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A CONSISTENCE-STABILITY APPROACH TO HYDRODYNAMIC LIMIT OF INTERACTING PARTICLE SYSTEMS ON LATTICES

ANGELIKI MENEGAKI AND CLÉMENT MOUHOT

ABSTRACT. This is a review based on the presentation done at the seminar Laurent Schwartz in December 2021. It is announcing results in the forthcoming [MMM22]. This work presents a new simple quantitative method for proving the hydrodynamic limit of a class of interacting particle systems on lattices. We present here this method in a simplified setting, for the zero-range process and the Ginzburg-Landau process with Kawasaki dynamics, in the parabolic scaling and in dimension 1. The rate of convergence is quantitative and uniform in time. The proof relies on a consistence-stability approach in Wasserstein distance, and it avoids the use of both the so-called “block estimates”.

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1. THE GENERAL METHOD

We consider the hydrodynamic limit of interacting particle systems on a lattice. The problem is to show that under an appropriate scaling of time and space, the local particle densities of a stochastic lattice gas converge to the solution of a macroscopic partial differential equation. We first present our method abstractly and then sketch applications to three concrete models: the simple-exclusion process (SEP), the zero-range process (ZRP) and the Ginzburg Landau process with Kawasaki dynamics (GLK). The hydrodynamic limit is known at a qualitative level for all these models under both hyperbolic and parabolic scalings for the SEP and ZRP and under parabolic scaling for the GLK, see [GPV88, Yau91, Rez91, KL99]. However finding quantitative error estimates had remained an important question, as well as understanding the long-time behaviour of the hydrodynamic limit. First results towards quantitative error, in the particular case of the Ginzburg-Landau process with Kawasaki dynamics in dimension 1, were obtained in the two-parts work [DMOWa, DMOWb], which builds upon partial progresses in [GOVW09].

1.1. Set up and notation. We denote by X the state space at a given site (number of particles, spin, etc.), which will in this paper be \mathbb{N} (ZRP) or \mathbb{R} (GLK). Consider the discrete torus \mathbb{T}_N^d and the corresponding phase space of particle configurations $X_N := X^{\mathbb{T}_N^d}$. Variables in \mathbb{T}_N^d are called *microscopic* and denoted by x, y, z , whereas variables in the limit continuous torus \mathbb{T}^d are called *macroscopic* and denoted by u ;

finally particle *configurations* in X_N are denoted by η . The canonical embedding $\mathbb{T}_N^d \rightarrow \mathbb{T}^d$, $x \mapsto x/N$ means the macroscopic distance between sites of the lattice is $1/N$. Given a particle configuration $\eta \in \mathsf{X}_N$, we define the *empirical measure*

$$(1.1) \quad \alpha_\eta^N := \sum_{x \in \mathbb{T}_N^d} \eta_x \delta_{x/N} \in \mathcal{M}_+(\mathbb{T}^d).$$

where η_x denotes the value of η at $x \in \mathbb{T}_N^d$, and $\mathcal{M}_+(\mathbb{T}^d)$ is the space of positive Radon measures on the torus, and \sum denotes the ‘‘average sum’’, here $N^{-d} \sum_{x \in \mathbb{T}_N^d}$.

At the microscopic level, the interacting particle system evolves through a stochastic process and the time-dependent probability measure describing the law of η is denoted by $\mu_t^N \in P(\mathsf{X}_N)$. We consider a linear operator $\mathcal{L}_N : C_b(\mathsf{X}_N) \rightarrow C_b(\mathsf{X}_N)$ generating uniquely a Feller semigroup $e^{t\mathcal{L}_N}$ on $P(\mathsf{X}_N)$ (see [Lig85, Chapter 1]) so that given $\mu_0^N \in P(\mathsf{X}_N)$ the solution $\mu_t^N = e^{t\mathcal{L}_N} \mu_0^N \in P(\mathsf{X}_N)$ satisfies

$$(1.2) \quad \forall \Phi \in C_b(\mathsf{X}_N), \quad \frac{d}{dt} \langle \Phi, \mu_t^N \rangle = \langle \mathcal{L}_N \Phi, \mu_t^N \rangle,$$

where $C_b(\mathsf{X}_N)$ denotes continuous bounded real-valued functions and $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $C_b(\mathsf{X}_N)$ and $P(\mathsf{X}_N)$.

At the macroscopic level, we consider a map $\mathcal{L}_\infty : \mathcal{M}_+(\mathbb{G}_\infty) \rightarrow \mathcal{M}_+(\mathbb{G}_\infty)$ (in general unbounded and nonlinear) and the evolution problem

$$(1.3) \quad \partial_t f_t = \mathcal{L}_\infty f_t, \quad f_{t=0} = f_0.$$

A measure $\mu^N \in P(\mathsf{X}_N)$ is called *invariant* for (1.2) if

$$\forall \Phi \in C_b(\mathsf{X}_N), \quad \langle \mu^N, \mathcal{L}_N \Phi \rangle = 0.$$

We also denote $\text{Lip}(\mathsf{X}_N)$ the Lipschitz functions $\Phi : \mathsf{X}_N \rightarrow \mathbb{R}$ with respect to the (normalised) ℓ^1 norm: for every $\eta, \zeta \in \mathsf{X}_N$, $|\Phi(\eta) - \Phi(\zeta)| \leq C_\Phi \sum_{x \in \mathbb{T}_N^d} |\eta_x - \zeta_x|$, and we denote the smallest such constant C_Φ by $[\Phi]_{\text{Lip}(\mathsf{X}_N)} \in \mathbb{R}_+$.

1.2. Abstract assumptions. We make the following assumptions on (1.2)-(1.3):

(H0) Local equilibrium structure. There are $n_\lambda : \text{Conv}(\mathsf{X}) \rightarrow \mathbb{R}_+$ depending on $\lambda \in \mathbb{R}$ (Conv denotes the convex hull) and $\sigma : \text{Conv}(\mathsf{X}) \rightarrow \mathbb{R}$ so that: (i) $n_\lambda^{\otimes \mathbb{T}_N^d}$ is invariant on X_N for each λ , and (ii) for any $\rho \in \text{Conv}(\mathsf{X})$, $\mathbb{E}_{n_{\sigma(\rho)}}[\eta_x] = \rho$. We then define, given a macroscopic profile f on \mathbb{T}^d , the *local Gibbs measure*

$$\vartheta_f^N(\eta) := \nu_{\sigma(f(\frac{\cdot}{N}))}^N(\eta) \quad \text{where} \quad \nu_F^N(\eta) := \prod_{x \in \mathbb{G}_N} n_{F(x)}(\eta(x)).$$

The two maps $\eta \mapsto \alpha_\eta^N$ and $f \mapsto \vartheta_f^N$ allow comparisons between the microscopic and macroscopic scales, as summarized in Figure 1.

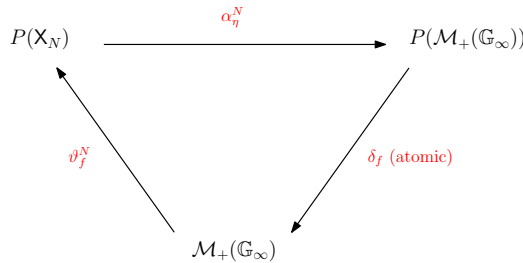


FIGURE 1. The functional setting.

