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## ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A STRONGLY NONLINEAR PARABOLIC PROBLEM

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**Résumé :** Nous étudions le problème d'évolution  $u_t + Au = 0$  dans  $(0, T) \times \mathbb{R}^N$ ,  $u(0) = u_0$  dans  $\mathbb{R}^N$ , avec  $N \geq 1$ ,  $0 < T \leq \infty$ ,  $Au = -\operatorname{div}(|Du|^{p-2} Du)$ ,  $Du$  étant le gradient de  $u$ ,  $1 < p < \infty$  et nous supposons que  $u_0$  appartient à un espace de fonctions intégrables. On prouve l'existence d'un temps fini d'extinction si  $N \geq 2$  et  $p < \frac{2N}{N+1}$ . Dans le cas contraire (si  $N = 1$  et  $p > 1$  ou si  $N \geq 2$  et  $p \geq \frac{2N}{N+1}$ ) on prouve la loi de conservation :  $\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx$  pour tout  $t > 0$ . On estime aussi la convergence vers zéro des intégrales  $\int_{\mathbb{R}^N} |u(t, x)|^m dx$ ,  $m > 1$  et on obtient certains effets régularisants.

**Summary :** The evolution problem  $u_t + Au = 0$  in  $(0, T) \times \mathbb{R}^N$ ,  $u(0) = u_0$  in  $\mathbb{R}^N$  is considered where  $N \geq 1$ ,  $0 < T \leq \infty$ ,  $Au = -\operatorname{div}(|Du|^{p-2} Du)$ , with  $Du$  the gradient of  $u$ ,  $1 < p < \infty$  and  $u$  is supposed to belong to some integrable space. If  $N \geq 2$  and  $p < \frac{2N}{N+1}$  the existence of a finite extinction time is shown. On the contrary, if  $N = 1$ ,  $p > 1$  or  $N \geq 2$ ,  $p \geq \frac{2N}{N+1}$  conservation of total mass holds, i.e.  $\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx$  for every  $t > 0$ . We prove also that the integrals  $\int_{\mathbb{R}^N} |u(t, x)|^m dx$ ,  $m > 1$  converge to zero as  $t$  goes to infinity, and some regularizing effects are shown.

## INTRODUCTION AND PRELIMINARIES

We shall consider the asymptotic behaviour in time of the solutions of

$$(P) \quad \begin{cases} u_t + Au = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases}$$

with  $N \geq 1$ ,  $1 < p < \infty$  and  $Au = - \sum_{i=1}^N \frac{\partial}{\partial x_i} (|Du|^{p-2} Du)$  where  $Du = \left( \frac{\partial u}{\partial x_i} \right)_i$  is the gradient of  $u$ . The operator  $A$  has been widely considered in the literature in P.D.E., and arises in several physical situations, such as one-dimensional non newtonian fluids and glaciology.

This behaviour depends strongly on  $p$  and  $N$ : in fact, if  $p \geq \frac{2N}{N+1}$  we show that the total mass  $\int_{\mathbb{R}^N} u(t, x) dx$  is conserved, i.e, is independent of time. On the contrary if  $p < \frac{2N}{N+1}$  we show that the solution corresponding to initial data  $u_0 \in L^m(\mathbb{R}^N)$ ,  $m = N(\frac{2}{p} - 1)$  vanishes in finite time. The existence of a finite extinction time was found by B enilan and Crandall [2] for the equation (E)  $u_t - \Delta u^m = 0$  in spatial domain  $\mathbb{R}^N$  <sup>(1)</sup> if and only if  $0 < m < \frac{N-2}{N}$ ,  $N \geq 3$ . As it is noted in [2], equation (E) in bounded domains with homogeneous Dirichlet conditions has also that property if  $0 < m < 1$ . The case  $N = 1$  was considered by Sabinina [8]. Several properties of solutions of (E) related to the ones we consider here can be found in Evans [5]. Finite extinction times for (E) <sub>$\beta$</sub>   $u_t - \Delta \beta(u) = 0$  with  $\beta$  maximal monotone graph and bounded domain are discussed in terms of  $\beta$  in [3].

We also consider the homogeneous Dirichlet problem

$$(P_\Omega) \quad \begin{cases} u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 & \text{in } (0, T) \times \Omega \\ u(x, t) = 0 & \text{in } (0, T) \times \partial\Omega \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

for  $\Omega \subset \mathbb{R}^N$  open and bounded. We show the existence of a finite extinction time if  $p < 2$ ,  $u_0 \in L^m(\Omega)$ , and  $m$  as above, completing a result of Bamberger [1]: he showed that effect for  $\frac{2N}{N+2} \leq p < 2$  and  $u_0 \in L^2(\Omega)$ . For  $p \geq 2$  it is easy to see that solutions with positive initial data do not vanish.

