ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 5^e série, tome 1, n° 3 (1979), p. 269-285
<http://www.numdam.org/item?id=AFST_1979_5_1_3_269_0>

Annales Faculté des Sciences Toulouse
Vol I, 1979, P. 269 à 285

ON MODIFYING CONSTRUCTED NORMAL NUMBERS

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Résumé : Soient $p_1 \leqslant p_2 \leqslant \ldots$ des entiers positifs, écrits en base $g \geqslant 2$, tels que le nombre réel $\alpha = {}_g 0{,}p_1\, p_2 \ldots$ soit normal. On démontre que $\alpha^* = {}_g 0{,}p_1^*\, p_2^* \ldots$, où $p_i^* = p_i + j_i$, $j_i \in \mathbb{N}^*$, $\log j_i = o(\log p_i)$, est aussi normal, théorème qui résout un problème posé par T. ŠALAT. Parmi d'autres résultats, on donne une généralisation aux mesures arbitraires de distribution g-adique, $\varkappa$, telles que $\varkappa(\{1\}) = 0$.

Summary : If $p_1 \leqslant p_2 \leqslant \ldots$ are natural numbers, written to base $g \geqslant 2$, such that the real number $\alpha = {}_g . p_1\, p_2 \ldots$ is normal, it is proved that $\alpha^* = {}_g . p_1^*\, p_2^* \ldots$ with $p_i^* = p_i + j_i$, $j_i \in \mathbb{N}^*$, $\log j_i = o(\log p_i)$, is also normal. This theorem solves a problem proposed by T. ŠALAT. Among other results, a generalisation to arbitrary g-adic distribution measures $\varkappa$ with $\varkappa(\{1\}) = 0$ is given.

1. PROBLEM.

Let $p_1, p_2, \ldots$ be an unbounded sequence of natural numbers which diverges to infinity and let $B_1, B_2, \ldots$ be the corresponding blocks of digits relative to a fixed (integral) base $g \geqslant 2$; furthermore, we suppose that the real number [1]

$$\alpha = {}_g . B_1\, B_2 \ldots \tag{1}$$

be normal. Mr T. ŠALAT has raised the question [2] whether the number α^*, obtained in the same way but with each p_i being replaced by $p_i^* = p_i + 1$, is also normal.

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(1) The pre-index g denotes a g-adic expansion.

(2) Communicated to the author by Mr F. SCHWEIGER.

This question appears to be well motivated inasmuch as the answer is affirmative in the «classical» cases of normal numbers constructed in the form (1) :

a) For Champernowne's normal number (cf. [1]), defined with $g = 10$ and $p_i = i$, the modified number is $\alpha^* = 10\alpha - 1$, normality being obvious.

b) For the normal numbers constructed by the method of DAVENPORT and ERDÖS [3] , where $p_i = p(i)$ for any non-constant polynomial p which maps $\mathbb{N}^*$ into $\mathbb{N}^*$ ([1]), the modified number α^* satisfies the same assumptions and is therefore normal.

c) Similary, normality of α^* follows immediately if α is constructed by the method of COPELAND and ERDÖS [2] which allows (p_i) to be any sequence with more than $n^{1-\epsilon}$ elements not exceeding n for every $\epsilon > 0$ and all $n \geqslant n_0(\epsilon)$, inasmuch as again α^* satisfies the same condition.

It is the main purpose of the present paper to show that the answer to the stated question, in a slightly more general form, is always affirmative (Theorem 1). More generally, it will be shown for which distribution measures $\varkappa$, assuming that α have the digit distribution $\varkappa$, the same is always true for α^* (Theorem 4). Furthermore, we give some classes of counter-examples which show that in Theorem 1 normality cannot be replaced by simple normality. In particular, Theorem 3 describes all possible pairs of digit frequency vectors which the pair of numbers α, α^* may possess.

The work contained in this paper was done during the author's stay at Bordeaux in the spring of 1978, and he wishes to express his thanks to the U.E.R. de Mathématiques et d'Informatique for all the hospitality and cooperation received.

2. NOTATION.

2.1. - With every integer $m \in \mathbb{N}$ we associate the block $B = \beta(m)$ of its g-adic digits, written in the customary order. We consider the set $\mathscr{B}_g$ of all g-adic blocks, including the empty block, to be denoted by F_ϕ, as a semi-group with respect to juxtaposition, thus regarding β as a mapping from $\mathbb{N}$ into $\mathscr{B}_g$.

Let $\mathscr{B}_g^0$ be the subset of $\mathscr{B}_g$ obtained by omitting F_ϕ and all blocks of lengths >1 which begin with the digit 0. If $\mathscr{B}_g$ is replaced by $\mathscr{B}_g^0$, the mapping β becomes bijective.

2.2. - For any block $F \in \mathscr{B}_g$, we denote the length by $\| F \|$, defining $\| F_\phi \| = 0$.

2.3. - Blocks of the form F F ... F (n times) shall be denoted by $F^{(n)}$. If F is the single digit g–1, we shall also write $(g-1)^{(n)} = G_n$.

([1]) We denote the set of natural number by $\mathbb{N}^*$, letting $\mathbb{N}^* \cup \{0\} = \mathbb{N}$.

2.4. - If $F \neq F_\phi$ we let $\rho(F)$ and $\sigma(F)$, respectively, be the largest integers $\geqslant 0$ such that F can be written in the form $F = F'0^{(\rho)} = F''(g-1)^{(\sigma)}$ (see 2.3) F, with $F', F'' \in \mathscr{B}_g$ (where evidently either $F' = F$ or $F'' = F$).