(H1) Microscopic stability. The semigroup $e^{t\mathcal{L}_N}$ satisfies

$$(1.4) \quad \forall \Phi \in \text{Lip}(\mathbb{X}_N), \quad [e^{t\mathcal{L}_N} \Phi]_{\text{Lip}(\mathbb{X}_N)} \leq [\Phi]_{\text{Lip}(\mathbb{X}_N)}.$$

(H2) Macroscopic stability. There is a Banach space $\mathfrak{B} \subset \mathcal{M}_+(\mathbb{G}_\infty)$ so that (1.3) is locally well-posed in \mathfrak{B} ; given the maximal time of existence $T_m \in (0, +\infty]$ we denote for $t \in [0, T_m)$, $R(t) := \|f_t - f_\infty\|_{\mathfrak{B}}$ when (1.3) has a unique stationary solution $f_\infty \in \mathfrak{B}$ with mass $\int_{\mathbb{T}^d} f_\infty = \int_{\mathbb{T}^d} f_0$, otherwise we denote $R(t) := \|f_t\|_{\mathfrak{B}}$.

(H3) Consistency. There is a *consistency error* $\epsilon(N) \rightarrow 0$ as $N \rightarrow \infty$ so that for $T \in [0, T_m)$

$$\begin{aligned} \frac{1}{T} \int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N} \Phi \right), \left[\mathcal{L}_N^* \left(\frac{d\vartheta_{f_s}^N}{d\nu_\infty^N} \right) - \frac{d}{ds} \left(\frac{d\vartheta_{f_s}^N}{d\nu_\infty^N} \right) \right] d\nu_\infty^N \right\rangle ds dt \\ \leq \epsilon(N) [\Phi]_{\text{Lip}(\mathbb{X}_N)} \int_0^t R(s) ds \end{aligned}$$

for any $\Phi \in \text{Lip}(\mathbb{X}_N)$, where ν_∞^N is an equilibrium measure.

1.3. The abstract strategy.

Theorem 1.1. *Consider (1.2)-(1.3) with the assumptions (H0)–(H1)–(H2)–(H3). Let $\phi \in C^\infty(\mathbb{T}^d)$, $\mu_0^N \in P_1(\mathbb{X}_N)$ for all $N \geq 1$, $f_0 \in \mathcal{B}$. Then*

$$(1.5) \quad \forall T \in [0, T_m), \quad \frac{1}{T} \int_0^T \|\mu_t^N - \vartheta_{f_t}^N\|_{\text{Lip}^*} dt \lesssim \epsilon(N) \int_0^T R(s) ds + \|\mu_0^N - \vartheta_{f_0}^N\|_{\text{Lip}^*}.$$

Remark 1. *Note that $\|\mu_t^N - \vartheta_{f_t}^N\|_{\text{Lip}^*} \rightarrow 0$ as $N \rightarrow \infty$ implies that the empirical measure (1.1) sampled from the law μ_t^N satisfies*

$$(1.6) \quad \forall \phi \in C_b(\mathbb{G}), \quad \forall \epsilon > 0, \quad \forall t \geq 0, \quad \lim_{N \rightarrow \infty} \mu_t^N (\{ |\langle \alpha_\eta^N, \phi \rangle - \langle f_t, \phi \rangle| > \epsilon \}) = 0$$

with a rate of convergence (thus recovering quantitatively results from [GPV88]):

$$\begin{aligned} & \mu_t^N (\{ |\langle \alpha_\eta^N, \phi \rangle - \langle f_t, \phi \rangle| > \epsilon \}) \\ & \leq \mu_t^N (\{ \langle \alpha_\eta^N, \phi \rangle \geq \langle f_t, \phi \rangle + \epsilon \}) + \mu_t^N (\{ \langle \alpha_\eta^N, \phi \rangle \leq \langle f_t, \phi \rangle - \epsilon \}) \\ & \leq \int_{\mathbb{X}_N} [F_\epsilon^+ (\langle \phi, \alpha_\eta^N \rangle) - F_\epsilon^+ (\langle \phi, f_t \rangle)] d\mu_t^N + \int_{\mathbb{X}_N} [F_\epsilon^- (\langle \phi, \alpha_\eta^N \rangle) - F_\epsilon^- (\langle \phi, f_t \rangle)] d\mu_t^N \end{aligned}$$

where F_ϵ^\pm are mollified version of the characteristic functions of respectively $\{z \geq \langle \phi, f_t \rangle + \epsilon\}$ and $\{z \leq \langle \phi, f_t \rangle - \epsilon\}$, which yields

$$\sup_{t \in [0, T]} \mu_t^N (\{ |\langle \alpha_\eta^N, \phi \rangle - \langle f_t, \phi \rangle| > \epsilon \}) \lesssim \epsilon^{-1} \|\mu_t^N - \vartheta_{f_t}^N\|_{\text{Lip}^*} + \epsilon^{-2} N^{-d}.$$

2. CONCRETE APPLICATIONS

We apply the abstract result to two archetypical models, the zero-range process (ZRP), and the Ginzburg-Landau process with Kawasaki dynamics (GLK).

2.1. The ZRP. In this case, the state space at each site is $\mathbb{X} = \mathbb{N}$. Given the choice of a *transition function* $p \in P(\mathbb{T}_N^d)$ with $p(0) = 0$ and a *jump rate function* $g : \mathbb{N} \rightarrow \mathbb{R}_+$, the base generator $\widehat{\mathcal{L}}_N$ writes

$$(2.1) \quad \forall \Phi \in C_b(\mathbb{X}_N), \quad \forall \eta \in \mathbb{X}_N, \quad \widehat{\mathcal{L}}_N \Phi(\eta) := \sum_{x, y \in \mathbb{T}_N^d} p(y-x) g(\eta_x) [\Phi(\eta^{xy}) - \Phi(\eta)]$$

where η^{xy} is defined as before. The local equilibrium structure of **(H0)** is given by

$$(2.2) \quad n_\lambda(k) := \frac{\lambda^k}{g(k)!Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \sum_{k=0}^{+\infty} \frac{\lambda^k}{g(k)!}$$

$$(2.3) \quad \sigma \text{ is defined implicitly by } \sigma(\rho) \frac{Z'(\sigma(\rho))}{Z(\sigma(\rho))} \equiv \rho$$

denoting $g(k)! := g(k)g(k-1)\cdots g(1)$. The pair (g, σ) thus constructed satisfies $\mathbb{E}_{n_{\sigma(\alpha)}}[g] = \sigma(\alpha)$. When $f \equiv \rho \in [0, +\infty)$ is constant, the local Gibbs measure $\vartheta_\rho^N = \nu_{\sigma(\rho)}^N$ is invariant with average number of particles ρ . The *mean transition rate* is defined by $\gamma := \sum_{x \in \mathbb{Z}^d} xp(x) \in \mathbb{R}^d$. When $\gamma \neq 0$, the first non-zero asymptotic dynamics as $N \rightarrow \infty$ is given by the hyperbolic scaling $\mathcal{L}_N := N\widehat{\mathcal{L}}_N$, and the corresponding expected limit equation is $\partial_t f = \gamma \cdot \nabla[\sigma(f)]$. When $\gamma = 0$, the first non-zero asymptotic dynamics as $N \rightarrow \infty$ is given by the parabolic scaling $\mathcal{L}_N := N^2\widehat{\mathcal{L}}_N$, and the corresponding limit equation is formally

$$(2.4) \quad \partial_t f = \Delta_a[\sigma(f)] \quad \text{with} \quad \Delta_a := \sum_{i,j=1}^d a_{ij} \partial_{i_j}^2 \quad \text{and} \quad a_{ij} := \sum_{x \in \mathbb{Z}^d} p(x) x_i x_j.$$

We make the following assumptions on the jump rate function $g : \mathbb{N} \rightarrow [0, \infty)$.

