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AN ELLIPTIC ANALOGUE OF THE FRANKLIN-SCHNEIDER THEOREM

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Résumé : Soit p une fonction elliptique de Weierstrass d'invariants g_2 et g_3 algébriques. Soient a et b des nombres complexes tels que ni a ni ab soit parmi les pôles de p. On donne une minoration pour l'approximation simultanée de p(a), b et p(ab) par des nombres algébriques, exprimée dans leur hauteur et leur degré. Par un contre-exemple, on montre qu'une certaine hypothèse sur les nombres β qui approximent b est nécessaire.

Summary : Let p be a Weierstrass elliptic function with algebraic invariants g_2 and g_3. Let a and b be complex numbers such that a and ab are not among the poles of p. A lower bound is given for the simultaneous approximation of p(a), b and p(ab) by algebraic numbers, expressed in their heights and degrees. By a counterexample it is shown that a certain hypothesis on the numbers β approximating b is necessary.

1 - INTRODUCTION

If a $\neq 0$ and b are complex numbers, the numbers a, b and a^b cannot simultaneously be approximated by algebraic numbers in such a way that the total approximation error is small in terms of the heights and degrees of these algebraic numbers, except in the case where all the

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numbers used to approximate b can be chosen rational. This result is known as the Franklin-Schneider theorem ; its sharpest version up until now is due to M. Waldschmidt [12], who proved the following : there exists a number C, effectively computable in terms of a, b and the determination of the logarithm of a used in defining a^b, such that for all triples (α,β,γ) of algebraic numbers with β irrational

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) > \exp(-CD^4 \log^3 H \log_2^{-2} H), \tag{1}$$

where $\log_2$ means loglog, D is the degree of the field $\mathbb{Q}(\alpha,\beta,\gamma)$ over $\mathbb{Q}$ and $H \geqslant e^e$ is a bound for the heights of α, β and γ. The condition that $\beta \notin \mathbb{Q}$ is crucial : in [2], the present author proved that, even if $b \notin \mathbb{Q}$, no bound of the type (1) exists if β is allowed to be rational.

The purpose of this paper is to prove elliptic analogues of both these statements. We fix the following

Notation. Let ω_1, ω_2 be complex numbers, linearly independent over $\mathbb{R}$; let Ω denote the set $\{ m_1 \omega_1 + m_2 \omega_2 : m_1, m_2 \in \mathbb{Z} \}$ and $\wp$ the Weierstrass elliptic function with period lattice Ω. Then $\wp$ satisfies a differential equation of the type

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

with $g_2, g_3 \in \mathbb{C}$; we shall assume everywhere that g_2, g_3 are algebraic. By e_1, e_2, e_3 we denote the roots of the equation $4X^3 - g_2X - g_3 = 0$; $\mathbb{K}$ will denote the field of complex multiplication of $\wp$, that is to say, $\mathbb{K} = \mathbb{Q}(\omega_2/\omega_1)$ if ω_2/ω_1 is a quadratic irrationality, $\mathbb{K} = \mathbb{Q}$ otherwise.

We now state the analogue of (1) we propose to prove.

THEOREM 1. *Suppose* $a, b \in \mathbb{C}$ *such that* a *and* ab *are not poles of* $\wp$. *Then there exists an effectively computable* $C \in \mathbb{R}$, *depending only on* $\wp$, a *and* b, *such that no triple* $(u,\beta,v) \in \mathbb{C}^3$ *satisfies* $\wp(u), \beta, \wp(v)$ *algebraic,* $\beta \notin \mathbb{K}$ *and*

$$\max(|\wp(a)-\wp(u)|, |b-\beta|, |\wp(ab)-\wp(v)|) < \exp(-CD^6\log^6 H \log_2^{-5} H) \tag{2}$$

while $[\mathbb{Q}(\wp(u), \beta, \wp(v)) : \mathbb{Q}] \leqslant D$ *and* $\max(e^e, H(\wp(u)), H(\beta), H(\wp(v))) \leqslant H$.

The proof of this theorem depends on a result on linear forms in algebraic points of $\wp$ (see Lemma 1 below) ; in the case of complex multiplication, i.e. when $\mathbb{K} \neq \mathbb{Q}$, it would also have been possible to deduce a slightly less sharp version of Theorem 1 from the results of M. Anderson announced in [9].

The condition $\beta \notin \mathbb{Q}$ that was necessary for (1) is replaced in Theorem 1 by $\beta \notin \mathbb{K}$. The necessity of the latter assumption follows from

THEOREM 2. *For every function* $g : \mathbb{N}^2 \to \mathbb{R}$ *there exist* $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{K}$, *such that* a *and* ab *are not poles of* p *and such that for every* $C \in \mathbb{R}$ *there exist infinitely many triples* $(u,\beta,v) \in \mathbb{C}^3$ *satisfying* $p(u)$, β, $p(v)$ *algebraic and*

$$\max(|p(a)-p(u)|, |b-\beta|, |p(ab)-p(v)|) < \exp(-Cg(D,H))$$

while $[\mathbb{Q}(p(u), \beta, p(v)) : \mathbb{Q}] \leqslant D$ *and* $\max(H(p(u)), H(\beta), H(p(v))) \leqslant H$.