For  $p > \frac{2N}{N+1}$  L. V eron [11] shows a smoothing and decay effect for the solutions

(1) with  $u_0 \in L^\beta(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  for a  $\beta = \beta(m, N)$ .

of  $(P_\Omega)$  : in fact, if  $N(\frac{2}{p}-1) < m_0 < m \leq \infty$  and  $u_0 \in L^{m_0}(\Omega)$ , then  $u(t, \cdot) \in L^m(\Omega)$  and in addition  $\|u\| \leq Ct^{-\delta} \cdot \|u_0\|_{m_0}^\sigma$  where  $\delta, \sigma$  depend on  $m, m_0, p$  and  $N$ . We adapt his proof for (P) to get similar results. We know that for  $m_0 = N(\frac{2}{p}-1)$  solutions vanish. For  $1 < m_0 < N(\frac{2}{p}-1)$  we prove a «backwards» effect : for  $t > 0$ ,  $u(t, \cdot) \in L^1(\mathbb{R}^N)$  and  $\|u\|_1 \leq Ct^{-\delta} \|u_0\|_{m_0}^\sigma$  with  $\delta, \sigma > 0$  as before.

We shall need some facts about the operator  $A$  in  $\mathbb{R}^N$  and in  $\Omega \subset \mathbb{R}^N$  bounded with homogeneous Dirichlet conditions : First, if  $J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |Du|^p$  when  $u \in L^2(\mathbb{R}^N)$  and  $|Du| \in L^p(\mathbb{R}^N)$ ,  $J(u) = +\infty$  otherwise,  $J$  is a convex l.s.c. proper functional in  $L^2(\mathbb{R}^N)$  whose subdifferential  $A$  is defined as  $Au = -\operatorname{div}(|Du|^{p-2} Du)$  in the domain  $D(A) = \left\{ u \in L^2(\mathbb{R}^N) : |Du| \in L^p(\mathbb{R}^N), \operatorname{div}(|Du|^{p-2} Du) \in L^2(\mathbb{R}^N) \text{ and for every } v \in D(J), \int_{\mathbb{R}^N} Au \cdot v = \int_{\mathbb{R}^N} |Du|^{p-2} Du \cdot Dv \right\}$ . If  $p \geq 2$ , the last condition may be omitted as it follows by density.  $A$  is accretive in  $L^1(\mathbb{R}^N)$  and  $L^\infty(\mathbb{R}^N)$ , hence in every  $L^p(\mathbb{R}^N)$ ,  $1 \leq p \leq \infty$  : in fact for  $t > 0$  and  $u_1, u_2 \in D(A) \cap L^p(\mathbb{R}^N)$ ,  $\| [u_1(t, \cdot) - u_2(t, \cdot)]^+ \|_p \leq \| [u_1(0, \cdot) - u_2(0, \cdot)]^+ \|_p$  where  $u^+ = \max(u, 0)$ . This implies a comparison principle that allows us to consider only nonnegative initial data and solutions ; for nonpositive data we consider  $-u$  instead of  $u$ . Defining for  $p \neq 2$ ,  $A_p = A \cap (L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N))$  we may close  $A_p$  to find  $\overline{A}_p$   $m$ -accretive in  $L^p(\mathbb{R}^N)$ . As  $\mathcal{D}(\mathbb{R}^N) \subset D(A_p)$ ,  $\overline{D(A_p)}^{L^p} = L^p(\mathbb{R}^N)$ .

The corresponding results for  $\Omega$  bounded and homogeneous Dirichlet conditions are well known ;  $Au = -\operatorname{div}(|Du|^{p-2} Du)$  and  $D(A) = \{ u \in W_0^{1,p}(\Omega) \cap L^2(\Omega) : Au \in L^2(\Omega) \}$ . On the other hand  $A_p$  is defined as  $m$ -accretive operator in  $L^p(\mathbb{R}^N)$  by restriction if  $p > 2$  and closure if  $p < 2$ .

We shall use the following inequality due to Nirenberg and Gagliardo (see [6], Th. 9.3.).

LEMMA 0. Let  $q, r$  be any numbers satisfying  $1 \leq q, r \leq \infty$  and  $u \in C_0^1(\mathbb{R}^N)$ . Then

$$\|u\|_p \leq C \|Du\|_r^a \|u\|_q^{1-a}$$

where  $\frac{1}{p} = a \cdot \frac{1}{r^*} + (1-a) \frac{1}{q}$  and  $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$  for all  $a$  in the interval  $0 \leq a \leq 1$ , with  $C = C(N, q, r, a)$ , with the following exception :  $r = N$  and  $a = 1$  (hence  $p = \infty$ ).

We remark that by density the result remains true for  $u \in L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$  such that  $Du \in L^r(\mathbb{R}^N)$  if  $r, q < \infty$  and  $q \leq r^*$  if  $r^*$  is positive. To show this, approach  $u$  by  $u^1$  bounded, then convolve  $u^1$  with a regular kernel to get  $u^2 \in C^\infty(\mathbb{R}^N)$  and finally cut  $u^2$  with a smooth function  $\xi_n$  which vanishes outside  $B_{2n}(0)$  and is equal to 1 on  $B_n(0)$  ; let us check this last step.

