foias and R. Temam [ 4 ], we have proved generic properties of the set of solutions of stationary hydromagnetic equations with viscosity and magnetic diffusivity. The main tool used is Smale's infinite-dimensional version of Sard's theorem.

### INTRODUCTION

In this paper we study the structure of the set of stationary solutions of equations for viscous incompressible conducting fluid with diffusivity in the presence of a magnetic field. The boundary condition for the velocity field is assumed to be homogeneous. We consider the following system of equations governing the stationary hydromagnetic flows :

$$(0.1) \quad -\nu \Delta u + \sum_{i=1}^n u_i D_i u - \sum_{i=1}^n B_i D_i B + \text{grad } p = f \text{ in } \Omega$$

$$(0.2) \quad -\lambda \Delta B + \sum_{i=1}^n u_i D_i B - \sum_{i=1}^n B_i D_i u = 0 \text{ in } \Omega$$

$$(0.3) \quad \operatorname{div} u = 0, \operatorname{div} B = 0 \quad \text{in } \Omega$$

$$(0.4) \quad u = 0, B = \phi, \text{ on } \Gamma, \text{ where } \phi \cdot n = 0 \text{ on } \Gamma.$$

Here  $\Omega$  is an open bounded set of  $\mathbb{R}^n$ ,  $n = 2$  or  $3$ , and  $\Gamma$  is its boundary.  $D_i = \partial / \partial x_i$  and  $p$  denotes the total pressure.  $u$  and  $B$  are the velocity and magnetic fields.

Without loss of generality, we take  $\lambda = \nu$ . The problem is reduced to the functional equation in  $u$  and  $B$ . We denote by  $S(f, \phi, \nu)$  the set of solutions  $\{u, B\}$  of the problem (0.1) - (0.4).

The methods of proving the properties of  $S(f, \phi, \nu)$  rely on those in C. Foias and R. Temam [4].

In section 1, we prove general properties of  $S(f, \phi, \nu)$ .

In section 2, we prove a generic property. The proof is based on an infinite-dimensional version of Sard's theorem due to Smale and some results developed in Section 1.

The case with nonhomogeneous boundary condition for velocity field can be treated as in [4].

Notations are as in [4].

## 1. GENERAL PROPERTIES OF THE SET $S(f, \phi, \nu)$

Regarding  $\Gamma$ , the boundary of  $\Omega$ , we assume that

$$(1.1) \quad \Gamma \text{ is a manifold of class } C^r \text{ of dimension } n - 1 \text{ and } \Omega \text{ is locally located on one side of } \Gamma \text{ (} r = 2 \text{ unless otherwise specified).}$$

$$(1.2) \quad \Gamma \text{ has a finite number of connected components.}$$

$$\text{Let } \mathcal{V} = \{ u \in \mathcal{D}(\Omega)^n, \operatorname{div} u = 0 \}$$

$$V = \text{closure of } \mathcal{V} \text{ in } H_0^1(\Omega)$$

$$= \{ u \in H_0^1(\Omega) / \operatorname{div} u = 0 \}$$

$$\mathcal{W} = \{ u \in \mathcal{D}(\Omega)^n / \operatorname{div} u = 0, u \cdot n = 0 \text{ on } \Gamma \}$$

$$W_S = \text{closure of } \mathcal{W} \text{ in } H^S(\Omega)$$

$$W_1 = W, \quad W_0 = H.$$

Then  $H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)$

$$= \{ u / u \in L^2(\Omega)^n, \text{div. } u = 0, u \cdot n = 0 \text{ on } \Gamma \}.$$

We have the following continuous injections :

$$V \subset H \subset V',$$

$$W_s \subset W \subset H \subset W' \subset W'_s.$$

### 1.1. - The Homogeneous Hydromagnetic Problem.

Given  $f$  and  $\phi$ , to find  $u$ ,  $B$  and  $p$  such that

$$(1.3) \quad -\nu \Delta u + \sum_{i=1}^n u_i D_i u - \sum_{i=1}^n B_i D_i B + \text{grad } p = f \text{ in } \Omega$$

$$(1.4) \quad -\nu \Delta B + \sum_{i=1}^n u_i D_i B - \sum_{i=1}^n B_i D_i u = 0 \text{ in } \Omega$$

$$(1.5) \quad \text{div. } u = 0, \text{div. } B = 0 \text{ in } \Omega$$

$$(1.6) \quad u = 0, B = \phi \text{ on } \Gamma \text{ where } B \cdot n = \phi \cdot n = 0 \text{ on } \Gamma.$$

We assume that  $f \in H^{-1}(\Omega)$ , the dual space of  $H_0^1(\Omega)$  and  $\phi$  is given in  $H^{1/2}(\Gamma)$  (the space of traces of functions in  $H^1(\Omega)$ ) with

$$(1.7) \quad \int_{\Gamma} \phi \cdot \bar{n} \, d\Gamma = 0$$

$\bar{n}$  being the unit outward normal on  $\Gamma$ . Then (1.7) is implied by (1.6).

