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Isogeometric analysis with C^1 functions on planar, unstructured quadrilateral meshes

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Abstract. In the context of isogeometric analysis, globally C^1 isogeometric spaces over unstructured quadrilateral meshes allow the direct solution of fourth order partial differential equations on complex geometries via their Galerkin discretization. The design of such smooth spaces has been intensively studied in the last five years, in particular for the case of planar domains, and is still task of current research. In this paper, we first give a short survey of the developed methods and especially focus on the approach [28]. There, the construction of a specific C^1 isogeometric spline space for the class of so-called analysis-suitable G^1 multi-patch parametrizations is presented. This particular class of parameterizations comprises exactly those multi-patch geometries, which ensure the design of C^1 spaces with optimal approximation properties, and allows the representation of complex planar multi-patch domains. We present known results in a coherent framework, and also extend the construction to parametrizations that are not analysis-suitable G^1 by allowing higher-degree splines in the neighborhood of the extraordinary vertices and edges. Finally, we present numerical tests that illustrate the behavior of the proposed method on representative examples.

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1. Introduction

Isogeometric analysis (IgA) is a framework for numerically solving partial differential equations (PDEs) whose basic idea is to bridge the gap between geometric modeling (that is, Computer-Aided Design) and numerical analysis (that is, Finite Element Analysis) by using the same (rational) spline function space for representing the geometry of the computational domain and for describing the solution of the PDE (cf. [5, 16, 23]). In contrast to finite elements, IgA allows a simple integration of smooth discretization spaces for the numerical simulation. While the design of such smooth spaces is trivial for single patch geometries, it is a challenging task for the case of multi-patch or manifold geometries.

The scope of this paper is to give a survey of the different existing methods for the construction of strongly enforced C^1 isogeometric spline spaces over planar, unstructured quadrilateral meshes (that is, planar multi-patch geometries composed of quadrilateral patches with possibly extraordinary vertices), see Section 2, with a special focus on the approach [28], see Section 4. We consider unstructured quadrilateral meshes allowing regular as well as extraordinary vertices and do not allow T-junctions in the mesh. Note that such a configuration can always be interpreted as a multi-patch geometry. The common goal of the developed techniques is to generate isogeometric spline spaces, which are not only exactly C^1-smooth within the single patches but also across the patch interfaces. The design of the C^1 isogeometric spaces is mainly based on the observation that an isogeometric function is C^1-smooth if and only if the associated graph surface is G^1-smooth (that is, geometric continuous of order 1), cf. [21]. The resulting global C^1-smoothness of the spaces then enables the solution of fourth order PDEs just via its weak form using a standard Galerkin discretization, see for example [4, 15, 25, 29, 44, 54] for the biharmonic equation, [3, 6, 34, 35, 36] for the Kirchhoff-Love shell formulation, [18, 19, 38] for the Cahn-Hilliard equation and [17, 46] for plane problems of first strain gradient elasticity.

A further possible strategy to impose C^1-smoothness across the interfaces of general multi-patch geometries is the use of subdivision surfaces (e.g., [11, 13, 14, 24, 43, 53, 58]). The surfaces are recursively generated via refinement schemes, and are described in the limit as the collection of infinitely many polynomial patches, e.g., in the case of Catmull-Clark subdivision of bicubic patches. We refer to [50] for further reading. Challenges of dealing with subdivision surfaces in IgA include the need for special techniques for the numerical integration [24] and the often reduced approximation power in the neighborhood of extraordinary vertices [43].

Instead of enforcing the C^1-continuity across the patch interfaces in a strong sense, the C^1-smoothness could also be achieved by coupling the neighboring patches in a weak sense. We do not cover this approach here, which is typically based on adding penalty terms to the weak formulation of the PDE [1, 22], or using Lagrange multipliers [1, 7]. These techniques are applicable to quite general multi-patch geometries with even non-matching meshes, but at the cost of obtaining an approximate C^1 solution. Moreover, the formulation of the problem, and as a result the system matrix, have to be adapted accordingly.

The outline of this paper is as follows. Section 2 describes the state of the art of such constructions over planar multi-patch domains. Section 3 further discusses the case of multi-patch parameterizations which are regular (that is, non-singular) and C^0 at the patch interfaces, and specifically describes so-called analysis-suitable G^1 parametrizations. In this setting we review the construction of C^1 isogeometric spaces, and their properties, in Section 4. Section 5 extends the construction beyond analysis-suitable G^1 parametrizations, by allowing higher-order splines around the extraordinary vertices. Numerical evidence of the optimal order of convergence of the proposed method is reported in Section 6.

2. The design of C^1 isogeometric spaces

We give an overview of existing strategies for the design of strongly enforced C^1 isogeometric spline spaces over unstructured quadrilateral meshes on planar domains. Such quadrilateral meshes can be understood in the context of multi-patch domains or spline manifolds. In the language of manifolds, we have given local charts, which usually overlap. The global smoothness is then determined by the smoothness within every chart. In the multi-patch framework, the patches do not overlap but share common interfaces. Hence, the smoothness is determined by the *smoothness within the patches* as well as the *smoothness across interfaces*. Since most CAD systems are built upon multi-patch structures we focus on this point of view.

The different techniques can be roughly classified into three approaches depending on the smoothness of the underlying parameterization $\mathbf{F}$ for the multi-patch domain Ω across its interfaces. Note that the smoothness of the isogeometric space depends on the smoothness of the underlying parameterization as well as on additional smoothness constraints in physical space. In the first case, these additional constraints need to be satisfied only at the extraordinary vertices, in the second case in a neighborhood of the vertices; and in the third case globally along the interfaces. Finally, for polar configurations the additional constraints need to be satisfied at the pole.

Multi-patch parameterizations which are C^1-smooth everywhere: In this setting, the parameterization of the multi-patch domain is assumed to be C^1-smooth everywhere. This immediately leads to a singularity appearing at every extraordinary vertex (EV). This was first studied in [48]. Consequently, the isogeometric functions are then C^1 everywhere away from the EV and possibly only C^0 at the EV, due to the singularity. To circumvent this issue, one has to enforce additional G^1 constraints at the EVs. This technique is based on the use of specific degenerate patches (e.g. D-patches [52]) in the neighborhood of an extraordinary vertex. These patches are obtained by collapsing some control points into one point and by a special configuration of some of the remaining control points which guarantee that the surface is G^1-smooth at the EV despite having a singularity there. The same approach allows to construct isogeometric functions that are C^1 everywhere on the multi-patch domain. Examples of this method are [45, 56, 57]. While the constructions [45, 57] are restricted to bicubic splines, the methodology [56] can be applied to bivariate splines of arbitrary bidegree (p, p) with $p \geq 3$. All three techniques can be used to construct sequences of nested isogeometric splines spaces.

A similar approach is to use subdivision surfaces to represent the isogeometric spaces. In subdivisions, the surface around an EV is composed of an infinite sequence of spline rings, where every ring is (at least) C^1 smooth. The shape of the surface at the EVs is guided by the refinement rules of the control mesh. For example, for Catmull-Clark subdivision this procedure generates a surface that is C^2 smooth everywhere and G^1 at the EVs. However, the approach suffers from a lack of approximation power near the EV. See [11, 43, 53, 58], where subdivision based isogeometric analysis was studied.

Multi-patch parameterizations which are C^1-smooth except in the vicinity of an extraordinary vertex: The core idea is to construct a parameterization of the multi-patch domain which is C^1-smooth in the regular regions of the mesh and only C^0-smooth in a neighborhood of the extraordinary vertices (see e.g. [10]). To obtain a globally C^1 isogeometric space, a G^1 surface construction is employed in the neighborhood of the EV. The same construction is used to generate the C^1 isogeometric functions over the multi-patch domain. This construction leads in general to a multi-patch surface which is even C^{p-1}-smooth away from an extraordinary vertex. One possibility is to use so-called G-splines, see [51]. To obtain surfaces of good shape also in the vicinity of an extraordinary vertex, the G^1-smoothness is obtained by using a suitable surface cap, which requires a slightly higher degree than the surrounding C^1 spline surface. The resulting smooth surfaces are then used to construct C^1 isogeometric splines spaces, but which are in general not nested, see e.g. [30, 44]. The method [44] employs the surface construction [31], which is based on a biquadratic C^1 spline surface and on a bicubic or biquartic G^1 surface cap depending on the valency of the corresponding extraordinary vertex. The paper [30] presents a new surface construction, where bicubic splines are complemented by biquartic splines in the neighborhood of extraordinary vertices. The methodology [33] can be seen as an extension of the above two techniques, and allows the construction of nested C^1 isogeometric spline spaces for a finite number of refinement steps. Another method which allows the construction of nested C^1 isogeometric spline spaces is [32]. Instead of further refining a G^1 surface of already good quality, the C^1 isogometric spline space over the surface is refined to obtain nested C^1 spaces but at the expense of keeping the higher degree in the entire initial neighborhood of an EV.

Multi-patch parameterizations which are (in general) just C^0-smooth at all interfaces: The main idea is to consider multi-patch parametrizations that are everywhere regular (non-singular) but only C^0 at the patch interfaces, and then construct C^1 isogeometric spaces over them. Again, the key issue for application to isogeometric analysis is to guarantee good approximation properties of these spaces. In [9, 40] the authors gave dimension formulas for meshes of arbitrary topology, consisting of quadrilateral polynomial patches and specific macro-elements. The techniques in [9, 40] work for splines of general bidegree (p, p) for some large enough p, and generate the C^1 basis functions by analyzing the module of syzygies of specific polynomial/spline functions. Extending from polynomials to general spline patches, dimensions were given and basis functions were constructed for bilinear two-patch domains in [29]. In [15] the reproduction properties of the C^1-smooth subspaces along an interface were studied for arbitrary B-spline patches. From the presented results, bounds for the dimension of the C^1-smooth subspaces of arbitrary geometries can be derived. Moreover, the specific class of analysis-suitable G^1 parametrizations was identified. In the last few years, a number of methods were developed which follow this approach, and which allow in most cases the design of nested C^1 isogeometric spline spaces. These techniques use particular classes of C^0 regular multi-patch parameterizations to obtain C^1 isogeometric spaces with good/optimal approximation properties:

- *(Mapped) bilinear multi-patch parameterizations (e.g.* [8, 25, 29, 39]*):* The aim is to explore C^1 isogeometric spaces over bilinear or mapped bilinear multi-patch geometries. The methods [8, 39] study the spaces of biquintic and for some specific cases also biquartic C^1 isogeometric Bézier functions, and generate basis functions which are implicitly given by minimal determining sets (cf. [37]) for the involved Bézier coefficients. In contrast to the other techniques using C^0 regular multi-patch parameterizations, the resulting spaces are not nested. In [25, 29], the case of bicubic and biquartic C^1 spline elements is considered. The resulting basis functions are explicitly given by simple formulae and possess a small local support. While the paper [29] deals with the case of two patches, the work in [25] is an extension to the multi-patch case.