2 - PROOF OF THEOREM 1

For a set K in the complex plane, K^o denotes the interior of K. By the size of an algebraic complex number, we mean the maximum of its denominator and the absolute values of its conjugates.

LEMMA 1. *For every compact subset* K *of* $\mathbb{C} \setminus \Omega$ *there exists an effectively computable* $C \in \mathbb{R}$, *depending only on* p *and* K, *with the following property. Let* $u, \beta, v \in \mathbb{C}$ *satisfy* $u, v \in K^o$, $p(u)$, β, $p(v)$ *algebraic and* $\beta \notin \mathbb{K}$. *Let* $A_1 \geqslant e^e$ *be an upper bound for* $\min(H(p(u)), H(p(v)))$; *let* $A_2 \geqslant A_1$ *be an upper bound for* $\max(H(p(u)), H(p(v)))$. *Let* $B \geqslant e$ *be an upper bound for* $H(\beta)$ *and take* $D := [\mathbb{Q}(p(u), \beta, p(v)) : \mathbb{Q}]$. *Then*

$$|\beta u - v| > \exp(-CD^6 \log A_1 \log A_2 \log^4(DB \log A_2) \log^{-5}(D \log A_1)). \tag{3}$$

Proof. I. Let (u,β,v) be a triple satisfying the conditions of the lemma. By $c_1, c_2,\ldots$ we shall denote effectively computable real numbers greater than 1 that depend only on p and K. Let x be some large real number ; further conditions on x will appear at later stages of the proof. Put $B' := xDB \log A_2$, $E := 4D^{1/2} \log^{1/2} A_1$ and assume

$$|\beta u - v| \leqslant \exp(-x^{16} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E). \tag{4}$$

From this we shall obtain a contradiction, which proves (3).

Define

$$L_1 := [x^8 D^3 \log A_2 \log^2 B' \log^{-3} E],$$
$$L_2 := [x^8 D^3 \log A_1 \log^2 B' \log^{-3} E]$$

in case $H(p(u)) \leqslant A_1$. If $H(p(u)) > A_1$, the definitions of L_1 and L_2 should be interchanged. By ξ we denote a primitive element for $\mathbb{Q}(p(u), \beta, p(v))$ of the form $m_1 p(u) + m_2 \beta + m_3 p(v)$, where $m_1, m_2, m_3 \in \mathbb{N} - 1$ and $m_1 + m_2 + m_3 \leqslant D^2$ (cf. [3], Lemma 1). Consider the auxiliary function

$$F(z) := \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\, \xi^\delta\, p^{\lambda_1}(z)\, p^{\lambda_2}(\beta z),$$

where the $p(\lambda_1,\lambda_2,\delta)$ are rational integers to be determined later. For $t \in \mathbb{N} - 1$, $s \in \mathbb{N}$ such that $su \notin \Omega$, $s\beta u \notin \Omega$ we have

$$(5) \qquad F^{(t)}(su) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta) \quad \sum_{\tau=0}^{t} \binom{t}{\tau} \beta^{t-\tau}. \\ \frac{d^\tau}{dz^\tau} p^{\lambda_1}(z)\,\Big|_{z=su} \frac{d^{t-\tau}}{dz^{t-\tau}} p^{\lambda_2}(z)\,\Big|_{z=s\beta u}.$$

Also define

$$F_s(z) := \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\, \xi^\delta\, p^{\lambda_1}(z)\, p^{\lambda_2}(\beta z - s\epsilon),$$

where $\epsilon := \beta u - v$. For $t \in \mathbb{N} - 1$, $s \in \mathbb{N}$ such that $su \notin \Omega$, $sv \notin \Omega$ we have

$$(6) \qquad F_s^{(t)}(su) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\, \xi^\delta \sum_{\tau=0}^{t} \binom{t}{\tau} \beta^{t-\tau}. \\ \frac{d^\tau}{dz^\tau} p^{\lambda_1}(z)\,\Big|_{z=su} \frac{d^{t-\tau}}{dz^{t-\tau}} p^{\lambda_2}(z)\,\Big|_{z=sv}.$$

Now put

$$S := [x^3 D \log B' \log^{-1} E].$$

As in [7], IX, § 1 an application of the box principle shows that we may assume there is a subset V of $\{1,\dots,S\}$ such that $\# V \geqslant c_1^{-1} S$ and $su, sv \in K^o + \Omega$ for all $s \in V$.
Moreover, it is no restriction to assume that the points $su : s \in V$ are all distinct. Indeed, this can either be brought about by interchanging u and v, or there exist $s_1, s_2 \in \{1,\dots,S\}$ and $m_1,\dots,m_4 \in \mathbb{Z}$, their absolute values bounded by $c_2 S$, such that

$$s_1 u = m_1 \omega_1 + m_2 \omega_2, \; s_2 v = m_3 \omega_1 + m_4 \omega_2.$$

However, in that case

$$|\beta u - v| = s_1^{-1} s_2^{-1} \,|(m_1 s_2 \beta - m_3 s_1)\, \omega_1 + (m_2 s_2 \beta - m_4 s_1)\, \omega_2|$$

and (3) then follows from Fel'dman's result [6]. Put

$$T := [x^{12} D^5 \log A_1 \log A_2 \log^3 B' \log^{-5} E]$$

and consider the system of linear equations

(7) $$F_s^{(t)}(su) = 0 : s \in V,\ t \in \{0,\ldots,T-1\}$$

in the $p(\lambda_1,\lambda_2,\delta)$. To solve this system, we shall use a method devised by M.Anderson.