(HZRP) The jump rate g satisfies $g(0) = 0$, $g(n) > 0$ for all $n > 0$, is non-decreasing, uniformly Lipschitz $\sup_{n \geq 0} |g(n+1) - g(n)| < +\infty$, and there are $n_0 > 0$ and $\beta > 0$ such that $g(n') - g(n) \geq \beta$ for any $n' \geq n + n_0$.

The main result on the ZRP is:

Theorem 2.1 (Hydrodynamic limit for the ZRP). *Consider $\widehat{\mathcal{L}}_N$ defined in (2.1) with g satisfying **(HZRP)**. Let $d = 1$, $f_0 \in C^3(\mathbb{T})$ with $f_0 \geq \delta > 0$, and $\mu_0^N \in P_1(\mathbb{X}_N)$ for all $N \geq 1$. Assume $\gamma = 0$, define $\mu_t^N = e^{tN^2\widehat{\mathcal{L}}_N}$ and $f_t \in C([0, T], C^3(\mathbb{T}^d))$ solution to (2.4), then the following convergence holds (with quantitative constants)*

$$(2.5) \quad \sup_{T \geq 0} \frac{1}{T} \int_0^T \|\mu_t^N - \vartheta_{f_t}^N\|_{\text{Lip}^*} dt \lesssim N^{-1/8} + \|\mu_0^N - \vartheta_{f_0}^N\|_{\text{Lip}^*}.$$

2.2. The GLK. In this case, the state space at each site is $\mathbb{X} = \mathbb{R}$. Given the choice of a *single-site potential* $V \in C^2(\mathbb{R})$, the base generator $\widehat{\mathcal{L}}_N$ writes

$$(2.6) \quad \widehat{\mathcal{L}}_N \Phi(\eta) := \frac{1}{2} \sum_{x \sim y \in \mathbb{T}_N^d} \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 - \frac{1}{2} \sum_{x \sim y \in \mathbb{T}_N^d} [V'(\eta_x) - V'(\eta_y)] \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)$$

where $x \sim y$ denotes neighbouring sites. The local equilibrium structure is given by

$$n_\lambda(r) := \frac{e^{\lambda r}}{Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \int_{\mathbb{R}} e^{\lambda r - V(r)} dr$$

$$\sigma \text{ is defined implicitly by } \frac{Z'(\sigma(\rho))}{Z(\sigma(\rho))} \equiv \rho.$$

When $f \equiv \rho \in \mathbb{R}$ is constant, the local Gibbs measure $\vartheta_\rho^N = \nu_{\sigma(\rho)}^N$ is invariant with average spin ρ . The hyperbolic scaling formally leads to zero and the parabolic scaling $\mathcal{L}_N := N^2\widehat{\mathcal{L}}_N$ formally leads to

$$(2.7) \quad \partial_t f = 2\Delta[\sigma(f)].$$

We assume that the single-site potential satisfies

(HGLK) The potential V is C^2 and decomposes as $V(u) = V_0(u) + V_1(u)$ with $V_0''(u) \geq \kappa$ for all $u \in \mathbb{R}$ for some $\kappa > 0$ and $\|V_1\|_{W^{1,\infty}(\mathbb{R})} \lesssim 1$.

This assumption is similar with those in [GOVW09, DMOWa, Fat13]. One can take for example a double-well potential, provided it is uniformly convex at infinity.

Theorem 2.2 (Hydrodynamic limit for the GLK). *Consider \mathcal{L}_N defined in (2.6) with V satisfying **(HGLK)**. Let $d = 1$, $f_0 \in C^3(\mathbb{T}^d)$ and $\mu_0^N \in P_1(\mathbb{X}_N)$ for all $N \geq 1$. Define $\mu_t^N = e^{tN^2\hat{\mathcal{L}}_N}$ and $f_t \in C([0, +\infty), C^3(\mathbb{T}^d))$ the global solution to (2.7), then the following convergence holds (with quantitative constants)*

$$(2.8) \quad \sup_{T \geq 0} \frac{1}{T} \int_0^T \|\mu_t^N - \vartheta_{f_t}^N\|_{\text{Lip}^*} dt \lesssim N^{-1/8} + \|\mu_0^N - \vartheta_{f_0}^N\|_{\text{Lip}^*}.$$

3. THE ABSTRACT STRATEGY

In this section we sketch the proof of Theorem 1.1. Let f_t be a solution to (1.3). Given $0 < \ell < N$, we denote by η^ℓ for the local ℓ -average $\eta_x^\ell := \sum_{|y-x| \leq \ell} \eta_y$.

Denote by $F_t^N := d\mu_t^N/d\nu_\infty^N$ and $G_t^N := d\vartheta_{f_t}^N/d\nu_\infty^N$ the densities with respect to ν_∞^N , and write

$$\frac{d}{dt} (F_t^N - G_t^N) = \mathcal{L}_N^* (F_t^N - G_t^N) + (\mathcal{L}_N^* G_t^N - \partial_t G_t^N)$$

so that Duhamel's formula yields

$$F_t^N - G_t^N = e^{t\mathcal{L}_N^*} (F_0^N - G_0^N) + \int_0^t e^{(t-s)\mathcal{L}_N^*} (\mathcal{L}_N^* G_s^N - \partial_s G_s^N) ds.$$

Take $\Phi \in \text{Lip}(\mathbb{X}_N)$ with $\|\Phi\|_{\text{Lip}(\mathbb{X}_N)} \leq 1$ and integrate the above equation to get

$$\begin{aligned} & \int_{\mathbb{X}_N} \Phi (F_t^N - G_t^N) d\nu_\infty^N \\ &= \underbrace{\int_{\mathbb{X}_N} (e^{t\mathcal{L}_N} \Phi) (F_0^N - G_0^N) d\nu_\infty^N}_{I_1(t)} + \underbrace{\int_{\mathbb{X}_N} \int_0^t (e^{(t-s)\mathcal{L}_N} \Phi) (\mathcal{L}_N G_s^N - \partial_s G_s^N) d\nu_\infty^N ds}_{I_2(t)}. \end{aligned}$$

(H1) implies $I_1(t) \lesssim \|\mu_0^N - \vartheta_{f_0}^N\|_{\text{Lip}^*}$ and **(H3)** implies $\frac{1}{T} \int_0^T I_2(t) dt \leq \epsilon(N) \int_0^T R(s) ds$, which implies the conclusion of Theorem 1.1.