2.5. - For any block $F \in \mathscr{B}_g$, $F \neq F_\phi$, and any real number $x \in (0,1]$ let $A_F(x,n)$ denote the number of copies of F occurring within the first n digits of the (non-terminating) g-adic expansion of x, with $A_{F_\phi}(x,n) = n$. Similary, if $B \in \mathscr{B}_g$, let $A_F(B)$ denote the number of copies of F which are embedded in B. Thus, normality of α may be expressed by means of the equations :

$$\lim_{n\to\infty} \frac{A_F(\alpha,n)}{n} = \frac{1}{g^{\|F\|}} \quad \text{for all } F \in \mathscr{B}_g. \tag{2}$$

The number α is called k-normal if (2) is satisfied for all F with $\|F\| = k$, and 1-normality is usually called simple normality.

2.6. - Any block $F \in \mathscr{B}_g$ determines the «basic interval» of numbers $x \in (0,1]$ whose g-adic expansion begins with F. For the sake of simplicity we shall denote this interval by F also, letting $F_\phi = (0,1]$.

2.7. - For any set $L \subseteq \mathbb{N}^*$ let L(n) denote the number of elements not exceeding n (n = 1,2,...).

3. MAIN RESULT.

We can now state our main result :

THEOREM 1. *Let* p_i, B_i (i = 1,2,...) *and* α *be defined as in (1), let* $j_1, j_2, \ldots, j_i \in \mathbb{N}$, be *numbers such that* $\log j_i = o(\log p_i)$, $p_i^* = p_i + j_i$, $\beta(p_i^*) = B_i^*$ (i = 1,2,...). *Then normality of* α *implies normality of the number*

$$\alpha^* = {}_g.\, B_1^*\, B_2^* \ldots \tag{3}$$

Proof. a) We define the numbers $\|B_i\| = q_i$, $\|B_i^*\| = q_i^*$, $s_i = q_1 + \ldots + q_i$, $s_i^* = q_1^* + \ldots + q_i^*$, $z_i = \|\beta(j_i)\|$ and the sets $S = \{s_1, s_2, \ldots\}$, $S^* = \{s_1^*, s_2^*, \ldots\}$, thus having

$$q_i \leqslant q_{i+1},\ q_i^* \leqslant q_{i+1}^*,\ q_i \leqslant q_i^*,\ p_i \leqslant p_i^* < p_i + g^{z_i} \quad (i = 1,2,\ldots). \tag{4}$$

Consequently, since $p_i \to \infty$, we obtain $s_{i+1} - s_i = q_{i+1} \nearrow \infty$ and $s_{i+1}^* - s_i^* \to \infty$, which implies that ([1])

$$S(n) = o(n) \text{ and } S^*(n) = o(n). \tag{5}$$

([1]) See 2.7.

b) If i is sufficiently large such that $q_i > z_i$ we rewrite B_i in the form

$$B_i = B_i' \; B_{i3}$$

where $\| B_{i3} \| = z_i$. Furthermore, let ([1]) $k_i = \sigma(B_i')$ and rewrite B_i' in the form $B_i' = B_{i1} \; B_{i2}$ with $\| B_{i2} \| = k_i$. Thus we have obtained a representation

$$B_i = B_{i1} \; B_{i2} \; B_{i3} ,$$

where B_{i1} is either empty or has a last digit different from g–1, and B_{i2} is either empty or consist of (g–1)'s only.

In this notation the transition from p_i to $p_i^* = p_i + j_i$ changes the block B_i into a block of the form

$$B_i^* = B_{i1}^* \; B_{i2}^* \; B_{i3}^*$$

with the following properties : $\| B_{i3}^* \| = \| B_{i3} \|$,

$$\beta^{-1}(B_{i3}^*) = \begin{cases} \beta^{-1}(B_{i3}) + j_i & \text{or} \\ \beta^{-1}(B_{i3}) + j_i - g^{z_i} \end{cases} \tag{6}$$

according as to whether or not the inequality

$$\beta^{-1}(B_{i3}) + j_i < g^{z_i} \tag{7}$$

holds

$$B_{i2}^* = 0^{(k_i)}, \; \| B_{i2}^* \| = \| B_{i2} \| \tag{8}$$

$$\beta^{-1}(B_{i1}^*) = \begin{cases} \beta^{-1}(B_{i1}) & \text{if (7) holds,} \\ \beta^{-1}(B_{i1}) + 1 & \text{otherwise.} \end{cases}$$

Example. g = 10, $B_i = 6089995301$, $j_i = 4711$, $k_i = 3$, $z_i = 4$, $B_i^* = 6090000012$, $B_{i1} = 608$, $B_{i2} = 999$, $B_{i3} = 5301$, $B_{i1}^* = 609$, $B_{i2}^* = 000$, $B_{i3}^* = 0012$.

c) If the n-th digit of α occurs within the block B_i, i.e. if $s_{i-1} < n \leqslant s_i$, then the «corresponding» digit of α^* has the index n' for which $s_i^* - n' = s_i - n$. In particular, we have $n' \geqslant n$ for all n, and the image set $\mathbb{N}^{*'}$ of $\mathbb{N}^*$ under this mapping has density one, its complement being contained in the set $\{s_1^* + 1, s_2^* + 1, \ldots\}$.

([1]) See 2.4.

d) The representation

$$\alpha = {}_g . B_{11}\, B_{12}\, B_{13}\, B_{21}\, B_{22}\, B_{23} \cdots$$

induces an obvious decomposition of the set $\mathbb{N}^*$ into three disjoint sets I_1, I_2 and I_3, where I_k consists of those indices whose position is occupied by one of the blocks B_{k1}, B_{k2}, ... (k = 1,2,3). We need the following two lemmas.

