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A uniqueness result for travelling waves in the Gross-Pitaevskii equation

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Abstract

This note is a summary of a series of papers [12], [13] and [14], done in collaboration with David Chiron. In them, we establish the uniqueness of the energy minimizer at fixed large momentum for the 2 dimensional Gross-Pitaevskii equation, up to the natural invariances of the problem. The minimizer is a nonradial travelling wave with a small speed, behaving like two well separated vortices. Here, we summarize the key steps of the proof, highlighting the arguments that can be used for similar problems in other equations.

1 Introduction and presentation of the results

We consider the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t u + \Delta u - (|u|^2 - 1)u = 0 \\ |u(x)| \to 1 \text{ when } |x| \to +\infty \end{cases}$$

in dimension 2 for $u : \mathbb{R}^2_x \to \mathbb{C}$. This equation is a physical model for Bose-Enstein condensate, superfluidity and nonlinear optics, see [1], [21], [33], [34]. Because of the condition at infinity, the trivial solution will be the constant 1 instead of 0. The equation is invariant by rotation, translation and multiplication by a complex of modulus one, but there are no scaling invariances. To this equation is associated the Ginzburg-Landau energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |u|^2)^2,$$

which is formally conserved by the flow.

The Gross-Pitaevskii equation is a type of defocusing Schrödinger equation. As such, it has been shown that it is globally well posed in time (see [18], [19], [20]), for instance in the energy space. If u has finite energy, up to multiplication by a complex of modulus one, we can choose that $u \to 1$ at $+\infty$, not only in modulus.

Another quantity formally conserved by the flow is the momentum. It is the quantity $\frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(i\nabla u \overline{u}) \in \mathbb{R}^2$, but a precise definition is delicate because of the condition at $+\infty$ on u, see [11], [31]. In this note, the momentum will always be defined and used in the case that it can be written as:

$$\vec{P}(u) = (P_1(u), P_2(u)) := \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(i\nabla u(\overline{u} - 1)).$$

Since global existence in time is assured, we are interested in the long time behavior of solutions of the Gross-Pitaevskii equation. For that reason, we focus on the construction and stability of stationary and travelling waves solutions of this equation.

1.1 Previous results on the Gross-Pitaevskii equation

1.1.1 Stationary solutions: vortices

We consider the stationary version of the Gross-Pitaevskii equation, that is the problem

$$\left\{\begin{array}{l}\Delta u-(|u|^2-1)u=0\\ |u(x)|\to 1 \ \text{ when } \ |x|\to+\infty.\end{array}\right.$$

It has been shown by Brezis-Merle-Rivière in [9] that any stationary solution of finite energy is constant. However, it is possible to construct solutions with infinite energy, and they will play a role in the description of finite energy travelling waves.

We look for solution of the particular form

$$V_n(x)=\rho_n(r)e^{in\theta}$$

for $n\in\mathbb{Z}^*$ and with (r,θ) the polar coordinates of $x\in\mathbb{R}^2$. In that form, the equation reduces to the ODE problem

$$\left\{\begin{array}{l}r^2\rho_n''(r)+r\rho_n'(r)-n^2\rho_n(r)+r^2\rho_n(r)(1-\rho_n^2(r))=0\\ \rho_n(0)=0,\ \rho_n(+\infty)=1.\end{array}\right.$$

The condition at $r=0$ is chosen so that V_n is smooth at 0. Remark that $V_n=\overline{V_{-n}}$ and we can therefore focus on the case $n\geqslant 1$.

By shooting methods, solutions of this equation have been constructed.

Theorem 1.1 ([10], [27]) *For all $n\in\mathbb{Z}^*$, there exists $V_n(x)=\rho_n(r)e^{in\theta}$ such that*

$$\Delta V_n-(|V_n|^2-1)V_n=0$$

with $\rho_n(0)=0,\rho_n(+\infty)=1$.

Equivalents at all orders when $r\to 0$ and $r\to+\infty$ are also computed in [10], [27]. At infinity, the decay is algebraic, as we have $\rho_n(r)-1\sim -n^2/2r^2$ when $r\to+\infty$. As previously stated, $E(V_n)=+\infty$ because $\nabla V_n\sim\frac{in}{r}V_n\vec{e}_\theta\notin L^2(\mathbb{R}^2)$ at infinity.

For $n=1$, the function ρ_1 is strictly increasing and concave. Also, it has been recently shown in [26] that the vortices of degree ±1 are orbitally stable, and it is conjectured that higher degree vortices should be unstable.

1.1.2 The energy minimisation problem

We are interested here in constructing travelling waves solutions, that is functions satisfying

$$\left\{\begin{array}{l}(\mathrm{TW}_c)(u):=-ic\partial_{x_2}u-\Delta u-(1-|u|^2)u=0\\ |u(x)|\to 1\,\text{when}\,|x|\to+\infty\end{array}\right.$$

with finite energy. The speed is taken in the direction $\overrightarrow{e_2}$ without any loss of generality, since the Gross-Pitaevskii equation is invariant by rotation. One general method to construct such solutions is to look for minimizer of the energy at fixed momentum: in the space

$$W(\mathbb{R}^2):=\{1\}+\{v;\nabla v,\mathfrak{Re}(v)\in L^2(\mathbb{R}^2),\mathfrak{Im}(v)\in L^4(\mathbb{R}^2),\mathfrak{Re}(\nabla v)\in L^{4/3}(\mathbb{R}^2)\},$$

consider the problem

$$E_{\min}(p):=\inf\{E(v);v\in W(\mathbb{R}^2),P_2(v)=p\}.$$

The space $W(\mathbb{R}^2)$ is chosen such that E and $\vec{P}$ are well defined on it. We cannot replace $W(\mathbb{R}^2)$ with the more natural space $\{1\}+H^1(\mathbb{R}^2)$, because it has been shown by Gravejat in [23] that travelling waves cannot be in it. A consequence is that the minimum cannot be reached on $\{1\}+H^1(\mathbb{R}^2)$. The $L^p(\mathbb{R}^2)$ spaces choosen for $\mathfrak{Im}(v),\mathfrak{Re}(\nabla v)$ do not have a particular importance, other choices could have been made.

It has been shown by Bethuel-Gravejat-Saut in [5] that this problem admits a minimizer, and that this minimizer is a travelling wave.

Theorem 1.2 ([5]) *For any $p > 0$, there exists a non constant finite energy solution $u_p \in W(\mathbb{R}^2)$ to the equation $(\mathrm{TW}_{c(p)})(u) = 0$ for some speed $c(p) > 0$, such that $\vec{P}(u_p) = (0, p)$ and*

$$E(u_p) = E_{\min}(p).$$

Remark that the translation and the multiplication by a complex of modulus one does not change the energy and the momentum. That is, if $u_p \in W(\mathbb{R}^2)$ is a minimizer, then for $a \in \mathbb{R}^2, \gamma \in \mathbb{R}$, $v = u_p(. - a)e^{i\gamma} \in W(\mathbb{R}^2)$ verifies

$$E(v) = E(u_p) = E_{\min}(p), \vec{P}(v) = (0, p)$$

and is therefore also a minimizer.