Lemma 5.1 of [10] states that for every $w \in \mathbb{C}$, there exist polynomials $\Phi_w, \Phi_w^* \in \mathbb{Z}[X_1,\ldots,X_5]$, their heights and degrees bounded by an absolute constant, such that

$$\wp(z+w) = \frac{\Phi_w^*}{\Phi_w}\left(\wp(z+\frac{\omega_1}{2}), \wp'(z+\frac{\omega_1}{2}), \wp(w), \wp'(w), \wp(\frac{\omega_1}{2})\right)$$

as a meromorphic function in z, while

$$\Phi_w(\wp(\frac{\omega_1}{2}), \wp'(\frac{\omega_1}{2}), \wp(w), \wp'(w), \wp(\frac{\omega_1}{2})) \neq 0.$$

Lemma 7.1 of [8], which remains valid without complex multiplication, states that for every $s \in \mathbb{N}$, there exist coprime polynomials Ψ_s, Ψ_s^* of degree at most s^2 such that

$$\wp(sz) = \frac{\Psi_s^*}{\Psi_s}(\wp(z))$$

as a meromorphic function in z ; the coefficients of Ψ_s, Ψ_s^* are themselves polynomials in $g_2/4$, g_3, with a degree at most s^2 and rational integer coefficients not larger than $c_3^{s^2}$. Now define

$$\phi_{s,w}(z) := \phi_w^{s^2}\left(\wp(z+\frac{\omega_1}{2}), \wp'(z+\frac{\omega_1}{2}), \wp(w), \wp'(w), \wp(\frac{\omega_1}{2})\right) \Psi_s(\wp(z+w))$$

and

$$\psi_{s,w}(z) := \phi_{s,w}(z)\, \wp(sz+sw) ;$$

then for $s \in V$ we have

(8) $$F_s^{(t)}(su) = \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\xi^\delta\, s^{-t}.$$

$$\frac{d^t}{dz^t}\left(\phi_{s,u}^{-\lambda_1}(z)\, \psi_{s,u}^{\lambda_1}(z)\, \phi_{s,v}^{-\lambda_2}(\beta z)\, \psi_{s,v}^{\lambda_2}(\beta z)\right)\Big|_{z=0}.$$

Therefore we have found a solution of (7) if we choose the $p(\lambda_1,\lambda_2,\delta)$ in such a way that

(9) $$f_{s,t} := \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\, \xi^\delta \frac{d^t}{dz^t} (\psi_{s,u}^{\lambda_1}(z)\, \psi_{s,v}^{\lambda_2}(\beta z))\, |_{z=0} = 0 :$$

$s \in V,\ t \in \{0,\dots,T-1\}$.

The number of equations in (9) is at most

$$ST \leqslant c_4 x^{15} D^6 \log A_1 \log A_2 \log^4 B' \log^{-6} E$$

while the number of unknowns is

$$(L_1+1)(L_2+1)D \geqslant c_5^{-1} x^{16} D^7 \log A_1 \log A_2 \log^4 B' \log^{-6} E.$$

According to Lemma 5.2 of [10], the expression

$$\frac{d^t}{dz^t} \psi_{s,u}^{\lambda_1}(z)\, |_{z=0}$$

can be written as a polynomial in $\wp(u)$ and $\wp'(u)$ of degree at most $c_6 \lambda_1 s^2$ in each of the variables ; the coefficients of this polynomial belong to $\mathbb{Q}(\wp(\omega_1/2), \wp''(\omega_1/2))$, and their size is bounded by $\exp(c_7(\lambda_1 s^2 + t \log t))$. Moreover, they have a common denominator of the form m^n, where m depends only on $\wp$ and $n \leqslant c_8(\lambda_1 s^2 + t \log t)$. Thus the coefficients of the system of linear equations (9) lie in a field of degree at most $c_9 D$ and their size and common denominator are bounded by

$$(H(\xi)+1)^D T^{c_{10} T} (H(\wp(u))+1)^{c_{11} L_1 S^2} (H(\wp(v))+1)^{c_{11} L_2 S^2} B^T \leqslant$$

$$\exp(c_{12} x^{14} D^5 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

According to Lemma 1.3.1 of [11], if $x > 2c_4c_5$, this implies the existence of $p(\lambda_1,\lambda_2,\delta) \in \mathbb{Z}$, not all zero, such that (9) and thereby (7) hold, while

$$P := \max |p(\lambda_1,\lambda_2,\delta)| \leqslant \exp(c_{13} x^{14} D^5 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