- *General analysis-suitable parameterizations (e.g.* [15, 26, 28]*):* The previous strategy was based on simple geometries such as bilinear parameterizations. The following approach uses a more general class of geometries, called analysis-suitable G^1 parameterizations, cf. [15] and Section 3.2, which includes the previous types of geometries. The class of analysis-suitable G^1 parameterizations contains exactly those geometries which allow the design of C^1 isogeometric spline spaces with optimal approximation properties. In [26], the space of C^1 isogeometric spline functions over analysis-suitable G^1 two-patch geometries was analyzed. The developed method is applicable to splines of general bidegree (p, p) with $p \geq 3$ and patch regularity $1 \leq r \leq p-2$, and constructs simple, explicitly given basis functions with a small local support. The work [28] extends the construction [26] to the case of analysis-suitable G^1 multi-patch parameterizations and will be discussed in detail in Section 4

- *Non-analysis-suitable parametrizations (e.g.* [12]*):* Instead of using a particular class of multi-patch parameterizations for the multi-patch domain, the method [12] increases locally along the patch interfaces the bidegree of the C^1 isogeometric spline functions to get spaces with good approximation properties. The constructed C^1 basis functions are implicitly given by means of minimal determining sets for the spline coefficients and possess in general large supports over one or more entire patch interfaces. This approach is a direct consequence of the results presented in [15] and extends the ideas of Theorems 1 and 3 therein. We will present further theoretical foundations in this paper.

Polar configurations: This approach is outside of the framework of unstructured multi-patch domains, as it can be interpreted as a regular mesh in polar coordinates. However, many ideas to study and construct smooth polar configurations can be carried over to extraordinary vertices. The approach is based on the use of polar splines to model the domains and to construct C^1 isogeometric spline spaces over these domains, see e.g. [42, 43, 55]. The method [43] employs the polar surface construction of [41], which generates a bicubic C^1 polar spline surface. In [55], a polar spline technology for splines of arbitrary bidegree (p,p) is presented. It can be used to generate polar spline surfaces which are C^s-smooth ($s \geq 0$) everywhere except at the polar point where the resulting surface is discontinuous. Furthermore, it was shown that this surface construction can be used to generate globally C^k isogeometric function spaces.

3. Multi-patch geometries and their analysis-suitable G^1 parameterization

We now focus on multi-patch parameterizations which are regular and C^0 at all interfaces. After some preliminaries and notation on multi-patch geometries, we will consider one specific class of geometries, called analysis-suitable G^1-multi-patch parameterizations (cf. [15]), which will be used throughout the paper. Furthermore, we will use in the following a slightly adapted notation and definitions mainly based on our work developed in [28] and [27].

3.1. Multi-patch domain

Let $p \geq 3$, $1 \leq r \leq p-2$ and $n \geq 1$. We denote by $\mathcal{S}_h^{p,r}$ the univariate spline space of degree p and continuity C^r on the parameter domain $[0,1]$ possessing the uniform open knot vector

$$\Big(\underbrace{0,\ldots,0}_{(p+1)-\text{times}}, \underbrace{\tfrac{1}{n}\cdots,\tfrac{1}{n}}_{(p-r)-\text{times}}, \underbrace{\tfrac{2}{n},\ldots,\tfrac{2}{n}}_{(p-r)-\text{times}}, \ldots, \underbrace{\tfrac{n-1}{n},\ldots,\tfrac{n-1}{n}}_{(p-r)-\text{times}}, \underbrace{1,\ldots,1}_{(p+1)-\text{times}} \Big)$$

with the mesh size $h = \frac{1}{n}$, and by $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$ with $\mathbf{p} = (p,p)$ and $\mathbf{r} = (r,r)$ the corresponding bivariate tensor-product spline space $\mathcal{S}_h^{p,r} \otimes \mathcal{S}_h^{p,r}$ on the parameter domain $[0,1]^2$. In addition, let b_j, $j = 0,\ldots,N-1$, with $N = p + (n-1)(p-r) + 1$, be the B-splines of $\mathcal{S}_h^{p,r}$, and let $b_{\mathbf{j}}$, $\mathbf{j} = (j_1,j_2) \in \{0,\ldots,N-1\}^2$, be the tensor-product B-splines of $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$, that is,

$$b_{(j_1,j_2)}(\xi_1,\xi_2) = b_{j_1}(\xi_1) b_{j_2}(\xi_2).$$

Consider an open domain $\Omega \subset \mathbb{R}^2$, which is given as the union of quadrilateral patches $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, interfaces $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, and inner vertices $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_\chi^\circ$, that is,

$$\Omega = \left(\bigcup_{i \in \mathcal{I}_\Omega} \Omega^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\Sigma^\circ} \Sigma^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\chi^\circ} \mathbf{x}^{(i)} \right).$$

We assume that all patches are mutually disjoint and that no hanging nodes exist. The boundary Γ of Ω, that is, $\Gamma = \partial\Omega$, is given as the collection of boundary edges $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\Gamma$, and boundary vertices $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_\chi^\Gamma$, that is,

$$\Gamma = \left(\bigcup_{i \in \mathcal{I}_\Sigma^\Gamma} \Sigma^{(i)} \right) \cup \left(\bigcup_{i \in \mathcal{I}_\chi^\Gamma} \mathbf{x}^{(i)} \right).$$

In addition, we assume that $\mathcal{I}_\Sigma^\circ \cap \mathcal{I}_\Sigma^\Gamma = \emptyset$ and $\mathcal{I}_\mathbf{x}^\circ \cap \mathcal{I}_\mathbf{x}^\Gamma = \emptyset$, and denote by $\mathcal{I}_\Sigma$ and $\mathcal{I}_\chi$ the index sets $\mathcal{I}_\Sigma = \mathcal{I}_\Sigma^\circ \,\dot\cup\, \mathcal{I}_\Sigma^\Gamma$ and $\mathcal{I}_\chi = \mathcal{I}_\chi^\circ \,\dot\cup\, \mathcal{I}_\chi^\Gamma$, respectively.

Each quadrilateral patch $\Omega^{(i)}$ is the open image of a bijective and regular geometry mapping

$$\mathbf{F}^{(i)} : [0,1]^2 \to \overline{\Omega^{(i)}} \subset \mathbb{R}^2,$$

with $\mathbf{F}^{(i)} \in \mathcal{S}_h^{\mathbf{p},\mathbf{r}} \times \mathcal{S}_h^{\mathbf{p},\mathbf{r}}$. We denote by $\mathbf{F}$ the resulting multi-patch geometry of Ω consisting of the single geometry mappings $\mathbf{F}^{(i)}$, $i \in \mathcal{I}_\Omega$.

3.2. Analysis-suitable G^1 parameterization: definition

Consider an interface $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$. Let $\Omega^{(i_1)}$ and $\Omega^{(i_2)}$, $i_1, i_2 \in \mathcal{I}_\Omega$, be the two neighboring patches with $\Sigma^{(i)} \subset \overline{\Omega^{(i_1)}} \cap \overline{\Omega^{(i_2)}}$. The two associated geometry mappings $\mathbf{F}^{(i_1)}$ and $\mathbf{F}^{(i_2)}$ can be always reparameterized (if necessary) into standard form (cf. [28]), which just means that the common interface $\Sigma^{(i)}$ is given by

$$\mathbf{F}^{(i_1)}(0,\xi) = \mathbf{F}^{(i_2)}(\xi,0), \ \xi \in (0,1), \tag{3.1}$$

see Fig. 3.1 (left).

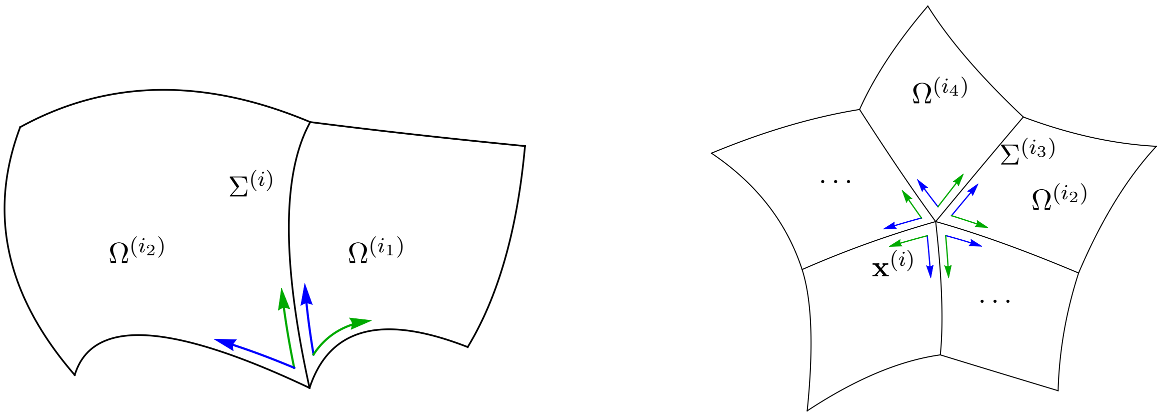


FIGURE 3.1. Representation in standard form (cf. [28]) of two patches $\Omega^{(i_1)}$ and $\Omega^{(i_2)}$ with the common interface $\Sigma^{(i)}$ (left) and of the patches $\Omega^{(i_2)}, \Omega^{(i_4)}, \ldots, \Omega^{(i_{2\nu})}$ possessing the common vertex $\mathbf{x}^{(i)}$ (right).

There exist uniquely determined functions $\alpha^{(i,i_1)} : [0,1] \to \mathbb{R}$, $\alpha^{(i,i_2)} : [0,1] \to \mathbb{R}$ and $\beta^{(i)} : [0,1] \to \mathbb{R}$ up to a common function $\gamma^{(i)}$ (with $\gamma^{(i)}(\xi) \neq 0$), which are given by

$$\begin{aligned}
\alpha^{(i,i_1)}(\xi) &= \gamma^{(i)}(\xi) \det \begin{bmatrix} \partial_1 \mathbf{F}^{(i_1)}(0,\xi) & \partial_2 \mathbf{F}^{(i_1)}(0,\xi) \end{bmatrix}, \\
\alpha^{(i,i_2)}(\xi) &= \gamma^{(i)}(\xi) \det \begin{bmatrix} \partial_1 \mathbf{F}^{(i_2)}(\xi,0) & \partial_2 \mathbf{F}^{(i_2)}(\xi,0) \end{bmatrix}, \\
\beta^{(i)}(\xi) &= \gamma^{(i)}(\xi) \det \begin{bmatrix} \partial_2 \mathbf{F}^{(i_2)}(\xi,0) & \partial_1 \mathbf{F}^{(i_1)}(0,\xi) \end{bmatrix},
\end{aligned} \tag{3.2}$$

and satisfy for all $\xi \in [0,1]$

$$\alpha^{(i,i_1)}(\xi)\alpha^{(i,i_2)}(\xi) > 0 \tag{3.3}$$

and

$$\alpha^{(i,i_1)}(\xi)\partial_2 \mathbf{F}^{(i_2)}(\xi,0) + \alpha^{(i,i_2)}(\xi)\partial_1 \mathbf{F}^{(i_1)}(0,\xi) + \beta^{(i)}(\xi)\partial_2 \mathbf{F}^{(i_1)}(0,\xi) = \mathbf{0}. \tag{3.4}$$

In addition, there exists non-unique functions $\beta^{(i,i_1)} : [0,1] \to \mathbb{R}$ and $\beta^{(i,i_2)} : [0,1] \to \mathbb{R}$ such that

$$\beta^{(i)}(\xi) = \alpha^{(i,i_1)}(\xi)\beta^{(i,i_2)}(\xi) + \alpha^{(i,i_2)}(\xi)\beta^{(i,i_1)}(\xi), \tag{3.5}$$

see e.g. [15, 49]. The functions $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$, $\beta^{(i,i_1)}$ and $\beta^{(i,i_2)}$ are called the gluing data[1] for the interface $\Sigma^{(i)}$.