An important conjecture is the uniqueness of this minimizer for all momentum up to these invariances. Our goal is to explain the proof of this uniqueness for large momentum, done in [12], [13] and [14]. The uniqueness for all momentum remains an open problem.

1.1.3 Properties of travelling waves

The study of travelling waves for the Gross-Pitaevskii equation started in the physical literature, with the works of Jones and Roberts (see [29], [30]).

We refer to the survey [8] for an overview of mathematical works on existence and properties of travelling waves of finite energy for the Gross-Pitaevskii equation in several dimensions. This includes more precise results on the minimization problem described in the previous subsection, but also construction of travelling waves by other methods, and general properties that all travelling waves must satisfied.

For instance, in [24], [25] by Gravejat, it has been shown that travelling waves of finite energy must have their speeds in $]0, \sqrt{2}[$, otherwise they are constant ($\sqrt{2}$ is the speed of sound in this problem). Recently, it has been shown by Bellazzini and Ruiz in [3] that travelling waves exists for almost all speeds in this range, but the existence for all speeds in $]0, \sqrt{2}[$ is still an open problem. Numerically, several branches of distinct travelling waves have been constructed for all subsonic speeds, see the works of Chiron-Scheid in [16].

Concerning the minimization problem, some properties of the minimizers are known in the limits $p \to 0$ (see [4]) and $p \to +\infty$. In particular, minimizers for small momentum have speeds close to $\sqrt{2}$, and the ones for large momentum have speeds close to 0. We will give some precise statement for the latter case in subsection 1.2.1.

1.2 Statement of the uniqueness result

We now state the main result of [13]. The rest of this note is devoted to explaining its proof.

Theorem 1.3 ([13], Theorem 1.4) *There exists $P_0 > 0$ such that, for $p > P_0$, if $u_1, u_2 \in W(\mathbb{R}^2)$ satisfy $p = P_2(u_1) = P_2(u_2)$ and $E(u_1) = E(u_2) = E_{\min}(p)$, then there exists $a \in \mathbb{R}^2, \gamma \in \mathbb{R}$ such that*

$$u_1 = u_2(. - a)e^{i\gamma}.$$

This is the uniqueness of the energy minimizer for fixed large momentum. That is, for any $p > 0$ large enough, there exists only one function with momentum p whose energy is $E_{\min}(p)$, up to the natural invariances of the problem. The proof of Theorem 1.3 can be decomposed in four steps, that are described in subsections 1.2.1 to 1.2.4. More details on each of these steps are given in section 2.

Let us explain the key difficulty to overcome: As we will see later on, minimizers of this problem are not radial. For problem with a radial minimizers, it is simpler, as we have very powerful tools to show the uniqueness (Cauchy theory, Wronskian...). To the best of our knowledge, there are no example of uniqueness result in this type of setting, that is without radiality or other equivalently strong properties of minimizers.

We expect Theorem 1.3 to hold for any $p > 0$ and not simply for large ones, but this seems to be a very difficult problem. There are numerical counterexamples for other nonlinearity and dimensions (see [15]), so to generalize Theorem 1.3 to all momentum, we would have in particular to understand what is special about the nonlinearity of Gross-Pitaevskii that is not true in general.

1.2.1 Properties of the minimizer when $p \to +\infty$

Theorem 1.2 tells us that the set of minimizers of the energy is non empty for any momentum $p > 0$. The goal of this section is to show a number of properties that any minimizer with a large momentum must satisfies.

The first result, proven by Bethuel-Gravejat-Saut, that is true for any momentum, is that any minimizer has to be even in x_1.

Proposition 1.4 ([5]) *For any $p > 0$, $u \in W(\mathbb{R}^2)$ satisfies $E(u) = E_{\min}(p), P_2(u) = p$, then there exists $X \in \mathbb{R}$ such that $u(. - X\vec{e}_1)$ is even in x_1.*

The proof is done by symmetrising any given minimizer around a well chosen line, and to show that the momentum does not change, but the energy strictly decreases if any change has been made to the function by this symmetrisation.

Next, using works of Bethuel-Gravejat-Saut and Sandier, we have an equivalent for $E_{\min}(p)$ when $p \to +\infty$.

Theorem 1.5 ([5], [35]) *The function $E_{\min} : \mathbb{R}^+ \to \mathbb{R}$ is concave, nondecreasing and $\sqrt{2}$-Lipschitz continuous. In addition, there exists $K > 0$ such that, for any $p \geqslant 1$, we have*

$$2\pi \ln(p) - K \leqslant E_{\min}(p) \leqslant 2\pi \ln(p) + K.$$

The proof of this result is rather involved, we refer to [5] and [35] for more details about it.

We continue. From Theorem 1.2, we know that minimizers are travelling waves. We can compute an equivalent of their speeds when the momentum goes to $+\infty$.

Proposition 1.6 ([7], [13]) *Any minimizer of the energy at fixed momentum $p > 0$ is a travelling wave of speed $c_p > 0$, and*

$$pc_p \to 2\pi$$

when $p \to +\infty$.

In particular, $c_p \to 0$ when $p \to +\infty$. Finally, still in the limit $p \to +\infty$, we can show that any minimizers has to have a specific shape.

Proposition 1.7 ([13]) *For any minimizer of the energy at fixed momentum $p > 0$ large enough, denoted u_p, there exists $d > 0$ with*

$$\frac{d}{p} \to \frac{1}{2\pi}$$

when $p \to +\infty$, such that, up to a translation,

$$|u - V_1(. - d\vec{e}_1)V_{-1}(. + d\vec{e}_1)| \to 0$$

in large balls around $\pm d\vec{e}_1$, and

$$||u| - 1| \to 0$$

outside of them when $p \to +\infty$.

This last result in not written in a very precise way here. To be exact, the properties we show for minimizers with large momentum are exactly the hypotheses of Proposition 1.10 below. The main arguments of the proof of Proposition 1.7 will be exposed in subsection 2.1.

Let us explain here what Proposition 1.7 means. First, we see that vortices are appearing in the limite $p \to +\infty$, however they have infinite energy (see subsection 1.1.1). The remark is that although indeed $E(V_1) = +\infty$ because of its behavior at $+\infty$, if we have a pair of vortices of opposite sign, their first order cancels out at $+\infty$ and the energy is finite. Naturally, when the distance between them goes to $+\infty$, so does the energy of the pair, that is

$$E(V_1(. - d\vec{e}_1)V_{-1}(. + d\vec{e}_1)) \to +\infty$$

when $d \to +\infty$, but this is consistent with the fact that $E_{\min}(p) \simeq 2\pi \ln(p)$ and $d \simeq \frac{p}{2\pi}$ when $p \to +\infty$. Also, the two vortices are multiplied and not added, because the product continue to satisfy the condition at $+\infty$.

