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CANNON-THURSTON MAPS, I-BOUNDED GEOMETRY AND A THEOREM OF MCMULLEN

Mahan Mj

ABSTRACT. — The notion of *i-bounded geometry* generalises simultaneously *bounded geometry* and the geometry of punctured torus Kleinian groups. We show that the limit set of a surface Kleinian group of *i-bounded geometry* is locally connected by constructing a natural Cannon-Thurston map.

1. Introduction

In [20] we prove the existence of Cannon-Thurston maps for arbitrary surface Kleinian groups without accidental parabolics. The proof proceeds by constructing a coarse model geometry, called *split geometry*, satisfied by all associated hyperbolic 3-manifolds. Our starting point in [20] is a model geometry constructed by Minsky in [14] and we proceed by forgetting some of the finer structure in [14] to establish that all surface Kleinian groups have associated hyperbolic 3-manifolds of split geometry. In [6], [22, 5] and [19] we completed the programme of proving the existence of Cannon-Thurston maps for arbitrary finitely generated Kleinian groups and describing point pre-images in terms of ending laminations.

The purpose of the present paper is to give an exposition of the existence of Cannon-Thurston maps for surface Kleinian groups without accidental parabolics satisfying a more restrictive model geometry called **i-bounded geometry** satisfied for instance by all punctured torus Kleinian groups. This gives a new proof of a result of McMullen [13].

The main pre-requisites for understanding the present paper are:

- (1) Generalities on hyperbolic metric spaces in the sense of Gromov [9] [4], [8], especially boundary theory in terms of asymptote classes of geodesics.

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- (2) The theory of simply and doubly degenerate Kleinian groups (Chapter 8 of [25])
- (3) Relative hyperbolicity and electric geometry [7, 9, 1].

A similar exposition in the special case of bounded geometry surface Kleinian groups had been given by the author in [24]. In [21] we give an exposition of more general model geometries leading up to split geometry used in [20].

1.1. Statement of Results

The main theorem of this paper is:

THEOREM 1.1 (7.6). — *Let M^h be a hyperbolic 3 manifold of **i -bounded geometry** homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$), where S^h is a hyperbolic surface of finite area. Let $i : S^h \rightarrow M^h$ be a type-preserving (i.e. taking parabolics to parabolics) homotopy equivalence. Then the inclusion $\tilde{i} : \widetilde{S^h} \rightarrow \widetilde{M^h}$ extends continuously to a map $\hat{i} : \widehat{S^h} \rightarrow \widehat{M^h}$. Hence the limit set of $\widetilde{S^h}$ is locally connected.*

The notion of *i -bounded geometry* generalises simultaneously *bounded geometry* and the geometry of punctured torus Kleinian groups. In particular, since punctured torus groups have *i -bounded geometry* by a result of Minsky [16], we have a new proof of the following Theorem of McMullen [13] as a consequence:

THEOREM 1.2 (McMullen [13]). — *Let M^h be a hyperbolic 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$), where S^h is a punctured torus. Let $i : S^h \rightarrow M^h$ be a type-preserving (i.e. taking parabolics to parabolics) homotopy equivalence. Then the inclusion $\tilde{i} : \widetilde{S^h} \rightarrow \widetilde{M^h}$ extends continuously to a map $\hat{i} : \widehat{S^h} \rightarrow \widehat{M^h}$. Hence the limit set of $\widetilde{S^h}$ is locally connected.*

i -bounded geometry can roughly be described as bounded geometry away from Margulis tubes. But this description is a little ambiguous. More precisely, we start with a collection of (uniformly) bounded geometry blocks $S \times I$ glued end to end. Next, for some blocks a curve is selected such that its representative on the lower end of the block has (uniformly) bounded length. Hyperbolic Dehn surgery is then performed along the geodesic representative within the block. Precise definitions will be given in Section 2.2.

We describe below a collection of examples of manifolds of *i -bounded geometry* for which Theorem 7.6 is known:

- (1) The cover corresponding to the fiber subgroup of a closed hyperbolic 3-manifold fibering over the circle (Cannon and Thurston [3]).
- (2) Hyperbolic 3 manifolds of bounded geometry, which correspond to simply or doubly degenerate Kleinian groups isomorphic to closed surface groups (Minsky [15]). (See also Section 4.3 of [18].)
- (3) Hyperbolic 3 manifolds of bounded geometry, arising from simply or doubly degenerate Kleinian groups corresponding to punctured surface groups (Bowditch [2]). (See also [23])
- (4) Punctured torus Kleinian groups (McMullen [13]).

1.2. Cannon-Thurston Maps and i -bounded geometry

Let S be a hyperbolic surface of finite area and let

$$\rho(\pi_1(S)) = H \subset PSL_2(\mathbb{C}) = \text{Isom}(\mathbb{H}^3)$$

be a representation, such that the quotient hyperbolic 3-manifold $M = \mathbb{H}^3/H$ is simply degenerate. Let \tilde{S} and \tilde{M} denote the universal covers of S and M respectively. Then \tilde{S} and \tilde{M} can be identified with \mathbb{H}^2 and \mathbb{H}^3 respectively. There exists a natural inclusion $i : \tilde{S} \rightarrow \tilde{M}$. Now let $\mathbb{D}^2 = \mathbb{H}^2 \cup \mathbb{S}_\infty^1$ and $\mathbb{D}^3 = \mathbb{H}^3 \cup \mathbb{S}_\infty^2$ denote the standard compactifications. The local connectivity of the limit set of \tilde{S} is equivalent to the existence of a continuous extension (a **Cannon-Thurston map**) $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$.

A word about the term **i -bounded geometry**. In the construction of a general model manifold (Section 9 of [14]), as a step towards the resolution of the Ending Lamination Conjecture, Minsky describes certain (complex) meridian coefficients which encode the complex structure for boundary torii of Margulis tubes. The uniform boundedness of these coefficients corresponds to *bounded geometry*. The manifolds that we discuss in this paper correspond to those which have a uniform bound on the **imaginary** part of these coefficients. Hence the term *i -bounded geometry*. Clearly, manifolds of *bounded geometry* have *i -bounded geometry*. In [16], Minsky further showed that punctured torus groups (and four-holed sphere groups) have *i -bounded geometry*. Roughly speaking, the number of twists gives the real part and the number of vertical annuli gives the imaginary part of the coefficients. Hence, in a manifold of *i -bounded geometry*, an arbitrarily large number of twists are allowed for each Margulis tube, but only a uniformly bounded number of vertical annuli.

As in [17, 18] and [23], our proof proceeds by constructing a *ladder-like* set $B_\lambda \subset \tilde{M}$ from a geodesic segment $\lambda \subset \tilde{S}$ and then a retraction Π_λ of

\widetilde{M} onto B_λ . We modify this construction in this paper and restrict our attention to one block, i.e. a copy of $\widetilde{S} \times I$ minus certain neighborhoods of geodesics and cusps and equip it with a model *pseudometric* which is zero along lifts of a simple closed geodesic.

To prevent cluttering, we restrict ourselves to closed surfaces first, and then indicate the modifications necessary for punctured surfaces.

2. Preliminaries

2.1. Hyperbolic Metric Spaces

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [9, 4, 8]. Let (X, d) be a hyperbolic metric space. The **Gromov boundary** of X , denoted by ∂X , is the collection of asymptote classes of geodesic rays.

A subset Z of X is said to be **k -quasiconvex** if any geodesic joining points of Z lies in a k -neighborhood of Z . A subset Z is **quasiconvex** if it is k -quasiconvex for some k .

A map f from one metric space (Y, d_Y) into another metric space (Z, d_Z) is said to be a **(K, ϵ) -quasi-isometric embedding** if

$$\frac{1}{K}(d_Y(y_1, y_2)) - \epsilon \leq d_Z(f(y_1), f(y_2)) \leq Kd_Y(y_1, y_2) + \epsilon$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$ then f is said to be a **quasi-isometry**. A (K, ϵ) -quasi-isometric embedding that is a quasi-isometry will be called a (K, ϵ) -quasi-isometry.

A **(K, ϵ) -quasigeodesic** is a (K, ϵ) -quasi-isometric embedding of a closed interval in \mathbb{R} . A (K, K) -quasigeodesic will also be called a K -quasigeodesic.

Let (X, d_X) be a hyperbolic metric space and Y be a subspace that is hyperbolic with the inherited path metric d_Y . By adjoining the Gromov boundaries ∂X and ∂Y to X and Y , one obtains their compactifications \widehat{X} and \widehat{Y} respectively.

Let $i : Y \rightarrow X$ denote inclusion.

DEFINITION 2.1. — *Let X and Y be hyperbolic metric spaces and $i : Y \rightarrow X$ be an embedding. A **Cannon-Thurston map** \hat{i} from \widehat{Y} to \widehat{X} is a continuous extension of i .*

The following lemma (Lemma 2.1 of [17]) says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if λ

lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X . For convenience of use later on, we state this somewhat differently.

LEMMA 2.2. — *A Cannon-Thurston map from \widehat{Y} to \widehat{X} exists if the following condition is satisfied:*

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ and for all geodesic segments λ lying outside an N -ball around $y_0 \in Y$ any geodesic segment in Γ_G joining the end-points of $i(\lambda)$ lies outside the $M(N)$ -ball around $i(y_0) \in X$.

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X .

2.2. i -bounded Geometry

We start with a hyperbolic surface S^h with or without punctures. The hyperbolic structure is arbitrary, but it is important that a choice be made. S will denote S^h minus a small enough neighborhood of the cusps.

Fix a *finite* collection \mathcal{C} of (geodesic representatives of) simple closed curves on S . $N_\epsilon(\sigma)$ will denote the ϵ -neighborhood of a geodesic $\sigma \in \mathcal{C}$.

$N_\epsilon(\sigma_i)$ will denote an ϵ neighborhood of $\sigma_i \subset S^h$ for some $\sigma_i \in \mathcal{C}$. ϵ and the neighborhood of the cusps in S^h are chosen small enough so that

- (1) $N_\epsilon(\sigma_i)$ is at least a distance of ϵ from the cusps.
- (2) No two lifts of $N_\epsilon(\sigma_i)$ to the universal cover $\widetilde{S^h}$ intersect.

Note that $S = S^h$ if S has no cusps. Restrict the metric on S^h to S and equip S with the resultant *path-metric*.

The Thin Building Block

For the construction of a thin block, I will denote the closed interval $[0, 3]$. Now put a product metric structure on $S \times I$, which restricts to the path-metric on S for each slice $S \times a, a \in I$ and the Euclidean metric on the I -factor. Let B_i^c denote $(S \times I - N_\epsilon(\sigma_i)) \times [1, 2]$. Equip B_i^c with the path-metric.

For each resultant torus component of the boundary of B_i^c , perform Dehn filling on some $(1, n_i)$ curve, which goes n_i times around the meridian and once round the longitude. n_i will be called the **twist coefficient**. The

metric on the solid torus Θ_i glued in is arranged in such a way that it is isometric to the quotient of a neighborhood of a bi-infinite hyperbolic geodesic by a hyperbolic isometry. Further, the $(1, n_i)$ -curve is required to bound a totally geodesic hyperbolic disk. In fact, we might as well foliate the boundary of Θ_i by translates (under hyperbolic isometries) of the meridian, and demand that each bounds a totally geodesic disk. Since there is no canonical way to smooth out the resulting metric, we leave it as such. Θ_i equipped with this metric will be called a **Margulis tube** in keeping with the analogy from hyperbolic space.

The resulting copy of $S \times I$ obtained, equipped with the metric just described, is called a **thin building block** and is denoted by B_i .

Thick Block

Fix constants D, ϵ and let $\mu = [p, q]$ be an ϵ -thick Teichmüller geodesic of length less than D . μ is ϵ -thick means that for any $x \in \mu$ and any closed geodesic η in the hyperbolic surface S_x^h over x , the length of η is greater than ϵ . Now let B^h denote the universal curve over μ reparametrized such that the length of μ is covered in unit time. Let B denote B^h minus a neighborhood of the cusps. Thus $B = S \times [0, 1]$ topologically.

A small enough neighborhood of the cusps of S^h is fixed. $S^h \times \{x\}, x \in [0, 1]$ is given the hyperbolic structure S_x^h corresponding to the point at distance $x d_{Teich}(p, q)$ from p along μ (d_{Teich} denotes Teichmüller metric). A neighborhood of the cusps of S^h having been fixed, we remove the images under the Teichmüller map (from S_0^h to S_x^h) of this neighborhood (having first fixed a neighborhood of the cusps of S_0^h as the image under the Teichmüller map from S^h).

The resultant manifold B (possibly with boundary) is given the path metric and is called a **thick building block**.

Note that after acting by an element of the mapping class group, we might as well assume that μ lies in some given compact region of Teichmüller space. This is because the marking on $S \times \{0\}$ is not important, but rather its position relative to $S \times \{1\}$. Further, since we shall be constructing models only upto quasi-isometry, we might as well assume that $S^h \times \{0\}$ and $S^h \times \{1\}$ lie in the orbit under the mapping class group of some fixed base surface. Hence μ can be further simplified to be a Teichmüller geodesic joining a pair (p, q) amongst a finite set of points in the orbit of a fixed hyperbolic surface S^h .

The Model Manifold

Note that the boundary of a thin block B_i consists of $S \times \{0, 3\}$ and the intrinsic path metric on each such $S \times \{0\}$ or $S \times \{3\}$ is equivalent to the path metric on S . Also, the boundary of a thick block B consists of $S \times \{0, 1\}$, where S_0^h, S_1^h lie in some given bounded region of Teichmuller space. The intrinsic path metrics on each such $S \times \{0\}$ or $S \times \{1\}$ is the path metric on S .