In the remainder of the paper, we will restrict ourselves to a specific class of multi-patch geometries, called analysis-suitable G^1 multi-patch parameterizations, which are needed to generate C^1 isogeometric spaces with optimal approximation properties, cf. [15].

[1]In the context of CAGD, these functions have been also called e.g. "shape parameters" [20] or "weight functions" [47].

Definition 3.1 (Analysis-suitable G^1 multi-patch parameterization, cf. [15, 28])**.** A multi-patch geometry $\mathbf{F}$ is called analysis-suitable G^1 (in short, AS-G^1), if for every interface $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$ there exist linear polynomials $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$, $\beta^{(i,i_1)}$ and $\beta^{(i,i_2)}$, with $\alpha^{(i,i_1)}$ and $\alpha^{(i,i_2)}$ relatively prime, such that (3.3), (3.4) and (3.5) hold.

Furthermore, for each interface $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, the linear gluing data $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$, $\beta^{(i,i_1)}$ and $\beta^{(i,i_2)}$ is selected by minimizing the terms

$$\|\alpha^{(i,i_1)} - 1\|^2_{L_2([0,1])} + \|\alpha^{(i,i_2)} - 1\|^2_{L_2([0,1])}$$

and

$$\|\beta^{(i,i_1)}\|^2_{L_2([0,1])} + \|\beta^{(i,i_2)}\|^2_{L_2([0,1])},$$

see [28], which implies in case of parametric continuity (that is, $\beta^{(i)} \equiv 0$ and $\alpha^{(i,i_1)} = \alpha^{(i,i_2)}$) $\beta^{(i,i_1)} = \beta^{(i,i_2)} \equiv 0$ and $\alpha^{(i,i_1)} \equiv \alpha^{(i,i_2)} \equiv 1$.

Remark 3.2. Given a C^1 isogeometric function space (as defined in Section 4.1) over a regular C^0 multi-patch parametrization. Then, analysis-suitable G^1 multi-patch parameterizations are the only configurations that allow optimal approximation properties under h-refinement with respect to the underlying norm of the considered problem. Note that optimal approximation is only possible under the additional condition that the spline space has reduced continuity $r \leq p-2$.

The reason for this is, that when the degree of the gluing data is assumed to be larger than one, there exist configurations such that the approximation order of function values and/or gradients along the interface is reduced. This is a direct consequence of Theorem 3 in [15].

Piecewise bilinear multi-patch parameterizations are one simple example of AS-G^1 multi-patch geometries (cf. [25, 29]), but the class of AS-G^1 multi-patch parameterizations is much wider, see e.g. [27]. It is interesting to note that similar linear gluing data also appear in the context of triangular meshes, see e.g. [47]. In Section 3.3, we will present two possible strategies to construct from given non-AS-G^1 multi-patch geometries parameterizations which are AS-G^1-continuous.

3.3. Analysis-suitable G^1 parameterization: construction

We describe the two approaches [27, 29] to generate from a given initial non-AS-G^1 multi-patch geometry $\widetilde{\mathbf{F}}$ an AS-G^1 multi-patch geometry $\mathbf{F}$. We assume that the associated parameterizations $\widetilde{\mathbf{F}}^{(i)}$, $i \in \mathcal{I}_\Omega$, of the non-AS-G^1 geometry $\widetilde{\mathbf{F}}$ belong to the space $\left(\mathcal{S}_{\widetilde{h}}^{\widetilde{\mathbf{p}},\widetilde{\mathbf{r}}}\right)^2$ and are regularly parameterized. The goal is to construct a multi-patch geometry $\mathbf{F}$ consisting of parameterizations $\mathbf{F}^{(i)} \in \left(\mathcal{S}_h^{\mathbf{p},\mathbf{r}}\right)^2$, $i \in \mathcal{I}_\Omega$, with $\mathcal{S}_h^{\mathbf{p},\mathbf{r}} \supseteq \mathcal{S}_{\widetilde{h}}^{\widetilde{\mathbf{p}},\widetilde{\mathbf{r}}}$, possessing B-spline representations of the form

$$\mathbf{F}^{(i)}(\xi_1,\xi_2) = \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} \mathbf{c}^{(i)}_{i_1,i_2} b_{(j_1,j_2)}(\xi_1,\xi_2), \ (\xi_1,\xi_2) \in [0,1]^2,$$

with control points $\mathbf{c}^{(i)}_{i_1,i_2} \in \mathbb{R}^2$, such that $\mathbf{F}$ is AS-G^1-continuous and that $\mathbf{F}$ approximates $\widetilde{\mathbf{F}}$ as good as possible. Below, we assume that for each edge $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma$, and for each vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_\chi$, the associated geometry mappings $\mathbf{F}^{(i_k)}$ are always given in standard form as shown in Fig. 3.1 and formally defined in (3.1) and (4.6). This is valid as well for the corresponding parameterizations $\widetilde{\mathbf{F}}^{(i_k)}$ of $\widetilde{\mathbf{F}}$.

3.3.1. *The piecewise bilinear fitting approach* [29]

Given a non-AS-G^1 multi-patch geometry $\widetilde{\mathbf{F}}$ with the associated parameterizations $\widetilde{\mathbf{F}}^{(i)}$, $i \in \mathcal{I}_\Omega$, we first choose a multi-patch geometry $\widehat{\mathbf{F}}$ consisting of bilinear parameterizations $\widehat{\mathbf{F}}^{(i)}$, $i \in \mathcal{I}_\Omega$, which roughly describes the initial geometry $\widetilde{\mathbf{F}}$. Then, following [29], we look for a suitable approximation $\mathbf{F}$ of $\widetilde{\mathbf{F}}$, of the form $\mathbf{F} = \mathbf{u} \circ \widehat{\mathbf{F}}$, where $\mathbf{u}$ is a C^1 isogeometric vector field representing a mapping of the bilinear geometry into the final one. By the construction of Section 4, an explicit basis for $\mathbf{u}$ is available. Since $\widehat{\mathbf{F}}$ is AS-G^1, and $\mathbf{F}$ has the same gluing data by construction, $\mathbf{F}$ is AS-G^1 as required. An example of a mapped piecewise bilinear multi-patch parameterization is given in [25, Example 6] or in [15, Appendix A], where the resulting domain is a multi-patch NURBS.

A drawback of the method is the limitation to multi-patch geometries which have to allow a rough estimation by a piecewise bilinear multi-patch parameterization. Furthermore, the approach cannot be used to generate AS-G^1 multi-patch geometries determining multi-patch domains with a smooth boundary. A more advanced technique, which provides amongst others the design of such multi-patch geometries, cf. [27, Example 3], is described in the following subsection.

3.3.2. *The AS-G^1 fitting approach* [27]

This method allows the construction of an AS-G^1 multi-patch geometry $\mathbf{F}$, which interpolates the boundary, the vertices and the first derivatives at the vertices of the initial non-AS-G^1 multi-patch geometry $\widetilde{\mathbf{F}}$, and which is as close as possible to $\widetilde{\mathbf{F}}$. The construction of $\mathbf{F}$ is divided into the following steps:

Step 1: For each interface $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, of the desired multi-patch geometry $\mathbf{F}$, we precompute the gluing data of $\mathbf{F}$ at the interface $\Sigma^{(i)}$, that is, $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$, $\beta^{(i,i_1)}$ and $\beta^{(i,i_2)}$, by linearizing the corresponding non-linear gluing data of $\widetilde{\mathbf{F}}$. Let $\widetilde{\alpha}^{(i,i_1)}$, $\widetilde{\alpha}^{(i,i_1)}$ and $\widetilde{\beta}^{(i)}$ be the gluing functions (3.2) for the parameterizations $\widetilde{\mathbf{F}}^{(i_1)}$ and $\widetilde{\mathbf{F}}^{(i_2)}$ for $\widetilde{\gamma}^{(i)}(\xi) \equiv 1$. Then, the linear functions $\alpha^{(i,i_1)}$ and $\alpha^{(i,i_2)}$ are obtained by

$$\alpha^{(i,i_k)}(\xi) = \widetilde{\alpha}^{(i,i_k)}(0)\,(1-\xi) + \widetilde{\alpha}^{(i,i_k)}(1)\,\xi,\ k = 1, 2,$$

and the linear functions

$$\beta^{(i,i_k)}(\xi) = b_0^{(i,i_k)}(1-\xi) + b_1^{(i,i_k)}\xi,\ b_0^{(i,i_k)}, b_1^{(i,i_k)} \in \mathbb{R},\ k = 1, 2,$$

are computed by minimizing the term

$$\int_0^1 \|\widetilde{\beta}^{(i)} - \underbrace{(\alpha^{(i,i_1)}\beta^{(i,i_2)} + \alpha^{(i,i_2)}\beta^{(i,i_1)})}_{\beta^{(i)}}\|^2 \mathrm{d}\xi$$
$$+\lambda_\beta \left(\int_0^1 \|\beta^{((i,i_1))}\|^2 \mathrm{d}\xi + \int_0^1 \|\beta^{(i,i_2)}\|^2 \mathrm{d}\xi \right) \to \min_{(b_0^{(i,i_1)}, b_1^{(i,i_1)}, b_0^{(i,i_2)}, b_0^{(i,i_2)})}$$

with respect to the linear constraints

$$\beta^{(i)}(0) = \widetilde{\beta}^{(i)}(0) \text{ and } \beta^{(i)}(1) = \widetilde{\beta}^{(i)}(1),$$

using a non-negative weight λ_β.

Step 2: We determine for the spline coefficients $\mathbf{c}_{j_1,j_2}^{(i)}$ of the multi-patch geometry $\mathbf{F}$ three different types of linear constraints, denoted by $\mathcal{L}_\Sigma^\circ$, $\mathcal{L}_\Sigma^\Gamma$ and $\mathcal{L}_\chi$, which will be used in Step 3 to construct the AS-G^1 multi-patch geometry $\mathbf{F}$. The constraints $\mathcal{L}_\Sigma^\circ$ are called AS-G^1 constraints and will ensure that the resulting multi-patch geometry $\mathbf{F}$ will be AS-G^1-continuous. For each interface $\Sigma^{(i)}$, we require that the geometry mappings $\mathbf{F}^{(i_1)}$ and $\mathbf{F}^{(i_2)}$ have to satisfy the condition (3.4) for the precomputed gluing data $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$, $\beta^{(i,i_1)}$ and $\beta^{(i,i_2)}$ from Step 1.

The so-called boundary constraints $\mathcal{L}_{\Sigma}^{\Gamma}$ will guarantee that the multi-patch geometry $\mathbf{F}$ will coincide with $\widetilde{\mathbf{F}}$ on each boundary edge $\Sigma^{(i)}$, $i \in \mathcal{I}_{\Sigma}^{\Gamma}$. Finally, the so-called vertex constraints $\mathcal{L}_{\chi}$ will ensure that $\mathbf{F}$ will interpolate the vertices and the first derivatives of $\widetilde{\mathbf{F}}$ at each vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_{\chi}$. All these constraints are linear, and are compatible with each other.