Recall that a vortex of degree ± 1 cancels out at exactly one point, its center, that is $V_{\pm 1}(0) = 0$. Proposition 1.7 tells us that we can find two points, well separated when p is large, such that any minimizers cancels at these two points, and behaves like the vortex, a stationary solution, near them. Outside of a neighborhood of these two points, the minimizers does not cancel, and in fact its modulus is close to 1.

This description, that seems to describe rather well the shape of minimizers, is in some sense rather weak. Indeed, two functions satisfying the hypotheses of Proposition 1.7 can be very different from one another, for two main reasons.

First, we only have an equivalent of d, namely $d = p\left((1 + o_{p\to+\infty}(1))/2\pi\right)$. For that reason we can imagine that there are two (or more) family of distinct minimizers with different values of d, for instance $d_1 = \frac{p}{2\pi} + \sqrt{p}$, $d_2 = \frac{p}{2\pi} - \sqrt{p}$. In this case, $d_{1,2}/p \to 1/2\pi$ but $d_1 - d_2 = 2\sqrt{p} \to +\infty$ when $p \to +\infty$.

Secondly, outside of a vicinity of the vortices, we only have an information on the modulus of minimizers. Two of them, even if they have the same position for their vortices, can have very different nonconstant phase far from the vortices and still satisfy Proposition 1.7.

For these reasons, it is possible to construct two functions satisfying the description of Proposition 1.7 that, even up to translation and shift of phase, are very different almost everywhere.

We have shown in this subsection that any minimizers of the energy at fixed large momentum is in a particular set of function, however this set is still large.

To show Theorem 1.3 from here, we would have to show that if we choose a function satisfying the properties described above, then any other function satisfying the same hypotheses will be, up to the invariances of the problem, equal to it. This is difficult because the size of the set of functions satisfying these hypotheses is very large, and not clearly endowed with a distance. Also, we do not have a specific candidate at this stage on which we have more information than the others. The goal of the next steps is to construct one such candidate.

1.2.2 Smooth branch of travelling waves for small speeds

In this section, we construct a travelling wave that, at this point, is unrelated to the minimization problem. We have seen in subsection 1.2.1 that minimizers behaves like two well separated vortices. Here, we construct, by a Lyapunov-Schmidt reduction, a branch of travelling waves, smooth with respect to the speed, that has this behavior. That is, we look for a travelling wave as a small perturbation of the approximate solution $V_1(. - d_c\vec{e}_1)V_{-1}(. + d_c\vec{e}_1)$ for some $d_c \simeq 1/c$.

Theorem 1.8 ([12], Theorem 1.1 and [14], Proposition 1.2) *There exists $c_0 > 0$ a small constant such that for any $0 < c \leqslant c_0$, there exists a solution of* (TW_c) *of the form*

$$Q_c := V_1(. - d_c\vec{e}_1)V_{-1}(. + d_c\vec{e}_1) + \Gamma_c,$$

where $d_c = (1 + o_{c\to0}(1))/c$ is a C^1 function of c. This solution has finite energy, $Q_c \to 1$ at infinity, and

$$\|\Gamma_c\|_{W^{1,\infty}(\mathbb{R}^2)} = o_{c\to0}(1).$$

In addition,

$$c \mapsto Q_c - 1 \in C^1(]0, c_0], W^{1,\infty}(\mathbb{R}^2))$$

and

$$\frac{d}{dc}(P_2(Q_c)) = \frac{-2\pi + o_{c\to0}(1)}{c^2} < 0,$$

hence the C^1 mapping $\mathcal{P}$ $:]0, c_0] \ni c \mapsto P_2(Q_c) \in \mathbb{R}$ is a strictly decreasing diffeomorphism from $]0, c_0]$ onto $[P_2(Q_{c_0}), +\infty[$.

It has been shown, in the thesis of the author (Theorem 1.5.2 there), that $c \mapsto Q_c - 1, E(Q_c), P_2(Q_c)$ are in fact C^∞ functions on $]0, c_1]$ for some small $c_1 > 0$.

Because of the way this branch is constructed, it has no reason to be a minimizer of the energy at this point, even though it has the same shape, and same relation between the energy and momentum (we can show that $|E(Q_c) - 2\pi\ln(P_2(Q_c))| \leqslant \Lambda$ for some universal constant $\Lambda > 0$ if c is small enough). Remark also that because Q_c

is close to the product of two vortices, we can show (see [14], Proposition 1.2) that it has exactly two zeros, close to $\pm d_c\vec{e}_1$.

Since the function Γ_c is constructed by a fixed point argument, its differentiability (and the one of d_c) will result from an implicit function theorem. This differentiability, which is not known for minimizers of the energy, will be a key point in the proof of the uniqueness. We can also be more precise on the size of the error Γ_c and its smallness. More details about this, and a sketch of the proof of Theorem 1.8, is proposed in subsection 2.2.

By the invariances of the equation, we have constructed a 5 dimensional family of travelling waves:

$$Q_{\vec{c}}(.-X)e^{i\gamma}, \vec{c}, X \in \mathbb{R}^2, \gamma \in \mathbb{R}.$$

1.2.3 Coercivity of the constructed branch

We study here the linearized operator around the branch of travelling waves of Theorem 1.8, done in [14]. This is necessary to establish a local uniqueness result on this branch. As in subsection 1.2.2, this is for now a priori unrelated to the minimization problem.

We decompose $(\mathrm{TW}_c)(Q_c+\varphi) = L_{Q_c}(\varphi) + \mathrm{NL}(\varphi)$, where $L_{Q_c}(\varphi)$ contains all the linear terms in φ, and is defined by

$$L_{Q_c}(\varphi) := -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

The question of which space of perturbations we should consider the linearized operator in is rather complicated. We are going to give some details in subsection 2.3, but for now, we consider $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\pm\tilde{d}_c\vec{e}_1\}, \mathbb{C})$, where $\pm\tilde{d}_c$ are the zeros of Q_c.

We define

$$\begin{aligned} B_{Q_c}(\varphi) &:= \int_{\mathbb{R}^2} \mathfrak{Re}(L_{Q_c}(\varphi)\overline{\varphi}) \\ &= \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - c\mathfrak{Re}(i\partial_{x_2}\varphi\overline{\varphi}), \end{aligned} \tag{1.1}$$

the quadratic form around Q_c. Our goal is to show some coercivity results on it.