The model manifold of **i-bounded geometry** is obtained from $S \times J$ (where J is a sub-interval of \mathbb{R} , which may be semi-infinite or bi-infinite. In the former case, we choose the usual normalisation $J = [0, \infty)$) by first choosing a sequence of blocks B_i (thick or thin) and corresponding intervals $I_i = [0, 3]$ or $[0, 1]$ according as B_i is thick or thin. The metric on $S \times I_i$ is then declared to be that on the building block B_i . Thus we have,

DEFINITION 2.3. — A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to be a model of **i-bounded geometry** if

- (1) there is a fiber preserving homeomorphism from M to $\tilde{S} \times J$ that lifts to a quasi-isometry of universal covers
- (2) there exists a sequence I_i of intervals (with disjoint interiors) and blocks B_i where the metric on $S \times I_i$ is the same as that on some building block B_i
- (3) $\bigcup_i I_i = J$

The figure below illustrates schematically what the model looks like. Filled squares correspond to torii along which hyperbolic Dehn surgery is performed. The blocks which have no filled squares are the *thick blocks* and those with filled squares are the *thin blocks*.

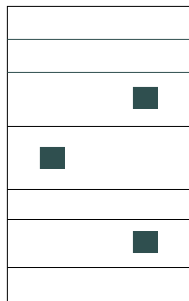


Figure 2.1. Model of *i*-bounded geometry (schematic)

DEFINITION 2.4. — *A manifold M homeomorphic to $S \times J$, where $J = [0, \infty)$ or $J = (-\infty, \infty)$, is said to have **i-bounded geometry** if there exists $K, \epsilon > 0$ such that the universal cover \widetilde{M} is K, ϵ quasi-isometric to a model manifold of i -bounded geometry.*

The Punctured Torus

In [16], Minsky constructs a model manifold for arbitrary punctured torus groups that motivates the above definitions. For him, S^h is the square punctured torus. \mathcal{C} consists of precisely two shortest curves a, b of equal length on S^h . \mathcal{C}_i is the singleton set $\{a\}$ for i even and the set $\{b\}$ for i odd. The numbers n corresponding to the surgery coefficients correspond to the number of Dehn twists performed about the i th curve. Thus, we see from Minsky's construction of the model manifold for punctured torus groups that all punctured torus groups give rise to manifolds of *i -bounded geometry*

Alternate Description of i -bounded geometry

We could weaken the definition of thin blocks in models of i -bounded geometry by requiring that a family \mathcal{C} of disjoint simple closed curves (rather than a single simple closed curve) are modified by hyperbolic Dehn surgery. This gives rise to an equivalent definition.

To see this, add on as many blocks (indexed by j) of $S \times I$ as there are curves in \mathcal{C} (this number is bounded in terms of the genus of S). Then isotope Margulis tubes to different levels by a bi-Lipschitz map away from the tubes. The universal covers of the original $S \times I$ and the new $S \times \bigcup_j I_j$ are quasi-isometric.

Hence, it does not multiply examples to allow a family \mathcal{C} rather than a single curve.

3. Relative Hyperbolicity

In this section, we shall recall first certain notions of relative hyperbolicity due to Farb [7]. Using these, we shall derive certain Lemmas that will be useful in studying the geometry of the universal covers of building blocks.

3.1. Electric Geometry

Given some σ , we construct a pseudometric, on S by defining

- the length of any path that lies along σ to be zero,
- the length of any path $[a, b]$ that misses all such geodesics in its interior (a, b) to be the hyperbolic length, and
- the length of any other path to be the sum of lengths of pieces of the above two kinds.

This allows us to define distances by taking the infimum of lengths of paths joining pairs of points and gives us a path pseudometric, which we call the **electric metric**. The electric metric also allows us to define geodesics. Let us call S equipped with the above pseudometric S_{el} .

We shall be interested in the universal cover \widetilde{S}_{el} of S_{el} . Paths in S_{el} and \widetilde{S}_{el} will be called electric paths (following Farb [7]). Geodesics and quasigeodesics in the electric metric will be called electric geodesics and electric quasigeodesics respectively.

DEFINITION 3.1. — γ is said to be an electric K, ϵ -quasigeodesic in \widetilde{S}_{el} **without backtracking** if γ is an electric K -quasigeodesic in \widetilde{S}_{el} and γ does not return to any any lift $N_\epsilon(\tilde{\sigma}) \subset \widetilde{S}_{el}$ of $N_\epsilon(\sigma)$ after leaving it.

A hyperbolic geodesic λ may follow a lift $\tilde{\sigma}$ for a long time *without/after/before/before and after* intersecting it. This is why in the definition of quasigeodesics without backtracking, we take $N_\epsilon(\tilde{\sigma})$ rather than $\tilde{\sigma}$ itself.

A similar definition can be given in the case of manifolds with cusps. Here electrocuted sets correspond to horodisks (lifts of cusps). More generally, we can consider X to be a convex subset of \mathbb{H}^n and \mathcal{H} to be a collection of uniformly separated horoballs in X based on points of ∂X (i.e. they are the intersection with X of certain horoballs in \mathbb{H}^n whose boundary point lies in ∂X). We present below two basic Lemmas due to Farb [7] in the general setup of hyperbolic metric spaces. Their specializations for \widetilde{S}_{el} are also indicated.

DEFINITION 3.2. — Let X be a convex subset of \mathbb{H}^n and \mathcal{H} a collection of uniformly separated horoballs H . Let Y be the complement $X - \{H : H \in \mathcal{H}\}$. γ is said to be an ambient K -quasigeodesic (resp. a K -quasigeodesic in X) **without backtracking** if

- γ is an ambient K -quasigeodesic in Y (resp. a K -quasigeodesic in X)
- γ is obtained from a K -quasigeodesic in X by replacing each maximal subsegment with end-points on a horosphere by a quasigeodesic

lying on the surface of the horosphere (resp. γ does not return to any horoball \mathbf{H} after leaving it).

We collect together certain facts about the electric metric that Farb proves in [7]. $N_R(Z)$ will denote the R -neighborhood about the subset Z in the hyperbolic metric. $N_R^e(Z)$ will denote the R -neighborhood about the subset Z in the electric metric.

LEMMA 3.3 (Lemma 4.5 and Proposition 4.6 of [7]). —

- (1) *Electric quasi-geodesics electrically track hyperbolic geodesics:* Given $P > 0$, there exists $K > 0$ with the following property: For some \tilde{S}_i , let β be any electric P -quasigeodesic without backtracking from x to y , and let γ be the hyperbolic geodesic from x to y . Then $\beta \subset N_K^e(\gamma)$.
- (2) *Hyperbolicity:* There exists δ such that each \tilde{S}_{e_i} is δ -hyperbolic, independent of the curve $\sigma \in \mathcal{C}$ whose lifts are electrocuted.

Note. — In [7], Farb proves the above Lemma for horospheres rather than geodesics. There is however, an easy trick to convert geodesics to horocycles as follows:

For each geodesic σ_i , construct a geometrically finite hyperbolic 3-manifold M_i , in which the curve σ_i is an (accidental) parabolic, i.e. one boundary component of M_i corresponds to the surface S^h and on the other component the curve become parabolic. Removing horoball neighborhoods of cusps, we get a manifold quasi-isometric to S (the quasi-isometry is uniform, i.e. independent of i , since only finitely many choices are possible for \mathcal{C}_i). The electric pseudo-metric obtained by putting the zero metric on horospheres is easily seen to be quasi-isometric to the metric on S_i . As a consequence, all of Farb's results go through with very minor modifications.

An alternate approach is to follow Lemmas 4.4 through 4.10 of [7] with geodesics replacing horospheres, and exactly the same arguments apply. However, we have to state the results a little carefully, to avoid some obvious errors. This is particularly necessary for the Lemmas 4.8 through 4.10 of [7], where the corresponding notion of “bounded penetration” for geodesics is a little different.

Farb gives more precise information regarding the tracking properties mentioned in Lemma 3.7 above. But before we go into this, we want to make the notion of *penetration* more precise. Hyperbolic geodesics may come arbitrarily close to an electrocuted geodesic without meeting it. However, from the first part of Lemma 3.7, electric geodesics lie electrically close to the hyperbolic geodesic. This is where we need a neighborhood of the

electrocuted geodesic rather than the geodesic itself. Recall that we chose an ϵ such that the ϵ -neighborhoods of the electrocuted geodesics are ϵ -separated. This allows us to give the following modified versions of Lemmas 4.8 - 4.10 of [7].

LEMMA 3.4. — *Let β be an electric P -quasigeodesic without backtracking and γ a hyperbolic geodesic, both joining x, y in some \widetilde{S}_{el} . Then, given ϵ there exists $D = D(P, \epsilon)$ such that*

- (1) *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an ϵ -neighborhood $N_\epsilon(\tilde{\sigma})$ of an electrocuted geodesic $\tilde{\sigma}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(\tilde{\sigma})$) from the first (entry) point to the last (exit) point is at most D .*
- (2) *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(\tilde{\sigma})$ then the length (measured in the intrinsic path-metric on $N_\epsilon(\tilde{\sigma})$) from the entry point of β to that of γ is at most D ; similarly for exit points.*

In fact, much of what Farb proved in [7] goes through under considerably weaker assumptions. In [7] the theorems were proven in the particular context of a pair (X, \mathcal{H}) , where X is a Hadamard space of pinched negative curvature with the interiors of a family of horoballs \mathcal{H} removed. Then \mathcal{H} can be regarded as a collection of horospheres in X separated by a minimum distance from each other. In this situation, X is not a hyperbolic metric space itself, but is hyperbolic relative to a collection of separated horospheres. Alternately, X may be regarded as hyperbolic space itself, and \mathcal{H} as a separated collection of horoballs. Note that this gives an equivalent description, and moreover, one that is easier to formulate abstractly.

Let X be a hyperbolic metric space and \mathcal{H} a collection of (uniformly) C -quasiconvex uniformly separated subsets, i.e. there exists $D > 0$ such that for $H_1, H_2 \in \mathcal{H}$, $d_X(H_1, H_2) \geq D$. In this situation X is hyperbolic relative to the collection \mathcal{H} (see [1]).

However, the property of *Bounded Horosphere Penetration (BHP)* or *Bounded Coset Penetration (BCP)* [7] was not abstracted out in Klarreich's proof as it was not necessary. What is essential for BCP (or BHP) to go through has been abstracted out by Bowditch [2, 1] in the case that the collection \mathcal{H} is a collection of geodesics or horocycles in a Farey graph. But though these things are available at the level of folklore, an explicit statement seems to be lacking. The crucial condition can be isolated as per the following definition:

DEFINITION 3.5. — A collection \mathcal{H} of uniformly C -quasiconvex sets in a δ -hyperbolic metric space X is said to be **mutually D -cobounded** if for all $H_i, H_j \in \mathcal{H}$, $\pi_i(H_j)$ has diameter less than D , where π_i denotes a nearest point projection of X onto H_i . A collection is **mutually cobounded** if it is mutually D -cobounded for some D .

Mutual coboundedness was proven for horoballs by Farb in Lemma 4.7 of [7] and by Bowditch in stating that the projection of the link of a vertex onto another [1] has bounded diameter in the link. However, the comparability of intersection patterns in this context needs to be stated a bit more carefully. We give the general version of Farb's theorem below and refer to [7, 1] and Klarreich [12] for proofs.

LEMMA 3.6 (See Lemma 4.5 and Proposition 4.6 of [7]). — Given δ, C, D there exists Δ such that if X is a δ -hyperbolic metric space with a collection \mathcal{H} of C -quasiconvex D -separated sets. then,

- (1) *Electric quasi-geodesics electrically track hyperbolic geodesics:* Given $P > 0$, there exists $K > 0$ with the following property: Let β be any electric P -quasigeodesic from x to y , and let γ be the hyperbolic geodesic from x to y . Then $\beta \subset N_K^e(\gamma)$.
- (2) γ lies in a hyperbolic K -neighborhood of $N_0(\beta)$, where $N_0(\beta)$ denotes the zero neighborhood of β in the electric metric.
- (3) *Hyperbolicity:* X is Δ -hyperbolic.

We shall have need to use Lemma 3.6 in the special case that $X = \widetilde{S}$ and where the electric metric on \widetilde{S}_{el} is obtained as at the beginning of this subsection.

- LEMMA 3.7. —
- (1) Given $P > 0$, there exists $K > 0$ with the following property: For some \widetilde{S}_i , let β be any electric P -quasigeodesic without backtracking from x to y , and let γ be the hyperbolic geodesic from x to y . Then $\beta \subset N_K^e(\gamma)$.
 - (2) There exists δ such that each \widetilde{S}_{el} is δ -hyperbolic, independent of the curve $\sigma \in \mathcal{C}$ whose lifts are electrocuted.

We shall need to give a general definition of geodesics and quasigeodesics without backtracking.

DEFINITION 3.8. — Given a collection \mathcal{H} of C -quasiconvex, D -separated sets and a number ϵ we shall say that a geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic) **without backtracking** with respect to ϵ neighborhoods if γ does not return to $N_\epsilon(H)$ after leaving it, for any $H \in \mathcal{H}$. A geodesic (resp. quasigeodesic) γ is a geodesic (resp. quasigeodesic)

without backtracking if it is a geodesic (resp. quasigeodesic) without backtracking with respect to ϵ neighborhoods for some $\epsilon \geq 0$.

Note. — For strictly convex sets, $\epsilon = 0$ suffices, whereas for convex sets any $\epsilon > 0$ is enough.