II. Put $T' := [x^2 T]$. We shall prove that, for our special choice of the $p(\lambda_1,\lambda_2,\delta)$, we have

(10) $$F_s^{(t)}(su) = 0 : s \in V,\ t \in \{0,\dots,T'-1\}.$$

For $s \in V$, $t \in \{0,\dots,T-1\}$, comparison of (5) and (6) yields

$$(11) \qquad |F^{(t)'}(su)| \leqslant P c_{14}^{D^2+L_1+L_2} T^{c_{15}T} S|\epsilon| \leqslant$$

$$\exp(-c_{16}^{-1} x^{16} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

Now define

$$G(z) := g(z)\, F(zu),$$

where

$$g(z) := \sigma^{L_1}(zu)\, \sigma^{L_2}(z\beta u)\ ;$$

here σ is the sigma-function of Weierstrass corresponding to Ω. Then G is an entire function and

$$(12) \qquad G^{(t)}(s) = \sum_{\tau=0}^{t} \binom{t}{\tau} u^{t-\tau} g^{(\tau)}(s) F^{(t-\tau)}(su)\ ;$$

as $|g^{(\tau)}(s)| \leqslant \tau!\ \exp(c_{17}((L_1+L_2)s^2 + \tau))$, by Cauchy's inequality, substitution of (11) shows that

$$(13) \qquad |G^{(t)}(s)| \leqslant \exp(-c_{18}^{-1} x^{16} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E) : s \in V,\ t \in \{0,\dots,T-1\}.$$

Lemma 2 of [5] states that

$$(14) \qquad \max_{|z|\leqslant 2S} |G(z)| \leqslant 2 \max_{|z|\leqslant 2ES} |G(z)| \cdot \left(\frac{4}{E}\right)^{c_1^{-1}ST} + 5\,.\,36^{ST} \max_{\substack{s \in V \\ t \in \{0,\dots,T-1\}}} \left|\frac{G^{(t)}(s)}{t!}\right|$$

For the factors in the right hand member of (14), we possess the following estimates :

$$\max_{|z|\leqslant 2ES} |G(z)| \leqslant P c_{19}^{D^2+(L_1+L_2)E^2S^2} \leqslant \exp(c_{20} x^{14} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E)\ ;$$

$$\left(\frac{4}{E}\right)^{c_1^{-1}ST} \leqslant \exp(-c_{21}^{-1} x^{15} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E)\ ;$$

$$5.36^{ST} \leqslant \exp(c_{22} x^{15} D^6 \log A_1 \log A_2 \log^4 B' \log^{-6} E)\ ;$$

$$\max_{\substack{s \in V \\ t \in \{0,\dots,T-1\}}} \left|\frac{G^{(t)}(s)}{t!}\right| \leqslant \exp(-c_{18}^{-1} x^{16} D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

Substitution in (14) yields

$$(15)\qquad \max_{|z|\leqslant 2S} |G(z)| \leqslant \exp(-c_{23}^{-1}x^{15}D^6\log A_1 \log A_2 \log^4 B' \log^{-5} E)$$

and thus, by Cauchy's inequality,

$$(16)\qquad \max_{\substack{s\in V\\ t\in\{0,\dots,T'-1\}}} |G^{(t)}(s)| \leqslant \exp(-c_{24}^{-1}x^{15}D^6\log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

Fix $s \in V$ and let t be the smallest number in $\{0,\dots,T'-1\}$ such that $f_{s,t} \neq 0$. From (8) it is then clear that $F^{(t-\tau)}(su) = 0$ for $\tau = 1,\dots,t$. For all terms with $\tau \neq 0$ in (12) we have an estimate of the form

$$(T')^{c_{25}T'} c_{26}^{(L_1+L_2)S^2} Pc_{27}^{D^2 S} |\epsilon| \leqslant \exp(-c_{28}^{-1}x^{16}D^6\log A_1 \log A_2 \log^4 B' \log^{-5} E),$$

while the left hand member is bounded by (16). This shows that

$$(17)\qquad |u^t g(s) F^{(t)}(su)| \leqslant \exp(-c_{29}^{-1}x^{15}D^6 \log A_1 \log A_2 \log^5 B' \log^{-5} E).$$

Now the definition of V, together with Lemma 7.1 of [8], implies that

$$|g(s)| \geqslant c_{30}^{-(L_1+L_2)S^2}$$

and therefore adjusting the value of c_{29} ensures that (17) also holds for $|F^{(t)}(su)|$. Thus

$$|F_s^{(t)}(su)| \leqslant Pc_{30}^{D^2+L_1+L_2}(T')^{c_{32}T'} S |\epsilon| + |F^{(t)}(su)| \leqslant$$

$$\exp(-c_{33}^{-1}x^{15}D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

According to (8), we have

$$f_{s,t} = \sum_{\tau=0}^{t} \binom{t}{\tau} s^{t-\tau} \frac{d^\tau}{dz^\tau} (\phi_{s,u}^{\lambda_1}(z)\phi_{s,v}^{\lambda_2}(\beta z))|_{z=0} F^{(t-\tau)}(su) ;$$

thus, by the choice of t,

$$f_{s,t} = s^t \phi_{s,u}^{\lambda_1}(0)\phi_{s,v}^{\lambda_2}(0)F_s^{(t)}(su).$$