One can find  $\Phi$  ([7], [11]) with  $\Phi \in H^1(\Omega)$ ,  $\text{div. } \Phi = 0$ ,  $\Phi = \phi$  on  $\Gamma$ . Hence the problem (1.3) - (1.6) is equivalent to finding  $\{u, \bar{B}\}$ ,  $\bar{B} = B - \Phi$ , solution of

$$(1.8) \quad -\nu \Delta u + \sum_{i=1}^n u_i D_i u - \sum_{i=1}^n \bar{B}_i D_i \bar{B} - \sum_{i=1}^n \Phi_i D_i \bar{B} - \sum_{i=1}^n \bar{B}_i D_i \Phi + \text{grad } p = \bar{f} \text{ in } \Omega$$

$$-\nu \Delta \bar{B} + \sum_{i=1}^n u_i D_i \bar{B} + \sum_{i=1}^n u_i D_i \Phi - \sum_{i=1}^n \bar{B}_i D_i u$$

$$(1.9) \quad - \sum_{i=1}^n \Phi_i D_i u = \nu \Delta \Phi \text{ in } \Omega$$

$$(1.10) \quad \text{div. } u = 0, \quad \text{div. } \bar{B} = 0 \quad \text{in } \Omega$$

$$(1.11) \quad u = 0, \quad \bar{B} = 0 \quad \text{on } \Gamma$$

where

$$(1.12) \quad \bar{f} = f + \sum_{i=1}^n \Phi_i D_i \Phi$$

In variational form, (1.8)–(1.11) become : to find  $u \in V$  and  $\bar{B} \in V$  such that

$$\nu((u,v)) + b(u,u,v) - b(\bar{B},\bar{B},v) - b(\Phi,\bar{B},v) - b(\bar{B},\Phi,v) = (\bar{f},v) \quad \text{for all } v \in V$$

and

$$\nu((\bar{B},c)) + b(u,\bar{B},c) + b(u,\Phi,c) - b(\bar{B},u,c) - b(\Phi,u,c) = -\nu((\Phi,c)) \quad \text{for all } c \in V.$$

Equivalently,

$$(1.13) \quad \nu Au + \mathcal{B}(u,u) - \mathcal{B}(\bar{B},\bar{B}) - \mathcal{B}(\Phi,\bar{B}) - \mathcal{B}(\bar{B},\Phi) = \bar{f}$$

$$(1.14) \quad \nu A\bar{B} + \mathcal{B}(u,\bar{B}) + \mathcal{B}(u,\Phi) - \mathcal{B}(\bar{B},u) - \mathcal{B}(\Phi,u) = \nu A\Phi$$

$\mathcal{B}$  is bilinear and  $A$  is linear

Let  $S(f,\phi,\nu)$  be the set of solutions  $\{u,\bar{B}\}$  of the problem (1.8)–(1.11) or (1.13)–(1.14). We now list some properties of this set which are easy consequences of wellknown results.

**THEOREM 1.1.** *Let (1.1) and (1.2) hold for  $\Omega$ . Let  $f$  be given in  $H^{-1}(\Omega)$  and  $\phi$  in  $H^{1/2}(\Gamma)$  with*

$$(1.15) \quad \int_{\Gamma_i} \phi \cdot n \, d\Gamma = 0, \quad i = 1, 2, \dots, k$$

which is satisfied by (1.6).

Then

(P<sub>1</sub>)  $S(f,\phi,\nu)$  is not empty

(P<sub>2</sub>)  $S(f,\phi,\nu)$  is closed and bounded in  $H^1(\Omega) \times H^1(\Omega)$  and compact in  $L^2(\Omega) \times L^2(\Omega)$ ,

$$(1.16) \quad \|u, \bar{B}\|_1 \leq \sigma_0 \quad \text{for all } \{u, \bar{B}\} \in S(f,\phi,\nu)$$

where  $\sigma_0 = \sigma_0(f,\phi,\nu,\Omega)$  depends on  $f,\phi,\nu,\Omega$ .

(P<sub>3</sub>)  $S(f,\phi,\nu)$  is reduced to one point (uniqueness) if

$$(1.17) \quad \nu - C_1 \sigma_0(f,\phi,\nu) > 0$$

$C_1$  depends on  $\Omega$

(P<sub>4</sub>) If  $f \in L^2(\Omega)$  and  $\phi \in H^{3/2}(\Gamma)$ , then

$$S(f, \phi, \nu) \subset H^2(\Omega) \times H^2(\Omega) \quad (\text{regularity}).$$

Here  $\|u, B\|_1$  is the product norm on  $V \times V$  defined, as usual, as  $(\|u\|_1^2 + \|B\|_1^2)^{1/2}$  which is equivalent to the norm  $\|\{u, B\}\|' = \|u\|_1 + \|B\|_1$ . (1.18)

*Proof.* The proof for (P<sub>1</sub>), the existence of solution  $\{u, B\}$  is given by easily extending the standard proof for existence for a single equation, say, Navier-Stokes equation ([6], [8]). See also [5], in which existence is proved assuming (1.15) as the boundary condition.