Step 3: Let $\mathbf{c}$ be the vector of all control points $\mathbf{c}_{j_1,j_2}^{(i)}$ of the multi-patch geometry $\mathbf{F}$. Then, the AS-G^1 multi-patch geometry $\mathbf{F}$ is finally constructed by minimizing the objective function

$$\mathcal{F}_2(\mathbf{c}) + \lambda_L \mathcal{F}_L(\mathbf{c}) + \lambda_U \mathcal{F}_U(\mathbf{c}) \to \min_{\mathbf{c}}$$

with respect to the linear constraints $\mathcal{L}_{\Sigma}^{\circ}$, $\mathcal{L}_{\Sigma}^{\Gamma}$ and $\mathcal{L}_{\chi}$, using non-negative weights λ_L and λ_U. While the quadratic functional $\mathcal{F}_2$ given by

$$\mathcal{F}_2(\mathbf{c}) = \sum_{i \in \mathcal{I}_{\Omega}} \int_{[0,1]^2} \|\mathbf{F}^{(i)} - \widetilde{\mathbf{F}}^{(i)}\|^2 \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_2,$$

will ensure that the resulting geometry $\mathbf{F}$ approximates $\widetilde{\mathbf{F}}$, the so-called parametric length functional $\mathcal{F}_L$ and the so-called uniformity functional $\mathcal{F}_U$ given by

$$\mathcal{F}_L(\mathbf{c}) = \sum_{i \in \mathcal{I}_{\Omega}} \int_{[0,1]^2} \left(\|\partial_1 \mathbf{F}^{(i)}\|^2 + \|\partial_2 \mathbf{F}^{(i)}\|^2 \right) \mathrm{d}\xi_1 \,\mathrm{d}\xi_2,$$

and

$$\mathcal{F}_U(\mathbf{c}) = \sum_{i \in \mathcal{I}_{\Omega}} \int_{[0,1]^2} \left(\|\partial_1^2 \mathbf{F}^{(i)}\|^2 + 2\|\partial_1 \partial_2 \mathbf{F}^{(i)}\|^2 + \|\partial_2^2 \mathbf{F}^{(i)}\|^2 \right) \mathrm{d}\xi_1 \,\mathrm{d}\xi_2,$$

respectively, will be needed to construct parameterizations of good quality.

In case that the quality of the resulting AS-G^1 multi-patch geometry $\mathbf{F}$ is not good enough, and some of the single parameterizations $\mathbf{F}^{(i)}$, $i \in \mathcal{I}_{\Omega}$, are even singular, the use of a sufficiently refined spline space $\mathcal{S}_h^{\mathbf{p},\mathbf{r}}$ solves this issue. In practice, the necessary refinement level of the spline space depends on the desired quality criteria specified by the user, hence it also depends on the choice of the parameters λ_L and λ_U. Two instances of constructed AS-G^1 multi-patch geometries using this approach are given in Fig. 6.1.

4. An isogeometric C^1 space

We first introduce the concept of C^1 isogeometric spline spaces over (general) multi-patch geometries, and then present the construction [28] to generate a particular C^1 isogeometric spline space over a given AS-G^1 multi-patch geometry.

4.1. Space of C^1 isogeometric functions

The space of C^1 isogeometric functions with respect to the multi-patch geometry $\mathbf{F}$ is defined as

$$\mathcal{V}^1 = \left\{ \varphi_h \in C^1(\overline{\Omega}) \,|\, \text{ for all } i \in \mathcal{I}_{\Omega},\ f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)} \in \mathcal{S}_h^{\mathbf{p},\mathbf{r}} \right\}. \tag{4.1}$$

This space can be characterized by the equivalence of the C^1-smoothness of an isogeometric function and the G^1-smoothness of its associated graph, or more precisely, $\varphi_h \in \mathcal{V}^1$ if and only if the graph of φ_h is G^1-smooth at all interfaces $\Sigma^{(i)}$, $i \in \mathcal{I}_{\Sigma}^{\circ}$, cf. [15, 21, 29]. Note that for an isogeometric function φ_h, its graph $\Phi \subset \Omega \times \mathbb{R}$ is the collection of the single graph surface patches

$$\Phi^{(i)} : [0,1]^2 \to \Omega^{(i)} \times \mathbb{R},\ \Phi^{(i)}(\xi_1, \xi_2) = \begin{bmatrix} \mathbf{F}^{(i)}(\xi_1,\xi_2) \\ f_h^{(i)}(\xi_1,\xi_2) \end{bmatrix},\ i \in \mathcal{I}_{\Omega},$$

with $f_h^{(i)} = \varphi_h \circ \mathbf{F}^{(i)}$. Then, an isogeometric function φ_h belongs to the space $\mathcal{V}^1$, if and only if for each interface $\Sigma^{(i)}$, $i \in \mathcal{I}_\Sigma^\circ$, assuming that the two associated neighboring geometry mappings $\mathbf{F}^{(i_1)}$ and $\mathbf{F}^{(i_2)}$ are given in standard form (3.1), there exist gluing data satisfying (3.3) and (3.5) such that conditions (3.1) and (3.4) are satisfied not only for the parametrizations $\mathbf{F}^{(i_1)}$, $\mathbf{F}^{(i_2)}$ but also for the graph surfaces $\Phi^{(i_1)}$, $\Phi^{(i_2)}$.

Since the two geometry mappings $\mathbf{F}^{(i_1)}$ and $\mathbf{F}^{(i_2)}$ already uniquely determine (up to a common function $\gamma^{(i)}$) the functions $\alpha^{(i,i_1)}$, $\alpha^{(i,i_2)}$ and $\beta^{(i)}$, compare Section 3.2, we obtain that $\varphi_h \in \mathcal{V}^1$ if and only if the last component of the graph surfaces satisfies the equations (3.1) and (3.4), that is, for $\xi \in [0,1]$

$$f_h^{(i_1)}(0,\xi_1) = f_h^{(i_2)}(\xi,0) = (\varphi_h|_{\Sigma^{(i)}}) \circ \mathbf{F}^{(i_k)}$$

and

$$\alpha^{(i,i_1)}(\xi)\partial_2 f_h^{(i_2)}(\xi,0) + \alpha^{(i,i_2)}(\xi)\partial_1 f_h^{(i_1)}(0,\xi) + \beta^{(i)}(\xi)\partial_2 f_h^{(i_1)}(0,\xi) = 0, \tag{4.2}$$

or equivalently to (4.2)

$$\frac{\partial_1 f_h^{(i_1)}(0,\xi) + \beta^{(i,i_1)}(\xi)\partial_2 f_h^{(i_1)}(0,\xi)}{\alpha^{(i,i_1)}(\xi)} = -\frac{\partial_2 f_h^{(i_2)}(\xi,0) + \beta^{(i,i_2)}(\xi)\partial_1 f_h^{(i_2)}(\xi,0)}{\alpha^{(i,i_2)}(\xi)} = (\nabla\varphi_h \cdot \mathbf{d}|_{\Sigma^{(i)}}) \circ \mathbf{F}^{(i_k)}, \tag{4.3}$$

where $\mathbf{d}$ is a suitable vector that is not tangential to the interface, see e.g. [15, 26]. These C^1-conditions were used to generate C^1 isogeometric spline spaces over general analysis-suitable G^1 multi-patch geometries, see [26] for the case of two-patches and [28] for the multi-patch case. In the following subsection, we will summarize the construction [28].

4.2. **The Argyris isogeometric space**

We give a survey of the method [28] for the design of a specific C^1 isogeometric spline space over a given AS-G^1 multi-patch geometry $\mathbf{F}$. The proposed C^1 space $\mathcal{A}$ is called Argyris (quadrilateral) isogeometric space, since it possesses similar degrees-of-freedom as the classical Argyris finite element space [2]. The Argyris triangular element enforces C^2 at all vertices and C^1 across edges. Hence, for degree five the degrees-of-freedom are six per vertex and one per edge. For higher degrees, there are more degrees-of-freedom per edge and some related to the triangle (interior) as well. This setting carries over to tensor-product spline patches as described in more detail in [28]. The space $\mathcal{A}$ is a subspace of the entire C^1 isogeometric space $\mathcal{V}^1$ maintaining the optimal order of approximation of the space $\mathcal{V}^1$ for the traces and normal derivatives along the interfaces, and is much easier to investigate and to construct than the space $\mathcal{V}^1$. E.g., the dimension of $\mathcal{A}$ does not depend on the geometry, which is in contrast to the dimension of the space $\mathcal{V}^1$, cf. [26] for the two-patch case. For the construction of $\mathcal{A}$, we need a minimal resolution within the patches given by $h \leq \frac{p-r-1}{4-r}$.

The C^1 isogeometric space $\mathcal{A}$ is constructed as the direct sum of subspaces referring to the single patch-interior, edge and vertex components, that is,

$$\mathcal{A} = \left(\bigoplus_{i\in\mathcal{I}_\Omega} \mathcal{A}^\circ_{\Omega^{(i)}}\right) \oplus \left(\bigoplus_{i\in\mathcal{I}_\Sigma} \mathcal{A}^\circ_{\Sigma^{(i)}}\right) \oplus \left(\bigoplus_{i\in\mathcal{I}_\chi} \mathcal{A}_{\mathbf{x}^{(i)}}\right).$$

The spaces $\mathcal{A}^\circ_{\Omega^{(i)}}$, $\mathcal{A}^\circ_{\Sigma^{(i)}}$ and $\mathcal{A}_{\mathbf{x}^{(i)}}$ are called patch-interior, edge and vertex function space, respectively. The patch-interior functions are completely supported within one patch, the edge functions have support along the edge and are restricted to two patches, whereas the vertex functions have support in a neighborhood of the vertex. The different types of functions are defined as follows:

Patch-interior function space $\mathcal{A}^{\circ}_{\Omega^{(i)}}$. Let $i \in \mathcal{I}_{\Omega}$. The space $\mathcal{A}^{\circ}_{\Omega^{(i)}}$ is given as

$$\mathcal{A}^{\circ}_{\Omega^{(i)}} = \operatorname{span}\{\mathrm{B}^{(i)}_{\mathbf{j}} : \mathbf{j} \in \{2, \ldots, N-3\}^2\}$$

with

$$\mathrm{B}^{(i)}_{\mathbf{j}}(\mathbf{x}) = \begin{cases} (b_{\mathbf{j}} \circ (\mathbf{F}^{(i)})^{-1})(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{\Omega^{(i)}}, \\ 0 & \text{otherwise.} \end{cases}$$