4. PROOF FOR THE ZRP

In this section we prove Theorem 2.1) (hydrodynamical limit for the ZRP). Note for this model $\mathcal{L}_N = \mathcal{L}_N^*$ is symmetric with respect to equilibrium measures. Given $f_t \in C^3(\mathbb{T}^d)$ with $f > \delta$, $\delta > 0$, and $\rho := \int_{\mathbb{T}^d} f$, the density of the local Gibbs measure relatively to the invariant measure with mass ρ is:

$$(4.1) \quad G_t^N(\eta) := \frac{d\vartheta_f^N(\eta)}{d\vartheta_\rho^N(\eta)} = \prod_{x \in \mathbb{T}_N^d} \left(\frac{\sigma(f_t(x/N))}{\sigma(\rho)} \right)^{\eta(x)} \left(\frac{Z(\sigma(f_t(x/N)))}{Z(\sigma(\rho))} \right)^{-1}.$$

where the function $\sigma(r)$ is defined by $\langle n_{\sigma(r)}, \eta(x) \rangle = r$ and the *partition function* $Z : [0, \lambda^*) \rightarrow \mathbb{R}$ is defined in (2.2), with $\lambda^* \in [0, +\infty]$ denoting the radius of convergence of the series.

It is proved in [KL99, Chapter 2, Section 3] that assumption **(HZRP)** on g implies that $\sigma = R^{-1} : [0, \infty) \rightarrow [0, \infty)$ is well-defined and strictly increasing, with

$$R(\lambda) = \lambda \partial_\lambda \log(Z(\lambda)) = \frac{1}{Z(\lambda)} \sum_{n \geq 0} \frac{n \lambda^n}{g(n)!}.$$

Then the building block n_ρ of the Gibbs measure satisfies $\langle n_{\sigma(\rho)}, g(\eta(x)) \rangle = \sigma(\rho)$. Moreover **(HZRP)** implies that the function σ is C^∞ with uniform bound on all derivatives on \mathbb{R}_+ , with Lipschitz constant less than g^* , see [KL99, Corollary 3.6], and with $\inf_{\lambda > 0} \lambda^{-1} \sigma(\lambda) > 0$ (in particular $\sigma'(0) > 0$). Finally **(HZRP)** also implies the following comparison principle: if one starts from two ordered configurations $\eta \leq \zeta$ (at all points $x \in \mathbb{T}_N^d$) then the evolution preserves this inequality at later times: $\eta_t \leq \zeta_t$. This implies that if for any $f^N \in C_b(\mathbf{X}_N)$ so that $f^N(\eta) \leq f^N(\zeta)$ for all $\eta \leq \zeta$ one has $\langle \mu_t^{N,1}, f^N \rangle \leq \langle \mu_t^{N,2}, f^N \rangle$, then at later times $\mu_t^{N,1} \prec \mu_t^{N,2}$. It is easy to deduce that the k th moments ($k \in \mathbb{N}$)

$$M_k [\mu_t^N] := \left\langle \mu_t^N, \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)^k \right\rangle$$

are uniformly bounded along time when $\mu_0^N \prec C \vartheta_\rho^N$ for some $C > 0$ and $\rho \in \mathbb{R}_+$.

4.1. Microscopic Stability – hypothesis (H1). We use again the “basic coupling” as in [Lig85, Rez91]. We define

$$(4.2) \quad \begin{aligned} \tilde{\mathcal{L}}_N \Psi(\eta, \zeta) &:= \sum_{x, y \in \mathbb{T}_N^d} p(y-x) \left(g(\eta_x) \wedge g(\zeta_x) \right) \left[\Psi(\eta^{xy}, \zeta^{xy}) - \Psi(\eta, \zeta) \right] \\ &+ \sum_{x, y \in \mathbb{T}_N^d} p(y-x) \left(g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) \right) \left[\Psi(\eta^{xy}, \zeta) - \Psi(\eta, \zeta) \right] \\ &+ \sum_{x, y \in \mathbb{T}_N^d} p(y-x) \left(g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) \right) \left[\Psi(\eta, \zeta^{xy}) - \Psi(\eta, \zeta) \right]. \end{aligned}$$

for a two-variable test function $\Psi(\eta, \zeta)$. Then $\tilde{\mathcal{L}}_N \Phi(\eta) = \hat{\mathcal{L}}_N \Phi(\eta)$ and $\tilde{\mathcal{L}}_N \Phi(\zeta) = \hat{\mathcal{L}}_N \Phi(\zeta)$, and **(H1)** follows from the fact that $e^{t \tilde{\mathcal{L}}_N}$ preserves sign and the inequality

$$\tilde{\mathcal{L}}_N \left(\sum_{z \in \mathbb{T}_N^d} |\eta_z - \zeta_z| \right) \leq 0.$$

To prove the latter inequality, we compute

$$\begin{aligned} \tilde{\mathcal{L}}_N \left(\sum_{z \in \mathbb{T}_N^d} |\eta_z - \zeta_z| \right) &= \sum_{x, y \in \mathbb{T}_N^d} p(y-x) \left(g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) \right) \\ &\quad \times \left[|\eta_x^{xy} - \zeta_x| + |\eta_y^{xy} - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right] \\ &+ \sum_{x, y \in \mathbb{T}_N^d} p(y-x) \left(g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) \right) \\ &\quad \times \left[|\eta_x - \zeta_x^{xy}| + |\eta_y - \zeta_y^{xy}| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right]. \end{aligned}$$

When $g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\eta_x - \zeta_x \geq 1$ and

$$\left[|\eta_x^{xy} - \zeta_x| + |\eta_y^{xy} - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right] \leq 0.$$

When $g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\zeta_x - \eta_x \geq 1$ and

$$\left[|\eta_x - \zeta_x^{xy}| + |\eta_y - \zeta_y^{xy}| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y| \right] \leq 0.$$

4.2. Macroscopic stability – hypothesis (H2). In the parabolic scaling the limit PDE is the nonlinear diffusion equation (2.4). We take $\mathcal{B} = C^3$ with its standard infinity Banach norm. The proof that this norm remains uniformly bounded in time is classical in dimension $d = 1$ (using the bounds on σ), and $f_t \in [\delta, 1 - \delta]$ for all times by maximal principle. Moreover $f_t \rightarrow \rho$ exponentially fast as $t \rightarrow \infty$ in \mathcal{B} .

4.3. Consistency estimate – hypothesis (H3). Note that the operator is self-adjoint, $\mathcal{L}_N^* = \mathcal{L}_N$, with respect to the equilibrium measures. We assume $\gamma = 0$.

Proposition 4.1. *Given $d = 1$ and the solution $f_t \in C^3(\mathbb{T}^d)$ to (2.4) with $f \geq \delta$, $\delta > 0$, and $\rho := \int_{\mathbb{T}^d} f$, and G_t^N defined in (4.1), we have for every $\Phi \in \text{Lip}(\mathbb{X}_N)$*

$$\frac{1}{T} \int_0^T I_t^N dt := \frac{1}{T} \int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N} \Phi \right), \left[\mathcal{L}_N G_s^N - \frac{d}{ds} G_s^N \right] d\nu_\infty^N \right\rangle ds dt = \mathcal{O}(N^{-1/8})$$

where the constant depends on the estimates in (H2).