The crucial step in the proof is to show that choice of above  $\Phi$  can be made so that

$$(1.19) \quad b(u, \Phi, B) \leq \frac{1}{2} \nu \|u\| \cdot \|B\| \quad \text{for all } u, B \in H_0^1(\Omega)$$

$H^1$ -norm of  $\Phi$  depends on  $\phi$  and  $\nu$ . (1)

We now prove (1.16) :

We take the scalar product of (1.13) with  $u$  and (1.14) with  $\bar{B}$ . Then we get

$$(1.20) \quad \nu \|u\|^2 - b(\bar{B}, \Phi, u) - b(\Phi, \bar{B}, u) - b(\bar{B}, \bar{B}, u) = (\bar{f}, u)$$

and

$$(1.21) \quad \nu \|\bar{B}\|^2 - b(\bar{B}, u, \bar{B}) + b(u, \Phi, \bar{B}) - b(\Phi, u, \bar{B}) = -\nu((\Phi, \bar{B}))$$

since  $b(u, u, u) = 0$  and  $b(u, B, B) = 0$ .

Also because  $b(u, v, w) = -b(u, w, v)$ , we get on adding (1.20) and (1.21)

$$\nu (\|u\|^2 + \|\bar{B}\|^2) - b(\bar{B}, \Phi, u) + b(u, \Phi, \bar{B}) = (\bar{f}, u) - \nu((\Phi, \bar{B}))$$

(1) Inequality (1.19) can be proved as follows using arguments as in [6], p. 109 :  
By Theo. 1.1. [5],  $\phi \in H^{1/2}$  and  $\int_{\Gamma_i} \phi \cdot \bar{n} \, d\Gamma = 0 \quad \forall i$  implies  $\phi = \text{curl } F|_{\Gamma}$ ,  $F \in H^2(\Omega)$ . i.e.  $\Phi|_{\Gamma} = \text{curl } F|_{\Gamma}$ .

Then as in [6], we can write  $\Phi = \text{curl}(b\xi)$ ,  $\xi$  constructed in [6].

Thus following [6], p. 109, we get

$$\left| \int_{\Omega} u_k \cdot \Phi \cdot B_{x_k} \, dx \right| \leq C_1 \|B\| \left[ 2 \left( \int_{\Omega} |\text{grad } u|^2 \, dx \right)^{1/2} + (C_2(\rho) \|u\|^2)^{1/2} \right]$$

(where  $C_2(\rho) \rightarrow 0$  as  $\rho \rightarrow 0$ )

$$\leq \|B\| (\epsilon \cdot C_3 + C_4(\rho)) \|u\|$$

choose  $\epsilon$  &  $\rho$  so small that  $\epsilon \cdot C_3 + C_4(\rho) \leq \frac{\nu}{2}$  and (1.19) follows.

So, by using (1.19), we have

$$\begin{aligned} \nu \|\{u, \bar{B}\}\|^2 &\leq |\bar{f}| \|u\| + \nu \|\Phi\| \cdot \|\bar{B}\| + \frac{1}{2} \nu \|u\| \cdot \|\bar{B}\| + \frac{1}{2} \nu \|u\| \cdot \|\bar{B}\| \\ &\leq |\bar{f}| \cdot \|u\| + C_2 \nu \|\bar{B}\| + \frac{1}{2} \nu (\|u\|^2 + \|\bar{B}\|^2) \\ &\quad \text{since } a \cdot b \leq \frac{a^2}{2} + \frac{b^2}{2} \end{aligned}$$

$$\text{Hence } \frac{1}{2} \nu \|\{u, \bar{B}\}\|^2 \leq (|\bar{f}| + C_3 \nu) \|\{u, \bar{B}\}\| \text{ using (1.18)}$$

$C_3$  depends on  $\phi$ .

$$\text{So } \|\{u, \bar{B} - \Phi\}\| = \|\{u, \bar{B}\}\| \leq \frac{2}{\nu} (|\bar{f}| + C_3 \nu) \text{ and (1.16) follows.}$$

The closedness of  $S$  in  $H^1(\Omega) \times H^1(\Omega)$  is elementary. Compactness of  $S$  in  $L^2(\Omega) \times L^2(\Omega)$  follows by Rellich Lemma applied to product space. For compactness result for Navier-Stokes (stationary) equation, see [6].

Uniqueness ( $P_3$ ) is standard by using estimates made above. (cf. [8], [11] for single equation).

Lastly, regularity ( $P_4$ ) follows by the arguments given in [4], by applying them to product-case (cf. [6], [11]).

Hereafter we assume that

$$(1.22) \quad f \in H, \phi \in H^{3/2}(\Gamma) \text{ and satisfies (1.13).}$$

From [5], it follows that the function  $\Phi$  can be chosen in  $H^2(\Omega)$  with

$$(1.23) \quad \text{div. } \Phi = 0, \Phi = \phi \text{ on } \Gamma$$

and satisfying (1.19).