The patch-interior functions $\mathrm{B}^{(i)}_{\mathbf{j}}$, $\mathbf{j} \in \{2, \ldots, N-3\}^2$, are the "standard" isogeometric function with a support entirely contained in $\Omega^{(i)}$ and have vanishing function values and vanishing gradients at the patch boundary $\partial\Omega^{(i)}$. This directly implies that $\mathrm{B}^{(i)}_{\mathbf{j}} \in C^1(\Omega)$ for $\mathbf{j} \in \{2, \ldots, N-3\}^2$. The dimension of the space $\mathcal{A}^{\circ}_{\Omega^{(i)}}$, $i \in \mathcal{I}_{\Omega}$, is given by

$$\dim(\mathcal{A}^{\circ}_{\Omega^{(i)}}) = ((p-r)(n-1) + p - 3)^2.$$

Edge function space $\mathcal{A}^{\circ}_{\Sigma^{(i)}}$. We consider first the case of an interface $\Sigma^{(i)}$, which means that $i \in \mathcal{I}^{\circ}_{\Sigma}$, and assume without loss of generality that the two associated neighboring geometry mappings $\mathbf{F}^{(i_1)}$ and $\mathbf{F}^{(i_2)}$ are given in standard form (3.1). Let b^{+}_{j}, $j = 0, \ldots, N_0 - 1$, with $N_0 = p + (n-1)(p-r-1) + 1$, be the B-splines of the univariate spline space $\mathcal{S}^{p,r+1}_{h}$, and let b^{-}_{j}, $j = 0, \ldots, N_1 - 1$, with $N_1 = p + (n-1)(p-r-1)$, be the B-splines of the univariate spline space $\mathcal{S}^{p-1,r}_{h}$. The space $\mathcal{A}^{\circ}_{\Sigma^{(i)}}$ is defined as

$$\mathcal{A}^{\circ}_{\Sigma^{(i)}} = \operatorname{span}\{\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)} : j_1 = 0, \ldots, N_{j_2} - 1,\, j_2 = 0, 1\}$$

with

$$\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)}(\mathbf{x}) = \begin{cases} \Big(\overline{f}^{(i,k)}_{(j_1,j_2)} \circ \Big(\mathbf{F}^{(k)}\Big)^{-1}\Big)(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{\Omega^{(k)}},\ k = i_1, i_2, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \overline{f}^{(i,i_1)}_{(j_1,0)}(\xi_1,\xi_2) &= b^{+}_{j_1}(\xi_2)(b_0(\xi_1) + b_1(\xi_1)) - \beta^{(i,i_1)}(\xi_2)(b^{+}_{j_1})'(\xi_2)\frac{h}{p}b_1(\xi_1), \\ \overline{f}^{(i,i_2)}_{(j_1,0)}(\xi_1,\xi_2) &= b^{+}_{j_1}(\xi_1)(b_0(\xi_2) + b_1(\xi_2)) - \beta^{(i,i_2)}(\xi_1)(b^{+}_{j_1})'(\xi_1)\frac{h}{p}b_1(\xi_2), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \overline{f}^{(i,i_1)}_{(j_1,1)}(\xi_1,\xi_2) &= \alpha^{(i,i_1)}(\xi_2)b^{-}_{j_1}(\xi_2)b_1(\xi_1), \\ \overline{f}^{(i,i_2)}_{(j_1,1)}(\xi_1,\xi_2) &= -\alpha^{(i,i_2)}(\xi_1)b^{-}_{j_1}(\xi_1)b_1(\xi_2). \end{aligned} \tag{4.5}$$

In case of a boundary edge $\Sigma^{(i)}$, $i \in \mathcal{I}^{\Gamma}_{\Sigma}$, the space $\mathcal{A}^{\circ}_{\Sigma^{(i)}}$ can be defined in a similar way. Assume that the associated geometry mapping is $\mathbf{F}^{(i_1)}$, and is parameterized as in (3.1) for the case of two patches. The gluing data $\alpha^{(i,i_1)}$ and $\beta^{(i,i_1)}$ can be simplified to $\alpha^{(i,i_1)}(\xi) = 1$ and $\beta^{(i,i_1)}(\xi) = 0$, which leads to functions

$$\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)}(\mathbf{x}) = \begin{cases} \Big(\overline{f}^{(i,i_1)}_{(j_1,j_2)} \circ \Big(\mathbf{F}^{(i_1)}\Big)^{-1}\Big)(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{\Omega^{(i_1)}}, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\overline{f}^{(i,i_1)}_{(j_1,0)}(\xi_1,\xi_2) = b^{+}_{j_1}(\xi_2)(b_0(\xi_1) + b_1(\xi_1)),$$

and

$$\overline{f}^{(i,i_1)}_{(j_1,1)}(\xi_1,\xi_2) = b^{-}_{j_1}(\xi_2)b_1(\xi_1).$$

The edge functions $\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)}$, $j_1 = 0, \ldots, N_{j_2} - 1, j_2 = 0, 1$, are constructed in such a way that they are C^1-smooth across the interface $\Sigma^{(i)}$, and that they span function values and cross derivative values along the edge, see [28] for details. In addition, the edge functions $\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)}$ possess a support, which is entirely contained in $\overline{\Omega^{(i_1)}} \cup \overline{\Omega^{(i_2)}}$ ($\overline{\Omega^{(i_1)}}$ if $\Sigma^{(i)}$ is a boundary edge) in an h-dependent neighborhood of $\Sigma^{(i)}$, and have vanishing derivatives up to second order at the endpoints (vertices) of the edge. This implies that $\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)} \in C^1(\Omega)$, $j_1 = 0,\ldots,N_{j_2}-1, j_2 = 0,1$. The dimension of the space $\mathcal{A}^{\circ}_{\Sigma^{(i)}}$, $i \in \mathcal{I}_{\Sigma}$, is given by

$$\dim(\mathcal{A}^{\circ}_{\Sigma^{(i)}}) = 2(p-r-1)(n-1)+p-9.$$

Vertex function space $\mathcal{A}_{\mathbf{x}^{(i)}}$. We consider a vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}_{\chi}$, and denote by ν the patch valence of the vertex $\mathbf{x}^{(i)}$. Let $\Sigma^{(i_1)}$, $\Omega^{(i_2)}$, $\Sigma^{(i_3)}$, …, $\Omega^{(i_{2\nu})}$, $\Sigma^{(i_{2\nu+1})}$, be the sequence of interfaces and patches around the vertex $\mathbf{x}^{(i)}$ in counterclockwise order, assuming in case of an inner vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}^{\circ}_{\chi}$, that $\Sigma^{(i_1)} = \Sigma^{(i_{2\nu+1})}$. The associated geometry mappings $\mathbf{F}^{(i_2)}$, $\mathbf{F}^{(i_4)}$, …, $\mathbf{F}^{(i_{2\nu})}$ containing the vertex $\mathbf{x}^{(i)}$ can be always reparameterized (if necessary) into standard form (cf. [28]), just meaning that we have

$$\mathbf{F}^{(i_{2k})}(0,\xi) = \mathbf{F}^{(i_{2k+2})}(\xi,0),\ \xi \in [0,1], \tag{4.6}$$

for $k \in \{1,\ldots,\nu-1\}$, and additionally

$$\mathbf{F}^{(i_{2\nu})}(0,\xi) = \mathbf{F}^{(i_2)}(\xi,0),\ \xi \in [0,1],$$

in case of an inner vertex $\mathbf{x}^{(i)}$, see Fig. 3.1 (right). This implies that

$$\mathbf{x}^{(i)} = \mathbf{F}^{(i_2)}(0,0) = \mathbf{F}^{(i_4)}(0,0) = \ldots = \mathbf{F}^{(i_{2\nu})}(0,0).$$

Considering a boundary vertex $\mathbf{x}^{(i)}$, $i \in \mathcal{I}^{\Gamma}_{\chi}$, we assume that the edges $\Sigma^{(i_1)}$ and $\Sigma^{(2i_{\nu}+1)}$ are the two boundary edges, for which the gluing data $\alpha^{(i_1,i_2)}$, $\alpha^{(i_{2\nu+1},i_{2\nu})}$ and $\beta^{(i_1,i_2)}$, $\beta^{(i_{2\nu+1},i_{2\nu})}$ can be simplified to $\alpha^{(i_1,i_2)}(\xi) = \alpha^{(i_{2\nu+1},i_{2\nu})}(\xi) = 1$ and $\beta^{(i_1,i_2)}(\xi) = \beta^{(i_{2\nu+1},i_{2\nu})}(\xi) = 0$.

Before defining the space $\mathcal{A}_{\mathbf{x}^{(i)}}$, $i \in \mathcal{I}_{\chi}$, we need further tools and definitions. We consider the basis transformations $\{b_0,b_1\}$ to $\{c_0,c_1\}$, $\{b_0^+,b_1^+,b_2^+\}$ to $\{c_0^+,c_1^+,c_2^+\}$ and from $\{b_0^-,b_1^-\}$ to $\{c_0^-,c_1^-\}$, with

$$\partial_\xi^j c_i(0) = \delta_i^j \quad \text{for } j=0,1, \quad \partial_\xi^j c_i^+(0) = \delta_i^j \quad \text{for } j=0,\ldots,2, \quad \text{and} \quad \partial_\xi^j c_i^-(0) = \delta_i^j \quad \text{for } j=0,1,$$

where δ_i^j is the Kronecker delta. For each edge $\Sigma^{(i_k)}$, $k \in \{1,3,\ldots,2\nu+1\}$, we use the abbreviated notations

$$\mathbf{t}^{(i_k)} = \partial_2 \mathbf{F}^{(i_{k-1})}(0,\xi) = \partial_1 \mathbf{F}^{(i_{k+1})}(\xi,0)$$

and

$$\begin{aligned}\mathbf{d}^{(i_k)}(\xi) &= \frac{1}{\alpha^{(i_k,i_{k-1})}(\xi)}\left(\partial_1 \mathbf{F}^{(i_{k-1})}(0,\xi) + \beta^{(i_k,i_{k-1})}(\xi)\,\partial_2 \mathbf{F}^{(i_{k-1})}(0,\xi)\right)\\ &= -\frac{1}{\alpha^{(i_k,i_{k+1})}(\xi)}\left(\partial_2 \mathbf{F}^{(i_{k+1})}(\xi,0) + \beta^{(i_k,i_{k+1})}(\xi)\,\partial_1 \mathbf{F}^{(i_{k+1})}(\xi,0)\right).\end{aligned}$$