Proof. We start by computing

$$\mathcal{L}_N G_s^N - \frac{d}{ds} G_s^N = \sum_{x \in \mathbb{T}_N^d} A_x^N G_s^N$$

with (note that $f_t \rightarrow \rho$ exponentially fast)

$$\begin{aligned} A_x^N &:= N^2 \sum_{y \in \mathbb{T}_N^d} p(y-x) g(\eta_x) \left(\frac{\sigma(f_t(y/N))}{\sigma(f_t(x/N))} - 1 \right) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) \\ &= \frac{g(\eta_x)}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) + \mathcal{O}(e^{-Cs}/N) \end{aligned}$$

for some $C > 0$. Since (conservation of mass)

$$\int_{\mathbb{X}_N} \left(\sum_{x \in \mathbb{T}_N^d} A_x^N G_s^N \right) d\nu_\infty^N = \int_{\mathbb{X}_N} \left(\sum_{x \in \mathbb{T}_N^d} A_x^N \right) d\vartheta_{f_s}^N = 0,$$

we can replace $\Phi_{t-s} := e^{(t-s)\mathcal{L}_N} \Phi$ by

$$\tilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N} \Phi - \mathbf{E}_{\vartheta_{f_s}^N} [e^{(t-s)\mathcal{L}_N} \Phi]$$

and use the Lipschitz bound on $e^{(t-s)\mathcal{L}_N} \Phi$ (microscopic stability) to get

$$I_t^N = \int_0^t \int_{\mathbb{X}_N} \tilde{\Phi}_{t,s}(\eta) \left(\sum_{x \in \mathbb{T}_N^d} \tilde{A}_x^N \right) d\vartheta_{f_s}^N + \mathcal{O}(1/N)$$

with \tilde{A}_x^N defined by (note that it has zero average against $d\vartheta_{f_s}^N$)

$$\tilde{A}_x^N := \{g(\eta_x) - \sigma(f_t(x/N)) - \sigma'(f_t(x/N))[\eta_x - f_t(x/N)]\} \frac{\Delta_a[\sigma(f)](x/N)}{\sigma(f_t(x/N))}.$$

We then form sub-sum over non-overlapping cubes of size $\ell \in \{1, \dots, N\}$ (this intermediate scale factor ℓ will be chosen later in terms of N). Let $\mathcal{R}_N^d \subset \mathbb{T}_N^d$ be a net of

centers of non-overlapping cubes of the form $\mathcal{C}_x := \{y \in \mathbb{T}_N^d : \|x - y\|_\infty \leq \ell\}$. Then

$$\begin{aligned} I_t^N &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \left(\sum_{y \in \mathcal{C}_x} \tilde{A}_y^N \right) d\vartheta_{f_s}^N + \mathcal{O}(1/N) \\ &= (2\ell + 1)^d \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \hat{A}_x^N d\vartheta_{f_s}^N + \mathcal{O}(1/N) \end{aligned}$$

with the \hat{A}_x^N defined by

$$\hat{A}_x^N := \{ \langle g(\eta) \rangle_{\mathcal{C}_x} - \sigma(f_t(x/N)) - \sigma'(f_t(x/N)) [\langle \eta \rangle_{\mathcal{C}_x} - f_t(x/N)] \} \frac{\Delta_a[\sigma(f)](x/N)}{\sigma(f_t(x/N))}$$

where $\langle F(\eta) \rangle_{\mathcal{C}_x}$, for $F = F(\eta_x)$, denotes taking the average over the cube \mathcal{C}_x . Note that the average of \hat{A}_x^N against $d\vartheta_{f_s}^N$ is $\mathcal{O}(e^{-Cs}\ell/N)$. Then

$$\begin{aligned} &\sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s} \hat{A}_x^N d\vartheta_{f_s}^N \\ &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \left(\tilde{\Phi}_{t,s} - \Pi_x^N \tilde{\Phi}_{t,s} \right) \hat{A}_x^N d\vartheta_{f_s}^N + \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \Pi_x^N \tilde{\Phi}_{t,s} \hat{A}_x^N d\vartheta_{f_s}^N \\ &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \left(\Phi_{t-s} - \Pi_x^N \Phi_{t-s} \right) \hat{A}_x^N d\vartheta_{f_s}^N \\ &\quad + \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \left(\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta_{f_s}^N} [\Pi_x^N \Phi_{t-s}] \right) \hat{A}_x^N d\vartheta_{f_s}^N =: J_t^1 + J_t^2 \end{aligned}$$

where Π_x^N projects on the local equilibrium with same mass in the cube \mathcal{C}_x (and does not touch the other site):

$$(4.3) \quad \begin{cases} \Pi_x^N \varphi(\eta) = [\Pi_x^N \varphi](\langle \eta \rangle_{\mathcal{C}_x}) = \int_{\Omega_{\langle \eta \rangle_{\mathcal{C}_x}}} \varphi(\tilde{\eta}) d\nu^{\ell, \langle \eta \rangle_{\mathcal{C}_x}}(\tilde{\eta}) \\ \Omega_m := \{ \tilde{\eta} : \langle \tilde{\eta} \rangle_{\mathcal{C}_x} = m \} \end{cases}$$

for a function φ on $\mathcal{X}^{\mathcal{C}_x}$. To estimate the first term J_t^1 we first approximate the measure $\vartheta_{f_s}^N$ on \mathcal{C}_x by the equilibrium measure with local mass $f_t(x/N)$, and denote it by $\bar{\vartheta}_{f_s}^N$ (note that the approximation is made differently for each cube and depends on x , even if it is written explicitly). This produces an error $\mathcal{O}(\ell^{d+1}/N)$ (using the Lipschitz regularity of Φ_{t-s} and the exponential convergence $f_t \rightarrow \rho$ to get uniform in time bounds). We then apply the Poincaré inequality [LSV96, Theorem 1.1] in the cube \mathcal{C}_x (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number $\|\hat{A}_x^N\|_{L^2(\bar{\vartheta}_{f_s}^N)} = \mathcal{O}(e^{-Cs}\ell^{-d/2})$:

$$\begin{aligned} J_t^1 &\leq \sum_{x \in \mathcal{R}_N^d} \int_0^t \|\Phi_{t-s} - \Pi_x^N \Phi_{t-s}\|_{L^2(\bar{\vartheta}_{f_s}^N)} \|\hat{A}_x^N\|_{L^2(\bar{\vartheta}_{f_s}^N)} ds + \mathcal{O}(\ell^{d+1}/N) \\ &\lesssim \ell^{1-d/2} \sum_{x \in \mathcal{R}_N^d} \int_0^t \sqrt{\bar{D}_x^\ell(\Phi_{t-s})} e^{-Cs} ds + \mathcal{O}(\ell^{d+1}/N) \\ &\lesssim \ell^{1-d/2} N^{d/2} \int_0^t \left(\sum_{x \in \mathcal{R}_N^d} \bar{D}_x^\ell(\Phi_{t-s}) \right)^{1/2} e^{-Cs} ds + \mathcal{O}(\ell^{d+1}/N) \end{aligned}$$

where $\overline{D}_x^\ell(\Phi)$ is the Dirichlet form on the cube \mathcal{C}_x with respect to the measure $\overline{\vartheta}_{f_s}^N$:

$$\overline{D}_x^\ell(\Phi) := \sum_{y,z \in \mathcal{C}_x} \int_{\mathcal{X}_N} p(z-y)g(\eta_y) [\Phi(\eta^{yz}) - \Phi(\eta)]^2 d\overline{\vartheta}_{f_s}^N.$$

Then we change back the measure $\overline{\vartheta}_{f_s}^N$ in each box, which produces (using the Lipschitz regularity of Φ) an error $\ell^{3/2}N^{-1/2}$, and we compute

$$\frac{1}{2N^2} \frac{d}{dt} \int_{\mathcal{X}_N} \Phi_{t-s}(\eta)^2 d\vartheta_{f_s}^N \leq - \sum_{x \in \mathcal{R}_N^d} D_x^\ell(\Phi_{t-s}) + \mathcal{O}(1/N^2)$$