Therefore

$$(18)\qquad |f_{s,t}| \leqslant S^{T'} c_{34}^{(L_1+L_2)S^2} |F^{(t)}(su)| \leqslant \exp(-c_{35}^{-1}x^{15}D^6 \log A_1 \log A_2 \log^4 B' \log^{-5} E).$$

However, once again by Lemma 5.2 of [10], $f_{s,t}$ is an algebraic number of degree at most $c_{36}D$ and of size at most

$$(L_1+1)(L_2+1)DP(H(\xi)+1)^D(T')^{c_{37}T'}(H(p(u))+1)^{c_{38}L_1S^2}(H(p(v))+1)^{c_{38}L_2S^2}B^{T'} \leqslant$$

$$\exp(-c_{39}x^{14}D^5\log A_1 \log A_2 \log^4 B' \log^{-5} E);$$

by formula (1.2.3) of [11], this implies

$$|f_{s,t}| > \exp(-c_{40}x^{14}D^6\log A_1 \log A_2 \log^4 B' \log^{-5} E),$$

so if $x > c_{35}c_{40}$ we obtain a contradiction with (18). This shows that

$$f_{s,t} = 0 : s \in V,\ t \in \{0,\ldots,T'-1\},$$

which, by (8), implies (10).

III. To obtain the final contradiction we shall use an argument involving resultants. It has been brought to the author's attention that this method is due to W.D. Brownawell and D.W. Masser, who will publish a detailed account of it in [4]. Put

$$p(\lambda_1,\lambda_2) := \sum_{\delta=0}^{D-1} p(\lambda_1,\lambda_2,\delta)\,\xi^\delta : \lambda_1 \in \{0,\ldots,L_1\},\ \lambda_2 \in \{0,\ldots,L_2\}$$

and

$$P(X,Y) := \sum_{\lambda_1=0}^{L_1} \sum_{\lambda_2=0}^{L_2} p(\lambda_1,\lambda_2)X^{\lambda_1}Y^{\lambda_2} \in \mathbb{C}[X,Y].$$

Suppose P is not identically zero. Let $P^*(X,Y)$ be an arbitrary non-constant irreducible factor of $P(X,Y)$ with algebraic coefficients and put

(19) $$N_s^* := \operatorname*{ord}_{z=su} P^*(p(z), p(\beta z - s\epsilon)) : s \in V.$$

Then, for every $s \in V$,

$$\operatorname*{ord}_{z=su} [p'(z)P_X^*(p(z),p(\beta z-s\epsilon)) + \beta p'(\beta z - s\epsilon)P_Y^*(p(z),p(\beta z - s\epsilon))] \geqslant N_s^* - 1,$$

where P_X^* and P_Y^* denote the derivatives of P^* with respect to the first and second variable respectively. Thus

(20) $$\operatorname*{ord}_{z=su} [(p'(z)P_X^*(p(z),p(\beta z-s\epsilon)))^2 - (\beta p'(\beta z - s\epsilon)P_Y^*(p(z),p(\beta z-s\epsilon)))^2] \geqslant N_s - 1.$$

Put

$$Q(X,Y) := (4X^3-g_2X-g_3)P_X^{*2}(X,Y) - \beta^2(4Y^3-g_2Y-g_3)P_Y^{*2}(X,Y) ;$$

then (20) may, by the differential equation for p, be written as

$$\operatorname*{ord}_{z=su} Q(p(z), p(\beta z - s\epsilon)) \geqslant N_s^* - 1. \tag{21}$$

We consider the case where neither P_X^* nor P_Y^* is identically zero ; from (19) and (21) we get

$$\operatorname*{ord}_{z=su} R(p(z)) \geqslant N_s^* - 1,$$

where R denotes the resultant of P* and Q with respect to the second variable. Let L_1, L_2 denote the degree of P* with respect to X, Y respectively ; then R is a polynomial of degree at most $c_{41}L_1^*L_2^*$. As

$$\operatorname*{ord}_{z=su} [p(z) - p(su)] \leqslant 2, \tag{22}$$

we find that either

$$\sum_{s \in V} (N_s - 1) \leqslant c_{42}L_1^*L_2^* \tag{23}$$

or R is identically zero.

Suppose (23) is not satisfied, so R is identically zero. As P* is irreducible, it follows that P* divides Q. Let Δ denote the resultant of P* and P_Y^* with respect to Y. If Δ were identically zero, it would follow from the irreducibility of P* that P* divides P_Y^*, thus that P_Y^* is identically zero ; therefore there is some $z_1 \in \mathbb{C} \setminus \Omega$ with $\Delta(p(z_1)) \neq 0$. In particular we may choose z_1 in such a way that $p(z_1)$ is transcendental. Clearly there exists a ζ with $P^*(p(z_1), \zeta) = 0$. If $P_Y^*(p(z_1), \zeta) = 0$, it would follow that $\Delta(p(z_1)) = 0$; thus $P_Y^*(p(z_1), \zeta) \neq 0$.