Assume  $u \in C^\infty(\mathbb{R}^N)$  and put  $u_n = u \zeta_n$ , where  $\zeta_n(x) = \zeta_0\left(\frac{|x|}{n}\right)$ ,  $0 \leq \zeta_0 \leq 1$ ,  $\zeta_0(x) = 1$  if  $|x| \leq 1$ ,  $\zeta_0(x) = 0$  if  $|x| \geq 2$  and  $|D\zeta_n(x)| \leq C$ . It is clear that  $u_n \rightarrow u$  in  $L^q(\mathbb{R}^N)$  and  $L^p(\mathbb{R}^N)$ . Also  $D_{u_n} = Du \cdot \zeta_n + u \cdot D\zeta_n$ .  $Du \cdot \zeta_n \rightarrow Du$  in  $L^r(\mathbb{R}^N)$  and we have to prove that  $u \cdot D\zeta_n \rightarrow 0$  in  $L^r(\mathbb{R}^N)$ . Then,  $C$  representing different constants independent of  $n$ :

$$\|u D\zeta_n\|_r^r = \int_{\mathbb{R}^N} |u|^r |D\zeta_n|^r \leq \frac{C}{n^r} \int_{n \leq |x| \leq 2n} |u|^r$$

if  $q \leq r$ ,  $\int |u|^r \leq \|u\|_\infty^{r-q} \cdot \int |u|^q$ , so  $\|u D\zeta_n\|_r^r \leq \frac{C \|u\|_\infty^{r-q}}{n^r} \cdot \|u\|_q^q \rightarrow 0$ ;

if  $r < q \leq r^*$ ,  $\int |u|^r \leq \left(\int |u|^q\right)^{r/q} \cdot \left(\int 1\right)^{1-q/q}$ , so:

$$\|u D\zeta_n\|_r^r \leq \frac{C}{n^r} \cdot \|u\|_{L^q(n \leq |x| \leq 2n)}^r \cdot n^{N(1-\frac{r}{q})} \rightarrow 0.$$

If  $r^* < 0$  the previous proof applies as well for every  $q$ ,  $1 \leq q < \infty$ .

Our plan is as follows: Sections 1, 2, 3 are devoted to problem (P). Section 1 studies the existence of a finite extinction time when  $p < \frac{2N}{N+1}$ ,  $u_0 \in L^m(\mathbb{R}^N)$ ,  $m = N(\frac{2}{p} - 1)$ . Section 2 is devoted to conservation of mass and Section 3 to the regularizing effects and decay of the integral norms  $\|u(t, \cdot)\|_m$  as  $t \rightarrow \infty$ . Finally Section 4 gathers the results on  $(P_\Omega)$ ,  $\Omega$  open and bounded.

## 1. - FINITE EXTINCTION TIME

We obtain the following result

**THEOREM 1.** Let  $N \geq 2$ ,  $1 < p < \frac{2N}{N+1}$  and let  $u_0 \in L^m(\mathbb{R}^N)$  where  $m = N(\frac{2}{p} - 1)$ . Then for every  $t > 0$   $u(t, \cdot) \in L^\infty(\mathbb{R}^N)$  and there exists  $t_0 > 0$  such that  $u(t, \cdot) = 0$  a.e. if  $t \geq t_0$ .

*Proof.* We may assume that  $u_0(x)$ ,  $u(t, x)$  are nonnegative. A formal proof to be justified later by discretization in time runs as follows: As  $p < \frac{2N}{N+1}$  if  $m = N(\frac{2}{p} - 1)$  we have  $m > 1$ . Let  $p^* = \frac{Np}{N-p}$  and  $q = \frac{m+p-2}{p}$ : then  $m = p^*q$ . Also for  $k \geq 0$  we write  $(u-k)_+ = \max(u-k, 0)$  and  $v = v_k = (u-k)_+^q$ . Multiply  $u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$  by  $m(u-k)_+^{m-1}$  and integrate over  $\mathbb{R}^N$  to obtain:

$$(1.1) \quad \frac{d}{dt} \int_{\mathbb{R}^N} (u-k)_+^m = m \int_{\mathbb{R}^N} u_t (u-k)_+^{m-1} = m \int_{\mathbb{R}^N} \operatorname{div}(|Du|^{p-2} Du) (u-k)_+^{m-1}$$

Integration by parts and Sobolev's inequality give

$$(1.2) \quad - \int_{\mathbb{R}^N} \operatorname{div}(|Du|^{p-2} Du)(u-k)_+^{m-1} = (m-1)\bar{q}^p \int_{\mathbb{R}^N} |Dv|^p \geq \\ \geq C_p (m-1)\bar{q}^p \left( \int_{\mathbb{R}^N} v^{p^*} \right)^{p/p^*}$$

Write  $E_{m,k}(t) = \int_{\mathbb{R}^N} (u-k)_+^m dx$ . (1.1) and (1.2) give

$$(1.3) \quad \frac{d}{dt} E_{m,k}(t) + C_p m(m-1)\bar{q}^p E_{m,k}^{p/p^*}(t) \leq 0$$

Integrating (1.3) gives

$$(1.4) \quad \begin{cases} E_{m,k}(t) \leq E_{m,k}(0) \left[ 1 - \frac{C_p m(m-1)p}{Nq^p (E_{m,k}(0))^{p/N}} \cdot t \right]^{\frac{N}{p}} & \text{for } 0 < t \leq t_{0,k} \\ E_{m,k}(t) = 0 & \text{for } t \geq t_{0,k} \end{cases}$$

where

$$t_{0,k} = \frac{Nq^p}{pC_p m(m-1)} E_{m,k}(0)^{p/N}$$

If we take  $k = 0$  the existence of a finite extinction time  $t_0 = t_{0,0}$  results. Given  $\bar{t} > 0$ , if we take  $k > 0$  large enough extinction of  $E_{m,k}(t)$  in time  $t_{0,k} \leq \bar{t}$  may be obtained. Hence  $u(t, \cdot) \in L^\infty(\mathbb{R}^N)$  for  $t > 0$ , a regularizing effect.