LEMMA 1. $$I_3(n) = o(n).$$

Proof. From the definitions of q_i and j_i we have :

$$q_i - 1 \leqslant \frac{\log p_i}{\log g} < q_i \quad \text{and} \quad z_i - 1 \leqslant \frac{\log j_i}{\log g} < z_i \quad (i = 1,2,\ldots).$$

Hence, if n is sufficiently large and $s_{i-1} < n \leqslant s_i$, it follows that

$$I_3(n) \leqslant \sum_{h=1}^{i} z_h \leqslant \sum_{h=1}^{i} \left(\frac{\log j_h}{\log g} + 1\right) = o\left(\sum_{h=1}^{i} \left(\frac{\log p_h}{\log g} + 1\right)\right) = o\left(\sum_{h=1}^{i} q_h + S(n)\right) = o(n)$$

where the relation (5) has been used.

LEMMA 2. $$I_2(n) = o(n).$$

Proof. If $\epsilon > 0$ is given, we choose an integer $\ell > 0$ such that $\frac{\ell}{g^\ell} < \epsilon$. Let $I_2^\ell \subseteq I_2$ be the set of those n which are associated with one of the last ℓ digits in some block B_{i2}. Then I_2^ℓ consists of chains of lengths $\leqslant \ell$ of consecutive integers each of which lies strictly between two consecutive elements of S. Hence, for any n

$$I_2^\ell(n) \leqslant \ell(S(n) + 1)$$

and thus, by (5),

(10) $$I_2^\ell(n) = o(n).$$

Let $I_{2\ell} = I_2 \setminus I_2^\ell$, then every $m \in I_{2\ell}$ is an index at which in the expansion (1) a copy of the block ([1]) G_ℓ begins. Therefore, it follows from (2) that

(11) $$I_{2\ell}(n) \leqslant A_{G\ell}(n+\ell-1) \leqslant \frac{\ell n}{g^\ell} + \epsilon n < 2\epsilon n$$

for all sufficiently large n.

([1]) See 2.3.

The assertion is implied by (10) and (11) since

$$I_2(n) = I_2^{\ell}(n) + I_{2\ell}(n).$$

e) In order to establish normality of α^* by means of the equations analogous to (2) it suffices to let n tend to infinity through the elements of $\mathbb{N}^{*'}$ (see step (c)). We consider a fixed block $F \neq F_\phi$ and an index $n' \in \mathbb{N}^{*'}$, and we subdivide the $A_F(\alpha^*,n')$ copies of F among the first n' digits of α^* into three subsets :

1) Let $A_F^1(\alpha^*,n)$ be the number of copies occurring within, but not at the end of some block B_{i1}^*. By virtue of (9), these copies correspond to copies of F in the blocks B_{i1}. Hence

$$A_F^1(\alpha^*,n') \leqslant A_F(\alpha,n) \leqslant A_F(\alpha,n'). \tag{12}$$

2) The number $A_F^2(\alpha^*,n')$ of copies of F which occur at the end of some block B_{i1}^* (for these copies there is no corresponding copy of F in the expansion (1)) is bounded by $S^*(n') + 1$, there being at most one such copy in each block B_i^*. Hence, by (5),

$$A_F^2(\alpha^*,n') = o(n'). \tag{13}$$

3) Each of the remaining copies of F overlaps with at least one of the blocks B_{i2}^* or B_{i3}^*. Hence, their number is

$$A_F^3(\alpha^*,n') \leqslant \| F \| (I_2(n') + I_3(n'))$$

and thus it follows from Lemmas 1 and 2 that

$$A_F^3(\alpha^*,n') = o(n'). \tag{14}$$

By combining (12), (13) and (14) we obtain

$$A_F(\alpha^*,n') = A_F^1(\alpha^*,n) + A_F^2(\alpha^*,n) + A_F^3(\alpha^*,n) \leqslant A_F(\alpha,n') + o(n')$$

and therefore, by (2), we have

$$\overline{\lim_{n' \to \infty}} \frac{A_F(\alpha^*,n')}{n'} \leqslant \lim_{n' \to \infty} \frac{A_F(\alpha,n')}{n'} = \frac{1}{g^{\| F \|}}. \tag{15}$$

Adding these inequalities for all blocks F of the same length q and observing that

$$\lim_{n' \to \infty} \frac{1}{n'} \sum_{\| F \| = q} A_F(\alpha^*,n') = 1$$

we find that equality must hold in (15) for every F. A simple argument shows now that the same equations are valid for the lower limits, and thus the assertion follows.

4. REMARKS.

4.1. - The assertion we have proved ceases to hold if normality is replaced by simple normality. For example, let $g \geqslant 2$, $B_i = (g-2)^{(i)}\ (g-3)^{(i)} \ldots 0^{(i)}\ (g-1)^{(i)}$ $(i = 1,2,\ldots)$, $j_1 = 1$ for all i. Then the integers $p_i = \beta^{-1}(B_i)$ are uniquely determined, satisfying $p_1 < p_2 < \ldots$. Inasmuch as every block B_i contains i zeros, i ones etc.., the number $\alpha = {}_g\,.\, B_1\, B_2 \ldots$ is simply normal to base g. But the number $\alpha^* = {}_g\,.\, B_1^*\, B_2^* \ldots$ involves the blocks

$$B_i^* = \beta(p_i+1) = (g-2)^{(i)}\ (g-3)^{(i)} \ldots 0^{(i-1)}1\ 0^{(i)} \quad (i = 1,2,\ldots).$$

A simple calculation shows that

$$\lim_{n\to\infty} \frac{A_0(\alpha^*,n)}{n} = \frac{2}{g} \quad \text{and} \quad \lim_{n\to\infty} \frac{A_{g-1}(\alpha^*,n)}{n} = 0.$$

Hence, the number α^* fails to be simply normal.