Item (2) in Lemma 3.6 is due to Klarreich [12], where the proof is given for β an electric geodesic, but the same proof goes through for electric quasigeodesics.

Note. — For Lemma 3.6, the hypothesis is that \mathcal{H} consists of uniformly quasiconvex, mutually separated sets. Mutual coboundedness has not yet been used. We introduce *co-boundedness* in the next lemma.

LEMMA 3.9. — *Suppose X is a δ -hyperbolic metric space with a collection \mathcal{H} of C -quasiconvex K -separated D -mutually cobounded subsets. There exists $\epsilon_0 = \epsilon_0(C, K, D, \delta)$ such that the following holds:*

Let β be an electric P -quasigeodesic without backtracking and γ a hyperbolic geodesic, both joining x, y . Then, given $\epsilon \geq \epsilon_0$ there exists $D = D(P, \epsilon)$ such that

- (1) *Similar Intersection Patterns 1: if precisely one of $\{\beta, \gamma\}$ meets an ϵ -neighborhood $N_\epsilon(H_1)$ of an electrocuted quasiconvex set $H_1 \in \mathcal{H}$, then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point to the exit point is at most D .*
- (2) *Similar Intersection Patterns 2: if both $\{\beta, \gamma\}$ meet some $N_\epsilon(H_1)$ then the length (measured in the intrinsic path-metric on $N_\epsilon(H_1)$) from the entry point of β to that of γ is at most D ; similarly for exit points.*

We summarise the two Lemmas 3.6 and 3.9 in forms that we shall use:

- If X is a hyperbolic metric space and \mathcal{H} a collection of uniformly quasiconvex separated subsets, then X is hyperbolic relative to the collection \mathcal{H} .
- If X is a hyperbolic metric space and \mathcal{H} a collection of uniformly quasiconvex mutually cobounded separated subsets, then X is hyperbolic relative to the collection \mathcal{H} and satisfies *Bounded Penetration*, i.e. hyperbolic geodesics and electric quasigeodesics have similar intersection patterns in the sense of Lemma 3.9.

The relevance of co-boundedness comes from the following Lemma which is essentially due to Farb [7].

LEMMA 3.10. — *Let M^h be a hyperbolic manifold of i -bounded geometry, with Margulis tubes $T_i \in \mathcal{T}$ and horoballs $H_j \in \mathcal{H}$. Then the lifts \widetilde{T}_i and \widetilde{H}_j are mutually co-bounded.*

The proof given in [7] is for a collection of separated horospheres, but the same proof works for neighborhoods of geodesics and horospheres as well.

A closely related theorem was proved by McMullen (Theorem 8.1 of [13]).

As usual, $N_R(Z)$ will denote the R -neighborhood of the set Z .

Let \mathcal{H} be a locally finite collection of horoballs in a convex subset X of \mathbb{H}^n (where the intersection of a horoball, which meets ∂X in a point, with X is called a horoball in X).

DEFINITION 3.11. — *The ϵ -neighborhood of a bi-infinite geodesic in \mathbb{H}^n will be called a **thickened geodesic**.*

THEOREM 3.12 ([13]). — *Let $\gamma : I \rightarrow X \setminus \bigcup \mathcal{H}$ be an ambient K -quasigeodesic (for X a convex subset of \mathbb{H}^n) and let \mathcal{H} denote a uniformly separated collection of horoballs and thickened geodesics. Let η be the hyperbolic geodesic with the same endpoints as γ . Let $\mathcal{H}(\eta)$ be the union of all the horoballs and thickened geodesics in \mathcal{H} meeting η . Then $\eta \cup \mathcal{H}(\eta)$ is (uniformly) quasiconvex and $\gamma(I) \subset B_R(\eta \cup \mathcal{H}(\eta))$, where R depends only on K .*

As in Lemmas 3.6 and 3.9, this theorem goes through for *mutually cobounded separated uniformly quasiconvex sets* **H**.

A special kind of *geodesic without backtracking* will be necessary for universal covers of surfaces with some electric metric.

Let λ_e be an electric geodesic in some (\widetilde{S}, d_e) for \widetilde{S} equipped with some electric metric obtained by electrocuting a collection of *mutually cobounded separated geodesics*. Then, each segment of λ_e between electrocuted geodesics is perpendicular to the electrocuted geodesics that it starts and ends at. We shall refer to these segments of λ_e as **complementary segments** because they lie in the complement of the electrocuted geodesics. Let a_η, b_η be the points at which λ_e enters and leaves the electrocuted (bi-infinite) geodesic η . Let $[a, b]_\eta$ denote the geodesic segment contained in η joining a, b . Segments like $[a, b]_\eta$ shall be referred to as **interpolating segments**. The union of the *complementary segments* along with the *interpolating segments* gives rise to a preferred representative of geodesics joining the end-points of λ_e ; in fact it is the unique quasigeodesic without backtracking whose underlying set represents an electric geodesic joining the end-points of λ_e . Such a representative of the class of λ_e shall be called the **canonical**

representative of λ_e . Further, the underlying set of the canonical representative in the *hyperbolic metric* shall be called the **electro-ambient representative** λ_q of λ_e . Since λ_q will turn out to be a hyperbolic quasigeodesic, we shall also call it an **electro-ambient quasigeodesic**. See Figure 3.1 below:

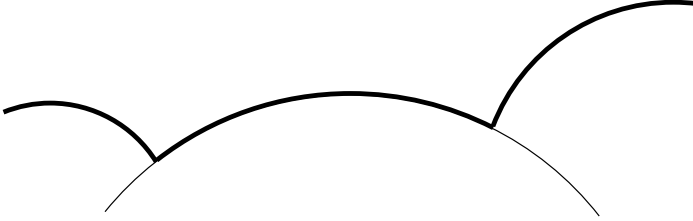


Figure 3.1. *Electro-ambient quasigeodesic*

Now, let λ_h denote the hyperbolic geodesic joining the end-points of λ_e . By Lemma 3.9, λ_h and λ_e , and hence λ_h and λ_q have similar intersection patterns with $N_\epsilon(\eta)$ for electrocutted geodesics η . Also, λ_h and λ_q track each other off $N_\epsilon(\eta)$. Further, each *interpolating segment* of λ_q being a *hyperbolic* geodesic, it follows (from the ‘ K -fellow-traveller’ property of hyperbolic geodesics starting and ending near each other) that each interpolating segment of λ_q lies within a $(K + 2\epsilon)$ neighborhood of λ_h . Again, since each segment of λ_q that does not meet an electrocutted geodesic that λ_h meets is of uniformly bounded (by C say) length, we have finally that λ_q lies within a $(K + C + 2\epsilon)$ neighborhood of λ_h . Finally, since λ_q is an electro-ambient representative, it does not backtrack. Hence we have the following:

LEMMA 3.13. — *There exists (K, ϵ) such that each electro-ambient representative of an electric geodesic is a (K, ϵ) hyperbolic quasigeodesic.*

In the above form, *electro-ambient quasigeodesics* are considered only in the context of surfaces and closed geodesics on them. This can be generalised considerably. Let X be a δ -hyperbolic metric space, and \mathcal{H} a family of C -quasiconvex, D -separated, k -cobounded collection of subsets. Then by Lemma 3.6, X_{el} obtained by electrocutting the subsets in \mathcal{H} is a $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. Now, let $\alpha = [a, b]$ be a hyperbolic geodesic in X and β be an electric P -quasigeodesic without backtracking joining a, b . Replace each maximal subsegment (with end-points p, q , say) of β lying within some $H \in \mathcal{H}$ by a hyperbolic *interpolating geodesic* $[p, q]$. The resulting **connected** path β_q is called an *electro-ambient quasigeodesic* in X . The following Lemmas justify the terminology.

LEMMA 3.14. — *Given δ, D, C, k, P as above, there exists C_3 such that the following holds: Let α, β be as above. Then α lies in a C_3 neighborhood of β_q*

Proof. — The proof idea is similar to that for surfaces and geodesics.

By Lemma 3.6, item (2), there exists C_0 such that α lies in a (hyperbolic) C_0 -neighborhood of $N_0(\beta_q)$. Further, by *bounded penetration* following from co-boundedness, there exists C_1 such that if some interpolating geodesic $[p, q]$ in H is of length greater than C_1 , then there exist $p_1, q_1 \in H \cap \alpha$ such that

$$d(p, p_1) \leq C_1$$

$$d(q, q_1) \leq C_1$$

$$d(p_1, q_1) \text{ is maximal over all pairs } u, v \in H \cap \alpha$$

Hence, by the fellow traveller property, there exists C_2 such that the hyperbolic geodesic $[p_1, q_1] \subset \alpha$ lies in a C_2 -neighborhood of $[p, q]$ and hence β_q .

Now, if $x \in \alpha$, x lies in a C_0 neighborhood of $N_0(\beta_q)$. Let $y \in N_0(\beta_q)$ be the point nearest to x . If y lies on $\beta - \mathcal{H}$, then $d(x, \beta_q) \leq C_0$. Else, y lies on some H . Two cases arise:

Case 1. — β and hence β_q do not penetrate H for more than C_1 . In this case, there exists $y \in \beta - \mathcal{H}$, such that $d(x, y) \leq C_0 + C_1$.

Case 2. — β and hence β_q do penetrate H for more than C_1 and therefore an interpolating geodesic $[p, q]$ of length greater than C_1 exists. Hence there exists a maximal subsegment of α within a C_2 neighborhood of $[p, q]$. From this it follows easily that x lies in a C_2 neighborhood of β_q .

Thus α lies in a (uniformly) bounded C_3 -neighborhood of β_q . (Here, $C_3 = C_0 + C_1 + C_2$ suffices). \square

In fact, more is true. β_q is a hyperbolic quasigeodesic. But to see this needs a bit more work. For the sake of concreteness, and to simplify the exposition, we assume that X is a complete simply connected manifold of pinched negative curvature. Let π_α denote the nearest point retraction onto α . Since β_q is connected, joins the end-points of α and π_α is continuous, $\pi_\alpha(\beta_q) = \alpha$.

CLAIM 3.1. — *There exists $D > 0$ such that any two points u, v with $\pi_\alpha(u) = \pi_\alpha(v) = w$ satisfy $d(u, v) \leq D$*

Proof of Claim. — The loop that goes from w to u by a hyperbolic path of length less than C_3 (from Lemma 3.14), then from u to v along β_q and

then back to w by a hyperbolic path of less than C_3 is a C_4 -quasigeodesic (for some uniform C_4). It can also be converted into a path without backtracking. Clearly, the geodesic joining the end-points of the path (a loop) has length zero. By Lemma 3.9 the path penetrates each $H \in \mathcal{H}$ by a uniformly bounded amount C_5 . Hence, there exists a uniform constant C_6 depending on C_5 and D , such that the loop has (hyperbolic) length less than C_6 . The Claim follows. \square

In the same way, it follows that given D_0 there exists D_1 , such that if $d(\pi_\alpha(u), \pi_\alpha(v)) \leq D_0$ then $d(u, v) \leq D_1$.

From the Claim above, it follows that β_q also must lie in a bounded neighborhood of α (else there will have to be long detours along β_q starting and ending at a distance less than $2C_3$ from each other). Further, β_q cannot have long pieces starting and ending close to each other for the same reason. Thus β_q lying in a bounded neighborhood of α must “progress”. In other words β_q must be a hyperbolic quasigeodesic. We state this formally below:

LEMMA 3.15. — *There exist K, ϵ depending on δ, D, C, k, P , such that β_q is a (K, ϵ) -quasigeodesic.*

In our proof of Lemma 3.14, we have used the hypothesis that the collection \mathcal{H} of qc sets is a mutually cobounded collection. However, this hypothesis can be relaxed. The proof is exactly the same as Klarreich’s proof of Proposition 4.3 of [12], which has been stated here as Item (2) in Lemma 3.6 above. We state this below and refer to Proposition 4.3 of [12] for the relevant details.

LEMMA 3.16. — *Given δ, C, D, P there exists C_3 such that the following holds.*

Let (X, d) be a δ -hyperbolic metric space and \mathcal{H} a family of C -quasiconvex, D -separated collection of quasiconvex subsets. Let (X, d_e) denote the electric space obtained by electrocuting elements of \mathcal{H} . Then, if α, β_q denote respectively a hyperbolic geodesic and an electro-ambient P -quasigeodesic with the same end-points, then α lies in a (hyperbolic) C_3 neighborhood of β_q .

Note. — The above Lemma generalises Klarreich’s Property (2) in Lemma 3.6 by replacing $N_0(\beta)$ with β_q . The former set can be quite large, but β_q is much smaller, containing only one geodesic segment in H rather than all of H . It is the introduction of the notion of electro-ambient quasigeodesic that makes for this generalisation. However, Lemma 3.15 is false in this generality. The idea is that two elements of \mathcal{H} might have geodesics

that are parallel (i.e. close to each other) for their entire length. Then an electro-ambient quasigeodesic might look like two adjacent edges of a thin triangle. This is precluded in Lemma 3.15 by the hypothesis of coboundedness. We shall not be needing this stronger Lemma 3.16 in this paper and it is included here for completeness.

Another Lemma which we shall be using follows from the proof of the Claim in the proof of Lemma 3.15 above.