To do so, we do a change of variable, $\varphi = Q_c\psi$. Then, after a few integration by parts, we have

$$\begin{aligned} B_{Q_c}(Q_c\psi) &= \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 \\ &+ \int_{\mathbb{R}^2} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c}).\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi). \end{aligned}$$

In this form, the quadratic form is easy to study far from $\pm\tilde{d}_c\vec{e}_1$, the zeros of Q_c. Indeed, for $\lambda > 0$, outside of $B(\tilde{d}_c\vec{e}_1,\lambda) \cup B(-\tilde{d}_c\vec{e}_1,\lambda)$, we check, using precise estimates on Q_c from [12], that

$$|Q_c| = 1 + o_{\lambda\to 0, c\to 0}(1), |\mathfrak{Im}(\nabla Q_c\overline{Q_c})| = o_{\lambda\to 0, c\to 0}(1).$$

Therefore, if for instance $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e}_1,\lambda)\cup B(-\tilde{d}_c\vec{e}_1,\lambda)), \mathbb{C})$ for some $\lambda > 0$ large and $c > 0$ small, then by using Cauchy-Schwarz inequality on the last two terms, we have

$$B_{Q_c}(Q_c\psi) \geqslant \kappa \int_{\mathbb{R}^2} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi),$$

that is a coercivity result on B_{Q_c} far from the zeros of Q_c for a semi norm. Now, close to the zeros of Q_c, the key remark is that there, Q_c is close to the vortices $V_{\pm 1}$ when $c \to 0$, and that the coercivity of the quadratic form for one vortex has been studied by Del Pino-Felmer-Kowalczyk in [17].

Let us make a quick summary of the results in [17]. For a vortex $V_{\pm 1}$, the quadratic form is coercive, up to three local orthogonality conditions, connected to the three dimensional invariance set of the problem (the translation in $\mathbb{R}^2$ and the multiplication by a complex of modulus one).

We deduce, by perturbative arguments, that for $c > 0$ small, the quadratic form around Q_c close to one of its zero is coercive up to three directions. Gluing together the quadratic form in the three areas $(B(\tilde{d}_c\vec{e}_1, \lambda), B(-\tilde{d}_c\vec{e}_1, \lambda)$ and $\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e}_1, \lambda) \cup B(-\tilde{d}_c\vec{e}_1, \lambda)))$, we deduce that B_{Q_c} is coercive up to six orthogonality conditions.

From subsection 1.2.1, we know that minimizers are even in x_1. So if we ask φ to have this symmetry as well, the number of orthogonality required becomes three. We have shown in [14] coercivity results without it (see for instance Theorem 1.5 and Proposition 1.12 there), but we do not need them for the proof of Theorem 1.3. Let us state the coercivity result we will use.

We define the space

$$H^{\exp}_{Q_c} := \left\{\varphi \in H^1_{\text{loc}}(\mathbb{R}^2), \|\varphi\|_{H^{\exp}_{Q_c}} < +\infty\right\},$$

where for $\varphi = Q_c\psi \in H^1_{\text{loc}}(\mathbb{R}^2)$ and $\tilde{r}_d := \min(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|)$, the minimum of the distances to the vortices, we define

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} := \|\varphi\|^2_{H^1(\{\tilde{r}_d \leqslant 10\})} + \int_{\{\tilde{r}_d \geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}\ln^2(\tilde{r})}.$$

Remark that

$$\int_{\{\tilde{r}_d \geqslant 5\}} \frac{|\psi|^2}{\tilde{r}\ln^2(\tilde{r})} \leqslant K\left(\int_{\{\tilde{r}_d \geqslant 5\}} |\nabla\psi|^2 + \|\varphi\|^2_{H^1(\{\tilde{r}_d \leqslant 10\})}\right),$$

and we add it so that $\|.\|_{H^{\exp}_{Q_c}}$ is now clearly a norm and not just a semi norm. We will come back in subsection 2.3 on why this is a good space to do the coercivity. We also define the symmetric functions in this space by

$$H^{\exp,s}_{Q_c} := \{\varphi \in H^{\exp}_{Q_c}, \forall(x_1, x_2) \in \mathbb{R}^2, \varphi(x_1, x_2) = \varphi(-x_1, x_2)\}.$$

Theorem 1.9 ([14], Theorem 1.13) *There exists $R, K, c_0 > 0$ such that, for $0 < c \leqslant c_0$, if a function $\varphi \in H^{\exp,s}_{Q_c}$ satisfies the three orthogonality conditions*

$$\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \mathfrak{Re}(\partial_c Q_c\overline{\varphi}) = \int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \mathfrak{Re}(\partial_{x_2} Q_c\overline{\varphi}) = 0,$$

$$\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \mathfrak{Re}(iQ_c\overline{\varphi}) = 0,$$

then

$$\frac{1}{K}\|\varphi\|^2_{H^{\exp}_{Q_c}} \geqslant B_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We will explain in subsection 2.3 why we have to consider these three specific directions for the coercivity, and why the orthogonalities are local. There, we will also give more details about spectral properties of L_{Q_c}, that are interesting in themselves, but not necessary for the proof of Theorem 1.3. We will also discuss the fact that B_{Q_c}, given by (1.1), is not well defined for $\varphi \in H^{\exp}_{Q_c}$.

The three orthogonality conditions are connected to the change of speed for the one with $\partial_c Q_c$, to the translation for $\partial_{x_2} Q_c$, and to the shift of phase for iQ_c.

In general, a coercivity result implies the local uniqueness of the branch in some space. This is done without symmetry in [14], Theorem 1.14 for instance. However, to show Theorem 1.3, we are going to need a uniqueness result in a given space of functions, which is very large, but with a specific shape.

1.2.4 Uniqueness result on the branch

In the previous two subsections, we constructed and studied a specific candidate to be the unique minimizer of the energy at fixed momentum. The choice is non intuitive, in the sense that we do not know at this point that the candidate Q_c is a minimizers, but we accept to loose this property, and to have instead information like the coercivity and the smoothness with respect to the speed/momentum, that we do not know on minimizers.

The uniqueness theorem we show is as follows.

Proposition 1.10 ([13], Proposition 1.8) *There exists $\lambda_* > 1$ such that, for any $\lambda \geqslant \lambda_*$, there exists $\varepsilon(\lambda) > 0$ such that if a function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with $E(u) < +\infty$ satisfies*

1. *u is even in x_1,*
2. *$u = V_1(x - d\vec{e}_1)V_{-1}(x + d\vec{e}_1) + \Gamma$, with $d > \frac{1}{\varepsilon(\lambda)}$, $\|\Gamma\|_{L^\infty(\{\tilde{r}_d \leqslant 2\lambda\})} \leqslant \varepsilon(\lambda)$,*
3. *$\||u| - 1\|_{L^\infty(\{\tilde{r}_d \geqslant \lambda\})} \leqslant 1/\lambda_*$,*
4. *$(\mathrm{TW}_c)(u) = 0$ for some $c > 0$ such that $|dc - 1| \leqslant \varepsilon(\lambda)$,*

then there exist $X \in \mathbb{R}^2, \gamma \in \mathbb{R}$ such that $u = Q_c(. - X\vec{e}_2)e^{i\gamma}$.

This proposition, combined with subsection 1.2.1, implies Theorem 1.3. Proposition 1.10 can be seen as a local uniqueness result on Q_c among travelling waves, and in fact it implies the local uniqueness in L^∞ for even functions in x_1:

Corollary 1.11 ([13], Corollary 1.10) *There exists $c_0, \varepsilon > 0$ such that, for $0 < c < c_0$, if a function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$ with $E(u) < +\infty$ satisfies*

1. *u is even in x_1*
2. *$(\mathrm{TW}_c)(u) = 0$ in the distributional sense*
3. *$\|u - Q_c\|_{L^\infty(\mathbb{R}^2)} \leqslant \varepsilon$,*

then there exist $X, \gamma \in \mathbb{R}$ such that $u = Q_c(. - X\vec{e}_2)e^{i\gamma}$.