4.2. - The example just given may be altered in such a way that the number α does not even possess an asymptotic distribution, as we shall demonstrate in the case g = 2. Indeed, let

$$\gamma = {}_2\,.\, f_1\, f_2 \ldots = {}_2\,.\, 0^{(1!)}\, 1^{(2!)}\, 0^{(3!)}\, 1^{(4!)} \ldots, \tag{16}$$

and define blocks B_i $(i = 1,2,\ldots)$ as

$$B_i = \begin{cases} 1^{(i)}\, 0^{(i)} & \text{if } f_i = 0, \\ 10^{(i)}\, 1^{(i-1)} & \text{if } f_i = 1. \end{cases}$$

Writing again $p_i = \beta^{-1}(B_i)$, the numbers p_i are uniquely determined (as each block B_i begins with the digit 1), and they form a monotonic sequence since $\| B_1 \| < \| B_2 \| < \ldots$. Hence, the number $\alpha = {}_2\,.\, B_1\, B_2 \ldots$ satisfies the conditions of Theorem 1, except that instead of being normal, it is simply normal only.

For the modified number $\alpha^* = {}_2\,.\, B_1^*\, B_2^* \ldots$ one has

$$B_i^* = \begin{cases} 1^{(i)}\, 0^{(i-1)}1 & \text{if } f_i = 0, \\ 10^{(i-1)}\, 10^{(i-1)} & \text{if } f_i = 1. \end{cases} \tag{17}$$

Now, let $\sum_{k=1}^{j} k! = n_j$ $(j=1,2,\ldots)$ and

$$\| B_1 \ldots B_{n_j} \| = \| B_1^* \ldots B_{n_j}^* \| = m_j \quad (j=1,2,\ldots).$$

These parameters satisfy the asymptotic equations

$$n_j \cong j! \tag{18}$$

and

$$m_j = 2 \sum_{i=1}^{n_j} i = n_j(n_j + 1) \cong n_j^2 \cong j!^2. \tag{19}$$

It will be convenient to rewrite the number α^* in the form

$$\alpha^* = {}_2 \cdot Q_1 Q_2 \ldots,$$

where

$$Q_1 = B_1^*, \; Q_k = B^*_{n_{k-1}+1} \ldots B^*_{n_k} \quad (k=2,3,\ldots). \tag{20}$$

Each of the blocks $f_1 f_2 \ldots f_{n_{2j}}$ in (16) ends with the sub-block $1^{(2j)}$; hence, the corresponding block in the expansion of α^*, consisting of the first m_{2j} digits, ends with a sub-block Q_{2j} in which all the B_i^* satisfy the second alternative of (17), thus having $A_1(B_i^*) = 2$.

Therefore,

$$A_1(\alpha^*, m_{2j}) \leqslant A_1(\alpha^*, m_{2j-1}) + A_1(Q_{2j}) \leqslant m_{2j-1} + 2(n_{2j} - n_{2j-1})$$

and thus

$$\frac{A_1(\alpha^*, m_{2j})}{m_{2j}} \leqslant \frac{m_{2j-1}}{m_{2j}} + \frac{2n_{2j}}{m_{2j}}.$$

Consequently, using (18) and (19), we obtain

$$\lim_{m \to \infty} \frac{A_1(\alpha^*, m)}{m} \leqslant \lim_{j \to \infty} \frac{A_1(\alpha^*, m_{2j})}{m_{2j}} = 0.$$

On the other hand, a similar argument shows that

$$A_1(\alpha^*, m_{2j-1}) \geqslant A_1(Q_{2j-1}) > \sum_{i=n_{2j-2}+1}^{n_{2j-1}} i \cong \frac{1}{2} n_{2j-1}(n_{2j-1} + 1) > \frac{1}{2} n_{2j-1}^2 \cong \frac{1}{2}(2j-1)!^2,$$

where the first alternative of (17) has been applied.

Therefore,

$$\overline{\lim_{m \to \infty}} \frac{A_1(\alpha^*, m)}{m} \geqslant \overline{\lim_{j \to \infty}} \frac{A_1(\alpha^*, m_{2j-1})}{m_{2j-1}} \geqslant \overline{\lim_{m \to \infty}} \frac{(2j-1)!^2}{2(2j-1)!^2} = \frac{1}{2}.$$

Since the opposite inequality is a trivial consequence of (17), we have

$$\overline{\lim_{m \to \infty}} \frac{A_1(\alpha^*, m)}{m} = \frac{1}{2} > \lim_{m \to \infty} \frac{A_1(\alpha^*, m)}{m} = 0,$$

which shows that no limit frequencies for the digits of α^* exists.

4.3. - Another modification of the example given in 4.1. demonstrates that the number α can be so chosen that it remains simply normal to base 2 but α^* has given lower and upper asymptotic digit frequencies, subjet only to the condition

$$\overline{\lim_{m\to\infty}} \frac{A_0(\alpha^*,m)}{m} \geqslant \frac{1}{2}$$

which is prompted by the fact that α is simply normal and the asymptotic frequency of zeros cannot be decreased by the transition from α to α^*. We restate this proposition as the following theorem.

THEOREM 2. *Given real numbers η and ζ satisfying*

$$\frac{1}{2} \leqq \eta < \zeta \leqq 1,$$

there exists a sequence $p_i \to \infty$ of natural numbers such that the number α constructed to base 2 in the sense of (1) is simply normal but, letting [1] *$j_i = 1$ for all i, the number α^* satisfies the equations*

$$\underline{\lim_{m\to\infty}} \frac{A_0(\alpha^*,m)}{m} = \eta\,, \quad \overline{\lim_{m\to\infty}} \frac{A_0(\alpha^*,m)}{m} = \zeta\,.$$