This formal proof can be made rigorous by means of the discrete scheme and Crandall-Liggett's results. Assume that  $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , let  $h > 0$  and define a discrete approximation to the solution of (P) thus :  $u_{i+1} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  is defined implicitly in terms of  $u_i$  by

$$(1.5) \quad \frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0$$

Now repeat the previous argument on (1.5) to obtain a discrete version of (1.3) and pass to the limit as  $h \rightarrow 0$ . The assumption on  $u_0$  can be weakened by approximation for  $t_{0,k}$  depends only on  $\|u_0\|_m$ . The details repeat those in [2] for  $u_t - \Delta u^m = 0$  and we omit them. Only the integration by parts needs some care : if  $m \geq 2$ ,  $u_0 \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then

$$(1.6) \quad - \int_{\mathbb{R}^N} Au_i u_i^{m-1} + (m-1) \int_{\mathbb{R}^N} |Du_i|^p u_i^{m-2} = 0$$

by the characterization of  $D(A)$ . If  $m < 2$  we have to linearize the function  $\phi(u) = u^{m-1}$  near the

origin to apply integration by parts. Passing to the limit it follows by Fatou in this case that

$$(1.7) \quad - \int_{\mathbb{R}^N} A u_i u_i^{m-1} + m \int_{\mathbb{R}^N} |D u_i|^p u_i^{m-2} \leq 0$$

For  $u_0$  as in the theorem the result follows by density for  $A$  is accretive #

## 2. - MASS CONSERVATION

We say that the mass conservation law (MCL) holds for (P) if for every  $t > 0$

$$\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx.$$

In this section the validity of MCL is discussed in terms of  $p$  :

**THEOREM 2.** *MCL holds for (P) if and only if  $N = 1, p > 1$  or  $N \geq 2, p \geq \frac{2N}{N+1}$ .*

In order to prove Theorem 2 we need some previous results. A variant of the following Lemma has been used in [10] :

**LEMMA 1.** *Let  $\Omega \subset \mathbb{R}^N$  be an open set and let  $u \in L^2(\mathbb{R}^N)$  be such that  $u \in D(A)$  and  $-Au = u$  a.e. in  $\Omega$ . Let  $\eta \in C^\infty(\Omega)$  be such that  $\text{supp}(\eta) \subset \subset \Omega$ ,  $\|\eta\|_\infty = 1$  and let  $\chi$  be the characteristic function of  $\text{supp}(D\eta)$ . Then*

$$(2.1) \quad \|\eta Du\|_p \leq p \|D\eta\|_\infty \cdot \|\chi u\|_p.$$

*Proof.* Multiply  $u = Au$  by  $u\eta^p$ , integrate over  $\mathbb{R}^N$ , integrate by parts ( $u \in D(A)$ ) and apply Hölder's inequality.

**LEMMA 2.** *Let  $\frac{2N}{N+1} \leq p \leq 2$  and let  $u$  be a solution of  $Au + u = f$ ,  $f \in L^1(\mathbb{R}^N)$ . Then*

$$\int_{\mathbb{R}^N} Au = 0.$$

*Proof.* By accretivity of  $A$  in  $L^1(\mathbb{R}^N)$ , we may restrict ourselves to consider  $f \in L^1_0(\mathbb{R}^N)$ . We obtain first an estimate for  $\|Du\|_p$  over the exterior of a ball : Assume  $\text{supp}(f) \subset B_R(0)$  and take  $n > R$ . Choose  $\eta_n \in C^\infty(\mathbb{R}^N)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 0$  if  $|x| \leq n$ ,  $\eta_n = 1$  if  $|x| \geq 2n$  and  $\|D\eta_n\|_\infty \leq \frac{C_1}{n}$ ,  $C_1 > 1$ . Put  $A_n = \{x \in \mathbb{R}^N : n \leq |x| \leq 2n\}$  and  $D_n = \{x \in \mathbb{R}^N : |x| \geq n\}$ . Then (2.1) gives in  $\Omega = \mathbb{R}^N - B_R(0)$  :

$$(2.2) \quad \|Du\|_{L^p(D_n)} \leq \frac{C}{n} \|\chi u\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{n} \|u\|_{L^p(A_n)}$$

Hereafter  $C$  denotes several positive constants depending only on  $p$  and  $N$  and not on  $n$ .

By virtue of [9], Corollary 2, the following estimate applies to  $u(x)$ , for  $|x| > R$ :

$$(2.3) \quad u(x) \leq C |x|^{-\frac{p}{2-p}}$$

Also by accretivity  $\|u\|_1 \leq \|f\|_1$ , so that

$$\|u\|_{L^p(A_n)}^p \leq \|u\|_{L^1(A_n)} \cdot \|u\|_{L^\infty(A_n)}^{p-1} = o(1) \cdot n^{-\frac{p(p-1)}{2-p}}. \quad (1)$$

$$\|Du\|_{L^p(D_n)} = o(1) \cdot n^{-\frac{1}{2-p}}. \text{ Putting } \xi_n(x) = 1 - \eta_n(x) \text{ we have}$$

$$(2.4) \quad \left| \int_{A_n} Au \cdot \xi_n \right| \leq \int_{D_n} |Du|^{p-1} |D\xi_n| \leq \frac{o(1)}{n} \cdot n^{-\frac{p-1}{2-p}} \cdot n^{N/p} = o(n) \cdot n^{\frac{N}{p} - \frac{1}{2-p}}$$