$\Phi$  can be chosen with

$$\|\Phi\|_2 \leq \sigma \left( \|\phi\|_{H^{3/2}(\Gamma)}, \nu \right)$$

where  $\sigma(s, \nu)$  is increasing with respect to  $s$ , decreasing with respect to  $\nu$ . Thus  $\Phi$  remains bounded in  $H^2(\Omega)$  when  $\phi$  remains bounded in  $H^{3/2}(\Gamma)$  (1.24)

Same thing holds for  $\sigma_i(f, \phi, \nu)$  which appear in this following sections. They remain bounded when  $(f, \phi)$  remain bounded in  $H \times H^{3/2}(\Gamma)$ .

## 1.2. - A Priori Estimates

LEMMA 1.1.  $S(f, \phi, \nu)$  is bounded and closed in

$$H^2(\Omega) \times H^2(\Omega).$$

*Proof.* We have, from (1.13) and (1.14), the inequalities

$$\nu |Au| \leq |\mathcal{B}(u, u)| + |\mathcal{B}(\bar{B}, \bar{B})| + |\mathcal{B}(\Phi, \bar{B})| + |\mathcal{B}(\bar{B}, \Phi)| + |\bar{f}|,$$

$$\nu |\bar{A}\bar{B}| \leq |\mathcal{B}(u, \bar{B})| + |\mathcal{B}(u, \Phi)| + |\mathcal{B}(\bar{B}, u)| + |\mathcal{B}(\Phi, u)| + \nu |A\Phi|$$

By Lemma 1.1 [4] and remark thereafter,

$$|\mathcal{B}(u, v)| \leq c |u|_1^{5/4} |v|_2^{3/4}$$

$$|\mathcal{B}(u, v)| \leq c |v|_2 |u|_1$$

and also

$$\leq c |u|_2 |v|_1.$$

Hence

$$\begin{aligned} \nu |Au| &\leq c'_1 |u|_1^{5/4} |u|_2^{3/4} + c'_2 |\bar{B}|_1^{5/4} |\bar{B}|_2^{3/4} \\ &\quad + c'_3 |\Phi|_2 |\bar{B}|_1 + |\bar{f}| \end{aligned}$$

and

$$\nu |\bar{A}\bar{B}| \leq c'_4 |u|_1^{5/4} |\bar{B}|_2^{3/4} + c'_5 |\Phi|_2 |u|_1 + c'_6 |\bar{B}|_1^{5/4} |u|_2^{3/4} + c'_7 \nu |\Phi|_2$$

We have

$$(1.25) \quad C_0^{-1} |u|_2 \leq |Au| \leq C_0 |u|_2 \quad \forall u \in D_A$$

Also from (1.16),

$$|\bar{B}|_1 = |B - \Phi|_1 \leq |B|_1 + |\Phi|_1 \leq \sigma_0 + |\Phi|_1 = \sigma_1(f, \phi, \nu)$$

Also

$$|u|_1 \leq \sigma_0 \leq \sigma_1(f, \phi, \nu)$$

Hence we get

$$\nu |Au| \leq C'_1 \sigma^{5/4} |Au|^{3/4} + C'_2 \sigma^{5/4} |\bar{A}\bar{B}|^{3/4} + C'_3 \sigma_1 |\Phi|_2 + |\bar{f}|$$



and

$$\nu |\overline{AB}| \leq C_4 \sigma_1^{5/4} |\overline{AB}|^{3/4} + C_5 |\Phi|_2 \sigma_1 + C_6 \cdot \sigma_1^{5/4} |Au|^{3/4} + C_7 \nu |\Phi|_2$$

Adding these two inequalities,

$$\nu (|Au| + |\overline{AB}|) \leq C_4 \sigma_1^{5/4} (|Au|^{3/4} + |\overline{AB}|^{3/4}) + |\overline{f}| + C_5 \sigma_1 |\Phi|_2$$

Applying Young's inequality,

$$C_4 \sigma_1^{5/4} |Au|^{3/4} \leq \frac{C_6 \sigma_1^5}{\nu^3} + \frac{\nu}{2} |Au|$$

and

$$C_4 \sigma_1^{5/4} |\overline{AB}|^{3/4} \leq \frac{C_7 \sigma_1^5}{\nu^3} + \frac{\nu}{2} |\overline{AB}|$$

Hence

$$\nu (|Au| + |\overline{AB}|) \leq \frac{C_8 \sigma_1^5}{\nu^3} + |\overline{f}| + C_5 \sigma_1 |\Phi|_2 + \frac{\nu}{2} (|Au| + |\overline{AB}|)$$

so  $\frac{\nu}{2} |\{Au, \overline{AB}\}| \leq C_9 \left( \frac{\sigma_1^5}{\nu^3} + |\overline{f}| + \sigma_1 \cdot |\Phi|_2 \right)$  using (1.18).