The implicit function theorem now states that there exists a holomorphic function h, defined on a neighbourhood U of z_1, such that $h(z_1) = \zeta$, while $P^*(p(z), h(z)) = 0$ and $P_Y^*(p(z), h(z)) \neq 0$ for $z \in U$. Differentiation shows that for $z \in U$ we have

$$p'(z)P_X^*(p(z),h(z)) + h'(z)P_Y^*(p(z),h(z)) = 0,$$

thus

$$(p'(z)P_X^*(p(z),h(z)))^2 - (h'(z)P_Y^*(p(z),h(z)))^2 = 0$$

and

$$(4\wp^3(z)-g_2\wp(z)-g_3)P_X^{*2}(\wp(z),h(z)) - (h'(z))^2 P_Y^{*2}(\wp(z),h(z)) = 0.$$

On the other hand, $Q(\wp(z),h(z)) = 0$, so we find that

$$(h'(z))^2 = \beta^2(4h^3(z)-g_2h(z)-g_3) \tag{24}$$

for $z \in U$. Because $\wp(z_1)$ is transcendental, $h(z_1) = \zeta \notin \{e_1,e_2,e_3\}$ and thus, by (24), $h'(z_1) \neq 0$; it is no restriction to assume $h'(z) \neq 0$ for all $z \in U$. Differentiation of (24) now yields

$$h''(z) = \beta^2(6h^2(z)-g_2/2) : z \in U,$$

an equation that shows that the coefficients h_n of the Taylor development of h around z_1 satisfy

$$n(n-1)h_n = 6\beta^2 \sum_{k=0}^{n-2} h_k h_{n-2-k} : n \geqslant 3. \tag{25}$$

It is clear that at most two analytic functions h on U can simultaneously satisfy (25) and

$$h_0 = \zeta\ ,\ h_1^2 = \beta^2(4\zeta^3-g_2\zeta-g_3),\ h_2 = \beta^2(3\zeta^2-g_2/4). \tag{26}$$

But as $\zeta \notin \{e_1,e_2,e_3\}$, there are $z_2, z'_2 \in \mathbb{C}$ such that

$$\wp(\beta z_1-z_2) = \wp(\beta z_1-z'_2) = \zeta\ ,\ z_2 \not\equiv z'_2 \pmod{\Omega}.$$

The functions $z \mapsto \wp(\beta z-z_2)$ and $z \mapsto \wp(\beta z-z_2^1)$ are different, yet they both satisfy (25) and (26) ; therefore either $h(z) = \wp(\beta z-z_2)$ for $z \in U$ or $h(z) = \wp(\beta z-z_2^1)$ for $z \in U$. It is no restriction to assume that the first equality holds. Thus $P^*(\wp(z),\wp(\beta z-z_2)) = 0$ for $z \in U$, and by analytic continuation for every z that is not a pole of either elliptic function. Here we obtain a contradiction with the algebraic independence over $\mathbb{C}$ of the elliptic functions involved, and we have proved that (22) holds for every irreducible factor P^* of P with the property that neither P_X^* nor P_Y^* are identically zero. If P_Y^* is identically zero but P_X^* is not, it follows immediately from (22) and the distinctness modulo Ω of the points su that

$$\sum_{s \in V} (N_s^*-1) \leqslant c_{43}.$$

If P_X^* is identically zero but P_Y^* is not, it likewise follows from (22) that

$$\sum_{s\in V} (N_s^*-1) \leqslant c_{44}S\ ;$$

note that the number of P^* fitting this last description is at most L_2. If we now put

$$N_s := \operatorname*{ord}_{z=su} P(\wp(z), \wp(\beta z - s\quad)) : s \in V,$$

it is immediately clear that

$$\sum_{s\in V} N_s \leqslant c_{45}L_1L_2 \leqslant c_{45}x^{16}D^6 \log A_1 \log A_2 \log^4 B' \log^{-6} E.$$

However, (10) states that $N_s \geqslant T'$ for $s \in V$ and thus

$$\sum_{s\in V} N_s \geqslant c_1^{-1}\, ST' \geqslant c_{46}^{-1}x^{17}D^6 \log A_1 \log A_2 \log^4 B' \log^{-6} E.$$

If $x > c_{45}c_{46}$, we obtain a contradiction which shows that P must be identically zero ; as the coefficients $p(\lambda_1,\lambda_2,\delta)$ are not all zero, this implies the existence of a linear dependence relation between $1, \xi, \xi^2, \ldots, \xi^{D-1}$. However, $\mathrm{dg}\, \xi = D$ and so we have obtained the final contradiction that completes the proof. ∎

Using Lemma 1, we shall now give a proof of Theorem 1. By $c_1, c_2, \ldots$ we shall denote effectively computable real numbers greater than 1 that depend only on, a and b. By C we shall denote some real number greater than 1 ; additional restrictions on the choice of C will appear below.