Since  $\int_{\mathbb{R}^N} Au = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} Au \xi_n$ , the desired result follows whenever

$$\frac{N}{p} - \frac{1}{2-p} \leq 0 \text{ i.e. } p \geq \frac{2N}{N+1} \quad \#$$

We say that the finite propagation property (PF) holds for (P) if for every admissible initial datum  $u_0(x)$  having compact support in  $\mathbb{R}^N$ , the corresponding solution  $u(t,x)$  is such that for every  $t > 0$   $u(t, \cdot)$  has compact support in  $\mathbb{R}^N$ . It is known that (PF) holds for (P) if and only if  $p > 2$  (see [4]). There exists a simple relation between (FP) and (MCL) :

LEMMA 3. *If  $p > 2$ , then (MCL) holds.*

*Proof.* Let  $u(x,t)$  be a solution of (P) such that  $u(x,0) = u_0(x)$  has compact support. If  $t > 0$  we know that there exists  $n$  such that  $\text{supp } u(t') \subset B_n(0)$  for  $0 \leq t' \leq t$ . Take  $\xi_n$  as before. Then for  $t'$  fixed :

$$\int_{\mathbb{R}^N} Au = \int_{\mathbb{R}^N} Au \cdot \xi_n = \int_{\mathbb{R}^N} |Du|^{p-2} Du \cdot D\xi_n = 0$$

Hence  $\int_{\mathbb{R}^N} u_t dx = 0$  and it follows that  $\int_{\mathbb{R}^N} u(t,x) = \int_{\mathbb{R}^N} u_0(x)$ . This last assertion can be justified by means of the discrete scheme as before.

If  $\text{supp}(u_0)$  is not compact, approximate  $u_0$  by  $\{u_{0_n}\}$ , a sequence of initial data with compact support #

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(1) Here  $o(1)$  denotes a quantity that goes to 0 as  $n \rightarrow \infty$ .



*Proof (of Theorem 2).* If  $N = 1$ ,  $p > 1$  or  $N \geq 2$ ,  $2 > p \geq \frac{2N}{N+1}$  the result follows from Lemma 2 applied to the discrete scheme

$$\frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0$$

for then  $\int_{\mathbb{R}^N} u_i = \int_{\mathbb{R}^N} u_{i+1}$ . If  $p > 2$  it follows from Lemma 3 in the same way. The case  $p = 2$  is classic (and it falls within the scope of [2]).

For the negative part it is sufficient to remind Theorem 1, for (MCL) is incompatible with extinction #

### 3. DECAY OF THE INTEGRAL NORMS. REGULARIZING EFFECT

Our first result is the extension to  $\mathbb{R}^N$  of the work of L. Véron [11] for the case  $\Omega$  bounded.

**THEOREM 3.** Let  $p > \frac{2N}{N+m_0}$ ,  $u_0 \in L^{m_0}(\mathbb{R}^N)$  with  $m_0 \geq 1$ . If  $t > 0$ ,  $u(t, \cdot) \in L^m(\mathbb{R}^N)$  for every  $m$  such that  $m_0 \leq m \leq \infty$ . In addition, the following estimate holds :

$$(3.1) \quad \|u(t, \cdot)\|_m \leq \frac{C}{t^\delta} \cdot \|u_0\|_{m_0}^\sigma \text{ for some constant } C = C(m, m_0, N, p), \text{ where}$$

$$(3.2) \quad \left\{ \begin{array}{l} \delta = \frac{N(m - m_0)}{m(m_0 p + N(p-2))} \text{ if } m < +\infty, \delta = \frac{N}{m_0 p + N(p-2)} \text{ if } m = +\infty. \\ \sigma = \frac{m_0(m p + N(p-2))}{m(m_0 p + N(p-2))} \text{ if } m < +\infty, \sigma = \frac{m_0 p}{m_0 p + N(p-2)} \text{ if } m = +\infty. \end{array} \right.$$

*Proof.* The case  $m = m_0$  follows from the accretivity property ; it suffices to show the case  $m = +\infty$ , the intermediate cases being obtained from these by interpolation. Assume (for simplicity) that  $u \geq 0$  ; for  $p \leq N$  we adapt the iterative procedure of L. Véron [11] as follows. Define the sequences  $m_n$ ,  $r_n$  by :

$$(3.3) \quad m_n = \gamma^n \cdot m_0 \quad \text{with} \quad 1 < \gamma < \frac{N}{N-1}, \quad m_0 \left( \frac{\gamma p}{N(\gamma-1)} - 1 \right) > \frac{1}{\gamma-1}$$

$$(3.4) \quad \frac{r_n + p-2}{m_n} = \frac{r_{n-1}}{m_{n-1}} - \frac{p}{N}$$

Note that from (3.3) and (3.4) it follows :

$$(3.5) \quad r_n = \frac{\gamma p}{N(\gamma-1)} m_{n-1} + \frac{p-2}{\gamma-1} = \frac{\gamma^n p}{N(\gamma-1)} m_0 + \frac{p-2}{\gamma-1}.$$