LEMMA 3.17. — *Given D_0 there exists D_1 such that if α be a loop without backtracking in X_{el} with electric length less than D_0 and further, if $\alpha \cap H$ is a geodesic for each $H \in \mathcal{H}$, then the hyperbolic length of α is less than D_1 .*

3.2. Dehn twists are electric isometries

Let S_i be a surface whose path-pseudometric is obtained from a (fixed) hyperbolic metric by electrocuting the geodesic σ_i in \mathcal{C} . We can think of the Dehn twists as supported in the ϵ -neighborhood $N_\epsilon(\sigma_i)$ and that these neighborhoods have been given the zero-metric. Denote the resultant electric metric on S_i by ρ_i

We want to show that any power of a Dehn twist about σ_i induces an isometry of the surface S_i equipped with ρ_i . Consider any two points $x, y \in S$. Let α be any path in general position with respect to σ joining x, y . Look at the action of Dehn twist tw about some curve $\sigma_i \in \mathcal{C}$ on α . Let α meet σ in p_1, \dots, p_k . Let $tw(\alpha)$ be the path obtained from α by keeping it unchanged off σ and for each intersection point p_i , we compose α with a path lying on σ starting and ending at p_i and traversing σ once in the direction of the Dehn twist. Since the restriction $\rho_i|_\sigma = 0$, α and $tw(\alpha)$ have the same length. Hence, the length of the shortest path (geodesic) in the homotopy class (rel. end-points) of $tw(\alpha)$ is less than or equal to the length of the geodesic representative of the class of α .

Again, let β be any path in the homotopy class of $tw(\alpha)$. Then by acting by the reverse Dehn twist tw^{-1} about σ , we find by an identical argument that the geodesic representative of the homotopy class of α , which is the same as that of $tw^{-1}(\beta)$ has length less than or equal to the length of the geodesic representative of β .

Since β and $tw(\alpha)$ are homotopic rel. endpoints, we conclude that α and $tw(\alpha)$ have geodesic representatives of the same length.

This proves:

LEMMA 3.18. — *Let tw_i^n denote a power of a Dehn twist about the curve $\sigma_i \in \mathcal{C}$ and ρ_i denote the electric metric on S_i . Then tw_i^n induces an isometry of (S_i, ρ_i) . In particular, we may arrange tw_i^n to take geodesics to geodesics.*

The last statement in Lemma 3.18 has been put because geodesics are not uniquely defined in the usual sense in the electric metric. But a preferred path does exist, viz. the path which does not backtrack (or double back) on any $\sigma \in \mathcal{C}_i$, i.e. restricted to σ the path is a geodesic in the ordinary sense.

Everything in the above can be lifted to the universal cover \tilde{S}_i . We let \widetilde{tw} denote the lift of tw to \tilde{S}_i . This gives:

LEMMA 3.19. — *Let \widetilde{tw}_i^n denote a lift of tw_i^n to \tilde{S}_i . Let $\tilde{\rho}_i$ denote the lifted electric metric on \tilde{S}_i . Then \widetilde{tw}_i^n induces an isometry of $(\tilde{S}_i, \tilde{\rho}_i)$. In particular, we may arrange \widetilde{tw}_i^n to take geodesics to geodesics.*

3.3. Nearest-point Projections

We need the following basic lemmas from [18]. The following Lemma says nearest point projections in a δ -hyperbolic metric space do not increase distances much.

LEMMA 3.20 (Lemma 3.1 of [18]). — *Let (Y, d) be a δ -hyperbolic metric space and let $\mu \subset Y$ be a C -quasiconvex subset, e.g. a geodesic segment. Let $\pi : Y \rightarrow \mu$ map $y \in Y$ to a point on μ nearest to y . Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ, C .*

The next lemma says that quasi-isometries and nearest-point projections on hyperbolic metric spaces “almost commute”.

LEMMA 3.21 (Lemma 3.5 of [18]). — *Suppose (Y_1, d_1) and (Y_2, d_2) are δ -hyperbolic. Let μ_1 be some geodesic segment in Y_1 joining a, b and let p be any vertex of Y_1 . Also let q be a vertex on μ_1 such that $d_1(p, q) \leq d_2(p, x)$ for $x \in \mu_1$. Let ϕ be a (K, ϵ) -quasiisometric embedding from Y_1 to Y_2 . Let μ_2 be a geodesic segment in Y_2 joining $\phi(a)$ to $\phi(b)$. Let r be a point on μ_2 such that $d_2(\phi(p), r) \leq d_2(\phi(p), x)$ for $x \in \mu_2$. Then $d_2(r, \phi(q)) \leq C_4$ for some constant C_4 depending only on K, ϵ and δ .*

Sketch of Proof (See [18] for details). — $[p, q] \cup \mu_1$ is called a **tripod**. Then $[p, q] \cup [q, b]$, $[p, q] \cup [q, a]$ and $[a, b]$ are all quasigeodesics. Hence after acting by ϕ they map to quasigeodesics. In particular, $\phi(q)$ must lie close

to the image under ϕ of each of $[p, q] \cup [q, b]$, $[p, q] \cup [q, a]$ and $[a, b]$. Hence it must lie close to each of $[\phi(a), \phi(b)]$, $[\phi(a), \phi(p)]$ and $[\phi(b), \phi(p)]$. Again, if $\phi(a), \phi(b), \phi(p), z$ form the four points of a *tripod* (where z is a nearest point projection of $\phi(p)$ onto the geodesic joining $\phi(a), \phi(b)$), then z too must lie close to each of $[\phi(a), \phi(b)]$, $[\phi(a), \phi(p)]$ and $[\phi(b), \phi(p)]$.

The result follows by thinness of hyperbolic triangles. \square

For our purposes we shall need the above Lemma for quasi-isometries from \widetilde{S}_a to \widetilde{S}_b for two different hyperbolic structures on the same surface. We shall also need it for the electrocutted surfaces obtained in Lemma 3.7.

Yet another property that we shall require for nearest point projections is that nearest point projections in the electric metric and in the hyperbolic metric almost agree. Let $\widetilde{S} = Y$ be the universal cover of a surface with the hyperbolic metric minus a neighborhood of cusps. Equip Y with the path metric d as usual. Then Y is either the hyperbolic plane (if S has no cusps) or else is quasi-isometric to a tree (the Cayley graph of a free group). Let σ be a closed geodesic on S . Let d_e denote the electric metric on Y obtained by electrocutting the lifts of σ . Now, let $\mu = [a, b]$ be a hyperbolic geodesic on (Y, d) and let μ_q denote the electro-ambient quasigeodesic joining a, b . Let π denote the nearest point projection in (Y, d) . Tentatively, let π_e denote the nearest point projection in (Y, d_e) . Note that π_e is not well-defined. It is defined upto a bounded amount of discrepancy in the electric metric d_e . But we would like to make π_e well-defined upto a bounded amount of discrepancy in the hyperbolic metric d .

DEFINITION 3.22. — *Let $y \in Y$ and μ_q be an electro-ambient representative of an electric geodesic μ_e in (Y, d_e) . Then $\pi_e(y) = z \in \mu_q$ if the ordered pair $\{d_e(y, \pi_e(y)), d(y, \pi_e(y))\}$ is minimised at z .*

Note that this gives us a definition of π_e which is ambiguous by a finite amount of discrepancy not only in the electric metric but also in the hyperbolic metric.

LEMMA 3.23. — *There exists $C > 0$ such that the following holds. Let μ be a hyperbolic geodesic joining a, b . Let μ_e be the canonical representative of the electric geodesic joining a, b . Also let μ_q be the electro-ambient representative of μ_e . Let π_h denote the nearest point projection of \mathbb{H}^2 onto μ . $d(\pi_h(y), \pi_e(y))$ is uniformly bounded.*

Proof. — The proof is similar to that of Lemma 3.21, i.e. Lemma 3.5 of [18].

$[u, v]_h$ and $[u, v]_e$ will denote respectively the hyperbolic geodesic and the canonical representative of the electric geodesic joining u, v

$[y, \pi_e(y)] \cup [\pi_e(y), a]$ is an electric quasigeodesic without backtracking. Hence as in the proof of Lemma 3.21, $[y, \pi_e(y)] \cup [\pi_e(y), a]$ lies in a bounded neighborhood of $[y, a]_h$. In particular $\pi_e(y)$ lies in a bounded (hyperbolic) neighborhood of $[y, a]_h$. By an identical argument $\pi_e(y)$ lies in a bounded neighborhood of $[y, b]_h$. Again, since $\pi_e(y)$ lies on μ_e , therefore by Lemma 3.13, $\pi_e(y)$ lies in a bounded neighborhood of μ . Hence there exists $C > 0$ such that $\pi_e(y) \in N_C([y, a]_h) \cap N_C([y, b]_h) \cap N_C(\mu)$.

Again, $[y, \pi_h(y)] \cup [\pi_h(y), a]$ is a hyperbolic quasigeodesic. Hence

$$[y, \pi_h(y)] \cup [\pi_h(y), a]$$

lies in a bounded neighborhood of $[y, a]_h$. In particular $\pi_h(y)$ lies in a bounded (hyperbolic) neighborhood of $[y, a]_h$. By an identical argument $\pi_h(y)$ lies in a bounded neighborhood of $[y, b]_h$. Again, since $\pi_h(y)$ lies on μ , therefore, trivially $\pi_h(y)$ lies in a bounded neighborhood of μ . Hence there exists $D > 0$ such that

$$\pi_h(y) \in N_D([y, a]_h) \cap N_D([y, b]_h) \cap N_D(\mu).$$

Next, by hyperbolicity (thin-triangles) $N_D([y, a]_h) \cap N_D([y, b]_h) \cap N_D(\mu)$ and $N_C([y, a]_h) \cap N_C([y, b]_h) \cap N_C(\mu)$ have diameter bounded by some D_1 depending on D, C and choosing $D = C = \max(C, D)$, we get

$$d(\pi_h(y), \pi_e(y)) \leq D_1.$$

□

4. Universal Covers of Building Blocks and Electric Geometry

For most of this section (except the last subsection) we shall restrict our attention to closed surfaces and models corresponding to them. Let $S = S^h$ be a closed surface with some hyperbolic structure. For surfaces with punctures S will denote S^h minus a neighborhood of cusps. This will call for some modifications of the exposition, but *not* the overall construction. Hence, for ease of exposition, we postpone dealing with cusps till the last subsection of this section.

4.1. Graph Model of Building Blocks

Thin Blocks

Given a geodesic segment $\lambda \subset \tilde{S}$ and a basic *thin building block* B , let $\lambda = [a, b] \subset \tilde{S} \times \{0\}$ be a geodesic segment, where $\tilde{S} \times \{0\} \tilde{B}$, and B is obtained from $S \times I$ by hyperbolic $(1, n)$ Dehn surgery on $N_\epsilon(\sigma) \times [1, 2]$.

We shall now build a graph model for \tilde{B} which will be quasi-isometric to an electrocuted version of the original model, where lifts of the curves $\sigma \in \mathcal{C}$ which correspond to cores of Margulis tubes are electrocuted.

On $\tilde{S} \times \{0\}$ and $\tilde{S} \times \{3\}$ put the hyperbolic metric obtained from $S = S^h$. On $\tilde{S} \times \{1\}$ and $\tilde{S} \times \{2\}$ put the electric metric obtained by electrocuting the lifts of σ . This constructs 4 “sheets” of \tilde{S} comprising the “horizontal skeleton” of the “graph model” of \tilde{B} . Now for the vertical strands. On each vertical element of the form $x \times [0, 1]$ and $x \times [2, 3]$ put the Euclidean metric.

The resulting copy of \tilde{B} will be called the **graph model of a thin block**.

Next, let ϕ denote the map induced on \tilde{S} by tw_σ^n , the n -fold Dehn twist along σ . Join each $x \times \{1\}$ to $\phi(x) \times \{2\}$ by a Euclidean segment of length 1.

Thick Block

For a thick block $B = \tilde{S} \times [0, 1]$, recall that B is the universal curve over a “thick” Teichmüller geodesic $\lambda_{Teich} = [a, b]$ of length less than some fixed $D > 0$. Each $S \times \{x\}$ is identified with the hyperbolic surface over $(a + \frac{x}{b-a})$ (assuming that the Teichmüller geodesic is parametrized by arc-length).

Here $S \times \{0\}$ is identified with the hyperbolic surface corresponding to a , $S \times \{1\}$ is identified with the hyperbolic surface corresponding to b and each (x, a) is joined to (x, b) by a segment of length 1.

The resulting model of \tilde{B} is called a **graph model of a thick block**.

Admissible Paths

Admissible paths consist of the following:

- (1) Horizontal segments along some $\tilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (thin blocks) or $i = \{0, 1\}$ (thick blocks).
- (2) Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for thin blocks or $x \times [0, 1]$ for thick blocks.
- (3) Vertical segments of length 1 joining $x \times \{1\}$ to $\phi(x) \times \{2\}$ for thin blocks.

4.2. Construction of Quasiconvex Sets for Building Blocks

In the next section, we will construct a set B_λ containing λ and a retraction Π_λ of \widetilde{M} onto it. Π_λ will have the property that it does not stretch distances much. This will show that B_λ is quasi-isometrically embedded in \widetilde{M} .

In this subsection, we describe the construction of B_λ restricted to a building block B .

Construction of $B_\lambda(B)$ - Thick Block

Let the thick block be the universal curve over a Teichmüller geodesic $[\alpha, \beta]$. Let S_α denote the hyperbolic surface over α and S_β denote the hyperbolic surface over β .

First, let $\lambda = [a, b]$ be a geodesic segment in \widetilde{S} . Let λ_{B_0} denote $\lambda \times \{0\}$.

Next, let ϕ be the lift of the “identity” map from \widetilde{S}_α to \widetilde{S}_β . Let Φ denote the induced map on geodesics and let $\Phi(\lambda)$ denote the hyperbolic geodesic joining $\phi(a), \phi(b)$. Let λ_{B_1} denote $\Phi(\lambda) \times \{1\}$.