The vector $\mathbf{d}^{(i_k)}$ plays the role of the transversal vector $\mathbf{d}$ in (4.3). Note that in case of a boundary edge in each case one term does not exist. Given the vector $\Phi = (\phi_{0,0},\phi_{1,0},\phi_{0,1},\phi_{2,0},\phi_{1,1},\phi_{0,2})$, which describes the C^2 interpolation data of an isogeometric function ϕ at the vertex $\mathbf{x}^{(i)}$ determined by the function value $\phi(\mathbf{x}^{(i)}) = \phi_{0,0}$, the gradient $\nabla\phi = (\phi_{1,0},\phi_{0,1})$ and the Hessian

$$H\phi = \begin{pmatrix} \phi_{2,0} & \phi_{1,1} \\ \phi_{1,1} & \phi_{0,2} \end{pmatrix},$$

we define for each patch $\Omega^{(i_k)}$, $k \in \{2, 4 \ldots, 2\nu\}$, the functions

$$\overset{\star}{f}{}_{\Phi}^{(i_{k+1},i_k)}(\xi_1, \xi_2) = \sum_{j=0}^{2} d_{0,j}^{(i_{k+1},i_k)} \left(c_j^{+}(\xi_2) c_0(\xi_1) - \beta^{(i_{k+1},i_k)}(\xi_2)(c_j^{+})'(\xi_2) c_1(\xi_1) \right)$$

$$+ \sum_{j=0}^{1} d_{1,j}^{(i_{k+1},i_k)} \alpha^{(i_{k+1},i_k)}(\xi_2) c_j^{-}(\xi_2) c_1(\xi_1),$$

$$\overset{\star}{f}{}_{\Phi}^{(i_{k-1},i_k)}(\xi_1, \xi_2) = \sum_{j=0}^{2} d_{0,j}^{(i_{k-1},i_k)} \left(c_j^{+}(\xi_1) c_0(\xi_2) - \beta^{(i_{k-1},i_k)}(\xi_1)(c_j^{+})'(\xi_1) c_1(\xi_2) \right)$$

$$- \sum_{j=0}^{1} d_{1,j}^{(i_{k-1},i_k)} \alpha^{(i_{k-1},i_k)}(\xi_1) c_j^{-}(\xi_1) c_1(\xi_2),$$

and

$$\overset{\star}{f}{}_{\Phi}^{(i_k)}(\xi_1, \xi_2) = \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} d_{j_1,j_2}^{(i_k)} c_{j_1}(\xi_1) c_{j_2}(\xi_2),$$

with

$$d_{0,0}^{(i_\ell,i_k)} = \phi_{0,0},\ d_{0,1}^{(i_\ell,i_k)} = \nabla\phi\, \mathbf{t}^{(i_\ell)}(0),\ d_{0,2}^{(i_\ell,i_k)} = (\mathbf{t}^{(i_\ell)}(0))^T\, H\phi\, \mathbf{t}^{(i_\ell)}(0) + \nabla\phi\, (\mathbf{t}^{(i_\ell)})'(0),$$

$$d_{1,0}^{(i_\ell,i_k)} = \nabla\phi\, \mathbf{d}^{(i_\ell)}(0),\ d_{1,1}^{(i_\ell,i_k)} = (\mathbf{t}^{(i_\ell)}(0))^T\, H\phi\, \mathbf{d}^{(i_\ell)}(0) + \nabla\phi\, (\mathbf{d}^{(i_\ell)})'(0),$$

for $\ell = k-1, k+1$, and

$$d_{0,0}^{(i_k)} = \phi_{0,0},\ d_{1,0}^{(i_k)} = \nabla\phi\, \mathbf{t}^{(i_{k-1})}(0),\ d_{0,1}^{(i_k)} = \nabla\phi\, \mathbf{t}^{(i_{k+1})}(0)$$

$$d_{1,1}^{(i_k)} = (\mathbf{t}^{(i_{k-1})}(0))^T\, H\phi\, \mathbf{t}^{(i_{k+1})}(0) + \nabla\phi\, \partial_1\partial_2 \mathbf{F}^{(i_k)}(0,0).$$

Let

$$\Phi_{(0,0)} = (1,0,0,0,0,0),\ \Phi_{(1,0)} = (0,1,0,0,0,0),\ \Phi_{(0,1)} = (0,0,1,0,0,0),$$

and

$$\Phi_{(2,0)} = (0,0,0,1,0,0),\ \Phi_{(1,1)} = (0,0,0,0,1,0),\ \Phi_{(0,2)} = (0,0,0,0,0,1),$$

then the space $\mathcal{A}_{\mathbf{x}^{(i)}}$ is defined as

$$\mathcal{A}_{\mathbf{x}^{(i)}} = \operatorname{span}\{\overset{\star}{\mathrm{B}}{}_{(j_1,j_2)}^{(i)} :\ 0 \le j_1, j_2 \le 2,\ j_1 + j_2 \le 2\}$$

with

$$\overset{\star}{\mathrm{B}}{}_{(j_1,j_2)}^{(i)}(\mathbf{x}) = \begin{cases} \sigma^{j_1+j_2} \left(\left(\overset{\star}{f}{}_{\Phi_{(j_1,j_2)}}^{(i_{k-1},i_k)} + \overset{\star}{f}{}_{\Phi_{(j_1,j_2)}}^{(i_{k+1},i_k)} - \overset{\star}{f}{}_{\Phi_{(j_1,j_2)}}^{(i_k)} \right) \circ \left(\mathbf{F}^{(i_k)} \right)^{-1} \right)(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{\Omega^{(i_k)}},\ k = 2,4,\ldots,2\nu, \\ 0 & \text{otherwise,} \end{cases}$$

where the factor

$$\sigma = \left(\frac{h}{p\,\nu} \sum_{\ell=1}^{\nu} \|\nabla \mathbf{F}^{(i_{2\ell})}(0,0)\| \right)^{-1}$$

is used to uniformly scale the functions with respect to the L^∞-norm. The vertex functions $\overset{\star}{\mathrm{B}}{}_{(j_1,j_2)}^{(i)}$, $0 \le j_1, j_2 \le 2$, $j_1 + j_2$, are constructed in such a way that they are C^1-smooth across all interfaces $\Sigma^{(i_{2k+1})}$, $k = 0, \ldots, \nu$, and that they span the function value and all derivatives up to second order at the vertex $\mathbf{x}^{(i)}$, see [28] for details. Furthermore, the support of an vertex function $\overset{\star}{\mathrm{B}}{}_{(j_1,j_2)}^{(i)}$ is entirely contained in $\cup_{k=1}^{\nu} \overline{\Omega^{i_{2k}}}$ in an h-dependent neighborhood of the vertex $\mathbf{x}^{(i)}$. Therefore, we obtain that

$\overset{\star}{\mathrm{B}}{}^{(i)}_{(j_1,j_2)} \in C^1(\Omega)$ and $\overset{\star}{\mathrm{B}}{}^{(i)}_{(j_1,j_2)} \in C^2(\mathbf{x}^{(i)})$ for $0 \leq j_1, j_2 \leq 2$, $j_1 + j_2$. The dimension of the space $\mathcal{A}_{\mathbf{x}^{(i)}}$, $i \in \mathcal{I}_\chi$, is equal to

$$\dim(\mathcal{A}_{\mathbf{x}^{(i)}}) = 6.$$

As already mentioned above, the dimension of the entire space $\mathcal{A}$ does not depend on the geometry and is finally given by the sum of dimensions of all subspaces $\mathcal{A}_{\Omega^{(i)}}$, $\mathcal{A}_{\Sigma^{(i)}}$ and $\mathcal{A}_{\mathbf{x}^{(i)}}$, that is,

$$\dim(\mathcal{A}) = |\mathcal{I}_\Omega| \cdot ((p-r)(n-1)+p-3)^2 + |\mathcal{I}_\Sigma| \cdot (2(p-r-1)(n-1)+p-9) + |\mathcal{I}_\chi| \cdot 6.$$

Moreover, all constructed patch-interior functions $\mathrm{B}^{(i)}_{(j_1,j_2)}$, edge functions $\overline{\mathrm{B}}^{(i)}_{(j_1,j_2)}$ and vertex functions $\overset{\star}{\mathrm{B}}{}^{(i)}_{(j_1,j_2)}$ from above form a basis of the space $\mathcal{A}$. This is a direct consequence of the definition of the single functions and their supports.

5. Beyond analysis-suitable G^1 parameterizations

It is possible to extend the C^1 basis construction to domains that are not analysis-suitable G^1. This is done by locally increasing the degree. This approach was employed for interfaces in the multi-patch framework in [12], based on the findings in [15]. More extensive research was done in [30, 31, 33, 44] for unstructured quadrilateral meshes, where higher degree elements where used in local regions around extraordinary vertices.

The mixed degree construction is based on the following observation. The definitions of the edge basis functions in (4.4) and (4.5) are not confined to the gluing data being linear functions. However, we have the following lemma.

Lemma 5.1. *Given an interface $\Sigma^{(i)}$ between patches $\Omega^{(i_1)}$ and $\Omega^{(i_2)}$. Let $\alpha^{(i,i_1)}, \beta^{(i,i_1)} \in \mathcal{S}_h^{p^*,r^*}$ and b_0^*, b_1^* be the first two basis functions in $\mathcal{S}_h^{p^+,r^-}$, where $p^+ = \max(p, p+p^*-1)$ and $r^- = \min(r, r^*)$. Then the edge functions*

$$\overline{f}^{(i,i_1)}_{(j_1,0)}(\xi_1,\xi_2) = b^+_{j_1}(\xi_2)(b_0^*(\xi_1) + b_1^*(\xi_1)) - \beta^{(i,i_1)}(\xi_2)(b^+_{j_1})'(\xi_2)\frac{h}{p}b_1^*(\xi_1)$$

and

$$\overline{f}^{(i,i_1)}_{(j_1,1)}(\xi_1,\xi_2) = \alpha^{(i,i_1)}(\xi_2) b^-_{j_1}(\xi_2) b_1^*(\xi_1)$$

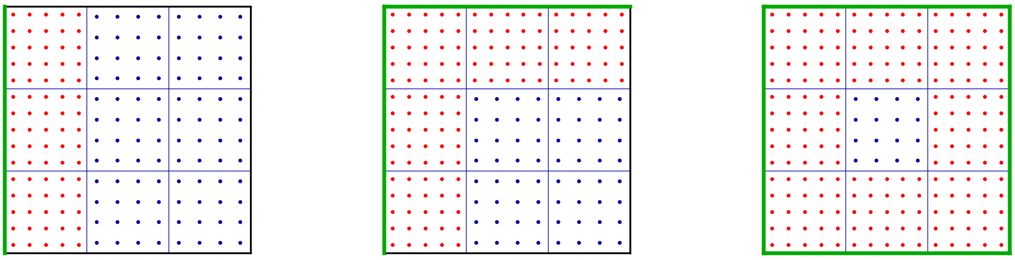
satisfy

$$\overline{f}^{(i,i_1)}_{(j_1,0)}, \overline{f}^{(i,i_1)}_{(j_1,1)} \in \mathcal{S}_h^{\mathbf{p}^+,\mathbf{r}^-},$$

with $\mathbf{p}^+ = (p^+, p^+)$ and $\mathbf{r}^- = (r^-, r^-)$. Analogously, we have the same for the patch $\Omega^{(i_2)}$.

A simple consequence is, that if there exists quadratic gluing data, then all edge functions are in the space $\mathcal{S}_h^{\mathbf{p+1},\mathbf{r}}$. This significantly increases the flexibility of the multi-patch geometry. Note that the biquadratic G-splines defined in [51] possess quadratic gluing data. Configurations of suitable spline-like patches of mixed degree are given in Figure 5.1. The blue dots signify Bézier coefficients of degree 3, whereas the red dots correspond to Bézier coefficients of degree 4. The blue line are mesh lines where parametric continuity of order C^1 or C^2 is described whereas the green edges at the boundary are of G^1 smoothness across patches and the black edges are either boundary edges, or edges where the patch can be extended with parametric continuity (at least C^1). Note that one may prescribe different continuity for different regions of the patch, e.g., only C^1 close to the interfaces and C^2 in the interior.

In this configuration of mixed degree 3 and 4, the coefficients corresponding to the inner elements (not neighboring the interfaces) do not influence the function value or value of first derivatives at the interfaces. Hence, the edge and vertex functions are completely determined by the Bézier elements of degree 4.



(A) One interface without special vertex construction.

(B) Two interfaces; coupled at the top-left vertex.

(C) Four interfaces; coupled at all vertices.

FIGURE 5.1. Spline-like patches of mixed degree (coefficients for $p = 4$ in red, for $p = 3$ in blue). The G^1 interfaces are depicted in green.