Proof. We consider the same number γ as defined in (16) but we change the definition of the blocks B_i as follows : for $j = 1,2,\ldots$ we let

$$v_j = \max\left\{1, [\frac{j}{2}]\right\},\ w_j = \max\left\{1, [(\zeta - \frac{1}{2})j]\right\},\ u_j = \max\left\{1, j-v_j-w_j\right\},$$
$$y_j = \max\left\{1, [(\eta - \frac{1}{2})j]\right\},\ x_j = \max\left\{1, j-v_j-y_j\right\},$$

and if $n_{j-1} < i \leqslant n_j$, we define

$$B_i = \begin{cases} 1^{(u_j)}\,0^{(v_j)}\,1^{(w_j)} & \text{if } f_i = 0, \\ 1^{(x_j)}\,0^{(v_j)}\,1^{(y_j)} & \text{if } f_i = 1. \end{cases}$$

Then the integers $p_i = \beta^{-1}(B_i)$ are easily seen to satisfy the condition $p_i \to \infty$ and the number $\alpha = {}_2 . B_1 B_2 \ldots$ is simply normal since

$$A_0(B_i) = v_j \cong \frac{1}{2} \| B_i \|,\ A_1(B_i) \cong \frac{1}{2} \| B_i \|.$$

The modified blocks B_i^* are

(21)
$$B_i^* = \begin{cases} 1^{(u_j)}\,0^{(v_j-1)}\,10^{(w_j)} & \text{if } f_i = 0, \\ 1^{(x_j)}\,0^{(v_j-1)}\,10^{(y_j)} & \text{if } f_i = 1,\ i > 1. \end{cases}$$

([1]) The theorem is also true for arbitrary sequences j_i with $\log j_i = 0\ (\log p_i)$ but the proof would be more complicated.

Using again the notation (20) we find by a straight-forward calculation that the lower limit $\underline{\lim}_{m\to\infty} \frac{A_0(\alpha^*,m)}{m}$ is obtained as m runs through the ending positions of the blocks Q_{2j} (j = 1,2,...). These values of m correspond to the end-points of long chains of blocks B_i^* for which the first alternative of (21) is valid, and since

$$A_0(B_i^*) = v_j + w_j - 1 \cong \zeta \parallel B_i^* \parallel ,$$

in this case, the result is

$$\underline{\lim}_{m\to\infty} \frac{A_0(\alpha^*,m)}{m} = \zeta .$$

Similary, using the second alternative of (21) and the blocks Q_{2j-1}, thus having

$$A_0(B_i^*) = v_j + y_j - 1 \cong \eta \parallel B_i^* \parallel ,$$

one obtains

$$\underline{\lim}_{m\to\infty} \frac{A_0(\alpha^*,m)}{m} = \eta .$$

4.4. - The result just proved can be generalised to an arbitrary base g. Instead of doing this we shall replace the condition that the number α be simply normal by requiring that α and α^* have given digit frequencies.

THEOREM 3. *Let* $g \geqslant 2$ *be an integer and let* g*-adic digit frequency vectors* ([1]) $\zeta = (\zeta_0,\zeta_1, \ldots, \zeta_{g-1})$, $\zeta^* = (\zeta_0^*, \zeta_1^*, \ldots, \zeta_{g-1}^*)$ *be given. Then there exists a sequence* $p_1 \leqslant p_2 \leqslant \ldots$ *such that, letting* ([2]) $j_i = 1$ *for all* i *the numbers* α *and* α^* *in the sense of Theorem 1 have digit frequencies* ζ *and* ζ^*, *if and only if* ([3])

$$\zeta_0 \leqslant \zeta_0^*, \tag{22}$$

$$\zeta_k = \zeta_k^* \quad (k = 1,\ldots, g-2). \tag{23}$$

COROLLARY. *The equations (23) imply the relation*

$$\zeta_0 + \zeta_{g-1} = \zeta_0^* + \zeta_{g-1}^* . \tag{24}$$

Proof. a.1. - Assume that numbers α and α^* whith the stated properties exist. In order to compare $A_0(B_i^*)$ with $A_0(B_i)$ (i = 1,2,...) we distinguish (in the notation of section 3) the following five cases ([4]) :

([1]) I.e. real numbers satisfying $0 \leqslant \zeta_k \leqslant 1$, $0 \leqslant \zeta_k^* \leqslant 1$, $\sum_{k=0}^{g-1} \zeta_k = \sum_{k=0}^{g-1} \zeta_k^* = 1$.

([2]) The theorem is also true for arbitrary sequences j_i with $\log j_i = 0$ ($\log p_i$) but the proof would be more complicated.

([3]) It should be observed that, for g = 2, (24) is trivial and (23) is void.

([4]) Observing that in case 3 one has $\sigma(B_i) + \rho(B_{i1}) \leqslant \parallel B_i \parallel$, we find that this list of cases is complete and non-overlapping. For g = 2, cases 1 and 4 do not occur.

1) $\sigma(B_i) = 0$, $\rho(B_i) = 0$ (example : 19805)

2) $\sigma(B_i) = 0$, $\rho(B_i) > 0$ (example : 19800)

3) $\sigma(B_i) > 0$, $\rho(B_{i1}) > 0$ (example : 900999 ; $B_{i1} = 900$)

4) $\| B_i \| > \sigma(B_i) > 0$, $\rho(B_{i1}) = 0$ (example : 108999, $B_{i1} = 108$)

5) $\| B_i \| = \sigma(B_i)$ (example : 99999).