Now we claim that, if we write  $v = u^{q_{n-1}}$  with  $q_n = \frac{m_n + p - 2}{p}$ , Nirenberg-Gagliardo's inequality applies to  $v$ . Namely one has :

$$(3.6) \quad \|v\|_{\frac{m_n}{q_{n-1}}}^{\frac{r_n + p - 2}{q_{n-1}}} \leq C \cdot \|Dv\|_p^p \cdot \|v\|_{\frac{m_{n-1}}{q_{n-1}}}^{\frac{r_n - m_{n-1}}{q_{n-1}}}$$

That is a consequence of the following facts : i) As it was pointed out in Theorem 1, we can suppose  $u \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  hence  $v \in L^{\frac{m_n}{q_{n-1}}}(\mathbb{R}^N) \cap L^{\frac{m_{n-1}}{q_{n-1}}}(\mathbb{R}^N)$  for each  $m_{n-1} > 1$ , for then  $\frac{m_{n-1}}{q_{n-1}}$  is always greater than one, ii)  $Dv \in L^p(\mathbb{R}^N)$  as a consequence of the validity of formula (1.7), iii) Nirenberg-Gagliardo's inequality (Lemma 0) applies with the present regularity, as it was observed at the introduction.

We shall give a formal proof, just as at the first part of Theorem 1 (rigorous justification by means of the discrete schema approximation is made in the same way as there). Assume first  $p < N$ . Multiply the equation  $u_t - Au = 0$  by  $m_{n-1} u^{\frac{m_{n-1}}{q_{n-1}}}$  and integrate over  $\mathbb{R}^N$  to get

$$(3.7) \quad \frac{d}{dt} \left( \int_{\mathbb{R}^N} v^{\frac{m_{n-1}}{q_{n-1}}} \right) + C_{m,n} \left( \int_{\mathbb{R}^N} |Dv|^p \right) \leq 0$$

Next multiply (3.7) by  $\|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n - m_{n-1}}$  and use (3.6). It follows that

$$(3.8) \quad \|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n - m_{n-1}} \cdot \frac{d}{dt} \left( \|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{m_{n-1}} \right) + C \|u\|_{\frac{m_n}{q_n}}^{r_n + p - 2} \leq 0$$

where  $C$  involves  $C_{m,n}$  and the constant in (3.6), which depends only on  $N$  and  $p$ . Take  $t_n = t(1 - \frac{1}{2^n})$  and integrate (3.8) in  $[t_{n-1}, t_n]$ . In this way we obtain :

$$(3.9) \quad \|u(t_n)\|_{\frac{m_n}{q_n}}^{r_n + p - 2} \leq \frac{2^n}{C \cdot t} \|u(t_{n-1})\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n}$$

The previous argument remains true if we replace  $u$  by  $u_k = (u-k)_+$  for some  $k > 0$ . But then  $|\Omega_{k,t}| = \text{meas} \{x : u_k(t) > 0\}$  is finite and

$$\|u_k(t)\|_\infty = \lim_{m_n \rightarrow \infty} \sup \|u_k(t)\|_{m_n} \leq \lim_{m_n \rightarrow \infty} \sup \|u_k(t_n)\|_{m_n}.$$

Now (3.1), (3.2) follow from two facts : a)  $\lim_{m_n \rightarrow \infty} \sup \|u_k(t_n)\|_{m_n}$  can be evaluated now just in the same way as in [11], which implies estimates (3.1) (3.2) for  $u_k$ . b) These estimates do not depend on  $k$ , and consequently we can pass to the limit and obtain the desired results for  $k = 0$ .

When  $p = N$ , choose  $\{\beta_n\}$  such that

$$(3.10) \quad \beta_n = q_n + m_n \left(1 - \frac{1}{N}\right)$$

Write  $w_{n-1} = u^{\beta_{n-1}}$ . Then  $D(u^{q_{n-1}}) = \frac{q_{n-1}}{\beta_{n-1}} \cdot w_{n-1}^{\frac{q_{n-1} - \beta_{n-1}}{\beta_{n-1}}} \cdot Dw_{n-1}$ , i.e.,

$$Dw_{n-1} = \frac{\beta_{n-1}}{q_{n-1}} \cdot D(u^{q_{n-1}}) \cdot w_{n-1}^{\frac{\beta_{n-1} - q_{n-1}}{\beta_{n-1}}}. \text{ Now by Hölder}$$

$$(3.11) \quad \left( \int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \leq \left( \frac{\beta_{n-1}}{q_{n-1}} \right)^N \cdot \left( \int_{\mathbb{R}^N} |D(u^{q_{n-1}})|^N \right) \cdot \left( \int_{\mathbb{R}^N} u^{m_{n-1}} \right)^{N-1}$$

On the other hand, by Sobolev

$$(3.12) \quad \left( \int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \geq C_N \left( \int_{\mathbb{R}^N} u^{\frac{N\beta_{n-1}}{N-1}} \right)^{N-1}$$

Now multiply (3.7) by  $\|u\|_{m_{n-1}}^{r_n - m_{n-1}}$ , use (3.11), (3.12) and a standard interpolation argument to get :

$$(3.13) \quad \|u\|_{m_{n-1}}^{r_n - m_{n-1}} \frac{d}{dt} \left[ \|u\|_{m_{n-1}}^{m_{n-1}} \right] + C \cdot \|u\|_{m_n}^{r_n + N - 2} \leq 0$$

where  $C = C_{m,n} \cdot \left( \frac{\beta_{n-1}}{q_{n-1}} \right)^N \cdot C_N$ . (3.13) is the analogous of (3.8) and we can now argue as in the previous case.