Hence

$$|\{Au, \overline{AB}\}| \leq \frac{C_{10}}{\nu} \left( \frac{\sigma_1^5}{\nu^3} + |\overline{f}| + \sigma_1 \cdot |\Phi|_2 \right) = \sigma_2(f, \phi, \nu).$$

Using (1.25) again,

$$|\{u, \overline{B}\}|_2 \leq C_0 \cdot \sigma_2(f, \phi, \nu)$$

and  $S(f, \phi, \nu)$  is bounded in  $H^2(\Omega) \times H^2(\Omega)$ .

Since the set is closed in  $H^1(\Omega) \times H^1(\Omega)$  and injection of  $H^2(\Omega) \rightarrow H^1(\Omega)$  is dense and continuous, it is closed in  $H^2(\Omega) \times H^2(\Omega)$ . The lemma is proved:

Before we state next lemma, we introduce, together with  $P_m$  of [4], one more projection operator  $P_m^*$  as follows: There are eigen vectors  $c_j$  in  $W_s$  such that

$$((c_j, c))_s = \lambda_j^* (c_j, c) \quad \forall c \in W_s.$$

$\{c_j\}$  then form basis of  $W_s$  and hence basis of  $W \cap (L^n(\Omega))^n$ .  $W \cap (L^n(\Omega))^n = W$  if  $n \leq 4$ .

Also injection of  $W_s \rightarrow H$  is compact, so  $\{c_j\}$  form basis of  $H$ . Then we may assume that

$$0 < \lambda_1^* \leq \lambda_2^* \leq \dots \quad \text{and} \quad \lambda_m^* \geq c \cdot m^{2/n}, \quad m \geq 1;$$

$c$  is a constant depending only on  $\Omega$ .

Then  $P_m^*$ ,  $m \geq 1$ , denotes the orthogonal projection onto the space spanned by  $c_1, c_2, \dots, c_j$ , either in  $H$ ,  $W$  or  $W'$ .

LEMMA 1.2. *There exists a constant  $\sigma_3$  depending on  $f, \phi, \Omega, v$  such that if*

$$(1.26) \quad \lambda_{m+1} \geq \sigma_3, \quad \lambda_{m+1}^* \geq \sigma_3$$

( $m$  to be chosen max. of  $m_1, m_2$  with  $\lambda_{m_1+1} \geq \sigma_3$  and  $\lambda_{m_2+1}^* \geq \sigma_3$ ) then for every pair  $\{u, B\}, \{v, c\}$  belonging to  $S(f, \phi, v)$ , we have

$$(1.27) \quad |\{u, B\} - \{v, c\}| \equiv |u - v| + |B - c| \leq \sigma_4 (|P_m(u - v)| + |P_m^*(B - c)|)$$

$$(1.28) \quad |\{Au, AB\} - \{Av, Ac\}| \leq \sigma_4 (|AP_m(u - v)| + |AP_m^*(B - c)|)$$

where  $\sigma_4$  is simply related to  $\sigma_3$ .

*Proof.* We have from (0.1),

$$v((u, \theta)) + b(u, u, \theta) - b(B, B, \theta) = (f, \theta)$$

$$v((v, \theta)) + b(v, v, \theta) - b(c, c, \theta) = (f, \theta) \quad \forall \theta \in V$$

or

$$(1.29) \quad v((w, \theta)) + b(u, u, \theta) - b(v, v, \theta) - b(B, B, \theta) + b(c, c, \theta) = 0$$

where

$$w = u - v \in D_A.$$

Similarly, from (0.2),

$$(1.30) \quad v((F, \psi)) + b(u, B, \psi) - b(v, c, \psi) - b(B, u, \psi) + b(c, v, \psi) = 0 \quad \forall \psi \in W$$

where

$$F = B - C \in D_A.$$

Let

$$Q_m = I - P_m, \quad Q_m^* = I - P_m^*$$

and put  $\theta = Q_m w$  in (1.29),  $\psi = Q_m^* F$  in (1.30).

Then (1.29) gives

$$\begin{aligned}
 \nu \| Q_m w \|^2 &= -b(u, u, Q_m w) + b(v, v, Q_m w) + b(B, B, Q_m w) - b(c, c, Q_m w) \\
 &= -b(u, w, Q_m w) - b(w, v, Q_m w) + b(B, F, Q_m w) + b(F, c, Q_m w) \\
 (1.31) \quad &= -b(u, P_m w, Q_m w) - b(P_m w, v, Q_m w) - b(Q_m w, v, Q_m w) \\
 &\quad + b(B, P_m^* F, Q_m w) + b(B, Q_m^* F, Q_m w) + b(P_m^* F, c, Q_m w) + b(Q_m^* F, c, Q_m w)
 \end{aligned}$$

where we have used properties of  $b$  and that

$$(P_m w, Q_m w) = 0.$$

Similar arrangements in (1.30) give

$$\begin{aligned}
 \nu \| Q_m^* F \|^2 &= -b(P_m w, B, Q_m^* F) - b(Q_m w, B, Q_m^* F) - b(v, P_m^* F, Q_m^* F) + b(B, P_m w, Q_m^* F) \\
 (1.32) \quad &\quad + b(B, Q_m w, Q_m^* F) + b(P_m^* F, v, Q_m^* F) + b(Q_m^* F, v, Q_m^* F)
 \end{aligned}$$