However Proposition 1.10 is much stronger than this corollary, because it applied to some functions u that can satisfy $\|u - Q_c\|_{L^\infty(\mathbb{R}^2)} \geqslant 1/2$, as long as u has a two vortex structure, even if the distance between them is not the same as the vortices of Q_c.

Let us sketch the proof of Proposition 1.10. Consider u satisfying the hypotheses of the proposition. Then, choose $c' > 0$ such that $Q_{c'}$ has the same position of the vortices than u. This uses the fact that $c \to Q_c$ is smooth, and thus given two points far away from each other, there exists an element of this family composed of two vortices at these points. Remark that we do not have necessary that $c = c'$ at this stage, but $(c - c')/c$ is small if p is large.

Near the vortices, write $u = Q_{c'} + \varphi$, and because both u and $Q_{c'}$ behaves like the same vortices, we have that φ is small there, in any H^s norm by standard elliptic arguments. By modulation, choose (c'', X, γ) close to $(c', 0, 0)$ such that if we consider $Q := Q_{c''}(. - X\vec{e}_2)e^{i\gamma}$ instead of $Q_{c'}$, then φ satisfies the three orthogonality conditions of Theorem 1.9. Using this coercivity result, and the fact that locally φ is small in any norm, this part will not be a concern for the uniqueness result.

The difficult part is to understand the error far from the vortices. There, we only have $|u| \simeq 1 \simeq |Q|$, and not $u \simeq Q$, so φ is not small a priori. To solve this issue, we use the fact that both u and Q have modulus close to one to write the error in the form

$$u = Qe^{\psi}$$

instead of $u = Q + \varphi$. In this form, the condition $|u| \simeq 1 \simeq |Q|$ tells us that $\mathfrak{Re}(\psi)$ is small in L^∞, but $\mathfrak{Im}(\psi)$ can be very large (at this stage we can simply show that it is in L^∞, without any uniform bound on it).

At the linear level, writing the perturbation on this form does not change anything, and the coercivity still holds. However, we have change the shape of the nonlinear terms. The goal is to show that now, in this form, they can be estimated by the coercivity norm on ψ and $\mathfrak{Re}(\psi)$ in L^∞, without never needing to control $\mathfrak{Im}(\psi)$ in L^∞ at any point. This is done by doing integration by parts in any term where $\mathfrak{Im}(\psi)$ appears, since $\mathfrak{Im}(\nabla\psi)$ is part of the coercivity norm. This will require infinitely many integration by parts, more detailed are given in subsection 2.4. The fact that such a computation is possible relies on the structure of the linear Schrödinger equation for functions that can be written as an exponential, and the fact that the nonlinearity is of the form $F(|u|)u$.

1.2.5 Orbital stability and other properties

We know state a few corollaries of Theorem 1.3. Consider the semi distance

$$D_0(u,v) := \|\nabla u - \nabla v\|_{L^2(\mathbb{R}^2)} + \||u| - |v|\|_{L^2(\mathbb{R}^2)}.$$

Theorem 1.12 ([11], [13], Theorem 1.13) *There exists $p_0 > 0$ such that, for any $p \geqslant p_0$, the set $S_p = \{Q_{\mathcal{P}^{-1}(p)}(. - X)e^{i\gamma}, X \in \mathbb{R}^2, \gamma \in \mathbb{R}\}$ is orbitally stable for the semi-distance D_0.*

This is a consequence of the work of Chiron-Maris [11], where it has been shown that for any momentum, the set S_p of minimizers is orbitally stable for D_0. With Theorem 1.3, we can describe exactly what is this set for p large.

Next, we show that Q_c is a minimizer for another problem as well.

Theorem 1.13 ([11], [13], section 1.4) *Consider for $\kappa > 0$ the variational problem*

$$I_{\min}(\kappa) := \inf\left\{\frac{1}{4}\int_{\mathbb{R}^2}(1-|v|^2)^2 dx - P_2(v), v \text{ is s.t. } \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2 = \kappa\right\}.$$

Then, there exists $\kappa_0 > 0$ such that any minimizers for $\kappa \geqslant \kappa_0$ is of the form $Q_c(. - X)e^{i\gamma}$ for some $X \in \mathbb{R}^2, \gamma \in \mathbb{R}$.

This is also a consequence of [11], where it has been shown that there exists minimizers of the problem $I_{\min}(\kappa)$ for any $\kappa > 0$, and furthermore, that if $p \mapsto E_{\min}(p)$ is smooth, then minimizers of both problems coincides. This last part is a consequence of Theorems 1.3 and 1.8.

Finally, we can show that there are no travelling waves almost minimizing the energy that are even in x_1 other than Q_c.

Theorem 1.14 ([13], Theorem 1.11) *For any $\Lambda_0 > 0$, there exists $p_0(\Lambda_0) > 0$ such that, if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with $E(u) < +\infty$ satisfies*

1. *u is even in x_1,*
2. *$(\mathrm{TW}_c)(u) = 0$ for some $c > 0$,*
3. *$P_2(u) \geqslant p_0(\Lambda_0)$,*
4. *$E(u) \leqslant 2\pi \ln(P_2(u)) + \Lambda_0$,*

then there exists $X \in \mathbb{R}, \gamma \in \mathbb{R}$ such that

$$u = Q_c(. - X\vec{e}_2)e^{i\gamma}.$$

This result, which is stronger than Theorem 1.3, is also a consequence of subsection 1.2.1 and Proposition 1.10.

1.2.6 Summary

We summarize here, in a generic way, the key steps of the proof of the uniqueness.

Consider a minimization problem under a constraint. Suppose that, in some limit of some parameter of the constraint, you have information on what minimizers should look like. In particular, that they converges, in a weak sense, to some limit profile. This limit profile in itself is known, and in particular it is possible to show some coercivity result on its linearized operator.

The case of the Gross-Pitaevskii is kind of degenerate in this regard. When the moment is large, it does not really converges to some nice limit object (the vortex here), it converges to a pair of them, and they are separated by a distance that is diverging. However the idea is still the same.

To show the uniqueness of this hypothetical minimization problem, the following approach might work. Construct a smooth family of functions close to the limit profile, depending on some parameter connected to the constraint, by perturbative arguments. Since the limit profile is coercive, this is just a matter of finding the right space in which to do a fixed point. Then, since the limit profile is coercive and the family of functions is close to it in a strong sense, they will also be coercive, up to some determinable orthogonality conditions.

This coercivity naturally implies local uniqueness in some space. Now, to finish the proof of the uniqueness of minimizers, this space has to include all the profile that looks weakly like the limit profile. There are now two angle of attack to do so. Either improve the properties known of minimizers, or increase the size of the space in which the local uniqueness is proved.