When  $p > N$  we do not need to use the iterative procedure. For note that Nirenberg-Gagliardo's inequality reads :

$$(3.14) \quad \|v\|_{\infty} \leq C \|Dv\|_p^a \cdot \|v\|_{m/q}^{1-a} \text{ where } a = \frac{N(m+p-2)}{mp+N(p-2)}, \quad m > 1$$

(3.14) and (3.7) give

$$(3.15) \quad \|u\|_{m_n}^{\frac{(1-a)pq}{a}} \frac{d}{dt} (\|u\|_{m_n}^m) + C_m \left( \frac{1}{c} \right)^{\frac{p}{a}} \|u^q\|_{\infty}^{\frac{p}{a}} \leq 0, \quad q = \frac{m+p-2}{p}$$

Now note that from the inequality

$$\phi(t)^\omega \frac{d}{dt} \phi(t) + k \psi(t)^\theta \leq 0$$

it follows, integrating between 0 and t

$$(3.16) \quad \psi(t) \leq \left( \frac{1}{kt} \right)^{1/\theta} \cdot \frac{(\phi(0))^\theta}{\omega+1}$$

Use (3.16) with  $\psi(t) = \|u^q\|_\infty$ ,  $\phi(t) = \|u\|_m^m$ ,  $\omega = \frac{(1-a)pq}{a}$ ,  $\theta = \frac{p}{a}$  and (3.1), (3.2)

follow. Note that this argument includes the case  $N = 1$  which was discarded in [11] #

When  $1 < m_0 < N \left( \frac{2}{p} - 1 \right)$  we have the following result, concerning a «backwards» regularizing effect.

**THEOREM 4.** *Let  $1 \leq m_0 < N \left( \frac{2}{p} - 1 \right)$ ,  $u_0 \in L^{m_0}(\mathbb{R}^N)$ . If  $t > 0$ ,  $u(t, \cdot) \in L^m(\mathbb{R}^N)$  for every  $m$  such that  $1 \leq m \leq m_0$ . In addition the following estimate holds :*

$$(3.17) \quad \left\{ \begin{array}{l} \|u(t, \cdot)\|_m \leq \frac{C}{t^\delta} \|u_0\|_{m_0}^\sigma \text{ for some constant } C = C(m, m_0, N, p), \text{ where} \\ \delta = \frac{N(m_0 - m)}{m[N(2-p) - m_0 p]}, \quad \sigma = \frac{m_0[N(2-p) - mp]}{m[N(2-p) - m_0 p]} \end{array} \right.$$

*Proof.* Let us see first that  $u(t, \cdot) \in L^m(\mathbb{R}^N)$  for each  $m$  such that  $1 < m < m_0$  (the case  $m = m_0$  follows by accretivity). Remark that

$$(3.18) \quad \|v\|_{m/q} \leq C \|Dv\|_p^a \cdot \|v\|_{m_0/q}^{1-a},$$

where  $v, q$  are as in the last part of Theorem 3, the validity of (3.18) is justified as there, and

$a = \frac{N(m_0 - m)(m + p - 2)}{m[m_0(N - p) - N(m + p - 2)]}$ . Arguing as in Theorem 1 (with  $k = 0$ ), we arrive at

$$(3.19) \quad \frac{d}{dt} E_m(t) + k E_m(t)^{\frac{pq}{am}} \leq 0, \quad E_m(t) = \int_{\mathbb{R}^N} u^m(t, x) dx.$$

Now notice that solutions of the inequality  $f' + \alpha f^\gamma \leq 0$  with  $\gamma > 1$  satisfy  $f \leq \frac{1}{((\gamma-1)\alpha t)^{\frac{1}{\gamma-1}}}$ . This gives (3.17).

The case  $u(t, \cdot) \in L^1(\mathbb{R}^N)$  is obtained by modifying slightly the previous argument :

instead of (3.18) write

$$(3.20) \quad \|v\|_{1/q} \leq C \|Dv\|_p^a \cdot \|v\|_{m/q}^{1-a} \quad \text{with} \quad 1 < m < 3-p \left(1 + \frac{1}{N}\right),$$

$a = \frac{N(m-1)(p-1)}{[m(N-p)-N(p-1)]}$ . Corresponding to (3.19) we have

$$(3.21) \quad \|u\|_m^{(1-a)\frac{pq}{a}} \cdot \frac{d}{dt} (\|u\|_n^m) + C_m \left(\frac{1}{c}\right)^{\frac{p}{a}} \|u\|_1^{\frac{pq}{a}} \leq 0.$$

Now integrate (3.21) between 0 and  $t$  and use the fact that  $\|u(t)\|_m$  is not increasing in  $t$  to get the result #

#### 4. - BOUNDED DOMAINS

Concerning  $(P_\Omega)$  with  $\Omega$  bounded, it is known that there is a finite extinction time if  $u_0 \in L^2(\Omega)$  and  $\frac{2N}{N+2} \leq p < 2$  ([1]). In that paper, extinction of the  $L^2$  norm of the solution implies this result. The method of the proof of Theorem 1, based on the extinction of the  $L^m$  norm of solutions for some  $m > 1$ , enables us to extend the above mentioned result to get the following complete picture.