For the universal cover \widetilde{B} of the thick block B , define:

$$B_\lambda(B) = \bigcup_{i=0,1} \lambda_{B_i}$$

DEFINITION 4.1. — *Each $\widetilde{S} \times i$ for $i = 0, 1$ will be called a **horizontal sheet** of \widetilde{B} when B is a thick block.*

Construction of $B_\lambda(B)$ - Thin Block

First, recall that $\lambda = [a, b]$ is a geodesic segment in \widetilde{S} . Let λ_{B_0} denote $\lambda \times \{0\}$.

Next, let λ_{el} denote the electric geodesic joining a, b in the electric pseudo-metric on \widetilde{S} obtained by electrocuting lifts of σ . Let λ_{B_1} denote $\lambda_{el} \times \{1\}$.

Third, recall that ϕ is the lift of the Dehn twist tw_σ^n to \widetilde{S} equipped with the electric metric. Let Φ denote the induced map on geodesics, i.e. if $\mu = [x, y] \subset (\widetilde{S}, d_{el})$, then $\Phi(\mu) = [\phi(x), \phi(y)]$ is the geodesic joining $\phi(x), \phi(y)$. Let λ_{B_2} denote $\Phi(\lambda_{el}) \times \{2\}$.

Fourthly, let $\Phi(\lambda)$ denote the hyperbolic geodesic joining $\phi(a), \phi(b)$. Let λ_{B_3} denote $\Phi(\lambda) \times \{3\}$.

For the universal cover \widetilde{B} of the thin block B , define:

$$B_\lambda(B) = \bigcup_{i=0,\dots,3} \lambda_{B_i}$$

DEFINITION 4.2. — *Each $\tilde{S} \times i$ for $i = 0 \dots 3$ will be called a **horizontal sheet** of \tilde{B} when B is a thick block.*

Construction of $\Pi_{\lambda, B}$ - Thick Block

On $\tilde{S} \times \{0\}$, let Π_{B_0} denote nearest point projection onto λ_{B_0} in the path metric on $\tilde{S} \times \{0\}$.

On $\tilde{S} \times \{1\}$, let Π_{B_1} denote nearest point projection onto λ_{B_1} in the path metric on $\tilde{S} \times \{1\}$.

For the universal cover \tilde{B} of the thick block B , define:

$$\Pi_{\lambda, B}(x) = \Pi_{B_i}(x), x \in \tilde{S} \times \{i\}, i = 0, 1$$

Construction of $\Pi_{\lambda, B}$ - Thin Block

On $\tilde{S} \times \{0\}$, let Π_{B_0} denote nearest point projection onto λ_{B_0} . Here the nearest point projection is taken in the path metric on $\tilde{S} \times \{0\}$ which is a hyperbolic metric space.

On $\tilde{S} \times \{1\}$, let Π_{B_1} denote the nearest point projection onto λ_{B_1} . Here the nearest point projection is taken in the sense of the definition preceding Lemma 3.23, that is minimising the ordered pair (d_{el}, d_{hyp}) (where d_{el}, d_{hyp} refer to electric and hyperbolic metrics respectively.)

On $\tilde{S} \times \{2\}$, let Π_{B_2} denote the nearest point projection onto λ_{B_2} . Here, again the nearest point projection is taken in the sense of the definition preceding Lemma 3.23.

Again, on $\tilde{S} \times \{3\}$, let Π_{B_3} denote nearest point projection onto λ_{B_3} . Here the nearest point projection is taken in the path metric on $\tilde{S} \times \{3\}$ which is a hyperbolic metric space.

For the universal cover \tilde{B} of the thin block B , define:

$$\Pi_{\lambda, B}(x) = \Pi_{B_i}(x), x \in \tilde{S} \times \{i\}, i = 0, \dots, 3$$

$\Pi_{\lambda, B}$ is a retract - Thick Block

The proof for a thick block is exactly as in [18]. The crucial tool is Lemma 3.21.

LEMMA 4.3. — *There exists $C > 0$ such that the following holds.*

Let $x, y \in \tilde{S} \times \{0, 1\} \subset \tilde{B}$ for some thick block B . Then

$$d(\Pi_{\lambda, B}(x), \Pi_{\lambda, B}(y)) \leq Cd(x, y).$$

Proof. — It is enough to show this for the two following cases:

- (1) $x, y \in \tilde{S} \times \{0\}$ OR $x, y \in \tilde{S} \times \{1\}$.
- (2) x, y are of the form $(p, 0)$ and $(\phi(p), 1)$ which are connected by a vertical segment of length one (as per the construction of the model B).

Case 1 above follows from Lemma 3.20, and Case 2 from the fact that ϕ is a uniform quasi-isometry (depending on the uniform bound on the length of the Teichmüller geodesic over which B is the universal curve) and Lemma 3.21 which says that there exists $C_1 > 0$ such that if π be the nearest point retraction in \tilde{S} onto λ then $d(\phi(\pi(p)), \pi(\phi(p))) \leq C_1$. From this it follows that

$$d(\Pi_{\lambda, B}((p, 0)), \Pi_{\lambda, B}((\phi(p), 1))) \leq C_1 + 1$$

Choosing $C = C_1 + 1$ we are through. □

$\Pi_{\lambda, B}$ is a retract - Thin Block

The two main ingredients in this case are Lemmas 3.21 and 3.23.

Note. — In the Lemma 4.4 below, there is implicit a constant n , the twist coefficient of the Dehn twist that distinguishes B . But the constant C below is independent of n due to the fact that powers of Dehn twists are **uniform quasi-isometries** of the electric metric. In fact this is the reason why we introduce the electric metric in the first place, so as to ensure that the techniques of [18] and [17] go through here.

LEMMA 4.4. — *There exists $C > 0$ such that the following holds: Let $x, y \in \tilde{S} \times \{0, 1, 2, 3\} \subset \tilde{B}$ for some thin block B . Then*

$$d_e(\Pi_{\lambda, B}(x), \Pi_{\lambda, B}(y)) \leq C d_e(x, y).$$

Proof. — It is enough to show this for the two following cases:

- (1) $x, y \in \tilde{S} \times \{0\}$ OR $x, y \in \tilde{S} \times \{1\}$.
- (2) $x = (p, 0)$ and $y = (p, 1)$ for some p
- (3) x, y are of the form $(p, 1)$ and $(\phi(p), 2)$ which are connected by a vertical segment of length one (as per the construction of the model B)
- (4) $x = (p, 2)$ and $y = (p, 3)$ for some p .

Case 1. — As in Lemma 4.3 above, this follows from 3.20.

Case 2 and Case 4. — These follow from Lemma 3.23 which says that the hyperbolic and electric projections of p onto the hyperbolic geodesic $[a, b]$ and the electro-ambient geodesic $[a, b]_{ea}$ respectively “almost agree”. If π_h and π_e denote the hyperbolic and electric projections, then there exists $C_1 > 0$ such that $d(\pi_h(p), \pi_e(p)) \leq C_1$. Hence

$$d(\Pi_{\lambda, B}((p, i)), \Pi_{\lambda, B}((p, i + 1))) \leq C_1 + 1, \text{ for } i = 0, 2.$$

Case 3. — First, from Lemma 3.19 the (power of the) Dehn twist ϕ used in the construction of B is a (uniform) quasi-isometry of \tilde{S} equipped with the electric metric. Again, if π denotes the nearest point projection in the electric metric, then from Lemma 3.21, there exists $C_2 > 0$ such that $d_e(\phi(\pi(p)), \pi(\phi(p))) \leq C_2$. Here d_e denotes the electric metric. From this it follows that

$$d_e(\Pi_{\lambda, B}((p, 1)), \Pi_{\lambda, B}((\phi(p), 2))) \leq C_1 + 1$$

Choosing $C = \max(C_1 + 1, C_2 + 1)$ we are through. \square

4.3. Modifications for Surfaces With Punctures

We deal with the thin block first.

Thin Block

For S^h a hyperbolic surface with punctures, let S denote S^h minus some neighborhood of the cusps. Then the construction of the model B and hence the graph model of \tilde{B} for a thin block B goes through *mutatis mutandis* even with respect to notation. The construction of the quasi-convex set and the retraction are modified as follows.

λ will no longer be a hyperbolic geodesic but rather an ambient quasi-geodesic in \tilde{S} . The construction is taken from [23]. We start with a hyperbolic geodesic λ^h in S^h . Fix a neighborhood of the cusps lifting to an equivariant family of horoballs in the universal cover $\mathbb{H}^2 = \tilde{S}^h$. Since λ^h is a hyperbolic geodesic in \tilde{S}^h there are unique entry and exit points for each horoball that λ^h meets and hence unique Euclidean geodesics joining them on the corresponding horosphere. Replacing the segments of λ^h lying inside Z -horoballs by the corresponding Euclidean geodesics, we obtain an ambient quasigeodesic λ in \tilde{M}_0 by Theorem 3.12. See Figure below.

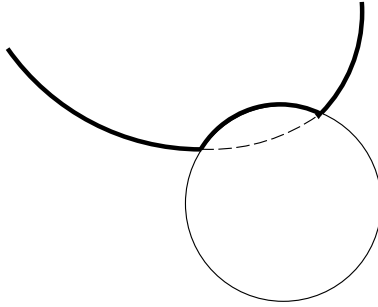


Figure 4.1. *Horo-ambient quasigeodesics*

Ambient quasigeodesics obtained by this kind of a construction will be termed **horo-ambient quasigeodesics** to distinguish them from *electro-ambient quasigeodesics* defined earlier. Starting with a horo-ambient quasigeodesic $\lambda \subset \tilde{S}$, we can proceed as before to construct $B_{\lambda,B}$, $\Pi_{\lambda,B}$ and prove Lemma 4.4 above. In fact the statement and proof of this Lemma goes through unchanged, with the only pro viso that for punctured surfaces, S and S^h are not the same and that λ is a horo-ambient quasigeodesic in general. (Note that if S has no punctures, a horo-ambient quasigeodesic is a hyperbolic geodesic).

Thick Block

Here B is obtained from the universal curve over a Teichmüller geodesic by removing a neighborhood of the cusps. Again, S is obtained from S^h by removing a neighborhood of the cusps. That the map ϕ is a uniform quasi-isometry is ensured by the fact that the corresponding copies of S^h are a uniformly bounded Teichmüller distance from each other, and that if ϕ denote a map between these copies of S^h , one can ensure that ϕ takes cusps to cusps.

The construction of the graph model for \tilde{B} , the construction of $B_{\lambda,B}$ and $\Pi_{\lambda,B}$ also go through as before with the pro viso that λ is a *horo-ambient quasigeodesic*. Lemma 4.3 goes through as before.

5. Construction of Quasiconvex Sets

5.1. Construction of B_λ and Π_λ

Given a manifold M of i -bounded geometry, we know that M is homeomorphic to $S \times J$ for $J = [0, \infty)$ or $(-\infty, \infty)$. By definition of i -bounded geometry, there exists a sequence I_i of intervals and blocks B_i where the metric on $S \times I_i$ coincides with that on some building block B_i . Denote:

- $B_{\mu, B_i} = B_{i\mu}$
- $\Pi_{\mu, B_i} = \Pi_{i\mu}$.

Now for a block $B = S \times I$ (thick or thin), a natural map Φ_B may be defined taking $\mu = B_{\mu, B} \cap \tilde{S} \times \{0\}$ to a geodesic $B_{\mu, B} \cap \tilde{S} \times \{k\} = \Phi_B(\mu)$ where $k = 1$ or 3 according as B is thick or thin. Let the map Φ_{B_i} be denoted as Φ_i for $i \geq 0$. For $i < 0$ we shall modify this by defining Φ_i to be the map that takes $\mu = B_{\mu, B_i} \cap \tilde{S} \times \{k\}$ to a geodesic $B_{\mu, B_i} \cap \tilde{S} \times \{0\} = \Phi_i(\mu)$ where $k = 1$ or 3 according as B is thick or thin.

We start with a reference block B_0 and a reference geodesic segment $\lambda = \lambda_0$ on the “lower surface” $\tilde{S} \times \{0\}$. Now inductively define:

- $\lambda_{i+1} = \Phi_i(\lambda_i)$ for $i \geq 0$
- $\lambda_{i-1} = \Phi_i(\lambda_i)$ for $i \leq 0$
- $B_{i\lambda} = B_{\lambda_i}(B_i)$
- $\Pi_{i\lambda} = \Pi_{\lambda_i, B_i}$
- $B_\lambda = \bigcup_i B_{i\lambda}$
- $\Pi_\lambda = \bigcup_i \Pi_{i\lambda}$.

Recall that each $\tilde{S} \times i$ for $i = 0 \cdots K$ is called a **horizontal sheet** of \tilde{B} , where $K = 1$ or 3 according as B is thick or thin. We will restrict our attention to the union of the horizontal sheets \widetilde{M}_H of \widetilde{M} with the induced metric.

Clearly, $B_\lambda \subset \widetilde{M}_H \subset \widetilde{M}$, and Π_λ is defined from \widetilde{M}_H to B_λ . Since \widetilde{M}_H is a “coarse net” in \widetilde{M} , we will be able to get all the coarse information we need by restricting ourselves to \widetilde{M}_H .

By Lemmas 4.3 and 4.4, we obtain the fact that each $\Pi_{i\lambda}$ is a retract. Hence assembling all these retracts together, we have the following basic theorem.

THEOREM 5.1. — *There exists $C > 0$ such that for any geodesic $\lambda = \lambda_0 \subset \tilde{S} \times \{0\} \subset \tilde{B}_0$, the retraction $\Pi_\lambda : \widetilde{M}_H \rightarrow B_\lambda$ satisfies.*

Then $d(\Pi_{\lambda, B}(x), \Pi_{\lambda, B}(y)) \leq Cd(x, y) + C$.