Considering a configuration as in Figure 5.1a containing only one interface, the edge basis as presented in Lemma 5.1 together with the patch interior basis obtained by resolving the C^k conditions in the interior give a complete basis of the C^1 smooth isogeometric function space.

When given configurations as in Figures 5.1b or 5.1c additional vertex functions can be defined by interpolation of C^2 data. The procedure is similar to the construction presented in Subsection 4.2. In all configurations, the patch interior basis is a tensor-product of suitable univariate basis functions.

The type of patches depicted in Figure 5.1b can be used to construct C^1 smooth isogeometric functions around extraordinary vertices. See Figure 5.2 for a possible construction. We refer to [30, 31, 33, 44], where such constructions were employed.

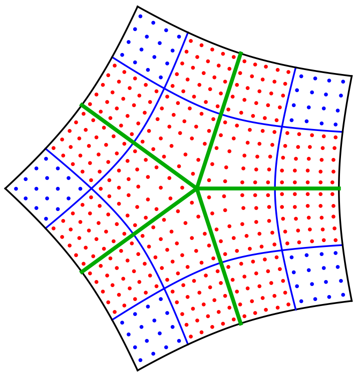


FIGURE 5.2. A possible construction for an extraordinary vertex of valency 5.

Another difficulty arises when refining the space. When performing a standard refinement step, the region where the degree is higher remains the same. Therefore the number of elements of higher degree scales with $O((\frac{1}{h})^2)$. This can be circumvented by locally reducing the degree again, which leads to the number of higher degree elements scaling as $O(\frac{1}{h})$. The process is sketched in Figure 5.3. Note that in this setting, the final (refined and reduced) space is not a superspace of the initial space. Hence, the spaces are not nested.

The constructions extend to higher degree as well as to more complex meshes. Many questions arise, that are worth to study in more detail; such as the definition of a basis forming a partition of unity, how to obtain nested spaces or how to efficiently construct domains and discretization spaces suitable for isogeometric analysis.

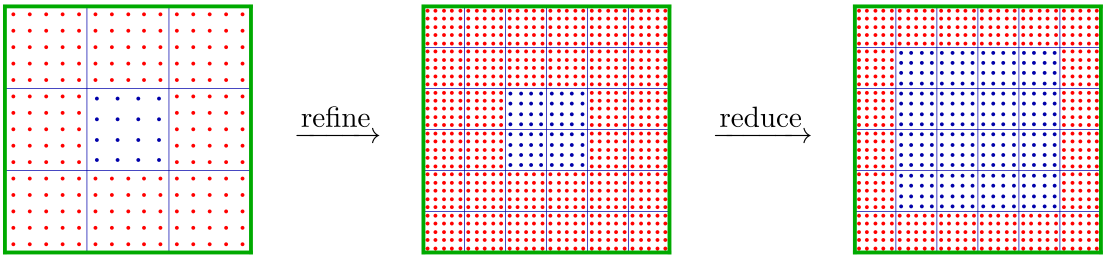


Figure 5.3. A mixed degree patch (left), its refinement (middle) and a patch of reduced degree (right).

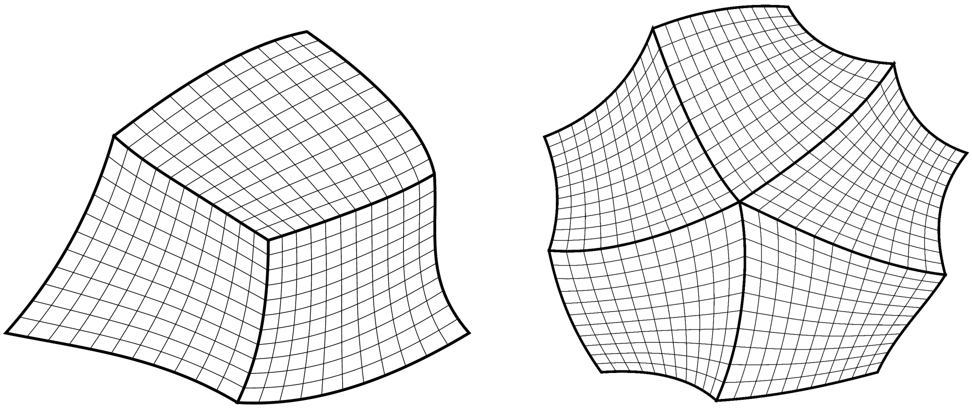


Figure 6.1. Two AS-G^1 multi-patch geometries constructed my means of AS-G^1 fitting approach [27], cf. Section 3.3.2.

6. Numerical examples

We consider the two multi-patch domains Ω shown in Fig. 6.1, which are described by AS-G^1 multi-patch geometries $\mathbf{F}$ consisting of parameterizations $\mathbf{F}^{(i)} \in \left(\mathcal{S}_{1/2}^{(3,3),(1,1)}\right)^2$. The two AS-$G^1$ multi-patch geometries $\mathbf{F}$ have been constructed from initial multi-patch geometries $\widetilde{\mathbf{F}}$ composed of bicubic Bézier patches $\widetilde{\mathbf{F}}^{(i)}$ by using the AS-G^1 fitting approach [27], cf. Section 3.3.2. While the AS-G^1 three-patch geometry (left) has been used in [28, Section 5], too, the AS-G^1 five-patch parameterization (right) is newly generated for this work. For both multi-patch parameterizations $\mathbf{F}$ we generate a sequence of C^1 Argyris spaces $\mathcal{A}_h$, $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, for $\mathbf{p} = (p,p) = (3,3), (4,4)$ and $\mathbf{r} = (r,r) = (1,1)$.

We employ the space family $\mathcal{A}_h$ to solve the biharmonic equation

$$\begin{cases} \triangle^2 u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g_1(\mathbf{x}) & \mathbf{x} \in \partial\Omega \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = g_2(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases} \tag{6.1}$$

by a standard Galerkin discretization. The functions g, g_1 and g_2 are selected to obtain the exact solution

$$u(\mathbf{x}) = u(x_1, x_2) = -4\cos\left(\frac{x_1}{2}\right)\sin\left(\frac{x_2}{2}\right)$$

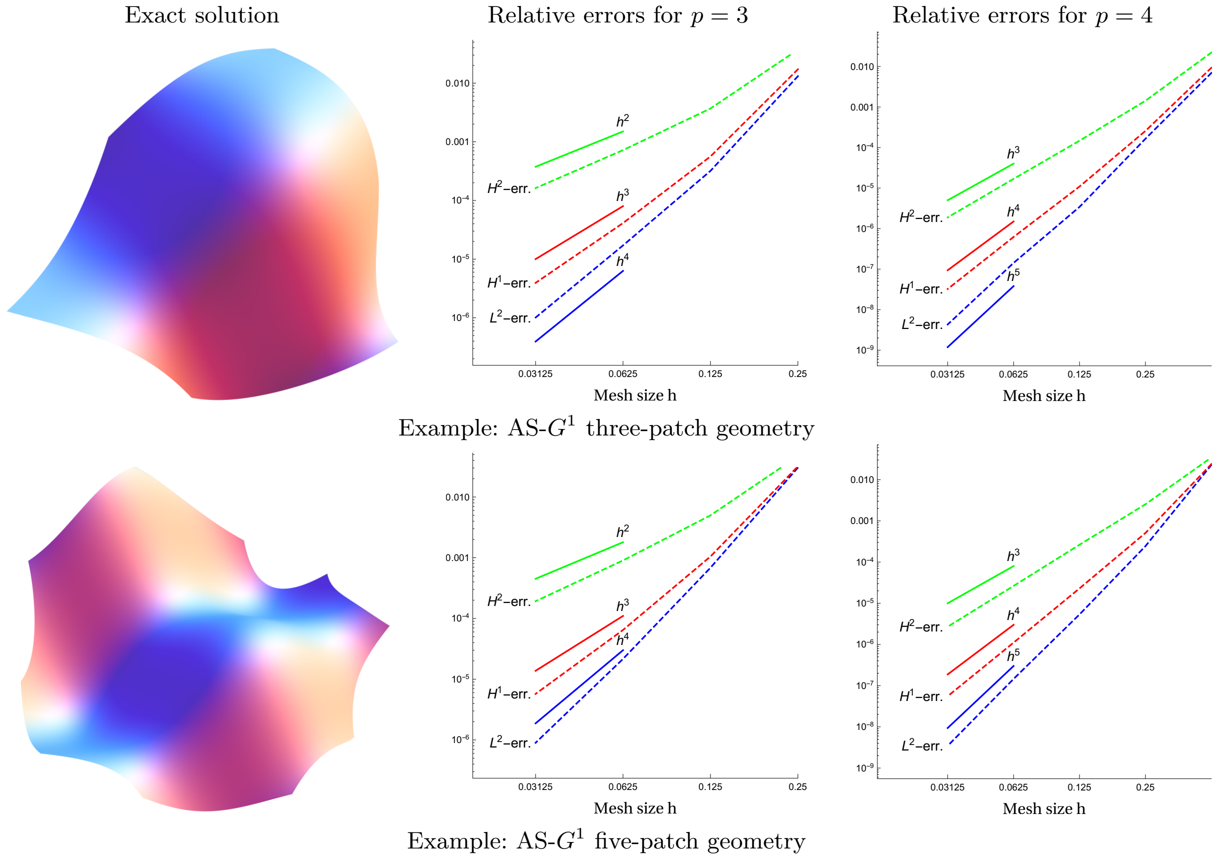


FIGURE 6.2. Solving the biharmonic equation (6.1) over the two AS-G^1 multi-patch geometries from Fig. 6.1: Exact solutions (first column) and the resulting relative L^2, H^1 and H^2 errors for $p = 3$ (second column) and $p = 4$ (third column).

on both domains. In particular, the boundary Dirichlet data g_1 and g_2 are L^2 projected and imposed strongly to the numerical solution u_h. The resulting relative L^2, H^1 and H^2 errors with their estimated convergence rates are presented in Fig. 6.2 (second and third column), and indicate rates of optimal order of $\mathcal{O}(h^{p+1})$, $\mathcal{O}(h^p)$ and $\mathcal{O}(h^{p-1})$, respectively.

7. Conclusion

In this paper we have listed and classified known methods to construct C^1-smooth isogeometric spaces over unstructured multi-patch domains. This is a research field that is attracting growing interest, at the confluence of geometric design and numerical analysis of partial differential equations. We have discussed, with more details, the case of multi-patch parametrizations that are regular and only C^0 at the patch interfaces, reviewing in a coherent framework some of the recent results that are more closely related to our research activity.

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References

[1] A. Apostolatos, M. Breitenberger, R. Wüchner, and K.-U. Bletzinger. Domain Decomposition Methods and Kirchhoff-Love Shell Multipatch Coupling in Isogeometric Analysis. In B. Jüttler and B. Simeon, editors, *Isogeometric Analysis and Applications 2014*, pages 73–101. Springer, 2015.

[2] J. H. Argyris, I. Fried, and D. W. Scharpf. The TUBA family of plate elements for the matrix displacement method. *The Aeronautical Journal*, 72(692):701–709, 1968.