Using Lemma 1.1 [4], (1.31) gives

$$\begin{aligned}
 \nu \| Q_m w \|^2 &\leq C_{11} (\|u\|_2 + \|v\|_2) \|Q_m w\| \cdot \|P_m w\| \\
 &\quad + C_{12} \|Q_m w\| \cdot \|v\|_2 \cdot \|Q_m w\| \\
 (1.33) \quad &\quad + C_{13} (\|B\|_2 + \|c\|_2) \|P_m^* F\| \cdot \|Q_m w\| \\
 &\quad + C_{14} (\|B\|_2 + \|c\|_2) \|Q_m^* F\| \cdot \|Q_m w\|
 \end{aligned}$$

and (1.32) gives

$$\begin{aligned}
 \nu \| Q_m^* F \|^2 &\leq C_{15} \|B\|_2 \|P_m w\| \cdot \|Q_m^* F\| + C_{16} \|v\|_2 \|P_m^* F\| \cdot \|Q_m^* F\| \\
 (1.34) \quad &\quad + C_{17} \|B\|_2 \|Q_m w\| \cdot \|Q_m^* F\| \\
 &\quad + C_{18} \|v\|_2 \|Q_m^* F\| \cdot \|Q_m^* F\|
 \end{aligned}$$

By Lemma 1.1,  $\|u\|_2$ ,  $\|v\|_2$ ,  $\|B\|_2$ ,  $\|c\|_2$  are all bounded by  $\sigma_2(f, \phi, \nu)$ .

Hence by (1.33),

$$\begin{aligned} \nu \|Q_m^w\|^2 &\leq C_{19} \sigma_2 (|P_m^w| + |Q_m^w|) \|Q_m^w\| \\ &\quad + C_{20} \sigma_2 (|P_m^*F| + |Q_m^*F|) \|Q_m^w\| \end{aligned}$$

and by (1.34),

$$\begin{aligned} \nu \|Q_m^*F\|^2 &\leq C_{21} \sigma_2 (|P_m^w| + |Q_m^w|) \|Q_m^*F\| \\ &\quad + C_{22} \sigma_2 (|P_m^*F| + |Q_m^*F|) \|Q_m^*F\| \end{aligned}$$

Adding these two inequalities, we get

$$\begin{aligned} \nu (\|Q_m^w\|^2 + \|Q_m^*F\|^2) &\leq \sigma_2 \cdot C_{23} (\|Q_m^w\| + \|Q_m^*F\|) \\ &\quad \cdot (|P_m^w| + |Q_m^w| + |P_m^*F| + |Q_m^*F|) \end{aligned}$$

or using equivalent product norm,

$$\nu \|\{Q_m^w, Q_m^*F\}\|^2 \leq \sigma_2 \cdot C_{24} \|\{Q_m^w, Q_m^*F\}\| \times (|\{P_m^w, P_m^*F\}| + |\{Q_m^w, Q_m^*F\}|)$$

Hence

$$(1.35) \quad \nu \|\{Q_m^w, Q_m^*F\}\| \leq \sigma_2 \cdot C_{24} (|\{P_m^w, P_m^*F\}| + |\{Q_m^w, Q_m^*F\}|)$$

Also we have

$$|Q_m^\theta| \leq (\lambda_{m+1})^{-1/2} \|Q_m^\theta\| \quad \forall \theta \in V$$

and

$$|Q_m^*\psi| \leq (\lambda_{m+1}^*)^{-1/2} \|Q_m^*\psi\| \quad \forall \psi \in W$$

So,

$$|\{Q_m^\theta, Q_m^*\psi\}| \leq \max \cdot \{\lambda_{m+1}^{-1/2}, \lambda_{m+1}^{*-1/2}\} \cdot \|\{Q_m^\theta, Q_m^*\psi\}\|$$

Let this max. be  $\lambda_{m+1}^{-1/2}$ .

Hence,

$$(1.36) \quad |\{Q_m^\theta, Q_m^*\psi\}| \leq \lambda_{m+1}^{-1/2} \cdot \|\{Q_m^\theta, Q_m^*\psi\}\|$$

Then (1.35) gives

$$\left( \nu - \frac{C_{24} \cdot \sigma_2}{\lambda_{m+1}^{1/2}} \right) \|\{Q_m^w, Q_m^*F\}\| \leq C_{24} \cdot \sigma_2 \cdot |\{P_m^w, P_m^*F\}|$$