Of course, this summary is very schematic and actual proofs could be much more involved. However, we believe that this approach can be applied to other problems, not necessarily dispersive.

2 Some additional elements of the proof

In this section, we explain some of the more technical parts of the proof.

2.1 About the limit $p \to +\infty$ in the energy minimisation

In this subsection, we discuss the proof of Proposition 1.7. The arguments given here are not fully rigorous, simply an overview of the steps of the proof. We refer to section 3 of [13] for the full details.

We consider $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with $E(u) < +\infty$ and $P_2(u) = p \gg 1$. We suppose that u is an energy minimizer, or at least has an energy close to the minimum, that is

$$E(u) \simeq 2\pi \ln(p).$$

Our goal is to show that u satisfies the properties of Proposition 1.7. We consider the rescaled function $\widehat{u}(x) = u(px)$, so that $P_2(\widehat{u}) = 1$.

First, using arguments from Alberti-Baldo-Orlandi and Jerrard-Soner [2], [28], we show that the Jacobian of $\widehat{u}$ converges to a sum of diracs when $p \to +\infty$, that is to $\sum d_k \delta_{y_k}$ for some $d_k \in \mathbb{Z}, y_k \in \mathbb{R}^2$. This sum can be empty. Furthermore, a vicinity of these points y_k contains an energy (at the level of $E(u)$) of size $\simeq \sum |d_k| \pi \ln(p)$.

Since $\sum d_k = 0$ (otherwise $E(u) = +\infty$) and $E(u) \simeq 2\pi \ln(p)$, that means that either this sum is empty, or contains two elements, $d_+ = -d_- = 1$.

Now, using arguments from Bethuel-Saut [7], we have that

$$\left| P_2(\widehat{u}) - \pi \sum d_k (y_k)_1 \right| \to 0$$

when $p \to +\infty$, where $(y_k)_1$ is the first component in cartesian coordinates of y_k. Since $P_2(\widehat{u}) = 1 \neq 0$, this implies in particular that the sum cannot be empty, it has therefore two elements, and $\pi((y_+)_1 - (y_-)_1) = 1 + o_{p \to +\infty}(1)$, which will give us the equivalent of the distance d between the two vortices.

Remark that, at first order, all the energy of $E(u) \simeq 2\pi \ln(p)$ is contains in a vicinity of y_+, y_- by the arguments of [2], [28]. Following now ideas from Bethuel-Orlandi-Smets [6], we show that if $||\widehat{u}| - 1| \geqslant \varepsilon$ at some point for some $\varepsilon > 0$, then a vicinity of this point has an energy of size $\eta(\varepsilon) \ln(p)$. Since no more energy is available, this implies that $|\widehat{u}|$ is close to 1 everywhere, except close to $y_\pm$.

Finally, near $y_\pm$, we have that $u \to u_\infty$ on $\mathbb{R}^2$ when $p \to +\infty$, where by passing to the limit, we check that $\Delta u_\infty = (|u_\infty|^2 - 1)u_\infty, |u_\infty| \to 1$ at infinity and $0 \neq \int_{\mathbb{R}^2} (|u_\infty|^2 - 1)^2 \leqslant 4\pi$. Using [9], stationary solutions of the Gross-Pitaevskii equation satisfies $\int_{\mathbb{R}^2} (|u_\infty|^2 - 1)^2 = 2\pi(\deg(u_\infty))^2$, where $\deg(u_\infty)$ is the degree of the solution at $+\infty$. Therefore, $\deg(u_\infty) = \pm 1$, and by a result from Mironescu [32], the only stationnary solution of degree ± 1 is $V_{\pm 1}$.

2.2 About the construction of a smooth branch

We detail here the proof of Theorem 1.8. Arguments in this subsection comes from [12]. Define for some $d \simeq 1/c$ the profile

$$V(x) := V_1(. - d\vec{e}_1)V_{-1}(. + d\vec{e}_1).$$

We do not have $(\mathrm{TW}_c)(V) = 0$, however this quantity is small. We look for a solution of $(\mathrm{TW}_c)(Q_c) = 0$ of the form (with $\Phi = V\Psi$)

$$Q_c = (1-\eta)(V + \Phi) + \eta V e^{\Psi},$$

where η is a cutoff function with value 0 near $\pm d\vec{e}_1$ and 1 far from them. We then decompose the equation $(\mathrm{TW}_c)(Q_c) = 0$ into

$$(\mathrm{TW}_c)(V) + L_V(\Psi) + \mathrm{NL}(\Psi) = 0,$$

where $L_V(\Psi)$ contains the linear terms in Ψ, and $\mathrm{NL}(\Psi)$ the nonlinear ones.

The goal is to inverse here the operator L_V, so that a fixed point can be made on the problem

$$\Psi = L_V^{-1}(-(\mathrm{TW}_c)(V) - \mathrm{NL}(\Psi))$$

on functions Ψ that are small, so $\mathrm{NL}(\Psi)$ is small compared to Ψ. Since we only want to construct a particular solution, we can impose what we want on Ψ. In particular, we ask Ψ to satisfies the symmetries $\Psi(x_1, x_2) = \Psi(-x_1, x_2)$ and $\Psi(x_1, -x_2) = \overline{\Psi}(x_1, x_2)$ (these symmetries are also satisfied by V).

We now look at the operator L_V. close to $\pm d\vec{e}_1$, it behaves like the linear operator around $V_{\pm 1}$, that we write $L_{V_{\pm 1}}$. This operator is studied in [17]. It is invertible, up to two orthogonality conditions.

Far from both vortices, $|V| \simeq 1$, and in that case, L_V has been studied in another context by Gravejat in [22]. In that region, we have

$$L_V(\psi) \simeq -ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi)$$

which can be understood using Fourier transform. We can show that it is invertible there without any orthogonality conditions.

For now, we have not detailed in which space we are inverting the operator. Our goal is to have a precise description of the error term Ψ, and for that reason we take a very precise norm. Define $\tilde{r} = \min(|x - d\vec{e}_1|, |x + d\vec{e}_1|)$, the minimum of the distance to the two vortices. We will invert L_V for function $\Psi = \Psi_1 + i\Psi_2$ in the space associated to the norm

$$\begin{aligned}
\|\Psi\|_{*,\sigma,d} \quad &:= \quad \|V\Psi\|_{C^2(\{\tilde{r}\leqslant 3\})} \\
&+ \quad \|\tilde{r}^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\Psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\
&+ \quad \|\tilde{r}^{\sigma}\Psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\Psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})}
\end{aligned}$$

for any $\sigma \in]0,1[$. Let us explain why we choose such a space.

At the end of the proof, we will show that there exists $K(\sigma) > 0$ and Ψ such that Q_c is indeed a travelling wave, and

$$\|\Psi\|_{*,\sigma,d} \leqslant K(\sigma)c^{1-\sigma}$$

for any $\sigma \in]0,1[$.