Note 5.2. — For Theorem 5.1 above, note that all that we really require is that the universal cover \widetilde{S} be a hyperbolic metric space. There is no restriction on \widetilde{M}_H . In fact, Theorem 5.1 would hold for general stacks of hyperbolic metric spaces with hyperbolic Dehn surgery performed on blocks.

Note 5.3. — M_H has been given built up out of **graph models of thick and thin blocks** and have sheets that are electrocuted.

5.2. Heights of Blocks

Instead of considering all the horizontal sheets, we would now like to consider only the **boundary horizontal sheets**, i.e. for a thick block we consider $\widetilde{S} \times \{0, 1\}$ and for a thin block we consider $\widetilde{S} \times \{0, 3\}$. The union of all boundary horizontal sheets will be denoted by M_{BH} .

OBSERVATION 5.1. — \widetilde{M}_{BH} is a “coarse net” in \widetilde{M} in the **graph model**, but not in the **model of i -bounded geometry**.

In the graph model, any point can be connected by a vertical segment of length ≤ 2 to one of the boundary horizontal sheets.

However, in the model of i -bounded geometry, there are points within Margulis tubes (say for instance, the center of the totally geodesic disk bounded by a meridian) which are at a distance of the order of $\ln(n_i)$ from the boundary horizontal sheets. Since n_i is arbitrary, \widetilde{M}_{BH} is no longer a “coarse net” in \widetilde{M} equipped with the model of i -bounded geometry.

OBSERVATION 5.2. — \widetilde{M}_H is defined only in the **graph model**, but not in the model of i -bounded geometry.

OBSERVATION 5.3. — The electric metric on the model of i -bounded geometry on \widetilde{M} obtained by electrocuting all lifts of Margulis tubes is quasi-isometric to the graph model of \widetilde{M} .

This follows from the fact that any lift of a Margulis tube has diameter 1 in the graph model of \widetilde{M} .

Bounded Height of Thick Block

Let $\mu \subset \widetilde{S} \times \{0\} \widetilde{B}_i$ be a geodesic in a (thick or thin) block. Then there exists a (K_i, ϵ_i) -quasi-isometry ψ_i ($= \phi_i$ for thick blocks) from $\widetilde{S} \times \{0\}$ to $\widetilde{S} \times \{1\}$ and Ψ_i is the induced map on geodesics. Hence, for any $x \in \mu$, $\psi_i(x)$ lies within some bounded distance C_i of $\Psi_i(\mu)$. But x is connected to $\psi_i(x)$ by:

Case 1. — **Thick Blocks:** a vertical segment of length 1

Case 2. — **Thin Blocks:** the union of

- (1) two vertical segments of length 1 between $\widetilde{S} \times \{i\}$ and $\widetilde{S} \times \{i+1\}$ for $i = 0, 2$
- (2) a horizontal segment of length bounded by (some uniform) C' (cf. Lemma 3.13) connecting $(x, 1)$ to a point on the electro-ambient geodesic $B_\lambda(B) \cap \widetilde{S} \times \{1\}$
- (3) a vertical segment of length one in the **graph model** connecting $(x, 1)$ to $(\phi(x), 2)$. Such a path has to travel *through the Margulis tube* in the model of **i-bounded geometry** and has length less than $g_0(n_i)$ for some function $g_0 : \mathbb{Z} \rightarrow \mathbb{N}$, and n_i the twist coefficient.
- (4) a horizontal segment of length less than C' (Lemma 3.13) connecting $(\phi_i(x), 3)$ to a point on the hyperbolic geodesic $B_\lambda(B) \cap \widetilde{S} \times \{3\}$

Thus x can be connected to a point on $x' \in \Psi_i(\mu)$ by a path of length less than $g(i) = 2 + 2C' + g_0(n_i)$. Here, in fact g_0 is at most linear in n_i but we shall not need this. Recall that λ_i is the geodesic on the lower horizontal surface of the block \widetilde{B}_i . The same can be done for blocks \widetilde{B}_{i-1} and going down from λ_i to λ_{i-1} . What we have thus shown is:

LEMMA 5.4. — *There exists a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i (resp. B_{i-1}), and $x \in \lambda_i$, there exists $x' \in \lambda_{i+1}$ (resp. λ_{i-1}) for $i \geq 0$ (resp. $i \leq 0$), satisfying:*

$$d(x, x') \leq g(i)$$

Modifications for Punctured Surfaces

For a punctured surface, the above argument has to be modified using some constructions from Lemma 5.1 of [23].

Given $\lambda^h \in \widetilde{S}^h$ we have already indicated how to construct a horo-ambient quasigeodesic λ in \widetilde{S} (where, recall that S is S^h minus a neighborhood of cusps). Let λ_c denote the union of the segments of λ that lie along cusps. Let $\lambda_b = \lambda - \lambda_c$. For punctured surfaces, recall that λ_i is a horo-ambient quasigeodesic on the lower horizontal surface of \widetilde{B}_i . λ_{ic} will denote the union of segments of λ_i lying along cusps and λ_{ib} will denote $\lambda_i - \lambda_{ic}$.

Lemma 5.1 of [23] says that there exists C_0 such that for any thick block B_i , and $x \in \lambda_{ib}$ there exists $x' \in \lambda_{i+1,b}$ such that $d(x, x') \leq C'$. Combining this with the argument given above for surfaces without punctures, we conclude:

LEMMA 5.5. — *There exists a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i (resp. B_{i-1}), and $x \in \lambda_{ib}$, there exists $x' \in \lambda_{i+1,b}$ (resp. $\lambda_{i-1,b}$) for $i \geq 0$ (resp. $i \leq 0$), satisfying:*

$$d(x, x') \leq g(i)$$

Note. — For a surface without punctures, λ and λ_b coincide.

6. Cannon-Thurston Maps for Surfaces Without Punctures

Unless explicitly mentioned otherwise, we shall assume till the end of this section that

- S is a closed surface. Hence $S^h = S$.
- there exists a hyperbolic manifold M and a homeomorphism from \widetilde{M} to $\widetilde{S} \times \mathbb{R}$. We identify \widetilde{M} with $\widetilde{S} \times \mathbb{R}$ via this homeomorphism.
- $\widetilde{S} \times \mathbb{R}$ admits a quasi-isometry g to a model manifold of *i -bounded geometry*
- g preserves the fibers over $\mathbb{Z} \subset \mathbb{R}$

Remarks. — 1) The above assumption is much stronger than what we need. It suffices to assume that \widetilde{M} is a Gromov-hyperbolic metric space. Further relaxations on the hypothesis may be considered while generalising the results of this paper to other hyperbolic metric spaces.

- 2) We have taken J to be \mathbb{R} here for concreteness. The other possibility of $J = \mathbb{R}^+$ can be treated in exactly the same way. We shall henceforth ignore the quasi-isometry g and think of \widetilde{M} itself as the universal cover of a model manifold of *i -bounded geometry*.

6.1. Admissible Paths

We want to define a collection of **B_λ -elementary admissible paths** lying in a bounded neighborhood of B_λ . B_λ is not connected. Hence, it does not make much sense to speak of the path-metric on B_λ . To remedy this we introduce a “thickening” (cf. [10]) of B_λ which is path-connected and where the paths are controlled. A **B_λ -admissible path** will be a composition of B_λ -elementary admissible paths.

Recall that admissible paths in the graph model of bounded geometry consist of the following :

- (1) Horizontal segments along some $\tilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (thin blocks) or $i = \{0, 1\}$ (thick blocks).
- (2) Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for thin blocks, where $x \in \tilde{S}$.
- (3) Vertical segments of length 1 joining $x \times \{1\}$ to $\phi(x) \times \{2\}$ for thin blocks.
- (4) Vertical segments of length 1 joining $x \times \{0\}$ to $\phi(x) \times \{1\}$ (or $x \times \{1\}$ if ϕ is regarded as the identity map at the topological level) for thick blocks.

We shall choose a subclass of these admissible paths to define B_λ -elementary admissible paths.

B_λ -elementary admissible paths in the thick block

Let $B = S \times [i, i+1]$ be a thick block, where each (x, i) is connected by a vertical segment of length 1 to $(\phi(x), i+1)$. Also Φ is the map on geodesics induced by ϕ . Let $B_\lambda \cap \tilde{B} = \lambda_i \cup \lambda_{i+1}$ where λ_i lies on $\tilde{S} \times \{i\}$ and λ_{i+1} lies on $\tilde{S} \times \{i+1\}$. π_j , for $j = i, i+1$ denote nearest-point projections of $\tilde{S} \times \{j\}$ onto λ_j . Next, since ϕ is a quasi-isometry, there exists $C > 0$ such that for all $(x, i) \in \lambda_i$, $(\phi(x), i+1)$ lies in a C -neighborhood of $\Phi(\lambda_i) = \lambda_{i+1}$. The same holds for ϕ^{-1} and points in λ_{i+1} , where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\tilde{S} \times \{i+1\}$ to $\tilde{S} \times \{i\}$. The **B_λ -elementary admissible paths** in \tilde{B} consist of the following:

- (1) Horizontal geodesic subsegments of λ_j , $j = \{i, i+1\}$.
- (2) Vertical segments of length 1 joining $x \times \{0\}$ to $\phi(x) \times \{1\}$.
- (3) Vertical segments of length 1 joining $y \times \{1\}$ to $\phi^{-1}(y) \times \{0\}$.
- (4) Horizontal geodesic segments lying in a C -neighborhood of λ_j , $j = i, i+1$.

B_λ -elementary admissible paths in the thin block

Let $B = S \times [i, i+3]$ be a thin block, where each $(x, i+1)$ is connected by a vertical segment of length 1 to $(\phi(x), i+2)$. Also Φ is the map on canonical representatives of electric geodesics induced by ϕ . Let $B_\lambda \cap \tilde{B} = \bigcup_{j=i \dots i+3} \lambda_j$ where λ_j lies on $\tilde{S} \times \{j\}$. π_j denotes nearest-point projection of $\tilde{S} \times \{j\}$ onto λ_j (in the appropriate sense - hyperbolic for $j = i, i+3$ and electric for $j = i+1, i+2$). Next, since ϕ is an electric isometry, but a hyperbolic quasi-isometry, there exists $C > 0$ (uniform constant) and $K = K(B)$ such that for all $(x, i) \in \lambda_i$, $(\phi(x), i+1)$ lies in an (electric) C -neighborhood and a hyperbolic K -neighborhood of $\Phi(\lambda_{i+1}) = \lambda_{i+2}$. The same holds for

ϕ^{-1} and points in λ_{i+2} , where ϕ^{-1} denotes the *quasi-isometric inverse* of ϕ from $\tilde{S} \times \{i + 2\}$ to $\tilde{S} \times \{i + 1\}$.

Again, since λ_{i+1} and λ_{i+2} are electro-ambient quasigeodesics, we further note that there exists $C > 0$ (assuming the same C for convenience) such that for all $(x, i) \in \lambda_i$, $(x, i + 1)$ lies in a (hyperbolic) C -neighborhood of λ_{i+1} . Similarly for all $(x, i + 2) \in \lambda_{i+2}$, $(x, i + 3)$ lies in a (hyperbolic) C -neighborhood of λ_{i+3} . The same holds if we go “down” from λ_{i+1} to λ_i or from λ_{i+3} to λ_{i+2} . The B_λ -**elementary admissible paths** in \tilde{B} consist of the following:

- (1) Horizontal subsegments of λ_j , $j = \{i, \dots, i + 3\}$.
- (2) Vertical segments of length 1 joining $x \times \{i + 1\}$ to $\phi(x) \times \{i + 2\}$, for $x \in \lambda_{i+1}$.
- (3) Vertical segments of length 1 joining $x \times \{j\}$ to $x \times \{j + 1\}$, for $j = i, i + 2$.
- (4) Horizontal geodesic segments lying in a *hyperbolic* C -neighborhood of λ_j , $j = i, \dots, i + 3$.
- (5) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi(x), i + 2)$ to a point on λ_{i+2} for $(x, i + 1) \in \lambda_{i+1}$.
- (6) Horizontal hyperbolic segments of *electric length* $\leq C$ and *hyperbolic length* $\leq K(B)$ joining points of the form $(\phi^{-1}(x), i + 1)$ to a point on λ_{i+1} for $(x, i + 2) \in \lambda_{i+2}$.
- (7) Hyperbolic geodesic segments lying entirely within some lift of a Margulis tube $N_\epsilon(\tilde{\sigma}) \times [1, 2]$ joining points $x, y \in \lambda_{i+1} \cup \lambda_{i+2}$.

DEFINITION 6.1. — *A B_λ -admissible path is a union of B_λ -elementary admissible paths.*

The following lemma follows from the above definition and Lemma 5.4.

LEMMA 6.2. — *There exists a function $g : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i , and x lying on a B_λ -admissible path in \tilde{B}_i , there exist $y \in \lambda_j$ and $z \in \lambda_k$ where $\lambda_j \subset B_\lambda$ and $\lambda_k \subset B_\lambda$ lie on the two boundary horizontal sheets, satisfying:*

$$d(x, y) \leq g(i)$$

$$d(x, z) \leq g(i)$$

Let $h(i) = \sum_{j=0 \dots i} g(j)$ be the sum of the values of $g(j)$ as j ranges from 0 to i (with the assumption that increments are by $+1$ for $i \geq 0$ and by -1 for $i \leq 0$). Then we have from Lemma 6.2 above,

COROLLARY 6.3. — *There exists a function $h : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i , and x lying on a B_λ -admissible path in \widetilde{B}_i , there exist $y \in \lambda_0 = \lambda$ such that:*

$$d(x, y) \leq h(i)$$

Important Note. — In the above Lemma 6.2 and Corollary 6.3, it is important to note that the distance d is **hyperbolic**, not electric. This is because the number $K(B_i)$ occurring in elementary paths of type 5 and 6 is a hyperbolic length depending only on i (in B_i).