If (1.26) holds with

$$\sigma_3 > \left( \frac{2C_{24} \cdot \sigma_2}{\nu} \right)^2$$

then

$$\lambda_{m+1}^{1/2} \geq \sigma_3^{1/2} > \frac{2C_{24} \cdot \sigma_2}{\nu}$$

or

$$\nu - \frac{C_{24} \cdot \sigma_2}{\lambda_{m+1}^{1/2}} > \frac{\nu}{2}$$

So

$$\frac{\nu}{2} \| \{Q_m w, Q_m^* F\} \| \leq C_{24} \cdot \sigma_2 | \{P_m w, P_m^* F\} |$$

Hence

$$\| \{Q_m w, Q_m^* F\} \| \leq \sigma_3^{1/2} | \{P_m w, P_m^* F\} |$$

Then (1.36) gives

$$\begin{aligned} |Q_m w, Q_m^* F| &\leq \lambda_{m+1}^{-1/2} \cdot \sigma_3^{1/2} | \{P_m w, P_m^* F\} | \\ &\leq \sigma_3 | \{P_m w, P_m^* F\} | \end{aligned}$$

( $\sigma_3$  chosen such that  $\geq 1$ )

and so

$$\begin{aligned} | \{w, F\} | &= ( | \{P_m w, P_m^* F\} |^2 + | \{Q_m w, Q_m^* F\} |^2 )^{1/2} \\ &\leq (1 + \sigma_3^2)^{1/2} | \{P_m w, P_m^* F\} | \\ &\leq (1 + \sigma_3^2)^{1/2} | \{P_m(u \sim \nu), P_m^*(B \sim c)\} | \end{aligned}$$

which gives (1.27) by using equivalent norms.

We now prove (1.28) : Replacing  $\theta$  by  $A Q_m w$  in (1.29) and  $\psi$  by  $A Q_m^* F$  in (1.30), we get

$$\nu |A Q_m w|^2 = -b(u, u, A Q_m w) + b(v, v, A Q_m w) + b(B, B, A Q_m w) - b(c, c, A Q_m w)$$

= (as before)

$$\begin{aligned} (1.37) \quad &= -b(P_m w, u, A Q_m w) - b(Q_m w, u, A Q_m w) - b(v, P_m w, A Q_m w) \\ &\quad - b(v, Q_m w, A Q_m w) + b(P_m^* F, B, A Q_m w) + b(Q_m^* F, B, A Q_m w) \\ &\quad + b(c, P_m^* F, A Q_m w) + b(c, Q_m^* F, A Q_m w) \end{aligned}$$

$$\leq C_{25} \cdot \sigma_2 \cdot ( \|A P_m w\| + \|Q_m w\| + \|A P_m^* F\| + \|Q_m^* F\| ) \cdot \|A Q_m w\|$$

similar to (1.33) ; and similar to (1.32) and (1.34) we get

$$(1.38) \quad \nu \|A Q_m^* F\|^2 \leq C_{26} \cdot \sigma_2 ( \|A P_m w\| + \|Q_m w\| + \|A P_m^* F\| + \|Q_m^* F\| ) \cdot \|A Q_m^* F\|$$

Here we have used Lemma 1.1 [4] and (1.25).

Hence on adding respective sides of (1.37) and (1.38), and using (1.18),

$$(1.39) \quad \nu |\{A Q_m w, A Q_m^* F\}| \leq C_{27} \cdot \sigma_2 \cdot ( |\{A P_m w, A P_m^* F\}| + \|\{Q_m w, Q_m^* F\}\| )$$

We have similar to (1.36),

$$\|\{Q_m \theta, Q_m^* \psi\}\| \leq \lambda_{m+1}^{-1/2} \cdot |\{A \theta, A \psi\}| \quad \forall \theta, \psi \in D_A.$$

Hence

$$\left( \nu - \frac{C_{27} \cdot \sigma_2}{\lambda_{m+1}^{1/2}} \right) |\{A Q_m w, A Q_m^* F\}| \leq C_{27} \cdot \sigma_2 |\{A P_m w, A P_m^* F\}|$$

choose

$$\sigma_3 = \left( \frac{2(C_{24} + C_{27}) \cdot \sigma_2}{\nu} \right)^2$$

so

$$\frac{\nu}{2} |\{A Q_m w, A Q_m^* F\}| \leq C_{27} \cdot \sigma_2 |\{A P_m w, A P_m^* F\}|$$

or

$$\begin{aligned} |\{A Q_m w, A Q_m^* F\}| &\leq \frac{2 C_{27} \cdot \sigma_2}{\nu} |\{A P_m w, A P_m^* F\}| \\ &\leq \sigma_3^{1/2} |\{A P_m w, A P_m^* F\}| \end{aligned}$$

Then

$$|\{A w, A F\}| \leq (1 + \sigma_3)^{1/2} |\{A P_m w, A P_m^* F\}|$$

and we get (1.28).

### 1.3. - Other Properties of $S(f, \phi, \nu)$

We are now in a position to prove

THEOREM 1.2.  $(P_{\mathcal{F}}) : S(f, \phi, \nu)$  is a compact subset of

$$H^2(\Omega) \times H^2(\Omega).$$

$(P_{\mathcal{G}}) : S(f, \phi, \nu)$  is homeomorphic to a compact subset of  $\mathbb{R}^m \times \mathbb{R}^m$ ,  $m$  sufficiently large so that (1.26) is satisfied.