First of all, having weights in $\tilde{r}$ is necessary if we want the constant $K(\sigma)$ to be independent of d, which is necessary for some computations to come.

Secondly, some equivalents at $+\infty$ have been computed for any travelling wave by Gravejat in [23]. If we compare it to the norm $\|.\|_{*,\sigma,d}$, it corresponds (for all the terms) to the case $\sigma = 1$. We therefore cannot expect to do better, and in fact we have to do a little worse to have some smallness to close the fix point argument. The other limit, that is $\sigma = 0^+$, is interested because we then have a lot of decay in L^∞ on Ψ, in particular if $\tilde{r} \simeq 1$.

Unfortunately, the choice of this norm makes the computations necessary to show that L_V is invertible rather technical. Indeed, we have to to elliptic estimates (for instance to invert the problem $-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = h$) in weighted L^∞ spaces, where the weight is $\tilde{r}$, which is not radial, and we want to show that the constants coming from the computations are independent of d, the distance between the vortices. In some sense this is more technical than difficult, but it requires an heavy amount of computations.

To finish the inversion of L_V, we have to deal with the four directions of its kernel. By the symmetries imposed on Ψ, only one of these directions survives. This is done by introducing a Lagrangian multiplier, that will be equal to 0 by choosing a specific value of the parameter d, that remained free up to this point.

We finish by saying a few words concerning the differentiability of the branch with respect to the speed. Since Ψ is defined by a fixed point and d by an implicit equation, we can show that they are differentiable with respect to the speed by an implicit function theorem. There are two main difficulties to do so. First, the norm $\|.\|_{*,\sigma,d}$ depends on d, and we have to show that if we fix some $d_* > 0$ large, the norms $\|.\|_{*,\sigma,d_*}$ and $\|.\|_{*,\sigma,d}$ are equivalents with constants independent of d, d_* if d is close to d_*. And secondly, this requires additional, more precise elliptic estimates, that are quite technical (see for instance the painful Proposition 4.5 of [12]).

2.3 About the coercivity of the branch

The Gross-Pitaevskii equation is invariant by translation and multiplication by a complex of modulus one. As such, the linearized operator around the travelling wave Q_c, that is

$$L_{Q_c}(\varphi) = -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c,$$

has three zeros, namely

$$L_{Q_c}(\partial_{x_1}Q_c) = L_{Q_c}(\partial_{x_2}Q_c) = L_{Q_c}(iQ_c) = 0.$$

Remark that, although $\partial_{x_1}Q_c, \partial_{x_2}Q_c$ are decaying at infinity, we have $iQ_c \to i$ at infinity, therefore $iQ_c \notin L^p(\mathbb{R}^2)$ for any $p \in [1, +\infty[$.

The natural definition for the quadratic form associated to L_{Q_c} would be

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - c\mathfrak{Re}(i\partial_{x_2}\varphi\overline{\varphi}),$$

and a natural space in which we should consider the function φ would be one such that the four quantities are integrable, and we can check (this is done in [14]) that such a space is

$$H_{Q_c} := \left\{\varphi \in H^1_{\text{loc}}(\mathbb{R}^2), \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |(1-|Q_c|^2)||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi) < +\infty\right\}.$$

Indeed, after some integration by parts, we show that $\int_{\mathbb{R}^2} \mathfrak{Re}(i\partial_{x_2}\varphi\overline{\varphi})$ is well defined in this space.

However, $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H_{Q_c}$ but $iQ_c \notin H_{Q_c}$ because of the slow decay of $|(1-|Q_c|^2)| \simeq 1/(1+r^2)$ at infinity. Having an element of the kernel not in the space of perturbation is generally a bad idea. For instance, because of this, the coercivity of B_{Q_c} in the space H_{Q_c} with its natural norm cannot hold, even with any number of local orthogonality conditions (see section 3.1 of [14]).

As explained in 1.2.3, if we write $\varphi = Q_c\psi$, compactly supported away from the zeros of Q_c, then by integrations by parts we have

$$\begin{aligned} B_{Q_c}(Q_c\psi) &= \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 \\ &+ \int_{\mathbb{R}^2} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c}) \cdot \mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi). \end{aligned}$$

Now remark here that for $\varphi = iQ_c$, that is $\psi = i$, all terms, even in absolute value, are now well defined (and they are all zero). These integration by parts created a cancellation at infinity. However, this change of variable is not good near the zeros of Q_c. We therefore take a cutoff function supported close to the vortices, decompose the quadratic form using this cutoff, and do the integration by parts only on part supported away from the vortices. This leads to the formulation (1.4) of [14] for the quadratic form, for which each term is integrable in the space

$$H^{\exp}_{Q_c} = \left\{\varphi \in H^1_{\text{loc}}(\mathbb{R}^2), \|\varphi\|_{H^{\exp}_{Q_c}} < +\infty\right\},$$

where for $\varphi = Q_c\psi \in H^1_{\text{loc}}(\mathbb{R}^2)$ and $\tilde{r}_d := \min(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|)$, the minimum of the distances to the vortices, we have

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} = \|\varphi\|^2_{H^1(\{\tilde{r}_d \leqslant 10\})} + \int_{\{\tilde{r}_d \geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}\ln^2(\tilde{r})}.$$

We check easily that $H_{Q_c} \subset H^{\exp}_{Q_c}$ and $iQ_c \in H^{\exp}_{Q_c}$.

Now, in addition to the three zeros $\partial_{x_1}Q_c, \partial_{x_2}Q_c$ and iQ_c of the linearized operator, there are two other interesting directions to consider. These are $\partial_c Q_c$ and $\partial_{c^\perp}Q_c := x^\perp.\nabla Q_c$, connected respectively to the variation of the speed and the invariance of Gross-Pitaevskii by rotation (that are both also in $H^{\exp}_{Q_c}$). We compute that

$$L_{Q_c}(\partial_c Q_c) = i\partial_{x_2}Q_c, L_{Q_c}(\partial_{c^\perp}Q_c) = -ic\partial_{x_1}Q_c,$$

and

$$B_{Q_c}(\partial_c Q_c) = \partial_c(P_2(Q_c)) = \frac{-2\pi + o_{c\to 0}(1)}{c^2} < 0,$$
$$B_{Q_c}(\partial_{c^\perp}Q_c) = cP_2(Q_c) = 2\pi + o_{c\to 0}(1) > 0.$$

We recall, following subsection 1.2.3, that the quadratic form B_{Q_c} is coercive, up to six orthogonality conditions. These are the two translations and the shift of phase of each vortices, that is the six directions

$$\partial_{x_1}V_1(x - d_c\vec{e}_1), \partial_{x_2}V_1(x - d_c\vec{e}_1), iV_1(x - d_c\vec{e}_1)$$

and

$$\partial_{x_1}V_{-1}(x + d_c\vec{e}_1), \partial_{x_2}V_{-1}(x + d_c\vec{e}_1), iV_{-1}(x + d_c\vec{e}_1)$$

Adapting results from [17] (see Proposition 1.3 of [14]), we check that we can take these orthogonality conditions locally around each respective vortices.