Next suppose that λ lies outside $B_N(p)$, the N -ball about a fixed reference point p on the boundary horizontal surface $\widetilde{S} \times \{0\} \subset \widetilde{B}_0$. Then by Corollary 6.3, any x lying on a B_λ -admissible path in \widetilde{B}_i satisfies

$$d(x, p) \geq N - h(i)$$

Also, since the electric, and hence hyperbolic “thickness” (the shortest distance between its boundary horizontal sheets) is ≥ 1 , we get,

$$d(x, p) \geq |i|$$

Assume for convenience that $i \geq 0$ (a similar argument works, reversing signs for $i < 0$). Then,

$$d(x, p) \geq \min\{i, N - h(i)\}$$

Let $h_1(i) = h(i) + i$. Then h_1 is a monotonically increasing function on the integers. If $h_1^{-1}(N)$ denote the largest positive integer n such that $h(n) \leq m$, then clearly, $h_1^{-1}(N) \rightarrow \infty$ as $N \rightarrow \infty$. We have thus shown:

LEMMA 6.4. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds: For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$ and any x on a B_λ -admissible path,*

$$d(\lambda, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

Aside for Punctured Surfaces

We mention parenthetically the versions of Corollary 6.3 and Lemma 6.4 here that will be useful for punctured surfaces in the next section.

COROLLARY 6.5. — *There exists a function $h : \mathbb{Z} \rightarrow \mathbb{N}$ such that for any block B_i , and x lying on λ_{ib} , there exist $y \in \lambda_{0b} = \lambda_b$ such that:*

$$d(x, y) \leq h(i)$$

LEMMA 6.6. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds: For any horo-ambient quasigeodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$ and any x on some λ_{ib} ,*

$$d(\lambda_b, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

In Lemma 6.6 we have used λ_b in place of λ as λ_b is constructed from λ^h by changing it along horocycles. However, another version of the above Lemma will also sometimes be useful. If we start with a λ^h that lies outside large balls about p , we can ensure that λ_b also lies outside large balls about p , for λ may approach p only along cusps. Hence we may replace the hypothesis that λ_b lie outside $B_N(p)$ by the hypothesis that λ^h lie outside $B_N(p)$:

LEMMA 6.7. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds: For any hyperbolic geodesic $\lambda^h \subset \widetilde{S}^h \times \{0\} \subset \widetilde{B}_0$, a fixed reference point $p \in \widetilde{S}^h \times \{0\} \subset \widetilde{B}_0$ and any x on some λ_{ib} ,*

$$d(\lambda^h, p) \geq N \Rightarrow d(x, p) \geq M(N).$$

6.2. Joining the Dots

Recall that **admissible paths** in a model manifold of bounded geometry consist of:

- (1) Horizontal segments along some $\widetilde{S} \times \{i\}$ for $i = \{0, 1, 2, 3\}$ (thin blocks) or $i = \{0, 1\}$ (thick blocks).
- (2) Vertical segments $x \times [0, 1]$ or $x \times [2, 3]$ for thin blocks.
- (3) Vertical segments of length 1 joining $x \times \{1\}$ to $\phi(x) \times \{2\}$ for thin blocks.
- (4) Vertical segments of length 1 joining $x \times \{0\}$ to $\phi(x) \times \{1\}$ for thick blocks.

Our strategy in this subsection is:

- Start with an electric geodesic β_e in \widetilde{M}_{el} joining the end-points of λ .
- Replace it by an *admissible quasigeodesic*, i.e. an admissible path that is a quasigeodesic.
- Project the intersection of the admissible quasigeodesic with the horizontal sheets onto B_λ .
- The result of step 3 above is disconnected. *Join the dots* using B_λ -admissible paths.

The end product is an electric quasigeodesic built up of B_λ admissible paths.

Now for the first two steps: Since \widetilde{B} (for a thick block B) has thickness 1, any path lying in a thick block can be perturbed to an admissible path lying in \widetilde{B} , changing the length by at most a bounded multiplicative factor. For B thin, we decompose paths into horizontal paths lying in some $\widetilde{S} \times \{j\}$, for $j = 0, \dots, 3$ and vertical paths of types (2) or (3) above. All this can be done as for thick blocks, changing lengths by a bounded multiplicative constant. The result is therefore an electric quasigeodesic. Without loss of generality, we can assume that the electric quasigeodesic is one without back-tracking (as this can be done without increasing the length of the geodesic - see [7] or [12] for instance). Abusing notation slightly, assume therefore that β_e is an admissible electric quasigeodesic without backtracking joining the end-points of λ .

Now act on β_e by Π_λ . From Theorem 5.1, we conclude, by restricting Π_λ to the horizontal sheets of \widetilde{M}_{el} that the image $\Pi_\lambda(\beta_e)$ is a ‘‘dotted electric quasigeodesic’’ lying entirely on B_λ . This completes step 3. Note that since β_e consists of admissible segments, we can arrange so that two nearest points on β_e which are not connected to each other are at a distance of one apart, i.e. they form the end-points of a vertical segment of type (2), (3) or (4). Let $\Pi_\lambda(\beta_e) \cap B_\lambda = \beta_d$, be the dotted quasigeodesic lying on B_λ . We want to join the dots in β_d converting it into a **connected** electric quasigeodesic built up of B_λ -**admissible paths**.

For vertical segments of type (4) joining p, q (say), $\Pi_\lambda(p), \Pi_\lambda(q)$ are a bounded hyperbolic distance apart. Hence, by the proof of Lemma 4.3, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by a B_λ -admissible path of length bounded by some C_0 (independent of B, λ).

For vertical segments of type (2) joining p, q , we note that $\Pi_\lambda(p), \Pi_\lambda(q)$ are a bounded hyperbolic distance apart. Hence, by the proof of Lemma 4.4, we can join $\Pi_\lambda(p), \Pi_\lambda(q)$ by a B_λ -admissible path of length bounded by some C_1 (independent of B, λ).

This leaves us to deal with case (3). Such a segment consists of a segment lying within a lift of a Margulis tube and a horizontal segment of length 1 lying outside. Decompose the bit within a Margulis tube into a horizontal segment lying on some horizontal surface and (possibly) a vertical segment of hyperbolic length 1. The image of the horizontal part of the path is again uniformly bounded in the electric metric. Further, by **co-boundedness**, we can ensure that the *hyperbolic* length of the image away from the lift of at most one Margulis tube lying in the zero neighborhood of B_λ is

bounded uniformly by some C_2 . The same can be ensured of vertical paths of hyperbolic length one lying inside lifts of Margulis tubes. These two pieces (images under Π_λ of horizontal paths inside lifts of Margulis tubes and vertical segments of length one inside lifts of Margulis tubes) can be replaced by B_λ -admissible paths of uniformly bounded electric length (since at most one lift of a Margulis tube lying in a zero neighborhood of B_λ is in the image for length $\geq C_2$.) Finally, the segment lying outside, being horizontal, its image is connected and of bounded length by Lemma 3.20.

After joining the dots, we can assume further that the quasigeodesic thus obtained does not backtrack (cf [7] and [12]).

Putting all this together, we conclude:

LEMMA 6.8. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds: For any geodesic $\lambda \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, there exists a connected electric quasigeodesic β_{adm} without backtracking, such that*

- β_{adm} is built up of B_λ -admissible paths.
- β_{adm} joins the end-points of λ .
- $d(\lambda, p) \geq N \Rightarrow d(\beta_{adm}, p) \geq M(N)$.

Proof. — The first two criteria follow from the discussion preceding this lemma. The last follows from Lemma 6.4 since the discussion above gives a quasigeodesic built up out of admissible paths. □

6.3. Proof of Theorem

Electric Geometry Revisited

We note the following properties of the pair (X, \mathcal{H}) where $X = \widetilde{M}$ and \mathcal{H} consists of the lifts of Margulis tubes in M to the universal cover. Each lift of a Margulis tube shall henceforth be termed an *extended Margulis tube*. There exist C, D, Δ such that

- (1) Each extended Margulis tube is C -quasiconvex.
- (2) Any two extended Margulis tubes are D -separated.
- (3) The collection \mathcal{H} is C -cobounded, i.e. the nearest point projection of any member of \mathcal{H} onto any other has diameter bounded by C .
- (4) $\widetilde{M}_{el} = X_{el}$ is Δ -hyperbolic, (where $\widetilde{M}_{el} = X_{el}$ is the electric metric on $\widetilde{M} = X$ obtained by electrocuting all extended Margulis tubes, i.e. all members of \mathcal{H}).

- (5) (X_{el}, \mathcal{H}) has the *Bounded Penetration Property*.
 (6) An electro-ambient quasigeodesic is a hyperbolic quasigeodesic.

The first property follows from the fact that each ϵ -neighborhood of a closed geodesic in a hyperbolic manifold is convex for sufficiently small ϵ .

The second follows from choosing ϵ sufficiently small.

The third follows from the uniform separation of (the convex) extended Margulis sets.

The fourth and fifth follow from Lemmas 3.6 and 3.9.

Property (6) follows from Lemma 3.15.

Note. — So far we have not used the hypothesis that \widetilde{M} and hence, (from Property (4) above, or by Lemma 3.6) \widetilde{M}_{el} are hyperbolic metric spaces. It is at this stage that we shall do so and assemble the proof of the main Theorem.

THEOREM 6.9. — *Let M be a 3 manifold homeomorphic to $S \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M has i -bounded geometry, where $S_0 \subset B_0$ is the lower horizontal surface of the building block B_0 . Then the inclusion $i : \widetilde{S} \rightarrow \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{S} \rightarrow \widehat{M}$. Hence the limit set of \widetilde{S} is locally connected.*

Proof. — Suppose $\lambda \subset \widetilde{S}$ lies outside a large N -ball about p . By Lemma 6.8 we obtain an electric quasigeodesic without backtracking β_{adm} built up of B_λ -admissible paths lying outside an $M(N)$ -ball about p (where $M(N) \rightarrow \infty$ as $N \rightarrow \infty$).

Suppose that β_{adm} is a (K, ϵ) quasigeodesic. Note that K, ϵ depend on “the Lipschitz constant” of Π_λ and hence only on \widetilde{S} and \widetilde{M} .

From Property (6) (or Lemma 3.14) we find that if β^h denote the hyperbolic geodesic in \widetilde{M} joining the end-points of λ , then β^h lies in a (uniform) C' neighborhood of β_{adm} .

Let $M_1(N) = M(N) - C'$. Then $M_1(N) \rightarrow \infty$ as $N \rightarrow \infty$. Further, the hyperbolic geodesic β^h lies outside an $M_1(N)$ -ball around p . Hence, by Lemma 2.2, the inclusion $i : \widetilde{S} \rightarrow \widetilde{M}$ extends continuously to a map $\hat{i} : \widehat{S} \rightarrow \widehat{M}$.

Since the continuous image of a compact locally connected set is locally connected (see [11]) and the (intrinsic) boundary of \widetilde{S} is a circle, we conclude that the limit set of \widetilde{S} is locally connected.

This proves the theorem. □

7. Cannon-Thurston Maps for Surfaces With Punctures

7.1. Modification of Construction for Punctured Surfaces

We summarise the modifications to be made to the construction in the previous sections, so as to make the results applicable for punctured surfaces:

- (1) λ is now a horo-ambient quasigeodesic built out of a hyperbolic geodesic λ^h
- (2) Π_λ and B_λ are constructed as before
- (3) Let β_a be an ambient admissible quasigeodesic, i.e. an ambient quasigeodesic built up of elementary admissible paths. Then

$$\Pi_\lambda(\beta_a \cap \widetilde{M_{BH}}) \subset B_\lambda.$$

- (4) Joining the dots on this projected image of β_a , we obtain finally via Lemma 6.6 • Suppose λ^h lies outside large balls about a fixed reference point p . There exists an ambient admissible electric quasigeodesic β_{amb} in $\widetilde{M_{el}}$ such that any horizontal piece of $\beta_{amb} \cap B_\lambda^b$ also lies outside large balls. Further, any piece of β_{amb} lying wholly inside the lift of a Margulis tube also lies outside large balls. (Note that $B_\lambda^b = \bigcup_i \lambda_{ib}$).

Next recall that M^h is a hyperbolic manifold with cusps, and that excising these cusps we get M which is naturally homeomorphic to $S \times J$ where $J = \mathbb{R}$ or $[0, \infty)$. Further we may assume that M is given the structure of a model of **i-bounded geometry**. (Here we are abusing notation slightly as M , strictly speaking, is only quasi-isometric to a model of i-bounded geometry.) Let M_{el}^h, M_{el} denote M^h, M with Margulis tubes electrocuted. Let \mathcal{H}_0 denote the collection of horoballs that correspond to the lifts of cusps in $\widetilde{M_{el}^h}$. Thus, $\widetilde{M_{el}} = \widetilde{M_{el}^h} - \{H : H \in \mathcal{H}_0\}$. $\widetilde{M_{el}^h}$ is hyperbolic by Lemma 3.6.