Suppose some triple (u,β,v) satisfies (2). First consider the case where $\wp(a) \notin \{e_1,e_2,e_3\}$. Then $\wp'(a) \neq 0$; according to § 3.3 in Chapter 4 of [1], there exists some c_1 such that $\wp$, restricted to the disk $|z-a| < c_1^{-1}$, has an analytic inverse, thus

(27) $$|z-a| \leqslant c_2\, |\wp(z) - \wp(a)|$$

for $|z-a| < c_1^{-1}$. Moreover, there exists a number c_3 such that, for every w with $|w-\wp(a)| < c_3^{-1}$, the equation $\wp(z) = w$ has exactly one root in the disk $|z-a| < c_1^{-1}$ (ibid., Theorem 11). Choose C so large that the right hand member of (2), and thereby $|\wp(a) - \wp(u)|$, is smaller than c_3^{-1} ; then there is exactly one u' with $|a-u'| < c_1^{-1}$ and $\wp(u') = \wp(u)$. By (26) we now have

(28) $$|a-u'| \leqslant c_2\, |\wp(a) - \wp(u)|.$$

In the case where $\wp(a) \in e_1, e_2, e_3$, (2) and Theorem 1.1 of [11] show that $\wp(a) = \wp(u)$ if C exceeds some c_4. Thus $\wp(u') = \wp(u)$ and (28) hold trivially if we take u' = a.

Similarly, we find v' with $\wp(v') = \wp(v)$ and

(29) $$|ab - v'| \leqslant c_5 \, |\wp(ab) - \wp(v)| .$$

Thus, combining (2), (28) and (29),

(30) $$|\beta u' - v'| \leqslant c_6 \exp(-CD^6 \log^6 H \log_2^{-5} H).$$

Let K be a compact subset of $\mathbb{C} \setminus \Omega$ containing a and ab in its interior and let c_7 be the constant from Lemma 1 corresponding to $\wp$ and K. Then (30) contradicts Lemma 1 if C is sufficiently large in terms of c_6 and c_7. ∎

3 - PROOF OF THEOREM 2

LEMMA 2. *For every* $g : \mathbb{N}^2 \to \mathbb{R}$, *there exist sequences* $(u_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, $(\epsilon_n)_{n=1}^{\infty}$, *such that for all* $n \in \mathbb{N}$ *the following statements are true :*

(31) $$u_n \in \left]\frac{1}{3}, \frac{1}{2}\right[\omega_1, \beta_n \in \left]\frac{3}{4}, \frac{3}{2}\right[\cap \mathbb{Q}, v_n = \beta_n u_n, \epsilon_n \in]0,1[,$$

$$\wp(u_n), \wp(v_n) \textit{ algebraic ;}$$

(32) $$\epsilon_{n+1} < \exp(-|g(D_n,H_n)|n), \textit{ where } D_n := [\mathbb{Q}(\wp(u_n), \wp(v_n)) : \mathbb{Q}],$$

$$H_n := \max(H(\wp(u_n), H(\beta_n), H(\wp(v_n))) ;$$

(33) $$\epsilon_{n+1} < \epsilon_n^2, \ \epsilon_{n+1} < \frac{1}{4} \operatorname{den}^{-4} \beta_n ;$$

(34) $$0 < |\beta_n - \beta_{n+1}| < \epsilon_{n+1}, \ |u_n - u_{n+1}| < \epsilon_{n+1} .$$

Proof. Define $u_1 := v_1 := \frac{2}{5}\omega_1$, $\beta_1 := 1$, $\epsilon_1 := \frac{1}{2}$. Then $\wp(u_1), \wp(v_1)$ are algebraic by Lemma 6.1 of [8] , which remains valid without complex multiplication. Now suppose $u_1, \dots, u_N, \beta_1, \dots, \beta_N v_1, \dots, v_N, \epsilon_1, \dots, \epsilon_N$ have been chosen in such a way that (31) holds for n = 1,...,N and (32), (33), (34) hold for n = 1,...,N−1, and proceed by introduction. Choose $\epsilon_{N+1} \in]0,1[$ so small that (32) and (33) hold for n = N. Take $u_{N+1} \in \left]\frac{1}{3}, \frac{1}{2}\right[\omega_1 \cap \mathbb{Q}\omega_1$

such that $|u_N - u_{N+1}| < \epsilon_{N+1}$ and $\beta_{N+1} \in]\frac{3}{4}, \frac{3}{2}[\cap \mathbb{Q}$ such that $0 < |\beta_N - \beta_{N+1}| < \epsilon_{N+1}$. Then (34) holds for n = N. Finally, put $v_{N+1} := \beta_{N+1} u_{N+1}$, then $v_{N+1} \in]\frac{1}{4}, \frac{3}{4}[\omega_1 \cap \mathbb{Q}\,\omega_1$. Again according to Lemma 6.1 of [8], we have $p(u_{N+1}), p(v_{N+1})$ algebraic ; thus (31) is satisfied for $n = N+1$. ■

LEMMA 3. *If two sequences* $(w_n)_{n=1}^{\infty}$ *of complex numbers and* $(\epsilon_n)_{n=1}^{\infty}$ *of positive real numbers satisfy* $|w_n - w_{n+1}| < \epsilon_{n+1} < \epsilon_n^2 < 1$, *then* $|w_m - w_n| < \epsilon_{n+1}^{1/2}$ *for almost all* n *and all* m > n.