(with D^ℓ denoting the Dirichlet form for $\vartheta_{f_s}^N$), where the last error accounts for the small default of self-adjointness. We deduce (in dimension $d = 1$) that

$$\int_0^T J_t^1 dt \lesssim T^{1/2} (\ell/N)^{1-d/2} + \mathcal{O}(T\ell^{d+1}/N).$$

To control the second term J_t^2 , we first use the *equivalence of ensemble* in [KL99, Appendix II, Corollary 1.7] on the measure $\overline{\vartheta}_{f_s}^N$ (together with the exponential tail estimates on the local Gibbs measure) to get

$$(4.4) \quad \langle g(\eta) \rangle_{\mathcal{C}_x} = \sigma(\langle \eta \rangle_{\mathcal{C}_x}) + \mathcal{O}(1/\ell^d).$$

Second we remark that the Lipschitz regularity of Φ_{t-s} implies that $\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta_{f_s}^N}[\Pi_x^N \Phi_{t-s}] = \mathcal{O}(\ell^d N^{-d})$, and since the average of \widehat{A}_x^N with respect to $\vartheta_{f_s}^N$ is $\mathcal{O}(\ell/N)$, we can write

$$J_t^2 = \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} (\Pi_x^N \Phi_{t-s}[\langle \eta \rangle_{\mathcal{C}_x}] - \Pi_x^N \Phi_{t-s}[f_s(x/N)]) \widehat{A}_x^N d\vartheta_{f_s}^N + \mathcal{O}(\ell/N).$$

Third, we remark that the Lipschitz regularity of Φ_{t-s} (with constant N^{-d}) implies a Lipschitz regularity of its averaged projection $\Pi_x^N \Phi_{t-s}$ with constant $\ell^d N^{-d}$, with respect to the local mass. Indeed, given $0 = m \leq m' < +\infty$, pick any pair of configuration (η_0, ζ_0) with $\langle \eta_0 \rangle_{\mathcal{C}_x} = m$, $\langle \zeta_0 \rangle_{\mathcal{C}_x} = m'$ and $\eta_0 \leq \zeta_0$ (such configuration trivially exists since $m \leq m'$). Then we consider the initial coupling $\delta_{(\eta_0, \zeta_0)}$ on $\Omega_m \times \Omega_{m'}$ which has ℓ^1 cost $m' - m$. Then we evolve it along the flow of the coupling operator $e^{t\mathcal{L}_N} \delta_{(\eta_0, \zeta_0)}$. The marginals respectively converge to $\nu^{\ell, m}$ and $\nu^{\ell, m'}$ (convergence to equilibrium of the original evolution). Since the evolution by the coupling operator does not increase the Wasserstein distance, we deduce $W_1(\nu^{\ell, m}, \nu^{\ell, m'}) \leq m' - m$. An optimal coupling Π associated to this distance thus satisfies

$$m' - m \leq \int_{\Omega_m \times \Omega_{m'}} \left(\sum_{x \in \mathbb{T}_N^d} |\eta_x - \zeta_x| \right) \Pi(\eta, \zeta) \leq m' - m$$

where the first inequality follows from Jensen's inequality. Thus the Jensen's inequality is saturated which implies that the cost does not change sign on the support of Π , i.e. $\eta \leq \zeta$ in the support. We then compute

$$\begin{aligned} \Pi_x^N \Phi_{t-s}(m') - \Pi_x^N \Phi_{t-s}(m) &= \int_{\Omega_{m'}} \Phi_{t-s}(\zeta) d\nu^{\ell, m'}(\zeta) - \int_{\Omega_m} \Phi_{t-s}(\eta) d\nu^{\ell, m}(\eta) \\ &= \int_{\Omega_m \times \Omega_{m'}} [\Phi_{t-s}(\zeta) - \Phi(\eta)] d\Pi(\eta, \zeta) \end{aligned}$$

and since $\eta \leq \zeta$ on the support of Π , $\|\zeta - \eta\|_{\ell^1(c_x)} = (m' - m)\ell^d$ and

$$|\Pi_x^N \Phi_{t-s}(m') - \Pi_x^N \Phi_{t-s}(m)| \leq \frac{\ell^d}{N^d} |m' - m|.$$

We deduce (using (5.3))

$$\begin{aligned} J_t^2 &\lesssim \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)| \times \\ &\quad |\sigma(\langle \eta \rangle_{c_x}) - \sigma(f_s(x/N)) - \sigma'(f_s(x/N)) [\langle \eta \rangle_{c_x} - f_s(x/N)]| d\vartheta_{f_s}^N e^{-Cs} ds \\ &\quad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)| d\vartheta_{f_s}^N e^{-Cs} ds + \mathcal{O}(\ell/N) \end{aligned}$$

which yields by Taylor formula, the approximation of $\vartheta_{f_s}^N$ by $\bar{\vartheta}_{f_s}^N$, and the law of large numbers

$$\begin{aligned} J_t^2 &\lesssim \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)|^3 d\vartheta_{f_s}^N e^{-Cs} ds \\ &\quad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)| d\vartheta_{f_s}^N e^{-Cs} ds + \mathcal{O}(\ell/N) e^{-Cs} ds \\ &\lesssim \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)|^3 d\bar{\vartheta}_{f_s}^N e^{-Cs} ds \\ &\quad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} |\langle \eta \rangle_{c_x} - f_s(x/N)| d\bar{\vartheta}_{f_s}^N e^{-Cs} ds + \mathcal{O}(\ell/N) \\ &\lesssim \mathcal{O}(\ell^{-3d/2}) + \mathcal{O}(\ell/N). \end{aligned}$$

Combining all estimates we get (optimizing $\ell := N^{1/4}$)

$$\frac{1}{T} \int_0^T I_t^N dt \lesssim \left(\frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{\ell^{d/2}} + \frac{\ell}{N} \right) \lesssim \frac{1}{N^{1/8}}. \quad \square$$

5. PROOF FOR THE GLK

In this section we prove Theorem 2.2 (hydrodynamic limit for the GLK). Note again that for this model $\mathcal{L}_N = \mathcal{L}_N^*$ is symmetric with respect to equilibrium measures. Given $f_t \in C^3(\mathbb{T}^d)$ and $\rho := \int_{\mathbb{T}^d} f \in \mathbb{R}$, the density of the local Gibbs measure relatively to the invariant measure with mass ρ is:

$$(5.1) \quad G_t^N(\eta) := \frac{d\vartheta_f^N(\eta)}{d\vartheta_\rho^N(\eta)} = \prod_{x \in \mathbb{T}_N^d} e^{[\sigma(f_t(x/N)) - \sigma(\rho)]\eta_x} \frac{Z(\sigma(\rho))}{Z(\sigma(f_t(x/N)))}.$$

where the function $\sigma(r)$ is defined by $\langle n_{\sigma(r)}, \eta_x \rangle = r$ and the *partition function* $Z(\lambda) = \int_{\mathbb{R}} e^{\lambda r - V(r)} dr$ is defined on \mathbb{R} . The uniform convexity of V at infinity easily implies bounds on all moments of the invariant measure

$$\int_{X_N} \sum_{x \in \mathbb{T}_N^d} \eta(x)^k d\vartheta_\rho^N(\eta) = C_k < \infty.$$

and it is known that **(HGLK)** implies that there exists $C > 0$ so that $0 < \frac{1}{C} \leq \sigma' \leq C < \infty$ (see [GOVW09, Lemma 41] and [DMOWa, Lemma 5.1]).