Near the vortices, we have

$$\partial_{x_1}Q_c = \partial_{x_1}V_1(x - d_c\vec{e}_1) + o_{c\to 0}(1), c^2\partial_c Q_c = -\partial_{x_1}V_1(x - d_c\vec{e}_1) + o_{c\to 0}(1)$$

and

$$\partial_{x_2}Q_c = \partial_{x_2}V_1(x - d_c\vec{e}_1) + o_{c\to 0}(1), c\partial_{c^\perp}Q_c = -\partial_{x_2}V_1(x - d_c\vec{e}_1) + o_{c\to 0}(1)$$

near $d_c\vec{e}_1$. Similar estimates can be made near $-d_c\vec{e}_1$. Since they are almost equal, we check that we can replace the local orthogonality on $\partial_{x_1}V_1, \partial_{x_2}V_1, \partial_{x_1}V_{-1}, \partial_{x_2}V_{-1}$ by $\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c$.

When $c \to 0$, the four directions are almost zeros of the quadratic form. However, with the new decomposition, we can say which ones are positive or negative. Indeed,

$$B_{Q_c}(\partial_{x_1}Q_c) = B_{Q_c}(\partial_{x_2}Q_c) = 0, \quad B_{Q_c}(\partial_c Q_c) < 0, B_{Q_c}(\partial_{c^\perp}Q_c) > 0.$$

Now, concerning the phases, the situation is a little more delicate. We need the two local orthogonality conditions on $iV_1(x - d_c\vec{e}_1)$ and $iV_{-1}(x + d_c\vec{e}_1)$, but we only have iQ_c as a direction. In some sense, we are missing a second direction on the phase.

A first way to solve this, which is enough for the proof of Theorem 1.3, is to consider only the coercivity on functions even in x_1. In that case, the two orthogonality conditions on the phase coincides.

It is also possible to have a coercivity result by removing one of the two orthogonality, but the cost is that the coercivity constant now depends on c, and can converge to 0 when $c \to 0$. We refer to [14], section 1.3 for more details about this case.

Let us conclude with some corollaries of the coercivity.

Corollary 2.1 ([14]) *For $c > 0$ small enough,*

$$\mathrm{Ker}_{H^{\exp}_{Q_c}}(L_{Q_c}) = \mathrm{Span}_{\mathbb{R}}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, iQ_c\}.$$

Furthermore, for $\varphi \in H^1(\mathbb{R}^2, \mathbb{C})$, if $\langle \varphi, i\partial_{x_2}Q_c\rangle = 0$, then

$$B_{Q_c}(\varphi) \geqslant 0.$$

Finally, $L_{Q_c} : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ has exactly one negative eigenvalue with eigenvector in $L^2(\mathbb{R}^2)$.

Remark that all these properties can be stated without needing Q_c to be part of a smooth branch of travelling waves, or have a small speed. We believe that the uniqueness of the energy minimizer hold for any momentum, and that Proposition 2.1 holds for any of these minimizers.

2.4 About the uniqueness result

We complete here the proof of Proposition 1.10. Decompose $u = Qe^{\psi}$ far from the vortices (we explained in 1.2.4 how to deal with the vicinity of vortices). Then, the equation $(\mathrm{TW}_c)(u) = 0$ becomes

$$0 = (\mathrm{TW}_c)(Q) + L_Q(Q\psi) + \mathrm{NL}(\psi),$$

where $L_Q(Q\psi)$ contains the linear terms in ψ, and

$$\mathrm{NL}(\psi) = -Q\nabla\psi.\nabla\psi + Q|Q|^2(e^{2\mathfrak{Re}(\psi)} - 1 - 2\mathfrak{Re}(\psi)).$$

Taking the scalar product with $Q\psi$ gives us

$$0 = \langle(\mathrm{TW}_c)(Q), Q\psi\rangle + \langle L_Q(Q\psi), Q\psi\rangle + \langle \mathrm{NL}(\psi), Q\psi\rangle.$$

We recall the coercivity norm (far from the zeros of Q)

$$\|\psi\|^2 = \|\mathfrak{Re}^2(\psi)\|^2_{L^2} + \|\nabla\psi\|^2_{L^2} + \left\|\frac{\psi}{(1+\tilde{r})^{3/2}}\right\|^2_{L^2},$$

and from section 1.2.3, we have

$$\langle L_Q(Q\psi), Q\psi\rangle \geqslant K\|\psi\|^2$$

by a good choice of the parameters in Q.

Now, with the hypotheses of Proposition 1.10, we can show that $\|\psi\|, \|\mathfrak{Im}(\psi)\|_{L^\infty} < +\infty$ but not that they are small. We simply have $\|\mathfrak{Re}(\psi)\|_{L^\infty} \leqslant \varepsilon$.

By doing a dichotomy on the size of $\|\psi\|$, we can check that we always have

$$|\langle(\mathrm{TW}_c)(Q), Q\psi\rangle| \leqslant o(1)\|\psi\|^2.$$

To control $\langle \mathrm{NL}(\psi), Q\psi\rangle$ by $o(1)\|\psi\|^2$ (the $o(1)$ is for $c \to 0$ and $\varepsilon \to 0$), we check that it is possible for all terms except

$$A_0 := \int_{\mathbb{R}^2} |Q|^2 \mathfrak{Im}(\nabla\psi.\nabla\psi)\mathfrak{Im}(\psi).$$

Define more generally

$$A_n := \int_{\mathbb{R}^2} |Q|^2 \mathfrak{Re}^n(\psi)\mathfrak{Im}(\nabla\psi.\nabla\psi)\mathfrak{Im}(\psi).$$

We have $\mathfrak{Im}(\nabla\psi.\nabla\psi) = 2\mathfrak{Im}(\nabla\psi).\mathfrak{Re}(\nabla\psi)$ and doing an integration by parts on $\mathfrak{Re}(\nabla\psi)$ in A_n, we check that we can control every term by $o(1)\|\psi\|^2$, except for the term that contains $\mathfrak{Im}(\Delta\psi)$. For this term, we use the equation $(\mathrm{TW}_c)(Qe^{\psi}) = 0$, which also contains the term $\Delta\psi$, to rewrite $\mathfrak{Im}(\Delta\psi)$ as a sum of a large number of terms, that can all be controlled by $o(1)\|\psi\|^2$ except one, but it turns out that this term is $2A_{n+1}$. To be specific, we check that there exists a universal constant $K > 0$ such that

$$|A_n| \leqslant 2|A_{n+1}| + K^n\|\mathfrak{Re}(\psi)\|^n_{L^\infty}\|\psi\|^2,$$

and $2^n A_n \to 0$ when $n \to +\infty$ by dominated convergence. Therefore, letting n, the number of integration by parts we do, go to $+\infty$, We deduce that A_0 can be controlled by $o(1)\|\psi\|^2$, concluding the proof.

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