Computing the blocks $B_i^* = \beta(\beta^{-1}(B_i) + 1)$ we obtain

$$(25) \qquad A_0(B_i^*) = \begin{cases} A_0(B_i) & \text{in case 1,} \\ A_0(B_i) - 1 & \text{in case 2,} \\ A_0(B_i) + \sigma(B_i) - 1 & \text{in case 3,} \\ A_0(B_i) + \sigma(B_i) & \text{in case 4,} \\ \| B_i \| > 0 = A_0(B_i) & \text{in case 5.} \end{cases}$$

Furthermore, we find

$$(26) \qquad \| B_i^* \| = \begin{cases} \| B_i \| & \text{in case 1,2,3,4} \\ \| B_i \| + 1 & \text{in case 5.} \end{cases}$$

A simple argument, using the relation $\| B_i^* \| = q_i^* \to \infty$ (see (4)) to handle the terms -1 in case 2, now shows that

$$\zeta_0^* = \lim_{n \to \infty} \frac{A_0(\alpha^*, n)}{n} = \lim_{n \to \infty} \frac{A_0(\alpha^*, s_i)}{s_i^*} \geqslant \lim_{i \to \infty} \frac{A_0(\alpha, s_i)}{s_i} = \zeta_0.$$

Hence, condition (22) is necessary.

a.2. - To prove the necessity of (23) we may assume that $g > 2$. Let k be a fixed digit with $1 \leqslant k \leqslant g-2$. Then, considering again the five cases introduced in a.1), we find always that $A_k(B_i^*) - A_k(B_i) = 0$ or ± 1, from which the equations (23) can be deduced.

b. - Conversely, let us assume that vectors ζ and ζ^* with the stated properties be given. Then we construct a pair of numbers α, α^* by modifying the construction of Theorem 2 as follows : we now introduce, for $j = 1,2,...$, the functions

$$v_{hj} = [\zeta_{hj}] \ (h = 1,...,g-2), \ v_{oj} = \max \{1, [\zeta_{oj}]\}$$
$$w_j = \max \{1, [(\zeta_0^* - \zeta_0)j]\},$$
$$u_j = j - \sum_{h=0}^{g-2} v_{hj} - w_j,$$

noting that, in view of (24),

$$u_j \cong (1 - \zeta_0 - \zeta_1 - \ldots - \zeta_{g-2} + \zeta_0 - \zeta_0^*)j = (\zeta_{g-1} + \zeta_0 - \zeta_0^*)j \tag{27}$$
$$= \zeta_{g-1}^* j .$$

If $n_{j-1} < i \leqslant n_j$, we define the block B_i as

$$B_i = 1^{(v_{1j})}\, 2^{(v_{2j})} \ldots (g-2)^{(v_{g-2j})}\, (g-1)^{(u_j)}\, 0^{(v_{oj})}\, (g-1)^{(w_j)},$$

letting again $\alpha = {}_g . B_1 B_2 \ldots$, $B_i^* = \beta(\beta^{-1}(B_i) + 1)$, and

$$\alpha^* = {}_g . B_1^* B_2^* \ldots$$

It is easy to show that the number α has the digit frequency vector ζ, using the relations $\| B_i \| = j \to \infty$, $v_{hj} \cong \zeta_h j$ $(h = 0,\ldots,g-2)$ and $u_j + w_j \cong (\zeta_{g-1}^* + \zeta_0^* - \zeta_0)j = \zeta_{g-1} j$, the last one being obtained by means of (24).

Similary, inasmuch as

$$B_i^* = 1^{(v_{1j})}\, 2^{(v_{2j})} \ldots (g-2)^{(v_{g-2j})}\, (g-1)^{(n_j)}\, 0^{(v_{oj}-1)}\, 1\, 0^{(w_j)} \quad (i=1,2,\ldots),$$

an obvious computation, using (27) and the relation

$$v_{oj} + w_j \cong \zeta_0^* j,$$

proves that the number α^* has the given digit frequency vector ζ^*.

5. GENERALISATION

Theorem 1 may be interpreted as stating that if the number α has Lebesgue measure λ as its g-adic distribution, the same is true for α^*. Hence, one may ask whether there are distribution measures other than λ having the same invariance property. This question is answered by the following proposition.

THEOREM 4. *Let $\varkappa$ be a g-adic distribution measure. Then any number α, constructed as in Theorem 1 and having the g-adic distribution* (1) *$\varkappa$, will yield a number α^* with the same distribution if and only if* (2) *$\varkappa(\{1\}) = 0$, i.e. if the point 1 is not an atom of $\varkappa$.*

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(1) I.e., for every block F, $\lim_{n\to\infty} \dfrac{A_F(\alpha,n)}{n} = \varkappa(F)$ in the sense of 2.6.

(2) It should be noted that this condition is violated by the examples discussed in section 4 whenever α^* and α have different distributions.

Proof. a) Let $\varkappa$ be a given g-adic distribution measure, i.e. a. Borel probability measure on [0,1] which is invariant under the g-adic shift operator $T_g\, x = \{g\, x\}$, satisfying the assumption $\varkappa(\{1\}) = 0$.

Parts a), b), c) and d) of the proof of theorem 1 remain valid, except that in the proof of Lemma 2 the following change is now necessary. Since ([1]) the intersection of the basic intervals G_ℓ is

$$\bigcap_{\ell=1}^{\infty} G_\ell = \{1\},$$

the assumption $\varkappa(\{1\}) = 0$ is equivalent to

$$\lim_{\ell \to \infty} \varkappa(G_\ell) = 0.$$

Hence, if $\epsilon > 0$ is given, there exists an ℓ such that $\varkappa(G_\ell) < \epsilon$. The remainder of the proof is unchanged, except that the inequality (11) has to be replaced by

$$I_{2\ell}(n) \leqslant A_{G_\ell}(n+\ell-1) \leqslant \varkappa(G_\ell)n + \epsilon\, n < 2\epsilon n.$$

The fact that α^* has distribution $\varkappa$ is then established by the argument of part e) in the proof of Theorem 1, with the term $\dfrac{1}{g^{\|F\|}}$ in (15) being replaced by $\varkappa(F)$.

b) Conversely, let $\varkappa$ be a g-adic distribution measure with $\varkappa(\{1\}) = \eta > 0$. We shall show that there exists a sequence p_i with $p_i \nearrow \infty$ such that, in the sense of the theorem, α has the distribution $\varkappa$ but α^* does not.