[3] F. Auricchio, L. Beirão da Veiga, A. Buffa, C. Lovadina, A. Reali, and G. Sangalli. A fully "locking-free" isogeometric approach for plane linear elasticity problems: a stream function formulation. *Comput. Methods Appl. Mech. Engrg.*, 197(1):160–172, 2007.

[4] A. Bartezzaghi, L. Dedè, and A. Quarteroni. Isogeometric Analysis of high order Partial Differential Equations on surfaces. *Comput. Methods Appl. Mech. Engrg.*, 295:446–469, 2015.

[5] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. Mathematical analysis of variational isogeometric methods. *Acta Numer.*, 23:157–287, 2014.

[6] D. J. Benson, Y. Bazilevs, M.-C. Hsu, and T. J. R. Hughes. A large deformation, rotation-free, isogeometric shell. *Comput. Methods Appl. Mech. Engrg.*, 200(13):1367–1378, 2011.

[7] A. Benvenuti. *Isogeometric Analysis for C^1-continuous Mortar Method.* PhD thesis, Corso di Dottorato in Matematica e Statistica, Università degli Studi di Pavia, 2017.

[8] M. Bercovier and T. Matskewich. *Smooth Bézier Surfaces over Unstructured Quadrilateral Meshes.* Lecture Notes of the Unione Matematica Italiana. Springer, 2017.

[9] A. Blidia, B. Mourrain, and N. Villamizar. G^1-smooth splines on quad meshes with 4-split macro-patch elements. *Comput. Aided Geom. Des.*, 52—53:106–125, 2017.

[10] F. Buchegger, B. Jüttler, and A. Mantzaflaris. Adaptively refined multi-patch B-splines with enhanced smoothness. *Appl. Math. Comput.*, 272:159–172, 2016.

[11] D. Burkhart, B. Hamann, and G. Umlauf. Iso-geometric analysis based on Catmull-Clark solid subdivision. *Computer Graphics Forum*, 29(5):1575–1784, 2010.

[12] C. L. Chan, C. Anitescu, and T. Rabczuk. Isogeometric analysis with strong multipatch C^1-coupling. *Comput. Aided Geom. Des.*, 62:294–310, 2018.

[13] F. Cirak, M. Ortiz, and P. Schröder. Subdivision surfaces: a new paradigm for thin-shell finite-element analysis. *Int. J. Numer. Meth. Engng.*, 47(12):2039–2072, 2000.

[14] F. Cirak, M. J. Scott, E. K. Antonsson, M. Ortiz, and P. Schröder. Integrated modeling, finite-element analysis, and engineering design for thin-shell structures using subdivision. *Computer-Aided Design*, 34(2):137–148, 2002.

[15] A. Collin, G. Sangalli, and T. Takacs. Analysis-suitable G^1 multi-patch parametrizations for C^1 isogeometric spaces. *Comput. Aided Geom. Des.*, 47:93–113, 2016.

[16] J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. John Wiley & Sons, 2009.

[17] P. Fischer, M. Klassen, J. Mergheim, P. Steinmann, and R. Müller. Isogeometric analysis of 2D gradient elasticity. *Comput. Mech.*, 47(3):325–334, 2011.

[18] H. Gómez, V. M. Calo, Y. Bazilevs, and T. J. R. Hughes. Isogeometric analysis of the Cahn–Hilliard phase-field model. *Comput. Methods Appl. Mech. Engrg.*, 197(49):4333–4352, 2008.

[19] H. Gómez, V. M. Calo, and T. J. R. Hughes. Isogeometric Analysis of Phase–Field Models: Application to the Cahn–Hilliard Equation. In *ECCOMAS Multidisciplinary Jubilee Symposium: New Computational Challenges in Materials, Structures, and Fluids*, pages 1–16. Springer, 2009.

[20] J. A. Gregory and J. M. Mahn. Geometric continuity and convex combination patches. *Comput. Aided Geom. Des.*, 4(1-2):79–89, 1987.

[21] D. Groisser and J. Peters. Matched G^k-constructions always yield C^k-continuous isogeometric elements. *Comput. Aided Geom. Des.*, 34:67–72, 2015.

[22] Y. Guo and M. Ruess. Nitsche's method for a coupling of isogeometric thin shells and blended shell structures. *Comput. Methods Appl. Mech. Engrg.*, 284:881–905, 2015.

[23] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39-41):4135–4195, 2005.

[24] B. Jüttler, A. Mantzaflaris, R. Perl, and M. Rumpf. On numerical integration in isogeometric subdivision methods for PDEs on surfaces. *Comput. Methods Appl. Mech. Engrg.*, 302:131–146, 2016.

[25] M. Kapl, F. Buchegger, M. Bercovier, and B. Jüttler. Isogeometric analysis with geometrically continuous functions on planar multi-patch geometries. *Comput. Methods Appl. Mech. Engrg.*, 316:209–234, 2017.

[26] M. Kapl, G. Sangalli, and T. Takacs. Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations. *Comput. Aided Geom. Des.*, 52–53:75–89, 2017.

[27] M. Kapl, G. Sangalli, and T. Takacs. Construction of analysis-suitable G^1 planar multi-patch parameterizations. *Computer-Aided Design*, 97:41–55, 2018.

[28] M. Kapl, G. Sangalli, and T. Takacs. An isogeometric C^1 subspace on unstructured multi-patch planar domains. *Comput. Aided Geom. Des.*, 69:55–75, 2019.

[29] M. Kapl, V. Vitrih, B. Jüttler, and K. Birner. Isogeometric Analysis with Geometrically Continuous Functions on Two-Patch Geometries. *Comput. Math. Appl.*, 70(7):1518–1538, 2015.

[30] K. Karčiauskas, T. Nguyen, and J. Peters. Generalizing bicubic splines for modeling and IGA with irregular layout. *Computer-Aided Design*, 70:23–35, 2016.

[31] K. Karčiauskas and J. Peters. Smooth multi-sided blending of biquadratic splines. *Computers & Graphics*, 46:172–185, 2015.

[32] K. Karčiauskas and J. Peters. Refinable G^1 functions on G^1 free-form surfaces. *Comput. Aided Geom. Des.*, 54:61–73, 2017.

[33] K. Karčiauskas and J. Peters. Refinable bi-quartics for design and analysis. *Computer-Aided Design*, pages 204–214, 2018.

[34] J. Kiendl, Y. Bazilevs, M.-C. Hsu, R. Wüchner, and K.-U. Bletzinger. The bending strip method for isogeometric analysis of Kirchhoff-Love shell structures comprised of multiple patches. *Comput. Methods Appl. Mech. Engrg.*, 199(35):2403–2416, 2010.

[35] J. Kiendl, K.-U. Bletzinger, J. Linhard, and R. Wüchner. Isogeometric shell analysis with Kirchhoff-Love elements. *Comput. Methods Appl. Mech. Engrg.*, 198(49):3902–3914, 2009.

[36] J. Kiendl, M.-C. Hsu, M. C. H. Wu, and A. Reali. Isogeometric Kirchhoff—Love shell formulations for general hyperelastic materials. *Comput. Methods Appl. Mech. Engrg.*, 291:280–303, 2015.

[37] M.-J. Lai and L. L. Schumaker. *Spline functions on triangulations*, volume 110 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 2007.

[38] J. Liu, L. Dedè, J. A. Evans, M. J. Borden, and T. J. R. Hughes. Isogeometric analysis of the advective Cahn-–Hilliard equation: Spinodal decomposition under shear flow. *J. Comput. Phys.*, 242:321–350, 2013.

[39] T. Matskewich. *Construction of C^1 surfaces by assembly of quadrilateral patches under arbitrary mesh topology.* PhD thesis, Hebrew University of Jerusalem, 2001.

[40] B. Mourrain, R. Vidunas, and N. Villamizar. Dimension and bases for geometrically continuous splines on surfaces of arbitrary topology. *Comput. Aided Geom. Des.*, 45:108–133, 2016.

[41] A. Myles, K. Karčiauskas, and J. Peters. Pairs of bi-cubic surface constructions supporting polar connectivity. *Comput. Aided Geom. Des.*, 25(8):621–630, 2008.

[42] A. Myles and J. Peters. C^2 splines covering polar configurations. *Computer-Aided Design*, 43(11):1322–1329, 2011.

[43] T. Nguyen, K. Karčiauskas, and J. Peters. A Comparative Study of Several Classical, Discrete Differential and Isogeometric Methods for Solving Poisson's Equation on the Disk. *Axioms*, 3(2):280–299, 2014.

[44] T. Nguyen, K. Karčiauskas, and J. Peters. C^1 finite elements on non-tensor-product 2d and 3d manifolds. *Appl. Math. Comput.*, 272:148–158, 2016.

[45] T. Nguyen and J. Peters. Refinable C^1 spline elements for irregular quad layout. *Comput. Aided Geom. Des.*, 43:123–130, 2016.

[46] J. Niiranen, S. Khakalo, V. Balobanov, and A. H. Niemi. Variational formulation and isogeometric analysis for fourth-order boundary value problems of gradient-elastic bar and plane strain/stress problems. *Comput. Methods Appl. Mech. Engrg.*, 308:182–211, 2016.

[47] J. Peters. Smooth mesh interpolation with cubic patches. *Computer-Aided Design*, 22(2):109–120, 1990.

[48] J. Peters. Parametrizing singularly to enclose data points by a smooth parametric surface. In *Proceedings of graphics interface*, volume 91, 1991.

[49] J. Peters. Geometric continuity. In *Handbook of computer aided geometric design*, pages 193–227. North-Holland, 2002.

[50] J. Peters and U. Reif. *Subdivision surfaces*, volume 3 of *Geometry and Computing*. Springer, 2008.

[51] U. Reif. Biquadratic G-spline surfaces. *Comput. Aided Geom. Des.*, 12(2):193–205, 1995.

[52] U. Reif. A refinable space of smooth spline surfaces of arbitrary topological genus. *J. Approx. Th.*, 90(2):174–199, 1997.

[53] A. Riffnaller-Schiefer, U. H. Augsdörfer, and D. W. Fellner. Isogeometric shell analysis with NURBS compatible subdivision surfaces. *Appl. Math. Comput.*, 272:139–147, 2016.

[54] A. Tagliabue, L. Dedè, and A. Quarteroni. Isogeometric Analysis and error estimates for high order partial differential equations in fluid dynamics. *Comput. Fluids*, 102:277–303, 2014.

[55] D. Toshniwal, H. Speleers, R. Hiemstra, and T. J. R. Hughes. Multi-degree smooth polar splines: A framework for geometric modeling and isogeometric analysis. *Comput. Methods Appl. Mech. Engrg.*, 316:1005–1061, 2017.

[56] D. Toshniwal, H. Speleers, and T. J. R. Hughes. Analysis-suitable spline spaces of arbitrary degree on unstructured quadrilateral meshes. Technical Report 16, Institute for Computational Engineering and Sciences (ICES), 2017.

[57] D. Toshniwal, H. Speleers, and T. J. R. Hughes. Smooth cubic spline spaces on unstructured quadrilateral meshes with particular emphasis on extraordinary points: Geometric design and isogeometric analysis considerations. *Comput. Methods Appl. Mech. Engrg.*, 327:411–458, 2017.

[58] Q. Zhang, M. Sabin, and F. Cirak. Subdivision surfaces with isogeometric analysis adapted refinement weights. *Computer-Aided Design*, 102:104–114, 2018.