5.1. Microscopic stability – hypothesis (H1). We consider a coupling of two Ginzburg-Landau processes with generator $\tilde{\mathcal{L}}_N : C_b(\mathbb{X}_N^2) \rightarrow C_b(\mathbb{X}_N^2)$ given by

$$(5.2) \quad \begin{aligned} \tilde{\mathcal{L}}_N \Psi(\eta, \zeta) := & \sum_{x \sim y} \left(\left[\left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^* \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \otimes 1 \right] \Psi(\eta, \zeta) \right. \\ & + \left[1 \otimes \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right)^* \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \right] \Psi(\eta, \zeta) \\ & \left. + K \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \otimes \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \Psi(\eta, \zeta) \right) \end{aligned}$$

where $K > 0$ is a constant to be chosen later and the adjoint is taken in $L^2(d\vartheta_\rho^N)$ so

$$\begin{aligned} \hat{\mathcal{L}}_N &= \sum_{x \sim y} \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 - (V'(\eta_x) - V'(\eta_y)) \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \\ &= \sum_{x \sim y} \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^* \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right). \end{aligned}$$

Then for any $p \in (1, 2]$ there is $K = K(p) > 0$ (depending on p) so that

$$\begin{aligned} \tilde{\mathcal{L}}_N \left(\sum_{x \in \mathbb{T}_N^d} |\eta_x - \zeta_x|^p \right) &= 2p(p-1)(2+4d) \sum_{x \in \mathbb{T}_N^d} |\eta_x - \zeta_x|^{p-2} \\ &\quad - 2(p-1) \sum_{x \sim y} [V'_0(\eta_x) - V'_0(\zeta_x)] (\eta_x - \zeta_x) |\eta_x - \zeta_x|^{p-1} \\ &\quad - 2(p-1) \sum_{x \sim y} [V'_1(\eta_x) - V'_1(\zeta_x)] (\eta_x - \zeta_x) |\eta_x - \zeta_x|^{p-1} \\ &\quad + Kp(p-1)(2+4d) \sum_{x \in \mathbb{T}_N^d} |\eta_x - \zeta_x|^{p-2} \leq 0 \end{aligned}$$

by using the assumptions on the potential: V_0 uniformly strictly convex and $V_1 \in W^{1,\infty}$. This implies the weak contraction of the evolution in W_p (p -Wasserstein distance) for any $p \in (1, 2]$, and thus by limit in W_1 . By duality this implies that the evolution is weakly contractive for the dual Lipschitz norm.

5.2. Macroscopic stability - hypothesis (H2). The limit equation is a one-dimensional nonlinear diffusion equation with uniform ellipticity bounds, and standard elliptic theory shows that the solution exists globally and converges exponentially fast to a constant in $C^3(\mathbb{T}^d)$.

5.3. Consistency estimate - hypothesis (H3).

Proposition 5.1. *Given $d = 1$ and the solution $f_t \in C^3(\mathbb{T}^d)$ to (2.4), and $\rho := \int_{\mathbb{T}^d} f$, and G_t^N defined in (5.1), we have for every $\Phi \in \text{Lip}(\mathbb{X}_N)$*

$$\frac{1}{T} \int_0^T I_t^N dt := \frac{1}{T} \int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N} \Phi \right), \left[\mathcal{L}_N G_s^N - \frac{d}{ds} G_s^N \right] d\nu_\infty^N \right\rangle ds dt = \mathcal{O}(N^{-1/8})$$

where the constant depends on the estimates in **(H2)**.

Proof. The proof follows the same structure as for the ZRP. We start by computing

$$\mathcal{L}_N G_s^N - \frac{d}{ds} G_s^N = \sum_{x \in \mathbb{T}_N^d} A_x^N G_s^N$$

with (note again that $f_t \rightarrow \rho$ exponentially fast)

$$\begin{aligned} A_x^N &:= \frac{N^2}{2} \sum_{y \sim x} \left[(\sigma(f_s(x/N)) - \sigma(f_s(y/N)))^2 \right. \\ &\quad \left. - (V'(\eta_x) - V'(\eta_y)) (\sigma(f_s(x/N)) - \sigma(f_s(y/N))) \right] \\ &\quad - \sum_x (\eta_x - f_s(x/N)) \sigma'(f_s(x/N)) \Delta[\sigma(f)](x/N) \\ &= \frac{N^2}{2} \sum_{y \sim x} \left[2\sigma(f_s(x/N)) (\sigma(f_s(x/N)) - \sigma(f_s(y/N))) \right. \\ &\quad \left. - 2V'(\eta_x) \sigma(f_s(x/N)) - \sigma(f_s(y/N)) \right] \\ &\quad - \sum_x (\eta_x - f_s(x/N)) \sigma'(f_s(x/N)) \Delta[\sigma(f)](x/N) \\ &= \Delta[\sigma(f)](x/N) \left[V'(\eta_x) - \sigma(f_s(x/N)) \right. \\ &\quad \left. - \sigma'(f_s(x/N)) (\eta_x - f_s(x/N)) \right] + \mathcal{O}(e^{-Cs}/N) \end{aligned}$$

for some $C > 0$. By conservation of mass we replace again $\Phi_{t-s} := e^{(t-s)\mathcal{L}_N} \Phi$ by

$$\tilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N} \Phi - \mathbf{E}_{\vartheta_{f_s}^N} [e^{(t-s)\mathcal{L}_N} \Phi]$$

and use the Lipschitz bound **(H1)** on $e^{(t-s)\mathcal{L}_N} \Phi$ to get

$$I_t^N = \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \left(\sum_{x \in \mathbb{T}_N^d} \tilde{A}_x^N \right) d\vartheta_{f_s}^N + \mathcal{O}(1/N)$$

with \tilde{A}_x^N defined by (note that it has zero average against $d\vartheta_{f_s}^N$)

$$\tilde{A}_x^N := \Delta[\sigma(f)](x/N) [V'(\eta_x) - \sigma(f(x/N)) - \sigma'(f_s(x/N)) (\eta_x - f(x/N))].$$

We again form sub-sum over non-overlapping cubes of size $\ell \in \{1, \dots, N\}$, with $\mathcal{R}_N^d \subset \mathbb{T}_N^d$ a net of centers of cubes $\mathcal{C}_x := \{y \in \mathbb{T}_N^d : \|x - y\|_\infty \leq \ell\}$. Then

$$\begin{aligned} I_t^N &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \left(\sum_{y \in \mathcal{C}_x} \tilde{A}_y^N \right) d\vartheta_{f_s}^N + \mathcal{O}(1/N) \\ &= (2\ell + 1)^d \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \tilde{\Phi}_{t,s}(\eta) \hat{A}_x^N d\vartheta_{f_s}^N + \mathcal{O}(1/N) \end{aligned}$$

with the \hat{A}_x^N defined by (and $\langle F(\eta) \rangle_{\mathcal{C}_x}$ again denotes the average over the cube \mathcal{C}_x)

$$\hat{A}_x^N := \Delta[\sigma(f)](x/N) [\langle V'(\eta) \rangle_{\mathcal{C}_x} - \sigma(f(x/N)) - \sigma'(f_s(x/N)) (\langle \eta \rangle_{\mathcal{C}_x} - f(x/N))].$$