b.1. - As the first step we consider a number α_1 with the distribution $\varkappa$ as constructed ([2]) by J. VILLE [6], following the exposition of A.G. POSTNIKOV [5], Chapter III. Given any sequence $\epsilon_1 > \epsilon_2 > \dots$, $\epsilon_k \to 0$, this construction defines recursively blocks $L_0, L_1, \dots$, in such a way that, letting

$$\alpha_1 = {}_g.\, L_0 L_1 L_2 \dots$$

and $\| L_0 L_1 \dots L_k \| = n_k$, any block $F \neq F_\phi$, $\| F \| \leqslant k$, satisfies the inequalities

$$(28) \qquad \left| \frac{A_F(\alpha_1, n)}{n} - \varkappa(F) \right| < \epsilon_k + \frac{k-1}{n_k} < 2\epsilon_k$$

for all $n \geqslant n_k$. The length $\| L_k \|$ may, at each stage of the construction, be chosen arbitrarily large in relation to the $\| L_{k-1} \|$. Hence, we may choose each n_{k+1} (k= 1,2,...) so large that, for all F with $\| F \| \leqslant k$,

$$(29) \qquad \left| \frac{A_F(L_{k+1})}{\| L_{k+1} \|} - \varkappa(F) \right| < 3\, \epsilon_k.$$

([1]) See 2.3 and 2.6.

([2]) An elegant construction of such numbers, using graph theory, was recently given by H. KÜHNLE [4].

For the same reason we may furthermore assume that

$$k / \| L_{k+1} \| \quad (k = 1,2,\ldots), \tag{30}$$

$$\| L_k \| \uparrow \infty \text{ and } k^2 = o(\| L_k \|). \tag{31}$$

Furthermore, we require that the first digit of L_0 be different from 0.

b.2. - By (29) we have

$$A_{G_k}(L_{k+1}) > (\varkappa(G_k) - 3\epsilon_k) \| L_{k+1} \| =: R(k) \qquad (k = 0,1,2,\ldots). \tag{32}$$

Hence, an application of Dirichlet's drawer principle shows that for each k there exists an arithmetic progression $\{ r_k, r_k+k, r_k+2k, \ldots \}$, $1 \leqslant r_k \leqslant k$, which contains more than $\frac{1}{k} R(k)$ numbers n such that some copy of G_k ends at the n-th digit of the block L_{k+1}.

Let L'_{k+1} be the block obtained from L_{k+1} $(k = 0,1,2,\ldots)$ by a cyclic shift by $k-r_k$ places to the right. In view of (30) it is possible to rewrite each block L'_{k+1} $(k = 1,2,\ldots)$ in the form

$$L'_{k+1} = Y'_{k1} Y'_{k2} \cdots Y'_{k\,h_k}, \tag{33}$$

where $\| Y'_{k1} \| = \ldots = \| Y'_{k\,h_k} \| = k$ and at least $\frac{1}{k} R(k)$ of these blocks are equal to G_k, to be called relevant copies.

Now, let a number α_2 be defined as

$$\alpha_2 = g \cdot L'_0 L'_1 \ldots \qquad (L'_0 = L_0).$$

It is easy to see that α_2 also has the digit distribution $\varkappa$. Indeed, if $\mathscr{U}$ denotes the set of places occupied in the g-adic expansion of α_1 by the first and the last digit of L_1, the first two and the last two digits of L_2, etc, then it is evident that, if a block F with $\| F \| = k > 0$ is given, the transition from α_1 to α_2 can only create or destroy copies of F to the right of the index n_k if they begin or terminate at places belonging to $\mathscr{U}$. Hence, we have, for all $n \geqslant n_k$,

$$| A_F(\alpha_1,n) - A_F(\alpha_2,n) | \leqslant 2\mathscr{U}(n) + n_k,$$

and since, by (31), $\mathscr{U}(n) = o(n)$, it follows that

$$\lim_{n\to\infty} \frac{A_F(\alpha_2,n)}{n} = \lim_{n\to\infty} \frac{A_F(\alpha_1,n)}{n} = \varkappa(F),$$

i.e. α_2 has the distribution $\varkappa$.

b.3. - The blocks L'_{k+1} $(k = 1,2,...)$ are changed as follows : we define blocks

$$Y''_{ki} = \begin{cases} 1\,Y \text{ if } Y'_{ki} \text{ is of the form } O\,Y, \\ Y'_{ki} \text{ otherwise,} \end{cases}$$

thus replacing (33) by

$$L''_{k+1} = Y''_{k1} \ldots Y''_{k\,h_k} \qquad (k = 1,2,...) \tag{34}$$

and then letting $\alpha_3 = {}_g\,.\, L''_2 L''_3 L''_4 \ldots$.

The first digits of all blocks Y'_{ki} determine a set $V = \{v_1, v_2, \ldots\}$ of indices for which $v_{m+1} - v_m = k$ whenever the index v_m occurs within the block L'_k. Hence, by (31), $V(m) = o(m)$ and thus α_3 again has the digit distribution $\varkappa$. Furthermore, it follows from the definition of L''_{k+1} that

$$A_{g-1}(L''_{k+1}) \leqslant A_{g-1}(L'_{k+1}) + \frac{1}{k} \| L'_{k+1} \| \quad (k = 1,2,...), \tag{35}$$

(with $A_{g-1}(L''_{k+1}) = A_{g-1}(L'_{k+1})$ if $g > 2$).