**THEOREM 5.** Assume that  $\Omega$  is bounded and regular. Let  $u_0 \in L^m(\Omega)$  where  $m \geq \max \left\{ N \left( \frac{2}{p} - 1 \right), 1 \right\}$  and  $p < 2$ . The corresponding solution of  $(P_\Omega)$  vanishes in a finite time  $t_0$ . If  $p \geq 2$  there are, for  $u_0 \in C^\infty(\Omega)$  and  $u_0 > 0$ , solutions which are strictly positive for every  $t > 0$ .

*Proof.* Let  $m > N \left( \frac{2}{p} - 1 \right)$  (the case  $m = N \left( \frac{2}{p} - 1 \right)$  is an easy modification of the proof in Theorem 1). We write again  $q = \frac{m+p-2}{p}$ ,  $v = u^q$ . By Hölder

$$(4.1) \quad \left( \int_\Omega u^m \right) \leq \left( \int_\Omega u^{p^*q} \right)^{\frac{m}{p^*q}} \cdot |\Omega|^{\frac{p^*q-m}{p^*q}}, \quad \text{where } p^* = \frac{Np}{N-p}, \quad |\Omega| = \text{meas}(\Omega).$$

Starting as in Theorem 1 (with  $k = 0$ ) we arrive at

$$(4.2) \quad \frac{d}{dt} \left( \int_\Omega u^m \right) + \frac{m(m-1)}{q^p} \left( \int_\Omega |Dv|^p \right) \leq 0$$

Next use Sobolev ( $\|Dv\|_p \geq c \|v\|_{p^*}$ ) and (4.1) to obtain

$$(4.3) \quad \frac{d}{dt} \left( \int_{\Omega} u^m \right) + \frac{cm(m-1)}{p^q |\Omega| \omega} \left( \int_{\Omega} u^m \right)^{\frac{pq}{m}} \leq 0, \text{ with } \omega = \frac{N-p}{N^2} \cdot \frac{N(p-2)+mp}{(m+p-2)}$$

From (4.3) we conclude that  $u$  vanishes at most at  $t_0$ , where

$$t_0 = \frac{(2-p)q^p}{cm^2(m-1)} \cdot |\Omega| \omega \cdot \|u_0\|_m^{2-p}$$

Assume now that  $\Omega$  is connected <sup>(1)</sup>. When  $p = 2$  the fact that for  $u_0 \geq 0$ ,  $u_0 \neq 0$  and  $t > 0$ ,  $u(t, \cdot) > 0$  follows from the strong maximum principle of L. Nirenberg (see [7]). If  $p > 2$  take  $\Omega = B_R(0)$  and  $g$  a positive eigenfunction corresponding to the first eigenvalue  $\lambda$  of  $-\Delta$  in  $B_R(0)$  with homogeneous Dirichlet conditions;  $g$  is radially symmetric,  $C^\infty$  and  $Ag \leq Cg$  for some  $C > 0$ . To check this last assertion, note that

$$\begin{aligned} -g'' - \frac{N-1}{r} g' &= g \text{ and hence } Ag = -\lambda(p-1) |g'|^{p-2} g'' - \frac{N-1}{r} |g'|^{p-2} g' = \\ &= \lambda(p-1) |g'|^{p-2} g' + (p-1) \frac{N-1}{r} |g'|^{p-2} g' - \frac{N-1}{r} |g'|^{p-2} g' \leq \lambda(p-1) |g'|^{p-2} g' \leq Cg. \end{aligned}$$

Now try as a subsolution  $\bar{v}(t, x) = T(t) g(x)$ , where  $T(t) = \frac{T_0}{(1+c(p-2)T_0^{p-2}t)^{1/p-2}}$  solves  $T'(t) + CT(t)^{p-1} = 0$ . It follows from the maximum principle that if  $u_0(x) \geq T_0 g(x)$ , the corresponding  $u(t, x)$  is greater or equal than  $\bar{v}(t, x)$  for each  $t > 0$  #

*Remark.* Observe that as a consequence of the decay of some  $m$ -norm,  $m > 1$  and  $\Omega$  being bounded, MCL never holds. When  $p \geq 2$  we have shown that for smooth initial data there is a retention property: if  $u_0 > 0$  in some  $\tilde{\Omega} \subset \Omega$ ,  $u(t, x) > 0$  in  $\bar{\Omega}$  for each  $t > 0$ .

We conclude by noting that the results of this paper are valid when  $Au$  is replaced by other similar nonlinear.

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

As a natural generalization we may consider operators like

$$Bu = \sum_{i=1}^N \frac{\partial}{\partial x_i} \beta_i \left( \frac{\partial u}{\partial x_i} \right)$$

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(1) For general  $\Omega$  argue on each connected component.

where  $\sum_{i=1}^N s_i \beta_i(s_i) \geq c |s|^p$  with  $s = (s_1, \dots, s_N)$ .

Some of the previous results have immediate counterparts. In particular Theorem 1 remains valid unchanged.

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