Now let β_{el}^h be the electric geodesic in the hyperbolic metric space $\widetilde{M_{el}^h}$ joining the end-points a, b of λ^h . Let $H(\beta_{el}^h)$ denote the union of β_{el}^h and the collection of horoballs in \mathcal{H}_0 that β_{el}^h meets. Then by Theorem 3.12 (using the fact stated there that the theorem goes through for separated mutually cobounded uniformly quasiconvex sets), we have

- $H(\beta_{el}^h)$ is quasiconvex in $\widetilde{M_{el}^h}$.
- β_{amb} lies in a bounded electric neighborhood of $H(\beta_{el}^h)$.

7.2. Electrically close implies hyperbolically close

In what follows we want to construct out of β_{amb} a hyperbolic quasi-geodesic γ in \widetilde{M}_{el}^h such that entry and exit points of γ with respect to $H \in \mathcal{H}_0$ lie outside large balls $B_N(p) \subset \widetilde{M}$ (here the metric is the *hyperbolic* metric). The strategy is as follows.

For any $H_i \in \mathcal{H}_0$ look at the part β_i of β_{amb} that lies close to H_i . We claim that if this piece is long, then after pruning it a bit at the ends if necessary, the pruned subsegment of β_i lies **hyperbolically close** to H_i . We make this precise below.

By Theorem 3.12, and as in [13], there exists $\Delta > 0$ such that β_{amb} lies in an (electric) Δ neighborhood of $H(\beta_{el}^h)$. Let H_1, \dots, H_k denote the horoballs in $H(\beta_{el}^h)$. Let β_i be the maximal subsegment of β_{amb} joining points of $N_\Delta^{el}(H_i) \cap \beta_{amb}$. Then there exists $D = D(\Delta)$ such that $\beta_i \subset N_D^{el}(H_i)$. Let a_i, b_i be the end-points of β_i and P_i denote nearest point projection onto H_i . $[x, y]_e$ will denote the *electric* geodesic joining x, y . $[P_i(x), P_i(y)]$ will denote the *hyperbolic* geodesic joining $P_i(x), P_i(y)$ within the horoball H_i .

Fixing $K \geq 0$ ($K = 4D$ will suffice for our purposes) let $c_i, d_i \in \beta_i$ be such that $\overline{c_i d_i}$, the subsegment of β_i joining c_i, d_i has length less than K . Suppose further that a_i, c_i, d_i, b_i occur in that order along the segment joining a_i, b_i . Then

$$[P_i(c_i), c_i]_e \cup \overline{c_i d_i} \cup [d_i, P_i(d_i)]_e \cup [P_i(d_i), P_i(c_i)] = \sigma$$

is a loop of electric length less than $C = C(K, D)$ ($= C(D)$ if $K = 4C$). This follows from the following observations:

- (1) $[P_i(c_i), c_i]_e, [P_i(d_i), d_i]_e$ have length less than or equal to D
- (2) $\overline{c_i d_i}$ has length less than K
- (3) $[P_i(c_i), P_i(d_i)]$ has length bounded in terms of K by Lemma 3.20.

Since σ has electric length less than C , we could conclude that σ has bounded hyperbolic length by Lemma 3.17 if in addition we knew that σ does not backtrack. (In particular we would be able to show that σ has bounded penetration.) However, we only know that each of the four components of σ individually does not backtrack. In fact,

$$[c_i, P_i(c_i)]_e \cup [P_i(c_i), P_i(d_i)] \cup [P_i(d_i), d_i]_e$$

is a path without backtracking. Therefore, backtracking, if it exists, is a consequence of overlap of initial segments of $[c_i, P_i(c_i)]_e$ and $\overline{c_i d_i}$ (or, $[d_i, P_i(d_i)]_e$ and $\overline{d_i, c_i}$). Clearly, such overlaps can have length at most D . Therefore, any such segment $\overline{c_i d_i}$ with $d_e(a_i, c_i) \geq D, d_e(b_i, d_i) \geq D$ must have bounded

penetration property (since the paths

$$[a_i, P_i(a_i)]_e \cup [P_i(a_i), P_i(b_i)] \cup [P_i(b_i), b_i]_e$$

and $\overline{a_i b_i}$ can have overlaps of length at most D at the beginning and end), i.e there exists $D_0 = D_0(D, K) \geq 0$ such that $\overline{c_i d_i} \cap T$ has hyperbolic length less than D_0 (where T is any lift of a Margulis tube).

Now, choose $x \in \overline{a_i b_i}$ such that $d_e(x, a_i) \geq 2D$ and $d_e(x, b_i) \geq 2D$. Choose c_i, d_i such that $d_e(x, c_i) = 2D, d_e(x, d_i) = 2D$ and a_i, c_i, d_i, b_i lie in that order along the path from a_i to b_i . Then using the loop

$$[P_i(c_i), c_i]_e \cup \overline{c_i, x} \cup [x, P_i(x)]_e \cup [P_i(x), P_i(d_i)]_e = \sigma$$

and the argument above, we conclude that $[x, P(x)]_e$ satisfies the bounded penetration property and hence $[x, P(x)]$ has bounded hyperbolic length. This is summarised in the following Lemma.

LEMMA 7.1. — *There exists $D_0 \geq 0$ such that the following holds. Let $\beta_i = \overline{a_i b_i}$ be as above and $x \in \overline{a_i b_i}$ with $d_e(x, a_i) \geq 2D, d_e(x, b_i) \geq 2D$. Then $d(x, H_i) \leq D_0$. (Note that $d(x, H_i)$ denotes hyperbolic distance.)*

Thus the subpath of β_i obtained by pruning pieces of (electric) length $2D$ from the beginning and the end lies in a bounded hyperbolic neighborhood of H_i (and not just in a bounded electric neighborhood of H_i).

7.3. Constructing an electric quasigeodesic

The argument in this subsection is a slight modification of the argument in [23] for punctured surfaces of bounded geometry. The slight modification is due to pruning electric quasigeodesics that follow a horoball for a considerable length.

Choose from the the collection of $H_i \in \mathcal{H}(\beta)$ the subcollection for which β_i has diameter greater than $4D$. We denote this subcollection as $\mathcal{H}_l(\beta)$ (l stand for “large”). Let H_{l1}, \dots, H_{lk} be the horoballs in this collection.

For the relevant subpaths $\beta_{l1}, \dots, \beta_{lk}$ of β we construct $\gamma_{l1}, \dots, \gamma_{lk}$ as follows.

Let $\alpha_{li} = \overline{c_{li} d_{li}} \subset \beta_{li} = \overline{a_{li} b_{li}}$ denote the subpath at distance less than or equal to D from H_{li} . By Lemma 7.1 we have $d_e(c_{li}, a_{li}) \leq 2D$ and $d_e(d_{li}, b_{li}) \leq 2D$. Let

$$\gamma_{li} = [c_{li}, P_{li}(c_{li})]_e \cup [P_{li}(c_{li}), P_{li}(d_{li})] \cup [P_{li}(d_{li}), d_{li}]_e$$

Let $\gamma = (\beta - \bigcup_i \beta_{li}) \cup \bigcup_i \gamma_{li}$.

Each β_{li} starts and ends (electrically) close to the entry and exit points u_{li}, v_{li} of β_{el}^h with respect to the horoball H_{li} .

Since c_{li}, d_{li} are close (bounded by $2D$) to a_{li}, b_{li} respectively, then from Lemma 3.10, Lemma 3.17 and Lemma 3.20 we find that there exists $D_1 \geq 0$ such that

$$\begin{aligned} d(P_{li}(c_{li}), P_{li}(a_{li})) &\leq D_1 \\ d(P_{li}(d_{li}), P_{li}(b_{li})) &\leq D_1 \end{aligned}$$

Note that d here is the *hyperbolic* distance. Hence the hyperbolic geodesic $[c_{li}, d_{li}]$ lies close to $[a_{li}, b_{li}]$ and hence to the hyperbolic geodesic $[u_{li}, v_{li}]$ (by fellow traveller property).

Thus we conclude

- γ lies in a bounded neighbourhood of the electric geodesic β_{el}^h .

Note. — The remaining β_i 's being less than $4D$ in length are therefore uniformly bounded. Hence their projections onto the corresponding H_i 's are also uniformly bounded in diameter. The length of $\beta_{el}^h \cap H_i$ for these H_i 's is also therefore uniformly bounded. (Else the projection onto $\beta_{el}^h \cup \mathcal{H}(\beta_{el}^h)$ would have to have jumps and hence not be “large-scale continuous”.)

Since γ is obtained from β_{amb} , γ tracks β_{el}^h off horoballs. Further, since entry and exit points of γ and β_{el}^h with respect to horoballs are a bounded distance apart, they are fellow travellers within horoballs. From this it follows easily that γ is an electric quasigeodesic.

γ therefore has two properties:

- (1) γ lies close to β_{el}^h and is an electric quasigeodesic.
- (2) All points of $\gamma \cap \widetilde{M}$ lie outside a large ball about the fixed reference point p if λ^h does.

The first property follows from the above discussion and the last is just a restatement of property (4) of Section 7.1 (the first subsection of the present section), coupled with the fact that entry points of γ into horoballs $H_i \in \mathcal{H}$ lie *hyperbolically* close to β_{amb} .

This gives rise to the following property of γ . Recall that building blocks are built from the truncated surface S , and that we fix a “starting block” B_0 . We identify $S \times \{0\}$ with the truncated surface obtained from S^h the hyperbolic reference surface.

PROPOSITION 7.2. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda^h \subset \widetilde{S}^h$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$,

there exists a connected electric quasigeodesic γ without backtracking, such that

- If λ^h lies outside $B_N(p)$, then every point x of $\gamma - \{H : H \in \mathcal{H}\}$ lies at a hyperbolic distance of at least $M(N)$ from p .

The above Proposition is a punctured surface version of Lemma 6.8.

Now, recall a Lemma from [23] (which has been proven there as a part of Theorem 5.9).

LEMMA 7.3. — *There exists a function $M_1(N)$ such that $M_1(N) \rightarrow \infty$ as $N \rightarrow \infty$ satisfying the following.*

Given a uniformly separated collection of horoballs \mathcal{H} and a point p lying outside them, let γ be a path without backtracking, such that

$$\gamma - \{H : H \in \mathcal{H}\}$$

lies outside $B_N(p)$. Further suppose that $\gamma \cap H$ is a (hyperbolic) geodesic, whenever $\gamma \cap H$ is non-empty. Then γ lies outside an $M_1(N)$ ball about p .

Combining Proposition 7.2 (for pieces of γ outside horoballs) and Lemma 7.3 above (for the geodesic segments within horoballs) we conclude:

PROPOSITION 7.4. — *There exists a function $M(N) : \mathbb{N} \rightarrow \mathbb{N}$ such that $M(N) \rightarrow \infty$ as $N \rightarrow \infty$ for which the following holds:*

For any geodesic $\lambda^h \subset \widetilde{S}^h$, and a fixed reference point $p \in \widetilde{S} \times \{0\} \subset \widetilde{B}_0$, there exists a connected electric quasigeodesic γ without backtracking, such that

- *If λ^h lies outside $B_N(p)$, then every point x of γ lies at a hyperbolic distance of at least $M(N)$ from p .*

7.4. From electric quasigeodesics to hyperbolic quasigeodesics

We have thus constructed an electric quasigeodesic γ without backtracking joining the end-points of λ^h every point of which lies outside a (hyperbolic) large ball about p . The last step is to promote γ to a *hyperbolic quasigeodesic*.

Since γ is built up of admissible paths within Margulis tubes, we might as well assume that γ is an electro-ambient quasigeodesic without backtracking.

LEMMA 7.5. — *The underlying path of γ is a hyperbolic quasigeodesic.*

Proof. — Margulis tubes satisfy the mutual co-boundedness property by Lemma 3.10. Hence by Lemma 3.15, γ is a hyperbolic quasigeodesic. \square

THEOREM 7.6. — *Let M^h be a 3 manifold homeomorphic to $S^h \times J$ (for $J = [0, \infty)$ or $(-\infty, \infty)$). Further suppose that M^h has i -bounded geometry. Let $S_0 \subset B_0$ be the lower horizontal surface of the building block B_0 in the manifold M obtained by removing cusps. Then the inclusion $i : \widetilde{S}^h \rightarrow \widetilde{M}^h$ extends continuously to a map $\hat{i} : \widehat{S}^h \rightarrow \widehat{M}^h$. Hence the limit set of \widetilde{S} is locally connected.*

Proof. — Suppose $\lambda^h \subset \widetilde{S}^h$ lies outside a large N -ball about p . By Lemma 7.4 and Lemma 7.5, we obtain a hyperbolic quasigeodesic γ lying outside an $M(N)$ -ball about p (where $M(N) \rightarrow \infty$ as $N \rightarrow \infty$).

If β^h denote the hyperbolic geodesic in \widetilde{M}^h joining the end-points of λ^h , then β^h lies in a (uniform) C' neighborhood of γ (since hyperbolic quasigeodesics starting and ending at the same points track each other throughout their lengths).

Let $M_1(N) = M(N) - C'$. Then $M_1(N) \rightarrow \infty$ as $N \rightarrow \infty$. Further, the hyperbolic geodesic β^h lies outside an $M_1(N)$ -ball around p . Hence, by Lemma 2.2, the inclusion $i : \widetilde{S}^h \rightarrow \widetilde{M}^h$ extends continuously to a map $\hat{i} : \widehat{S}^h \rightarrow \widehat{M}^h$.

Since the continuous image of a compact locally connected set is locally connected (see [11]) and the (intrinsic) boundary of \widetilde{S}^h is a circle, we conclude that the limit set of \widetilde{S}^h is locally connected.

This proves the theorem. \square

The proof of the above theorem is just a modification of Theorem 6.9, once Lemma 7.4 and Lemma 7.5 are in place.

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