b.4. - Finally, we renumber the blocks Y''_{ki} $(k = 1,2,\ldots\ ;\ i=1,\ldots,h_k)$ without changing order, as $B_1, B_2, \ldots$, thus rewriting α_3 as $\alpha = {}_g\,.\, B_1\, B_2 \ldots$. Then the number α satisfies the assumptions of the theorem since : 1) it has the distribution $\varkappa$, 2) none of the blocks B_i begins with 0, hence the integers $p_i = \beta^{-1}(B_i)$ are uniquely determined, 3) $p_i \nearrow \infty$ as $\| B_i \| \nearrow \infty$. Furthermore, each relevant copy of a block G_k $(k = 1,2,...)$ forms one of the blocks B_i. We shall also use the fact that the assumption $\log j_i = o(\log p_i)$ implies the relation

$$\| \beta(j_i) \| =: z_i = o(\| B_i \|). \tag{36}$$

b.5. - For the corresponding number $\alpha^* = {}_g\,.\, B_1^* B_2^* \ldots$ we also use the block de composition

$$\alpha^* = {}_g\,.\, \tilde{L}_2 \tilde{L}_3 \ldots,$$

where $\tilde{L}_k$ is obtained from L'_k by subjecting each of its blocks B_i to the * operation of Theorem 1. Each relevant copy of a block G_k which forms a block B_i with $k \geqslant z_i$, furnishes a block (see (6)) of the form $B_i^* = 1\,0^{(k-z_i)}\,B_{i3}$, $\| B_{i3} \| = z_i$, where B_{i3} is defined as in the proof of Theorem 1. We shall show that α^* has «much fewer» $(g-1)$'s than α. Indeed, letting $k - z_i - 1 = k_1$, we have, if B_i is embedded in L''_{k+1} with $k = k(i) \geqslant z_i$,

$$A_{g-1}(B_i^*) \leqslant A_{g-1}(B_i) - k_1. \tag{37}$$

Hence, applying (37) to each of the relevant copies of G_k and observing that they are contained in L'_{k+1} (and therefore, in L''_{k+1}), we have, for $k = 1,2,...$,

$$A_{g-1}(\tilde{L}_{k+1}) \leqslant A_{g-1}(L''_{k+1}) - \frac{k_1}{k} R(k)$$
$$\leqslant A_{g-1}(L'_{k+1}) + \frac{1}{k} \| L'_{k+1} \| - \frac{k_1}{k} R(k) \quad \text{by (35).}$$

Since it follows from the definition of L'_{k+1} that

$$| A_{g-1}(L'_{k+1}) - A_{g-1}(L_{k+1}) | < k \quad \text{and} \quad \| L_{k+1} \| = \| L_{k+1} \|,$$

this implies

$$A_{g-1}(\tilde{L}_{k+1}) < A_{g-1}(L_{k+1}) + k + \frac{1}{k} \| L_{k+1} \| - \frac{k_1}{k} R(k)$$
$$\leqslant (\varkappa(g-1) + 3\,\epsilon_k + \frac{1}{k}) \| L_{k+1} \| + k - \frac{k_1}{k} R(k) \quad \text{by (29), which is}$$
$$= \left\{ \varkappa(g-1) + 3\,\epsilon_k + \frac{1}{k} - \frac{k_1}{k} (\varkappa(G_k) - 3\,\epsilon_k) \right\} \| L_{k+1} \| + k \quad \text{by (32).}$$

Letting k tend to infinity, observing that $\epsilon_k \to 0$, $0 < \eta = \lim_{k \to \infty} \varkappa(G_k) \leqslant \varkappa(G_1)$, $k_1 \cong k$ by (36), $k = o(\| L_{k+1} \|)$ by (31) and $\| \tilde{L}_{k+1} \| \geqslant \| L_{k+1} \|$ by definition, we obtain the relation

$$\overline{\lim_{k \to \infty}} \frac{A_{g-1}(L_{k+1})}{\| \tilde{L}_{k+1} \|} \leqslant \varkappa(g-1) - \eta .$$

Therefore, $\overline{\lim_{n \to \infty}} \dfrac{A_{g-1}(\alpha^*, n)}{n} \leqslant \varkappa(g-1) - \eta < \varkappa(g-1)$, and thus α^* does not have the distribution $\varkappa$.

Remark. Theorem 4 may be applied to show that, in the sense of Theorem 2, for any $k \geqslant 1$ there exists even a k-normal number α such that α^* fails to be simply normal. It suffices to consider any g-adic distribution measure $\varkappa$ such that $\varkappa(F) = \dfrac{1}{g^{\| F \|}}$ for all F with $\| F \| = k$, but with $\varkappa(\{1\}) > 0$, and to construct a pair of numbers α, α^* by the method of Theorem 4. For example, if $g = 2$, $k = 2$, we can choose

$$\varkappa = \frac{1}{4}(\delta_0 + \delta_{\frac{1}{2}} + \delta_{\frac{2}{3}} + \delta_1),$$

where δ_x is the Dirac measure concentrated at x. It is easily seen that the measure $\varkappa$ is invariant under the binary shift operator T_2, and that any corresponding number α is 2-normal but α^* is not.

REFERENCES

[1] D.G. CHAMPERNOWNE. *«The construction of decimals normal in the scale of ten»*. Journ. London Math. Soc. 8 (1933), 254-260.

[2] A.H. COPELAND and P. ERDÖS. *«Note on normal numbers»*. Bull. Amer. Math. Soc. 52 (1946), 857-860.

[3] H. DAVENPORT and P. ERDÖS. *«Note on normal decimals»*. Canad. Journ. Math. 4 (1952), 58-63.

[4] H. KÜHNLE. *«Schnell konvergierende Ziffernfolgen bei speziellen zahlentheoretischen Transformationen»*. Diss. Univ. Stuttgart 1978.

[5] A.G. POSTNIKOV. *«Ergodic problems in the theory of congruences and of diophantine approximations»*. (Russian). Trudy Mat. Inst. Stekov 82 (1966). Engl. transl. Proc. Stekov Inst. Math. 82, Amer. Math. Soc. 1967.

[6] J. VILLE. *«Etude critique de la notion de collectif»*. Monogr. des Probabilités, 3, Gauthiers-Villars, Paris 1939.

(Manuscrit reçu le 19 mai 1979)