(Note again that the average of \widehat{A}_x^N against $d\vartheta_{f_s}^N$ is $\mathcal{O}(e^{-Cs}\ell/N)$.) Then

$$\begin{aligned}
 & \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \widetilde{\Phi}_{t,s} \widehat{A}_x^N d\vartheta_{f_s}^N \\
 &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} (\widetilde{\Phi}_{t,s} - \Pi_x^N \widetilde{\Phi}_{t,s}) \widehat{A}_x^N d\vartheta_{f_s}^N + \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} \Pi_x^N \widetilde{\Phi}_{t,s} \widehat{A}_x^N d\vartheta_{f_s}^N \\
 &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} (\Phi_{t-s} - \Pi_x^N \Phi_{t-s}) \widehat{A}_x^N d\vartheta_{f_s}^N \\
 & \quad + \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathcal{X}_N} (\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta_{f_s}^N} [\Pi_x^N \Phi_{t-s}]) \widehat{A}_x^N d\vartheta_{f_s}^N =: J_t^1 + J_t^2
 \end{aligned}$$

where Π_x^N again averages over Ω_m (and does not touch the other site) as in (4.3).

To estimate the first term J_t^1 we again approximate the measure $\vartheta_{f_s}^N$ on \mathcal{C}_x by the equilibrium measure with local mass $f_t(x/N)$, and denote it by $\bar{\vartheta}_{f_s}^N$ (note that the approximation is made differently for each cube and depends on x , even if it is written explicitly). This produces an error $\mathcal{O}(\ell^{d+1}/N)$ (using the Lipschitz regularity of Φ_{t-s} and the exponential convergence $f_t \rightarrow \rho$ to get uniform in time bounds). We then apply the Poincaré inequality [LY93, Theorem 2] in the cube \mathcal{C}_x (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number $\|\widehat{A}_x^N\|_{L^2(\bar{\vartheta}_{f_s}^N)} = \mathcal{O}(e^{-Cs}\ell^{-d/2})$:

$$\begin{aligned}
 J_t^1 &\leq \sum_{x \in \mathcal{R}_N^d} \int_0^t \|\Phi_{t-s} - \Pi_x^N \Phi_{t-s}\|_{L^2(\bar{\vartheta}_{f_s}^N)} \|\widehat{A}_x^N\|_{L^2(\bar{\vartheta}_{f_s}^N)} ds + \mathcal{O}(\ell^{d+1}/N) \\
 &\lesssim \ell^{1-\frac{d}{2}} \sum_{x \in \mathcal{R}_N^d} \int_0^t \sqrt{\overline{D}_x^\ell(\Phi_{t-s})} e^{-Cs} ds + \mathcal{O}(\ell^{d+1}/N) \\
 &\lesssim \ell^{1-\frac{d}{2}} N^{d/2} \int_0^t \left(\sum_{x \in \mathcal{R}_N^d} \overline{D}_x^\ell(\Phi_{t-s}) \right)^{1/2} e^{-Cs} ds + \mathcal{O}(\ell^{d+1}/N)
 \end{aligned}$$

where $\overline{D}_x^\ell(\Phi)$ is the Dirichlet form on the cube \mathcal{C}_x with respect to the measure $\bar{\vartheta}_{f_s}^N$:

$$\overline{D}_x^\ell(\Phi) := \sum_{y \sim z \in \mathcal{C}_x} \int_{\mathcal{X}_N} [\partial_{\eta_x} \Phi(\eta) - \partial_{\eta_y} \Phi(\eta)]^2 d\bar{\vartheta}_{f_s}^N.$$

Then we use the entropy production

$$\frac{1}{2N^2} \frac{d}{dt} \int_{\mathcal{X}_N} \Phi_{t-s}(\eta)^2 d\vartheta_{f_s}^N \leq - \sum_{x \in \mathcal{R}_N^d} D_x^\ell(\Phi_{t-s}) + \mathcal{O}(1/N^2)$$

as before to deduce that

$$\int_0^T J_t^1 dt \lesssim T^{1/2} (\ell/N)^{1-d/2} + \mathcal{O}(T\ell^{d+1}/N).$$

To control the second term J_t^2 , we first use the *equivalence of ensemble* in [LPY02, Corollary 5.3] on the measure $\bar{\vartheta}_{f_s}^N$ (together with the exponential tail estimates on the local Gibbs measure) to get

$$(5.3) \quad \langle V'(\eta) \rangle_{\mathcal{C}_x} = \sigma(\langle \eta \rangle_{\mathcal{C}_x}) + \mathcal{O}(1/\ell^d).$$

Second we remark that the Lipschitz regularity of Φ_{t-s} implies that $\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta_{f_s}^N} [\Pi_x^N \Phi_{t-s}] = \mathcal{O}(\ell^d N^{-d})$, and since the average of \widehat{A}_x^N with respect to $\vartheta_{f_s}^N$ is $\mathcal{O}(\ell/N)$, we can write

$$J_t^2 = \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{X_N} (\Pi_x^N \Phi_{t-s}[\langle \eta \rangle_{C_x}] - \Pi_x^N \Phi_{t-s}[f_s(x/N)]) \widehat{A}_x^N d\vartheta_{f_s}^N + \mathcal{O}(\ell/N).$$

Third, we prove again that the Lipschitz regularity of Φ_{t-s} (with constant N^{-d}) implies a Lipschitz regularity of its averaged projection $\Pi_x^N \Phi_{t-s}$ with constant $\ell^d N^{-d}$, with respect to the local mass. Indeed, given $0 = m < m' < +\infty$, pick any pair of configuration (η_0, ζ_0) with $\langle \eta_0 \rangle_{C_x} = m$, $\langle \zeta_0 \rangle_{C_x} = m'$ and $\eta_0 < \zeta_0$ (such configuration trivially exists since $m < m'$). Then consider the coupling on $\Omega_m \times \Omega_{m'}$ produced by a product of localised smooth distribution around δ_{η_0} and δ_{ζ_0} , so that the support only contains strictly ordered $\eta < \zeta$. Then we evolve it along the flow of the coupling operator $e^{t\tilde{\mathcal{L}}^N} \delta_{(\eta_0, \zeta_0)}$. The marginals respectively converge to $\nu^{\ell, m}$ and $\nu^{\ell, m'}$ (convergence to equilibrium of the original evolution). Arguing as for the ZRP, we deduce that $W_1(\nu^{\ell, m}, \nu^{\ell, m'}) = m' - m$, and a corresponding optimal coupling Π associated to this distance is so that the cost does not change sign on its support, i.e. $\eta \leq \zeta$ in the support. We deduce as for the ZRP that $\Pi_x^N \Phi_{t-s}$ is $\ell^d N^{-d}$ -Lipschitz.

We deduce (using (5.3)), the Taylor formula, the approximation of $\vartheta_{f_s}^N$ by $\bar{\vartheta}_{f_s}^N$, and the law of large numbers, the same estimate on J_t^2 as for the ZRP, and finally the same conclusion follows (optimizing $\ell := N^{1/4}$)

$$\frac{1}{T} \int_0^T I_t^N dt \lesssim \left(\frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{\ell^{d/2}} + \frac{\ell}{N} \right) \lesssim \frac{1}{N^{1/8}}. \quad \square$$

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