Proof. From $\epsilon_{n+1} < \epsilon_n^2 < 1$ it follows that $\lim \epsilon_n = 0$. Put

$$I_k := \{ z \in \mathbb{C} : |z - w_k| < \epsilon_{k+1}^{1/2} \}.$$

As for all m we know $w_m \in I_{m-1}$, it is sufficient to prove that for almost all n and all k > n one has $I_k \subset I_{k-1}$. Take k > n and $z \in I_k$, i.e. $|z - w_k| < \epsilon_{k+1}^{1/2}$. Then

$$|z - w_{k-1}| \leqslant |z - w_k| + |w_{k-1} - w_k| < \epsilon_{k+1}^{1/2} + \epsilon_k < 2\epsilon_k < \epsilon_k^{1/2}$$

if n is sufficiently large, so that in that case $z \in I_{k-1}$. ■

Using Lemmas 2 and 3, we shall now give a proof of Theorem 2. Construct sequences $(u_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, $(\epsilon_n)_{n=1}^{\infty}$ as in Lemma 2. According to Lemma 3, $(u_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ are Cauchy sequences and their limits a, b satisfy

(35) $$\max(|a - u_n|, |b - \beta_n|) \leqslant \epsilon_{n+1}^{1/2}$$

for almost all n. From (31) it follows that $a \in [\frac{1}{3}, \frac{1}{2}]\,\omega_1$, $b \in [\frac{3}{4}, \frac{3}{2}]$; therefore $ab \in [\frac{1}{4}, \frac{3}{4}]\,\omega_1$. We conclude that neither a nor ab are poles of p, and thereby (35) implies

(36) $$\max(|p(a) - p(u_n)|, |b - \beta_n|, |p(ab) - p(v_n)|) \leqslant c\,\epsilon_{n+1}^{1/2}$$

for almost all n, where c does not depend on n. In the notation of (32), the right hand member of (36) satisfies

$$c\,\epsilon_{n+1}^{1/2} < c \exp(-1/2\, |g(D_n, H_n)| n) < \exp(-Cg(D_n, H_n))$$

if n is sufficiently large in terms of C and c. Finally, (34) implies the existence of arbitrarily large n for which $\beta_n \neq b$; as, by (33) and (35), every β_n is a convergent of the continued fraction expansion of b and $\lim \beta_n = b$, it follows that b has infinitely many convergents. Thus $b \in \mathbb{R} \setminus \mathbb{Q}$ and therefore $b \notin \mathbb{K}$. ∎

It is clear that in case of complex multiplication it makes no essential difference if we replace β_n by $\beta_n' := \beta_n \ \omega_2 / \omega_1$; thus for Theorem 1 the condition $\beta \notin \mathbb{Q}$ is not sufficient and we really need $\beta \notin \mathbb{K}$.

REFERENCES

[1] L.V. AHLFORS. *«Complex analysis»*. 2nd edition. McGraw-Hill Book co., New-York, 1966.

[2] A. BIJLSMA. *«On the simultaneous approximation of* a, b *and* a^b*»*. Compositio Math. 35 (1977), 99-111.

[3] A. BIJLSMA & P.L. CIJSOUW. *«Dependence relations of logarithms of algebraic numbers»*.

[4] W.D. BROWNAWELL & D.W. MASSER. *«Multiplicity estimates for analytic functions (I)»*.

[5] P.L. CIJSOUW & M. WALDSCHMIDT. *«Linear forms and simultaneous approximations»*. Compositio Math. 34 (1977), 173-197.

[6] N.I. FEL'DMAN. *«The periods of elliptic functions»*. (in Russian). Acta Arith. 24 (1973/74), 477-489.

[7] S. LANG. *«Elliptic curves, diophantine analysis»*. Springer-Verlag, Berlin, 1978.

[8] D.W. MASSER. *«Elliptic functions and transcendence»*. Lecture Notes in Mathematics 437, Springer-Verlag, Berlin, 1975.

[9] D.W. MASSER. *«Some recent results in transcendence theory»*. Astérisque 61 (1979), 145-154.

[10] E. REYSSAT. *«Approximation algébrique de nombres liés aux fonctions elliptiques et exponentielle»*.

[11] M. WALDSCHMIDT. *«Nombres transcendants»*. Lecture Notes in Mathematics 402. Springer-Verlag, Berlin, 1974.

[12] M. WALDSCHMIDT. *«Simultaneous approximation of numbers connected with the exponential function»*. J. Austral. Math. Soc. (Ser. A) 25 (1978), 466-478.

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