*Proof.* By using lemma 1.2 above, proof of [4] can directly be extended in this case, using product spaces.

## 2. GENERIC PROPERTIES

We set

$$(2.1) \quad E_1 = \{u \in H^2(\Omega), \operatorname{div} u = 0, u = 0 \quad \text{on } \Gamma\}$$

$$(2.2) \quad E_2 = \{B \in H^2(\Omega), \operatorname{div} B = 0, B.n = 0 \quad \text{on } \Gamma\}$$

Let 
$$E = E_1 \times E_2.$$

Set 
$$F = F_1 \times F_2,$$

$$(2.3) \quad F_1 = H \times H$$

$$F_2 = \{\phi \in H^{3/2}(\Gamma), \phi . n = 0 \quad \text{on } \Gamma\}.$$

Then  $\int_{\Gamma_i} \phi . n \, d\Gamma = 0, i = 1, \dots, k$  is automatically satisfied.

$$\mathcal{E} : E \rightarrow F,$$

$$(2.4) \quad \mathcal{E}(u, B) = \{ \nu \mathcal{A} u + \mathcal{B}(u, u) - \mathcal{B}(B, B), \nu \mathcal{A} B + \mathcal{B}(u, B) - \mathcal{B}(B, u) ; B|_{\Gamma} \}$$

where  $\mathcal{A} u = -P_H \Delta u$ ,  $P_H$  the projector in  $L^2(\Omega)$  onto  $H$ .

We now prove

THEOREM 2.1. We assume that  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$  satisfying (1.1), (1.2). Then, for every  $\nu > 0$

$(P_{\mathcal{G}}) : There exists a dense open set  $\Theta \subset F$ , such that for every  $(f, 0, \phi) \in \Theta$ , the set  $S(f, \phi, \nu)$  is finite.$

For every connected component  $\Theta_\alpha$  of  $\Theta$ , the number of elements in  $S(f, \phi, \nu)$  for  $(f, 0, \phi) \in \Theta_\alpha$  is constant and every solution is a  $C^\infty$ -function of  $(f, 0, \phi)$ .

*Proof.* We apply Sard-Smale theorem ( [4] , [9] ) with  $E, F, \&$  given as above. Clearly  $\&$  is  $C^\infty$  and its Fréchet derivative is given by

$$(2.5) \quad \begin{aligned} & \langle \&'(u,B), (v,c) \rangle \\ & = \begin{pmatrix} \nu \mathcal{A} v + \mathcal{B}(u,c) + \mathcal{B}(v,u) & \nu \mathcal{A} c + \mathcal{B}(u,c) - \mathcal{B}(v,u) + \\ - \mathcal{B}(B,c) - \mathcal{B}(v,B) & + \mathcal{B}(v,B) - \mathcal{B}(B,c) \end{pmatrix} \cdot C|_\Gamma \end{aligned}$$

$\&'(u,B) \in \mathcal{L}(E,F)$  has the form  $\Lambda + K$ ,

$$\Lambda(v,c) = \{ \nu \mathcal{A} v, \nu \mathcal{A} c, c|_\Gamma \},$$

$$K(v,c) = \{ \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B), \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c), 0 \}$$

That  $\Lambda$  is an isomorphism from  $E$  to  $F$  follows from the classical result on Stoke's problem applied to product space. Also  $\mathcal{B}$  is bilinear continuous on  $H^2(\Omega) \times H^1(\Omega)$  and on  $H^1(\Omega) \times H^2(\Omega)$  with values in  $L^2(\Omega)$  or  $H$ . Hence for  $u \in E_1, B \in E_2$ ,

$$(v,c) \rightarrow \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B)$$

and

$$(v,c) \rightarrow \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c)$$

are linear continuous mappings from  $H^1 \times H^1$  into  $H$ . Hence

$$(v,c) \rightarrow \{ \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B), \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c) \}$$

is linear continuous from  $H^1 \times H^1$  into  $H \times H$ .

$K$  is therefore compact. Hence  $\dim. \text{Ker. } K$  and  $\dim. \text{Coker } K$  are both finite and they are equal. For  $\Lambda$ , being isomorphism, both these dimensions are zero. Hence we conclude that  $\&'(u,B)$  is a Fredholm operator of index zero.

Hence by Smale's theorem, the set  $\Theta$  of regular values  $\{f, 0, \phi\}$  of  $\&$  is dense in  $F$  and  $S(f, \phi, \nu)$  is discrete in  $E$  for all  $(f, 0, \phi)$  in  $\Theta$ . Since by  $(P_5)$ ,  $S(f, \phi, \nu)$  is compact in  $E$ , it is finite.

The proof of openness of  $\Theta$  and the last part of the theorem can be given in this case by easily extending the proof given in [4] , p. 161-162.



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